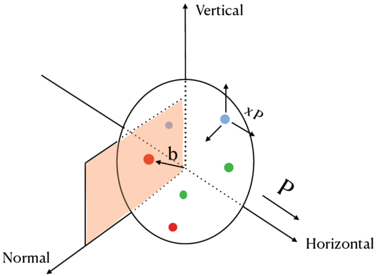
MSTT factorization: Bridging Large-x and Small-x

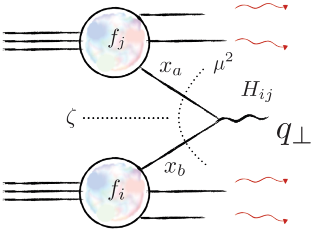
Shaswat Tiwari

North Carolina State University

Based on: S. Mukherjee, V. V. Skokov, A. Tarasov, S. T., "Unified description of DGLAP, CSS, & BFKL..."
Phys. Rev. D 109 (3) (2024) 034035

Motivation





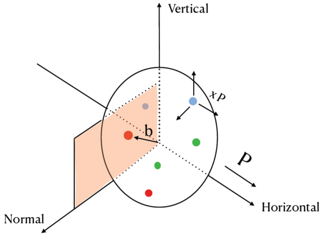
- TMD factorization connects TMDPDF(x, $b_\perp$) to cross-sections
- At large-x : collinear matching → **DGLAP** and **CSS**
- At small-x : eikonal expansion → **BFKL**
- **Understanding the general structure of TMDPDFs (large-x logs and small-x logs)**

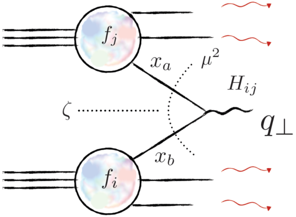
Unified description of DGLAP, CSS, and BFKL evolution: TMD factorization bridging large and small x

Swagato Mukherjee[1,*] Vladimir V. Skokov[2,†] Andrey Tarasov[2,3,‡] and Shaswat Tiwari[2,§]

- A general framework which gives the full UV and IR picture of TMDs
- Based on the background field method
- Reduces to the collinear matching formula and the eikonal expansion formula in the appropriate limits

TMD factorization





$$\Lambda^2_{QCD} << q^2_\perp << Q^2 \implies b_\perp \leq \Lambda^{-1}_{QCD}(\textbf{Large})$$

$$\frac{d\sigma}{dQdyd^2q_\perp} = H(Q,\mu) \int d^2b_\perp\, e^{iq_\perp.b_\perp}\, \mathrm{TMDPDF}_1(x_a, b_\perp, \mu, \zeta_a)\, \mathrm{TMDPDF}_2(x_b, b_\perp, \mu, \zeta_b) + \mathcal{O}(\frac{q^2_\perp}{Q^2})$$

Collins(2011)
TMD handbook(2304.03302)

Collinear matching

- **Complicated** dependence of $\text{TMDPDF}(x, b_\perp, \mu, \zeta)$ on μ and ζ
- Expand in $b\perp$:

$$\text{TMDPDF}(x, b_\perp, \mu, \zeta) = C_1(\mu, \zeta) \otimes \underbrace{\text{PDF}(x, \mu)}_{\textbf{Twist 2}} + b_\perp^2\, C_2(\mu, \zeta) \otimes \underbrace{\text{PDF}_3(x, \mu) + \ldots\ldots\ldots\ldots}_{\textbf{Higher twist content}}$$

- This is the collinear twist expansion

Need to resum all twists to get the full picture

Eg. Scimeme, Tarasov, Vladimirov (2019)

CSS framework: Keep leading twist

$$\text{TMDPDF}(x_B, b_\perp, \mu, \zeta) = \text{PDF}(x_B, \mu) - \frac{\alpha_s N_c}{\pi} \boxed{L_b^\mu \int_0^1 \frac{dz}{z} P_{gg}(z)} \text{PDF}(\frac{x_B}{z}, \mu)$$

DGLAP

$$+ \frac{\alpha_S N_c}{2\pi} \boxed{\left(-\frac{1}{2}(L_b^\mu)^2 + L_b^\mu \, ln \frac{\mu^2}{\zeta^2} - \frac{\pi^2}{12} \right)} \text{PDF}(x_B, \mu)$$

CSS

- Valid for: $b_\perp$ small
- But TMD region: $b_\perp$ large
- Introduce Non-perturbative function to extrapolate
- Large x phenomenology

CSS (N.P.B, 250, 1985)

Eikonal Expansion

- Expand in x_B

$$\text{TMDPDF}(x_B, b_\perp, \mu, \zeta) = \underbrace{\tilde{C}_1 \otimes \text{DIPOLE}(b_\perp, \zeta)}_{\textbf{BFKL logs}} + x_B \underbrace{\tilde{C}_2 \otimes D_2(b_\perp, \zeta) + \ldots\ldots..}_{\textbf{Sub-eikonal corrections}}$$

Need to resum all subeikonal corrections to get the full picture

Altinoluk, Arnesto, Beuf, Martinelli, Martinez, Salgado (2014)
Balitsky, Tarasov (2014-2016)
Chirilli (2019)
Cougoulic, Kovchegov, Tarasov, Tawabutr (2022)

Eikonal Expansion

- Expand in x_B

$$\text{TMDPDF}(x_B, b_\perp, \mu, \zeta) = \underbrace{\tilde{C}_1 \otimes \text{DIPOLE}(b_\perp, \zeta)}_{\textbf{BFKL logs}} + x_B \underbrace{\tilde{C}_2 \otimes D_2(b_\perp, \zeta) + \ldots\ldots..}_{\textbf{Sub-eikonal corrections}}$$

Hence we need a new scheme which keeps full dependence on x_B and $b_\perp$

Summary

MSTT factorization is based on the background field method

gluon TMDPDF($x_B, b_\perp$) operator

NLO(Dilute) background ↓ Introducing IR and UV cutoffs

TMDPDF($x_B, b_\perp$, scales) valid for all x_B and $b_\perp$

$x_B \to 0$ ↙ ↘ $b_\perp \to 0$

Eikonal expansion/BFKL

Collinear matching/CSS

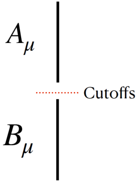
CSS (N.P.B, 250, 1985)
DGLAP (N.P.B, 126, 1977)
BFKL (P.L.B, 60, 1975)
MSTT (P.R.D 109, no.3, 034035 (2024))
SCET (P.L.B, 735, 2014)

Background field method

$$\langle P_1|O|P_2\rangle = \int DC\ \psi^*_{P_1}(C(t_i))\ O(C)\ \psi_{P_2}(C(t_f))\ e^{-iS_{QCD}(C)}$$

Split the field modes into different parts

$$\mathbf{C}_\mu = \mathbf{A}_\mu + \mathbf{B}_\mu$$



- Separation made using cutoffs
- Integrate A modes out with B fixed

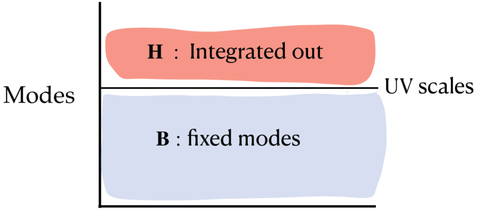
$$\langle P_1|O(C)|P_2\rangle = Coeff \times \langle P_1|\tilde{O}(B)|P_2\rangle$$

The exact structure depends on the particular factorization scheme

How we use the background field method

- Separate the hard modes:

$$\langle P_1|\text{Observable}|P_2\rangle = \text{Hard} \times \langle P_1|O(B)|P_2\rangle$$



Gives a factorization formula like the DY TMD factorization

$$\frac{d\sigma}{dQdyd^2q_\perp} = H(Q,\mu) \int d^2b_\perp\, e^{iq_\perp . b_\perp}\, \text{TMDPDF}_1(x_a, b_\perp, \mu, \zeta_a)\, \text{TMDPDF}_2(x_b, b_\perp, \mu, \zeta_b) + \mathcal{O}(\frac{q_\perp^2}{Q^2})$$

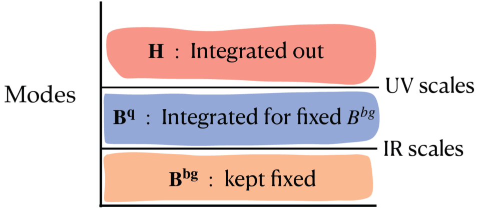
How we use the background field method

- Separate the hard modes:

$$\langle P_1|\text{Observable}|P_2\rangle = \text{Hard} \times \langle P_1|O(B)|P_2\rangle$$

- Separate into Quantum and background modes

$$\langle P_1|O(B)|P_2\rangle = \text{Coeff} \times \langle P_1|\tilde{O}(B^{bg})|P_2\rangle$$

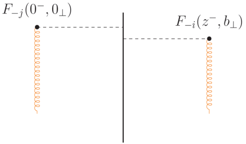


Coeff contains the logarithms which generate the evolution equations

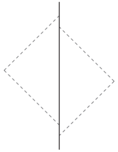
Definitions

We consider the matrix element of gluon TMD

$$B_{ij}(x_B, b_\perp) = \int_{-\infty}^{\infty} dz^- \, e^{-ix_B P^+ z^-} \langle P, S| \, F^m_{-i}(z^-, b_\perp) \, [z^-, \infty]^{ma}_b [\infty, 0^-]^{an}_0 F^n_{-j}(0^-, 0_\perp)|P, S\rangle$$



Gluon beam function



Soft function

$$\mathcal{S}(b_\perp) = \langle 0|Tr\Big[S^\dagger_{\bar{n}}(b_\perp) \, S_n(b_\perp) \, S^\dagger_n(0_\perp) S_{\bar{n}}(0_\perp)\Big]|0\rangle$$

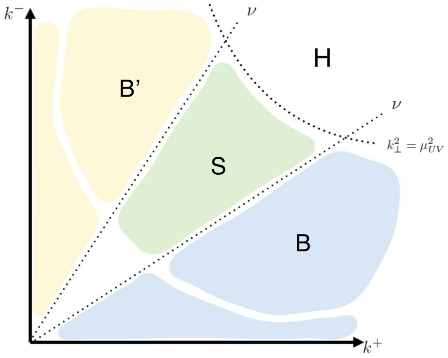
$$\boxed{f_{ij}(x_B, b_\perp) = B_{ij}(x_B, b_\perp) \sqrt{S(b_\perp)}}$$

TMD

$$f_{ij}(x_B, b_\perp) = B_{ij}(x_B, b_\perp)\sqrt{S(b_\perp)}$$

- μ_{UV} separates hard factor and TMD
- ν separates soft factor and TMD

UV scales : μ_{UV}, ν



Soft function is introduced to avoid double-counting

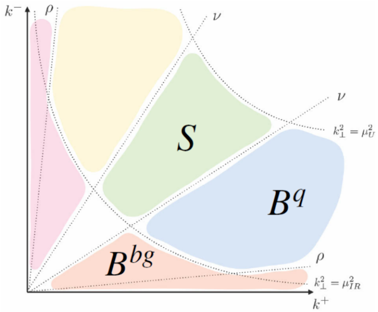
What we want

Separate TMD modes into a quantum part and a background part

Using background field method:

$$f_{ij}(B) = \text{Coeff} \otimes f_{ij}(B^{bg})$$

IR scales ρ, μ_{IR} separate B^q and B^{bg}



$$f_{ij}(\text{UV scales}) = \text{Coeff}(\text{UV scales, IR scales})\ f_{ij}(\text{IR scales})$$

Calculate the coefficient Coeff(UV scales, IR scales)

How are the scales introduced?

- By regulating divergences. We use renormalization method (not strict cutoffs)
- Dimensional regularization is used to regulate transverse integrals
- η regularization is used to regulate rapidity divergences

$$\int_0^\infty \frac{dk^-}{k^-} \to \int_0^\infty \frac{dk^-}{k^-} |k^+|^{-\eta}$$

- UV poles are removed by multiplying $B_{ij}(x, b_\perp)$ with Z_{UV} and $\sqrt{S(b_\perp)}$
- IR poles are absorbed in Background field operators

J. Y. Chiu, A. Jain, D. Neill and I. Z. Rothstein, JHEP ***05****, 084 (2012) doi:10.1007/JHEP05(2012)084 [arXiv:1202.0814 [hep-ph]].*

The procedure summarized

gluon beam function

NLO(Dilute) background $\downarrow$ Real + virtual $O(\alpha_S)$ corrections

1-loop Coefficient with UV and IR divergences

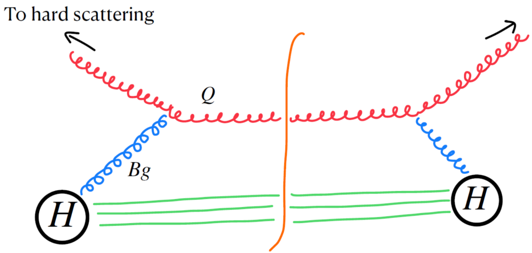
Regulate divergences $\downarrow$ Use Dim reg and η scheme

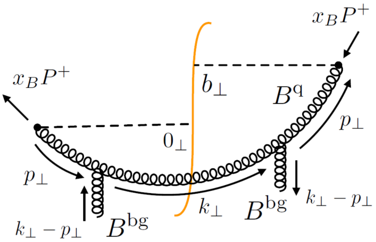
Coeff$^{\mathbf{1-loop}}$(UV scales, IR scales) with UV and IR poles

Multiply Z_{UV} and $\sqrt{S}$ to remove UV $\downarrow$ Absorb IR poles in background fields

$$f_{ij}(\textbf{UV scales}) = C^{\textbf{1 loop}}(\textbf{UV scales}, \textbf{IR scales}) \otimes f_{ij}(\textbf{IR scales})$$

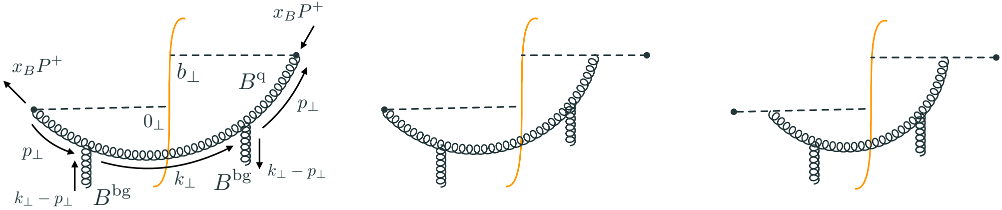
Real corrections





- Computed in 2-gluon background (dilute)
- B^{bg} provides non-zero transverse momentum
- This is different from collinear matching and SCET where background fields transfer zero transverse momentum

Real corrections



- **$\mathbf{k}_\perp$ divergence:**
  Dim reg: Get μ_{UV} and μ_{IR} scales

- **Rapidity UV ($\mathbf{k}^- \to \infty$)** divergence:
  η scheme: Get ν scale

Real corrections: Result

$$B_{ij}^{real}(x_B, b_\perp) = -\frac{4\alpha_s N_c}{(2\pi)^4} \int d^2p_\perp\, e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)} \int d^2k_\perp \,[\text{Finite diagrams}]$$

$$\times \int d^2z_\perp\, e^{-i(p_\perp - k_\perp)z_\perp}\, B_{lm}(\frac{x_B}{z}, z_\perp) - \frac{\alpha_s N_c}{\pi} \boxed{\left(\frac{1}{\epsilon_{IR}} + L_b^{\mu_{IR}}\right) \int_0^1 dz \left[\frac{1}{(1-z)_+} + \frac{1}{z}\right]} B_{ij}(\frac{x_B}{z}, b_\perp)$$

IR pole: Part of DGLAP

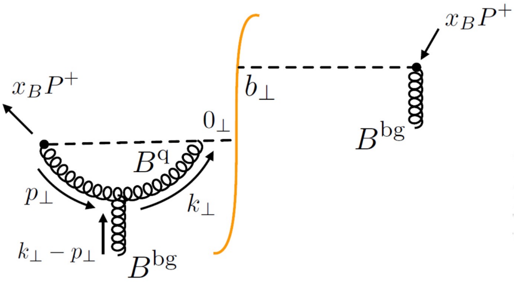
$$+ \frac{\alpha_s N_c}{\pi} \boxed{\left(\frac{1}{\epsilon_{UV}} + ln(\frac{b_\perp^2 \mu_{UV}^2}{2e^{-\gamma_E}})\right)\left(\frac{1}{\eta} + \ln(\frac{\nu}{x_B P^+})\right)}$$

UV Pole: CSS evolution

$$z = \frac{x_B}{x_B + \frac{k^+}{P^+}}$$

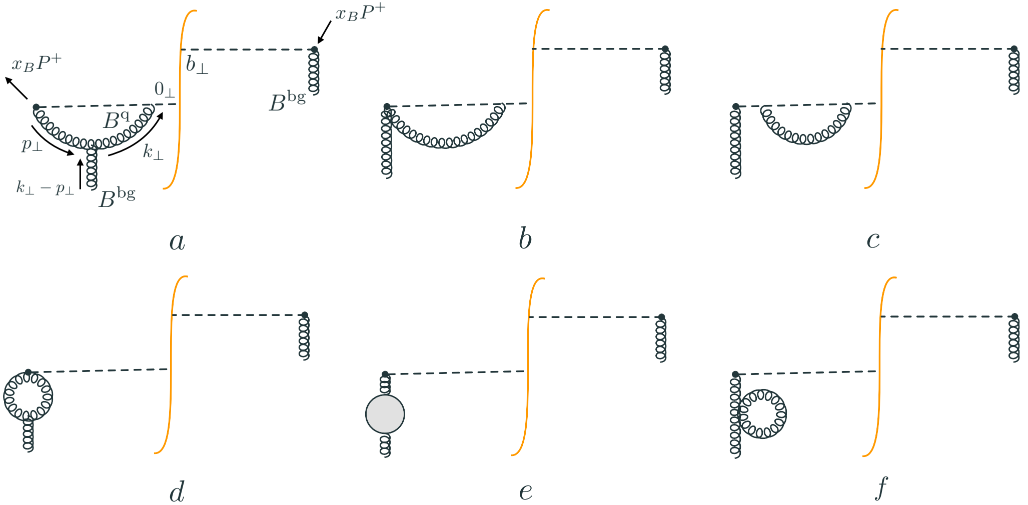
Virtual corrections

- **$\mathbf{k}_\perp$ divergence:**
  Dim reg: Get μ_{IR}, μ_{UV} scales
- **Rapidity IR ($\mathbf{k}^- \to 0$) divergence:**
  η scheme: Get ρ scale



- Vanishes for collinear/SCET where $k_\perp - p_\perp = 0$
  $\Longrightarrow$ **no rapidity IR divergence**

Virtual corrections



Virtual corrections: Result

$$B_{ij}^{virt}(x_B, b_\perp) = -\frac{\alpha_S N_c}{2\pi}\left(\frac{1}{\epsilon_{IR}^2} + \boxed{\frac{1}{\epsilon_{IR}}\left(\frac{1}{\zeta} + ln(\frac{\rho}{x_B P^+})\right) - \frac{\pi^2}{12}}\right)$$

Virtual part of the BFKL kernel

$$\times \int d^2 z_\perp \int đ^2 p_\perp \, e^{ip_\perp (b-z)_\perp} \left(\frac{\mu_{IR}^2}{p_\perp^2}\right)^{\epsilon_{IR}} \frac{g_{il} p_j p_m + p_i p_l g_{mj}}{p_\perp^2} B_{lm}^{bg}(x_B, z_\perp)$$

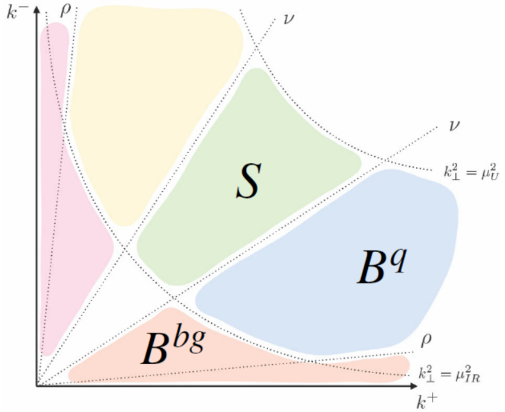
$$+ \frac{\alpha_s N_c}{2\pi}\left(\frac{1}{\epsilon_{UV}} \frac{\beta_0}{2N_c} + \frac{67}{18} - \frac{5N_f}{9N_c}\right) \int d^2 z_\perp \int đ^2 p_\perp \, e^{ip_\perp .(b-z)_\perp} \left(\frac{\mu_{UV}^2}{p_\perp^2}\right)^{\epsilon_{UV}} B_{ij}^{bg}(x_B, z_\perp)$$

- TMDPDFs contain BFKL logs
- Collinear/SCET miss these contribution since virtual diagrams vanish

Construction of full result

- Transverse UV pole, $1/\epsilon_{UV}$ is removed by Z_{UV}
- Rapidity UV pole, $1/\eta$ is removed by $S(b_\perp)$
- The IR poles, $1/\epsilon_{IR}$ and $1/\xi$ are absorbed into $B_{ij}^{bg}(x_B, b_\perp)$

UV scales : μ_{UV}, ν
IR scales : μ_{IR}, ρ



Full result

$$f_{ij}(x_B, b_\perp, \mu_{\rm UV}^2, \zeta) = f_{ij}(x_B, b_\perp, \mu_{\rm IR}^2, \rho) - 4\alpha_s N_c \int d\!\!{}^-{}^2 p_\perp e^{ip_\perp b_\perp} \int_0^1 \frac{dz}{z(1-z)} \int d\!\!{}^-{}^2 k_\perp \Big[\mathcal{R}^a_{ij;lm}(z, p_\perp, k_\perp)$$

$$+\mathcal{R}^b_{ij;lm}(z, p_\perp, k_\perp)\Big] \int d^2 z_\perp e^{-i(p_\perp - k_\perp)z_\perp} f_{lm}\Big(\frac{x_B}{z}, z_\perp, \mu_{\rm IR}^2, \rho\Big) + \boxed{\frac{\alpha_s N_c}{2\pi}\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}} \ln\frac{\mu_{\rm UV}^2}{\zeta^2} - \frac{\pi^2}{12}\Big)}$$

CSS

$$\times f_{ij}(x_B, b_\perp, \mu_{\rm IR}^2, \rho) - \boxed{\frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 dz \Big[\frac{1}{(1-z)_+} + \frac{1}{z}\Big]} f_{ij}\Big(\frac{x_B}{z}, b_\perp, \mu_{\rm IR}^2, \rho\Big) - \frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int d\!\!{}^-{}^2 p_\perp e^{ip_\perp (b-z)_\perp}$$

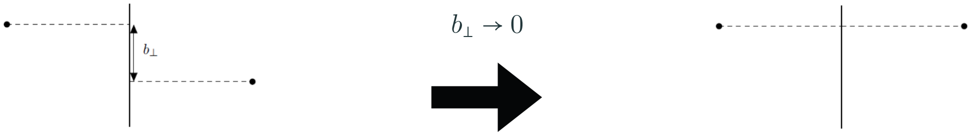
Part of the DGLAP

$$\times\Big(\frac{1}{2}\ln^2\frac{\mu_{\rm IR}^2}{p_\perp^2} + \boxed{\ln\frac{\mu_{\rm IR}^2}{p_\perp^2}\ln\frac{\rho}{\zeta}} - \frac{\pi^2}{12}\Big)\frac{g_{il}p_j p_m + p_i p_l g_{mj}}{p_\perp^2} f_{lm}(x_B, z_\perp, \mu_{\rm IR}^2, \rho)$$

Virtual BFKL

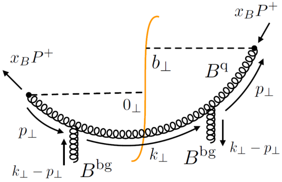
$$+\frac{\alpha_s N_c}{2\pi} \int d^2 z_\perp \int d\!\!{}^-{}^2 p_\perp e^{ip_\perp (b-z)_\perp} \Big(\frac{\beta_0}{2N_c}\ln\frac{\mu_{\rm UV}^2}{p_\perp^2} + \frac{67}{18} - \frac{5N_f}{9N_c}\Big) f_{ij}(x_B, z_\perp, \mu_{\rm IR}^2, \rho) + O(\alpha_s^2)\,.$$

Collinear matching

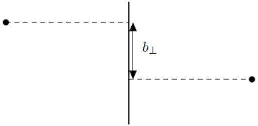


$$B^{bg}_{ij}(x_B, p_\perp) = \int d^2 b_\perp \, e^{-ip_\perp b_\perp} B^{bg}_{ij}(x_B, b_\perp) \xrightarrow[b_\perp = 0]{} \delta^2(p_\perp) \, B^{bg}_{ij}(x_B)$$

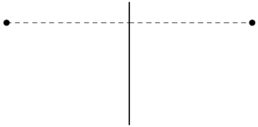
- No transverse momenta from background fields
- Additional factorization condition: $k_\perp^{\text{background}} << k_\perp^{\text{quantum}}$
- Collinear singularity appears in previously finite diagrams



Collinear matching



$b_\perp \to 0$



$$f_1(x_B, b_\perp, \mu_{\rm UV}^2, \zeta) = f_1(x_B, 0_\perp, \mu_{\rm IR}^2)$$

$$- \underbrace{\frac{\alpha_s N_c}{\pi} L_b^{\mu_{\rm IR}} \int_0^1 \frac{dz}{z} P_{gg}(z) f_1\Big(\frac{x_B}{z}, 0_\perp, \mu_{\rm IR}^2\Big)}_{\text{DGLAP}} + \underbrace{\frac{\alpha_s N_c}{2\pi}\Big(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}} \ln\frac{\mu_{\rm UV}^2}{\zeta^2} - \frac{\pi^2}{12}\Big) f_1(x_B, 0_\perp, \mu_{\rm IR}^2)}_{\text{CSS}} + \ldots ,$$

- Coincides with standard SCET results
- No virtual contributions present

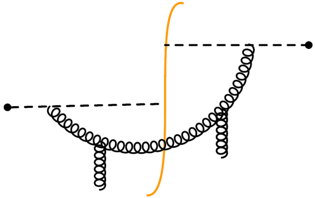
Eikonal expansion

Take the small x limit $x_B \to 0$

$$B^{bg}_{ij}(x_B, b_\perp) = \int_{-\infty}^{\infty} dz^-\, e^{-ix_B P^+ z^-} \int d^2 b_\perp\, e^{-ib_\perp . p_\perp} \langle P| F^m_{-i}(z^-, b_\perp)\, [z^-, \infty]^{ma}_b [\infty, 0^-]^{an}_0 F^n_{-j}(0^-, 0_\perp)|P\rangle^{bg}$$

$$\xrightarrow{x_B \to 0} \langle P| Tr(U_b \partial_i U_b^\dagger)(U_0 \partial_j U_0^\dagger)|P\rangle \implies f_{ij}(x_B, p_\perp) = \frac{p_i p_j}{p_\perp^2} h_1(x_b, p_\perp)$$

- At small-x TMDPDFs match to WW distributions
- Additional factorization condition: $k^-_{\text{quantum}} >> k^-_{\text{background}}$
- Rapidity IR divergence in real corrections as a result



$$f_1(x_B, p_\perp, \mu_{\rm UV}^2, \zeta) \simeq \mathcal{H}_1(p_\perp, \rho) + \underbrace{\boxed{\ln\frac{\rho}{\zeta}\int d\!\!\!{}^-{}^2 k_\perp K_{\rm BFKL}(p_\perp, k_\perp)\mathcal{H}_1(p_\perp - k_\perp, \rho)}}_{\text{BFKL}}$$

$$+\frac{\alpha_s N_c}{2\pi}\int d^2 b_\perp \underbrace{\boxed{\left(-\frac{1}{2}(L_b^{\mu_{\rm UV}})^2 + L_b^{\mu_{\rm UV}}\ln\frac{\mu_{\rm UV}^2}{\zeta^2} - \frac{\pi^2}{12}\right)}}_{\text{CSS}}\int d\!\!\!{}^-{}^2 k_\perp e^{ik_\perp b_\perp}\mathcal{H}_1(p_\perp - k_\perp, \rho)$$

$$+\frac{\alpha_s N_c}{2\pi}\left(\frac{\beta_0}{2N_c}\ln\frac{\mu_{\rm UV}^2}{p_\perp^2} + \frac{67}{18} - \frac{5N_f}{9N_c}\right)\mathcal{H}_1(p_\perp, \rho)\,.$$

Outlook

TMD phenomenology

- Generalization from collinear framework to MSTT framework
- CSS equation will change due to BFKL type logs
- NP function not needed at large $b_\perp$?

Lattice

- Compute matching between pseudo/quasi TMDPDFs and TMDPDFS with MSTT
- Changes expected due to BFKL type logs

Small-x

- TMDPDFs calculated with MSTT and extracted from phenomenology/lattice can be used to provide initial conditions for small-x evolution

Summary

MSTT factorization is based on the background field method

gluon TMDPDF($x_B, b_\perp$) operator

NLO(2-gluon) background ↓ Introducing IR and UV cutoffs

TMDPDF($x_B, b_\perp$, scales) valid for all x_B and $b_\perp$

$x_B \to 0$

$b_\perp \to 0$

Eikonal expansion/BFKL

Collinear matching/CSS

CSS (N.P.B, 250, 1985)
DGLAP (N.P.B, 126, 1977)
BFKL (P.L.B, 60, 1975)
MSTT (P.R.D 109, no.3, 034035 (2024))
SCET (P.L.B, 735, 2014)

Backup Slides

SCET and MSTT

- SCET uses modes with fixed scaling on-shell modes,

$$A^n \sim (p^+, p^-, p_\perp) \sim Q(\lambda^2, 1, \lambda)\,,\; A^{\bar{n}} \sim Q(1, \lambda^2, \lambda)\,,\; S \sim Q(\lambda, \lambda, \lambda)$$

  where $\lambda << 1$ is a small scale.

- Glauber SCET off-shell modes: $G \sim Q(\lambda^2, \lambda^2, \lambda)$

- MSTT distribution functions are defined by the cutoffs instead of fixed scaling

- Typical off-shell modes $\sim Q(\lambda^4, 1, \lambda)$

Separation of Rapidity divergences

- Rapidity divergences are typically of the form,

$$\int_0^\infty \frac{dk^-}{k^-}$$

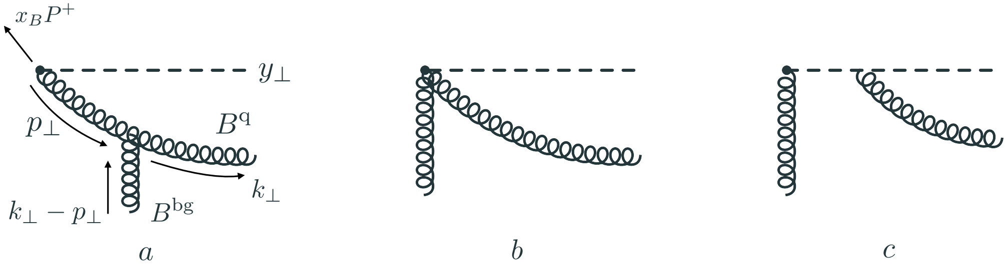
- Introduce the z-variable to separate rapidity divergence,

$$z = \frac{x_B}{x_B + \frac{k_\perp^2}{2k^- P^+}}$$

  It is the fraction of longitudinal (+) momenta coming from the hadron before the loop correction
- Separation of rapidity divergence

$$\int_0^\infty \frac{dk^-}{k^-} \rightarrow \underbrace{\int_0^1 \frac{dz}{z}}_{\text{Rapidity IR}} + \underbrace{\int_0^1 \frac{dz}{1-z}}_{\text{Rapidity UV}}$$

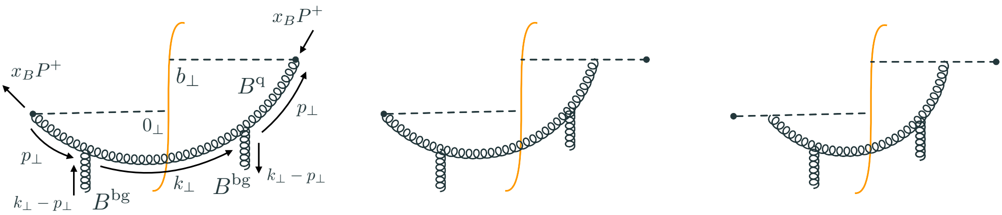
Real corrections: Lipatov vertex



$$L^{ab}_{\mu j}(k, y_\perp, x_B) \equiv i \lim_{k^2\to 0} k^2 \langle B^{{\rm q}a}_\mu(k) \int_{-\infty}^{\infty} dy^- e^{i x_B P^+ y^-} [\infty, y^-]_y^{bd} F^{{\rm q+bg};d}_{-j}(y^-, y_\perp) \rangle_{B^{\rm bg}}$$

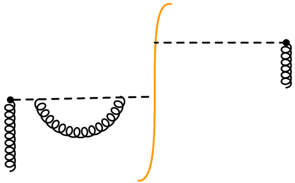
$$\langle \tilde{\mathcal{F}}^a_i(x_B, x_\perp) \mathcal{F}^a_j(x_B, y_\perp) \rangle^{\rm real} = -\int \frac{đk^-}{2k^-} \int đ^2 k_\perp \tilde{L}_i^{\ \mu ba}(k, x_\perp, x_B) L^{ab}_{\mu j}(k, y_\perp, x_B)$$

Rapidity Divergence in real corrections

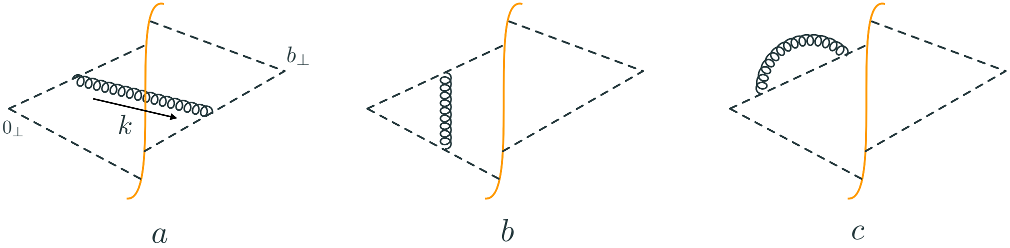


By kinematics of real correction: $z \geq x_B$

- MSTT only has Rapidity UV divergences in real corrections
- Simple computation shows that CSS log comes with μ_{IR}
- Need to combine with wilson line virtual correction diagram to get the shown μ_{UV} dependence



Soft factor and UV renormalization



$$S^{(1)}(b_\perp) = \frac{\alpha_s N_c}{2\pi}\left(\frac{2}{\epsilon_{UV}^2} + 4\Big(\frac{1}{\epsilon_{UV}} + L_b\Big)\left(-\frac{1}{\eta} + ln\frac{\mu}{\nu}\right) - L_b^2 - \frac{\pi^2}{6}\right)$$

$$Z_{UV} = 1 - \frac{\alpha_s N_c}{2\pi}\left[\frac{1}{\epsilon_{UV}^2} + \frac{1}{\epsilon_{UV}}\, ln\left(\frac{\mu_{UV}^2}{(x_B P^+)^2}\right)\right]$$