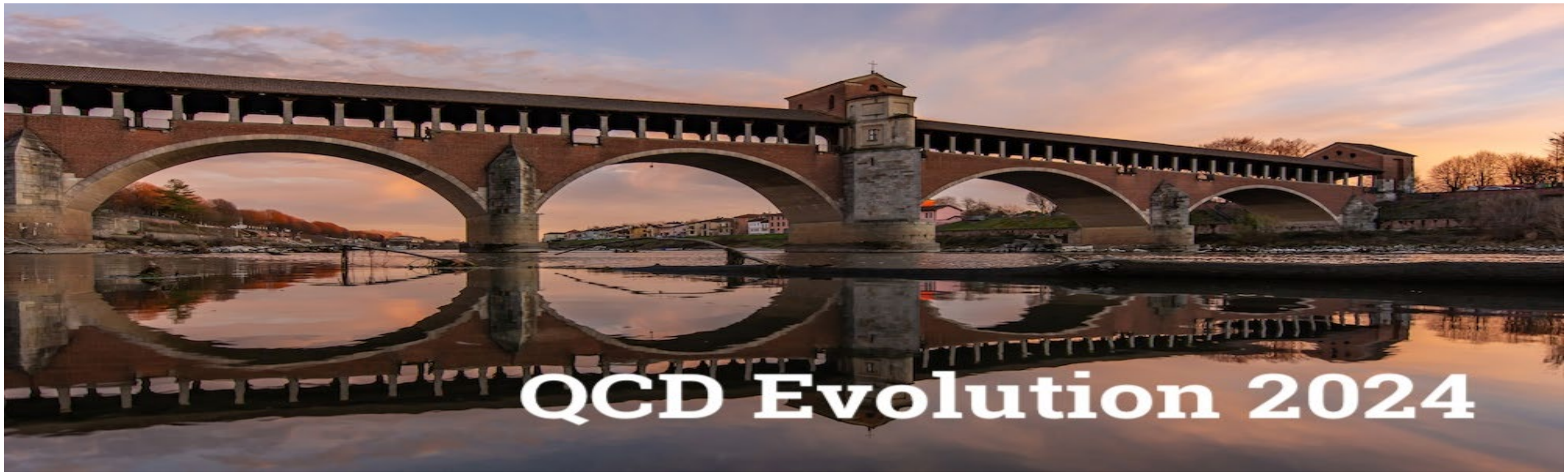

TRANSVERSE POLARIZATION OF Λ HYPERONS WITHIN A JET IN UNPOLARIZED HADRONIC COLLISIONS

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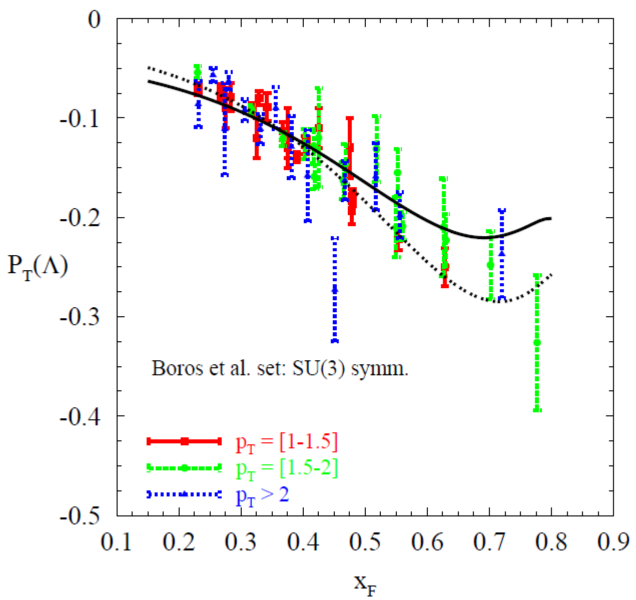
PRD 2023 & PLB 2024

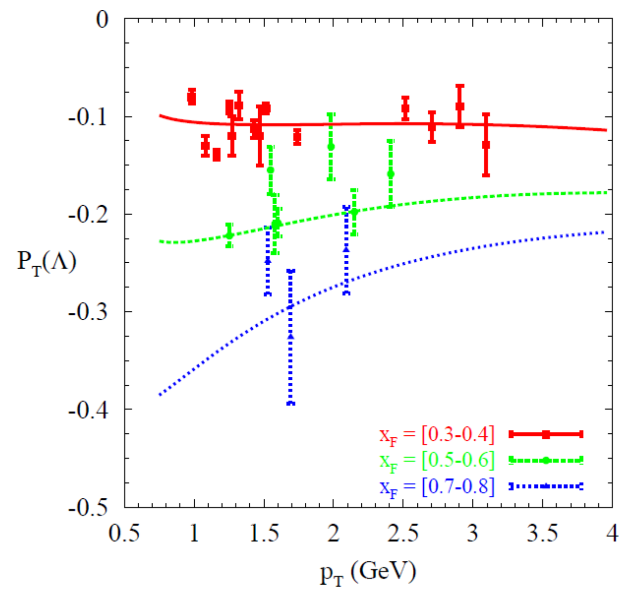
OUTLINE

- Motivations
- hadron-in-jet in unpolarized pp collisions
  - ✓ TMD approach: tool, issues…
  - ✓ Λ polarization: access to the quark and gluon polarizing FFs
  - ✓ Universality
- Present knowledge of the polarizing FF
  - ✓ Extraction from $e^+e^- \rightarrow h_1\ h_2 + X$ [Belle data]
  - ✓ the inclusive case
  - ✓ *SU(2)* symmetry and role of charm contribution
- Predictions and comparison against STAR data
- Concluding remarks

MOTIVATIONS AND MORE

- Transverse Λ polarization in *unpolarized pA* collisions [Bunce *et al.* '76]
- fixed target, sizeable, rising with x_F, p_T plateau…still challenging
- zero at midrapidity (for symmetry reasons!)
- First attempt within a TMD scheme [Anselmino, Boer, UD, Murgia '01]
- good description … no formal theory behind





MOTIVATIONS AND MORE

- Breakthrough: Belle e^+e^- data [Guan *et al.* (Belle Coll.) '19]

 - *2h,* TMD factorization [Collins '11; Echevarria, Idilbi, Scimemi '12]

 - *1h:* new TMD scheme? [Kang, Shao, Zhao '20; Boglione, Simonelli '21, '22, '23]

- Extraction of the polarizing FF (fixed scale)

- *2h* and *2h+1h* fits [UD, Murgia, Zaccheddu '20]

- *2h* fit [Callos, Kang, Terry '20]

- Analysis w/ CSS scheme [Li, Wang, Yang, Lu '21; Gamberg, Kang, Shao, Terry, Zhao '21; UD, Gamberg, Murgia, Zaccheddu '22]

- *SU(2)* symmetry issue: [Chen, Liang, Pan, Song, Wei '21; Chen, Liang, Song, Wei '22, '23; UD, Gamberg, Murgia, Zaccheddu '23]

$$A(p_A)\, B(p_B) \to \text{jet}(p_\text{j})\, \Lambda^\uparrow(p_\Lambda)\, X$$

- Complementary to SIDIS and e^+e^- processes:

- 2 ordered scales: p_{jT} (large) and $p_{\perp\Lambda}$ (small) [$\perp$ w.r.t. jet]

- TMD effects (relevant) only in the fragmentation mechanism: use of collinear PDFs

- Several studies for hadron-in-jet:

- Collins effect in pp collisions [Yuan 2008; UD, Murgia, Pisano '11, '17; Kang *et al.* '17] - w/ or w/o TMD evolution: general agreement

- TMD-Jet-FFs and their relation to TMDFFs [Kang *et al.* '20, '21, '23]

- New data from STAR [Gao *et al.* @SPIN2023]

$$A(p_A)\,B(p_B) \to \text{jet}(p_\text{j})\,\Lambda^\uparrow(p_\Lambda)\,X$$

- Factorization scale and jet-cone radius: proper scale $\mu_j = p_{\text{j}T} R$,
- Evolution up to $\mu = p_{\text{j}T}$ resumm. of single logs in jet param.
- At LO (no depend. on R) one can use $\mu = p_{\text{j}T}$ [Kang *et al* '17]

- Backward going jet in addition [Boer, Bomhof, Hwang, Mulders '07]
- to determine the partonic cm frame (different from the pp cm frame, depending event by event on x_1, x_2)
- ξ and $p_\perp$ in TMDFF are not Lorentz boost invariant (pp vs e^+e^-)
- At high energies differences are power suppressed $E_j/\sqrt{s}$: backward jet not relevant [Boer '10] …to be further explored

$$A(p_A)\,B(p_B) \to \mathrm{jet}(p_\mathrm{j})\,\Lambda^\uparrow(p_\Lambda)\,X$$

- TMD effects only in the fragmentation mechanism [leading contribution: other terms suppressed by partonic phases]
- Factorization scheme at LO [*AB* **cm frame**]

unpol collinear PDFs

$$d\sigma^{\uparrow(\downarrow)} \equiv E_\mathrm{j}\,\frac{d\sigma^{AB\to\mathrm{jet}\Lambda^\uparrow X}}{d^3\boldsymbol{p}_\mathrm{j}\,d\xi\,d^2\boldsymbol{p}_{\perp\Lambda}} = \sum_{a,b,c,d}\int dx_a\,dx_b\,\frac{\alpha_s^2}{\hat{s}}\,f_{a/A}(x_a)\,f_{b/B}(x_b)$$

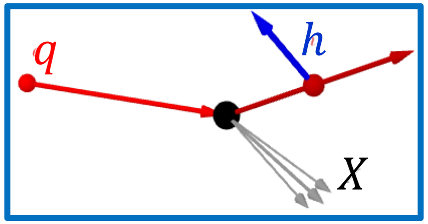
$$\times\,|\overline{M}_{ab\to cd}|^2\,\delta(\hat{s}+\hat{t}+\hat{u})\,\hat{D}_{\Lambda^\uparrow/c}(\xi,\boldsymbol{p}_{\perp\Lambda})\,,$$

unpol. partonic xsec

$$P_T^\Lambda(\boldsymbol{p}_\mathrm{j},\xi,\boldsymbol{p}_{\perp\Lambda}) = \frac{d\sigma^\uparrow - d\sigma^\downarrow}{d\sigma^\uparrow + d\sigma^\downarrow} = \frac{d\Delta\sigma}{d\sigma_\mathrm{unp}}$$

- Parton c can be a gluon!!!

The numerator involves the polarizing FF:



$$\Delta \hat{D}_{\Lambda^\uparrow/c}(\xi, \boldsymbol{p}_{\perp\Lambda}) = \hat{D}_{\Lambda^\uparrow/c}(\xi, \boldsymbol{p}_{\perp\Lambda}) - \hat{D}_{\Lambda^\downarrow/c}(\xi, \boldsymbol{p}_{\perp\Lambda})$$
$$= \Delta D_{\Lambda^\uparrow/c}(\xi, p_{\perp\Lambda})\, \hat{\boldsymbol{P}}^{\Lambda} \cdot (\hat{\boldsymbol{p}}_c \times \hat{\boldsymbol{p}}_{\perp\Lambda}),$$

$$\Delta D_{\Lambda^\uparrow/c}(\xi, p_{\perp\Lambda}) = \frac{p_{\perp\Lambda}}{\xi m_\Lambda} D_{1T}^{\perp c}(\xi, p_{\perp\Lambda}) \qquad \hat{\boldsymbol{P}}^{\Lambda} \cdot (\hat{\boldsymbol{p}}_c \times \hat{\boldsymbol{p}}_{\perp\Lambda}) = \sin(\tilde{\phi}_{S_\Lambda} - \tilde{\phi}_\Lambda)$$

the angles $\tilde{\phi}$ are defined in the parton helicity frame

$\tilde{\phi}_\Lambda$ is the Λ azimuthal angle around the jet

- Polarization transverse w.r.t. the jet-Λ plane: $\sin(\tilde{\phi}_{S_\Lambda} - \tilde{\phi}_\Lambda) = 1$

SCALING VARIABLES

$z_\Lambda = E_\Lambda / E_\mathrm{j}$ (energy fraction)

$z_p = |\boldsymbol{p}_\Lambda| / E_\mathrm{j}$ (momentum fraction)

$$z = \frac{\boldsymbol{p}_\Lambda \cdot \boldsymbol{p}_\mathrm{j}}{\boldsymbol{p}_\mathrm{j}^2} = \frac{\boldsymbol{p}_\Lambda \cdot \hat{\boldsymbol{p}}_\mathrm{j}}{E_\mathrm{j}} = \frac{\tilde{p}_{L\Lambda}}{E_\mathrm{j}}$$ (longitudinal momentum fraction)

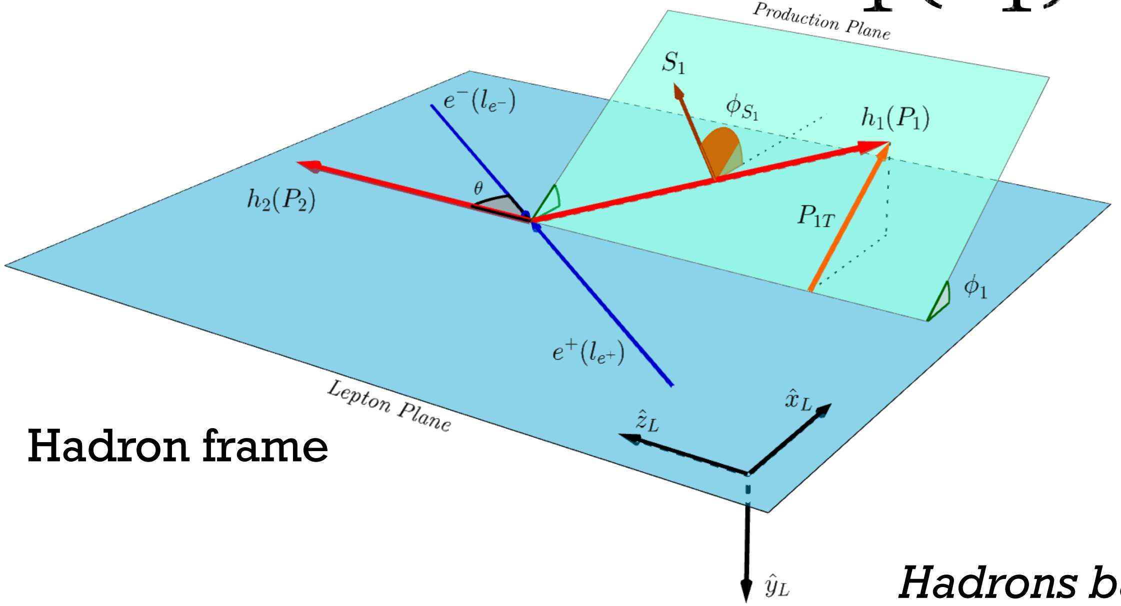
Exp. analysis

Mass effects properly taken into accout

INFORMATION ON THE POLARIZING FF

- Two data sets from Belle
- $e^+e^- \rightarrow h_1(P_1)\, h_2(P_2) + X$ and $e^+e^- \rightarrow h_1(P_1) + X$
- TMD factorization well established only for the first case
- CSS approach at NLL for the associated production + *SU(2)* symmetry issue
- Inclusive case not compatible with the associated case: attempt of a combined fit with two nonperturbative models (not a real fit): use of the model from the inclusive-data fit
- Few details for the associated production fit

$e^+e^- \rightarrow h_1(P_1)\, h_2(P_2) + X$



$$z_p = \frac{2|\boldsymbol{P}_h|}{Q} \simeq z\left(1 - \frac{M_h^2}{z^2Q^2}\right)$$

$$z_h = \frac{2P_h \cdot q}{Q^2} = \frac{2E_h}{Q} \simeq z\left(1 + \frac{M_h^2}{z^2Q^2}\right)$$

Hadron frame

Hadrons back-to-back

$$\boldsymbol{P}_{1T} = -\, z_1\, \boldsymbol{q}_T$$

$$\frac{d\sigma^{e^+e^- \to h_1(S_1) h_2\, X}}{2dy\, dz_{h_1}\, dz_{h_2}\, d^2\boldsymbol{q}_T} = \sigma_0^{e^+e^-}\left[F_{UU} - |S_{1T}|\sin(\phi_1 - \phi_{S_1}) F_{TU}^{\sin(\phi_1 - \phi_{S_1})} + \cdots\right]$$

$$F_{UU} \;=\; z_{p_1}^2 z_{p_2}^2 \mathcal{H}^{(e^+e^-)}(Q)\, \mathcal{F}[D_1 \bar{D}_1] \qquad \mathcal{H}^{(e^+e^-)}(Q)|_{\mathrm{LO}} = 1$$

$$F_{TU}^{\sin(\phi_1 - \phi_{S_1})} \;=\; z_{p_1}^2 z_{p_2}^2 \mathcal{H}^{(e^+e^-)}(Q)\, \mathcal{F}\left[\frac{\hat{\boldsymbol{h}} \cdot \boldsymbol{k}_T}{M_1} D_{1T}^{\perp} \bar{D}_1\right]$$

[Boer, Jakob, Mulders ‘97; Pitonyak, Schlegel, Metz ‘14; UD, Murgia, Zaccheddu ‘21]

$$e^+e^- \to h_1(P_1)\, h_2(P_2) + X$$

CSS approach… better in b_T space

$$F_{TU}^{\sin(\phi_1-\phi_{S_1})} = M_1 z_{p_1}^2 z_{p_2}^2 \mathcal{B}_1\left[\widetilde{D}_{1T}^{\perp(1)}\,\widetilde{\bar{D}}_1\right]$$
$$= M_1 z_{p_1}^2 z_{p_2}^2 \sum_q e_q^2 \int \frac{db_T}{2\pi}\, b_T^2 J_1(b_T\, q_T)\widetilde{D}_{1T}^{\perp(1)}(z_1, b_T)\widetilde{\bar{D}}_1(z_2, b_T)$$

$$\widetilde{D}_{1T}^{\perp(1)}(z, b_T) = -\frac{2}{M_1^2}\frac{\partial}{\partial b_T^2}\widetilde{D}_{1T}^{\perp}(z, b_T)$$

FT of the polarizing FF

$$e^+e^- \to h_1(P_1)\, h_2(P_2) + X$$

Polarization along ***n*** (⊥ to the production plane)

$$P_n^{h_1}(z_{h_1}, z_{h_2}) = \frac{M_1 \int dq_T\, q_T\, d\phi_1\, \mathcal{B}_1\left[\widetilde{D}_{1T}^{\perp(1)}\, \widetilde{\bar{D}}_1\right]}{\int dq_T\, q_T\, d\phi_1\, \mathcal{B}_0\left[\widetilde{D}_1\, \widetilde{\bar{D}}_1\right]}$$

$q_T < 0.27\, Q$

TMD fact.

Denominator *(known)*

(small-b_T matching onto)
collinear FFs

$$\mathcal{B}_0\left[\widetilde{D}_1\widetilde{\bar{D}}_1\right] = \frac{1}{z_1^2 z_2^2}\sum_q e_q^2 \int \frac{db_T}{2\pi}\, b_T\, J_0(b_T\, q_T)\, d_{h_1/q}(z_1;\bar{\mu}_b)\, d_{h_2/\bar{q}}(z_2;\bar{\mu}_b)$$

$$\times\, M_{D_1}(b_c(b_T), z_1)\, M_{D_2}(b_c(b_T), z_2)\, e^{-g_K(b_c(b_T);b_{\max})\ln\left(\frac{Q^2 z_1 z_2}{M_1 M_2}\right) - S_{\rm pert}(b_*;\bar{\mu}_b)}$$

Nonpert. funct.s

Nonpert.
CS Kernel

Perturbative
Sudakov factor

[Bacchetta, Delcarro, Pisano, Radici, Signori ‘17
Boglione, Simonelli ’21]

Numerator of P_n

$$\mathcal{B}_1\left[\widetilde{D}_{1T}^{\perp(1)}\widetilde{\bar{D}}_1\right] = \frac{1}{z_1^2 z_2^2}\sum_q e_q^2 \int \frac{db_T}{(2\pi)}\, b_T^2 J_1(b_T\, q_T)\, D_{1T}^{\perp(1)}(z_1;\bar{\mu}_b)\, d_{h_2/\bar{q}}(z_2;\bar{\mu}_b)$$

$$\times\, M_{D_1}^{\perp}(b_c(b_T), z_1)\, M_{D_2}(b_c(b_T), z_2)\, e^{-g_K(b_c(b_T);b_{\max})\ln\left(\frac{Q^2 z_1 z_2}{M_1 M_2}\right) - S_{\text{pert}}(b_*;\bar{\mu}_b)}$$

- Polarizing FF first moment

$$D_{1T,\Lambda/q}^{\perp(1)}(z;\mu_b) = \mathcal{N}_q^p(z)\, d_{\Lambda/q}(z;\mu_b)$$

$$q = u, d, s, \bar{u}, \bar{d}, \bar{s}$$

$$\mathcal{N}_q^{\text{p}}(z) = N_q z^{a_q}(1-z)^{b_q}\frac{(a_q+b_q)^{(a_q+b_q)}}{a_q^{a_q} b_q^{b_q}}$$

No a priori bound

- nonperturbative part

$$M_{D,\Lambda}^{\perp}(b_T, z) = \exp\left(-\frac{\langle p_\perp^2\rangle_{\text{p}} b_T^2}{4z_p^2}\right)$$

FIT OF BELLE DATA [NLL]
(ASSOCIATED PRODUCTION)

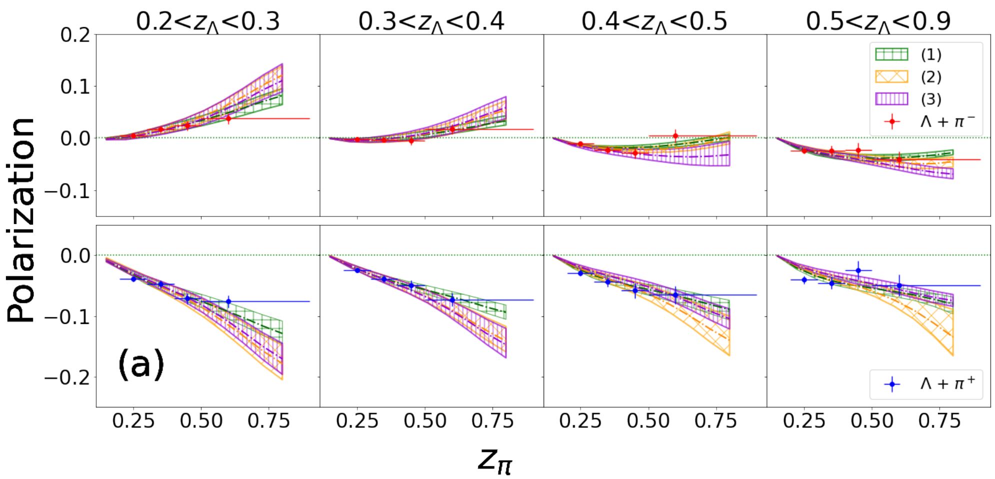
- Data selection: $\Lambda + \pi/K$: $z_{\pi.K}$ [0.5-0.9] bin excluded → 96 data points (128)

Scenarios considered: [UD, Gamberg, Murgia Zaccheddu '23]

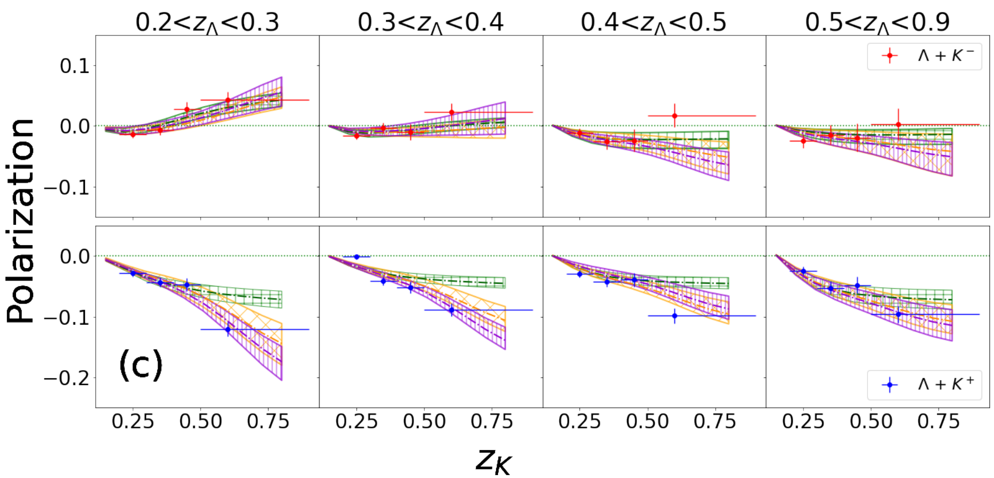
1. Only light quarks & No $SU(2)$ sym.
   pFFs for u, d, s, sea; [8 par]
2. Inclusion of charm in unpol. xsec, No $SU(2)$ sym.
   pFFs for u, d, s, sea; [9 par]
3. Inclusion of charm in unpol. xsec, $SU(2)$ sym.
   pFFs for $d = u$, $\bar{d} = \bar{u}$, $s, \bar{s}$ [9 par]

χ^2_{dof}
96 pts
1.17
1.26
1.36

DATA DESCRIPTION

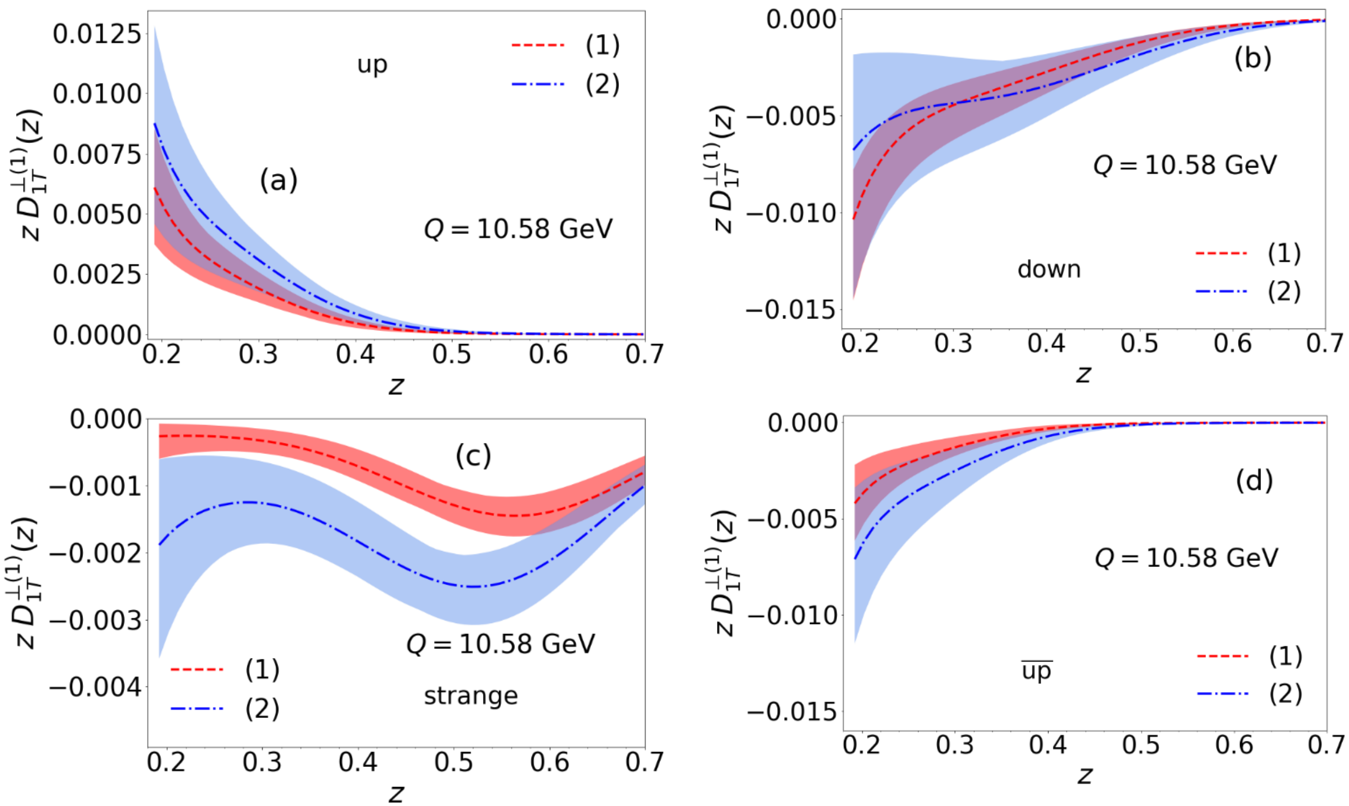


- Good agreement for $\Lambda\pi^\pm, (\overline{\Lambda}\pi^\pm)$ data, Sc. 1,2,3;



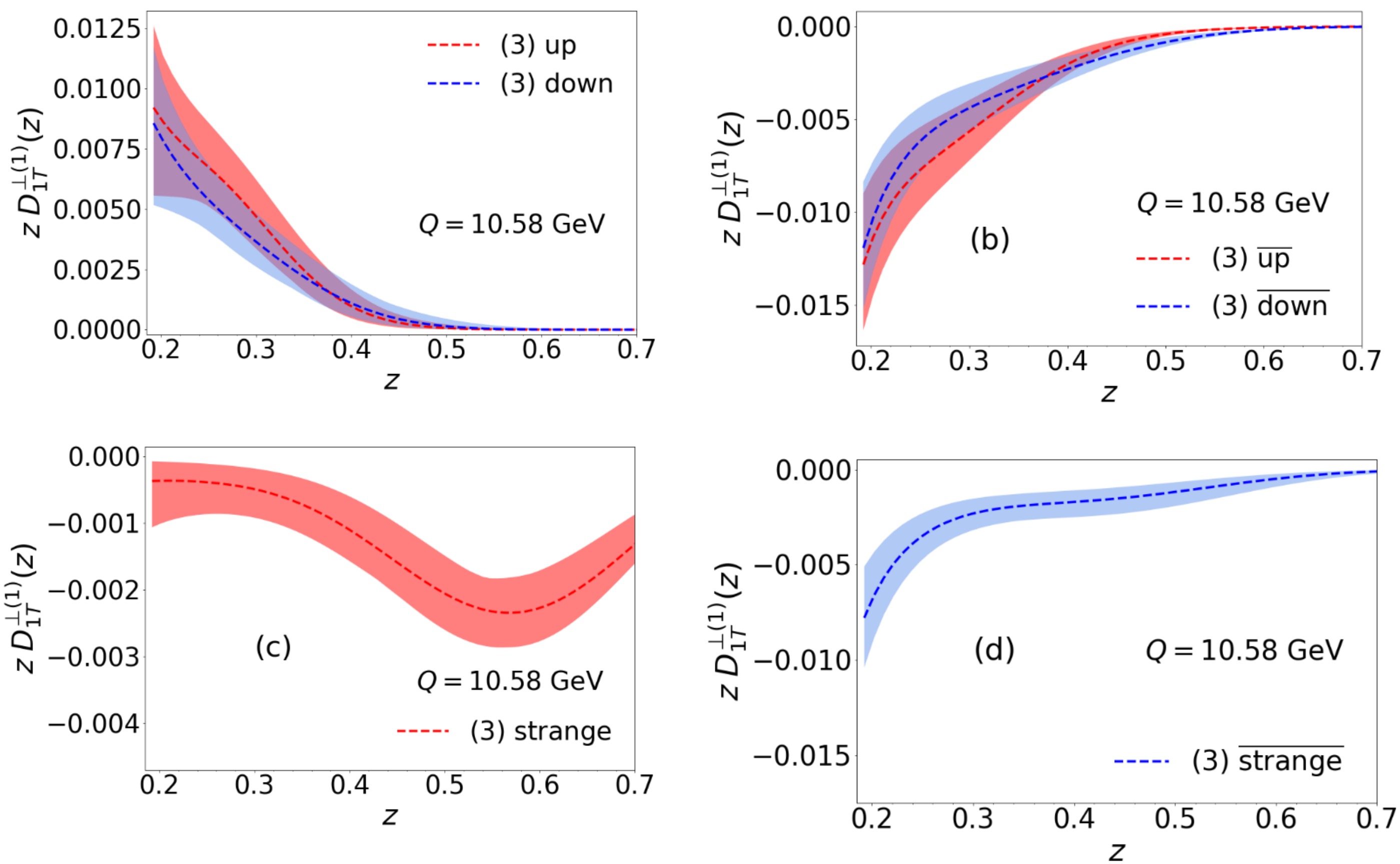
- Good agreement for $\Lambda K^-(\overline{\Lambda}K^+)$ data, Sc. 1,2,3;
- Sc. 1 (no charm/no *SU(2)*) cannot describe $\Lambda K^+(\overline{\Lambda}K^-)$ data with $z_K > 0{,}5$;
- Inclusion of charm allows for similar good fits (w/, w/o *SU(2)*)

FIRST MOMENTS: SC 1, 2 NO *SU(2)* SYMM.



- different magnitudes, Sc 2 bigger in size: charm contribution;
- u pFF positive; d pFF negative STRONG *SU(2)* violation
- Sc 1 and 2 compatible, except for strange
- Similar size for the Gaussian width.

FIRST MOMENTS: SC 3 WITH *SU(2)* SYMM.



- u, d pFFs **positive**; all others pFFs negative
- sea contribution larger in size w.r.t. Sc. 1 & 2

GENERAL REMARKS FROM e^+e^-

- Charm in unpol. xsec improves the quality of the fit (kaons)
- Polarizing FF for charm: tried but without any improvement
- Sc 2 and 3 (charm, w/ or w/o *SU(2)* symmetry) equally good
- If *SU(2)* not imposed, fits favor opposite pFFs for u, d

STRONG *SU(2)* symmetry violation

- Where/how can we check it?
- SIDIS…pp collisions

PREDICTIONS FOR pp COLLISIONS

[UD, Gamberg, Murgia, Zaccheddu '24]

- STAR preliminary data at 200 GeV [Gao @ SPIN2023]

$$p_{\perp\Lambda} \leq 1.6 \text{ GeV}/c, \quad 0 \leq z \leq 1,$$

$$8 \leq p_{\mathrm{jT}} \leq 25 \text{ GeV}/c \quad \text{with} \quad \langle p_{\mathrm{jT}} \rangle = 11 \text{ GeV}/c,$$

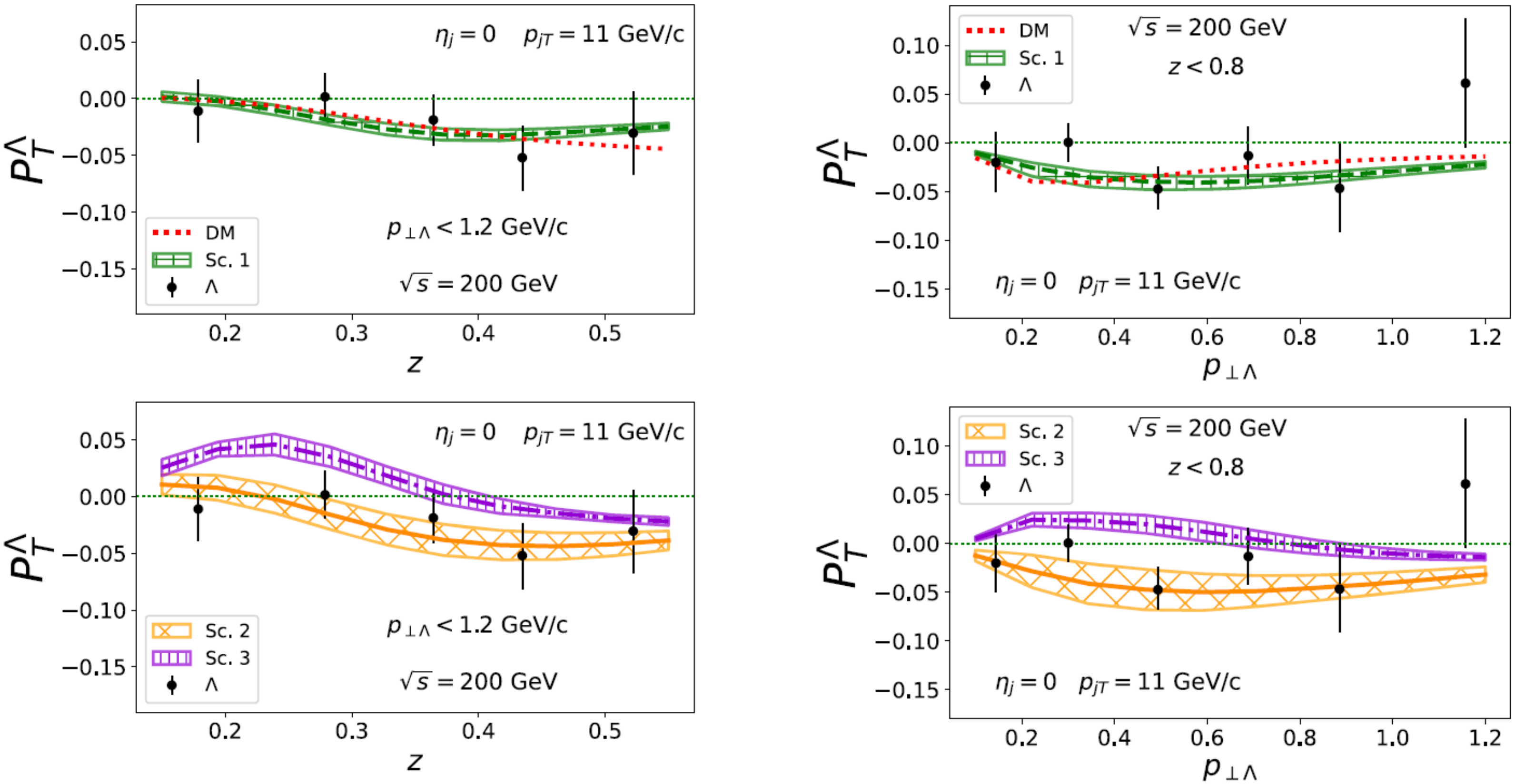
$$|\eta_{\mathrm{j}}| \leq 1.0, \; p_{T\Lambda} \leq 10 \text{ GeV}/c, \; |\eta_\Lambda| \leq 1.5,$$

- Theory estimates

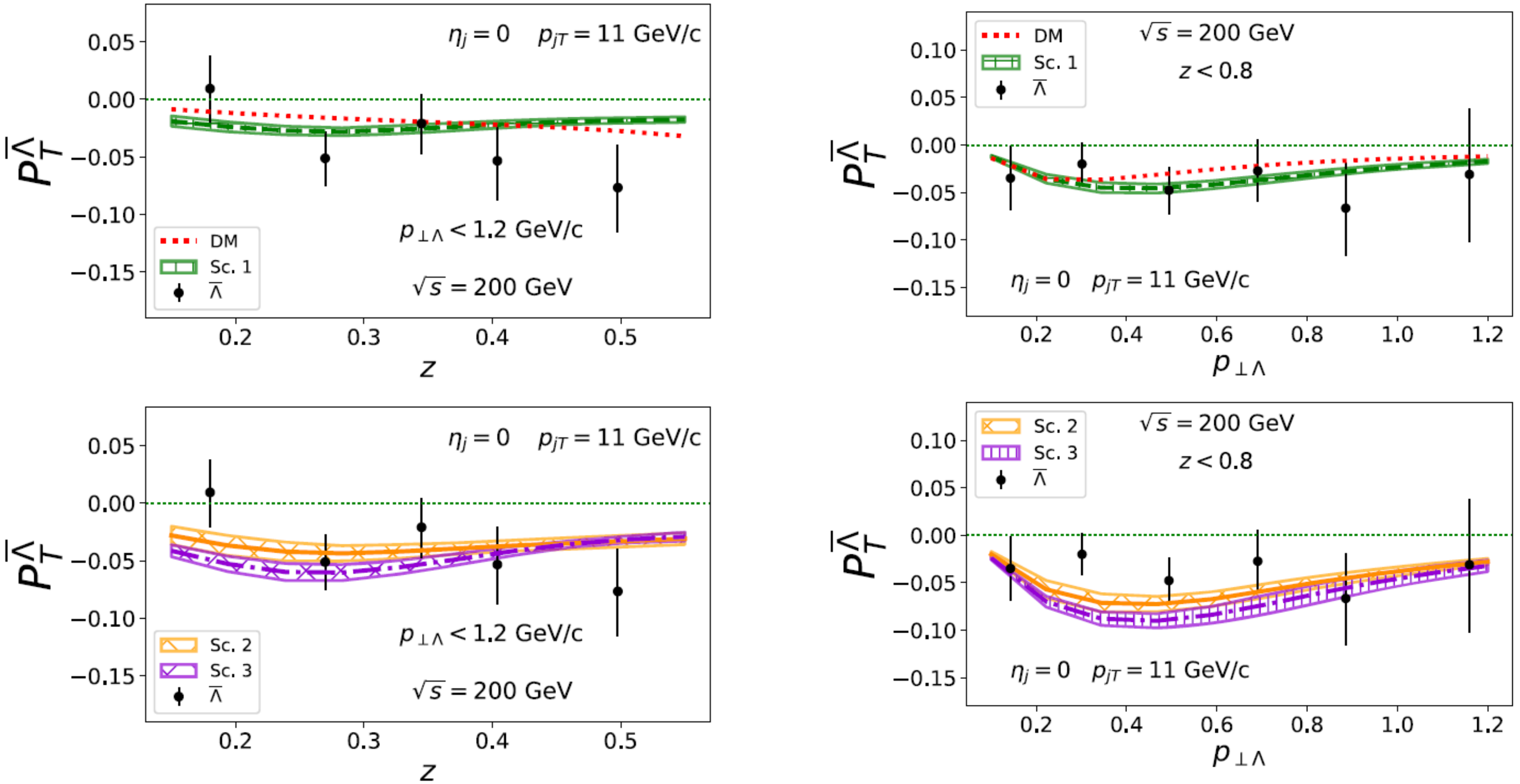
$$\eta_{\mathrm{j}} = 0 \text{ and } p_{\mathrm{jT}} = 11 \text{ GeV}/c \qquad z < 0.8, \quad p_{\perp\Lambda} \leq 1.2 \text{ GeV}/c.$$

- Jet reconstruction: anti-k_T algorithm, $R = 0.6.$
- 3 (ass. prod.)+1 (DM) scenarios; gluon polFF set to zero
- Inv. FT to k_T space; compression procedure for bands
  [Boglione, UD, Flore, Gonzalez, Murgia, Prokudin '24]
- similar study/results, Sc.1 via TMDJFF [Kang, Lee, Zhao '20]

RESULTS: Λ


$\eta_j = 0$ $p_{jT} = 11$ GeV/c
$p_{\perp\Lambda} < 1.2$ GeV/c
$\sqrt{s} = 200$ GeV
DM
Sc. 1
Λ
P_T^Λ
z
$\sqrt{s} = 200$ GeV
$z < 0.8$
$\eta_j = 0$ $p_{jT} = 11$ GeV/c
$p_{\perp\Lambda}$
Sc. 2
Sc. 3

RESULTS: $\overline{\Lambda}$



FEW REMARKS

- Large experimental error bars: no strong conclusion on the scenarios considered
- General agreement with data, in favor of the predicted universality of the polFF [Metz '02; Boer *et al.* '03; Collins, Metz '04; Yuan, Zhou '09; Boer *et al.* '10]
- Only Sc. 3 gives estimates somewhat far from the data
- Sc. 1 and, to a lesser extent, Sc. 2 seem to be able to describe the data fairly well [interesting to note: both NO *SU(2)*…]

GLUON POLARIZING FF

- Not an issue to parametrize/extract it
- Indirectly:

- role of the unpolarized gluon TMDFF* in the denominator: 50%
- quark contribution to the polarization: around 5% (as data)
- Only a gluon polFF reduced in size up to **10%** of its bound seems to be allowed

* nonperturbative Sudakov soft factor for gluon: $g_K^g = \frac{C_A}{C_F} g_K^q$

CONCLUDING REMARKS

- ❑ Λ-jet in pp collisions: tool to test universality of TMDFF and factoriz.
- ❑ TMD scheme with TMD effects only in the fragmentation mechanism (advantages, direct access to the polFF)
- ❑ estimates based on fits of Belle $e^+e^- \rightarrow \ h_1 \ h_2 + \ X$ data on transverse Λ polarization in fair agreement with STAR pp data: universality, *SU(2)* symmetry issue…
- ❑ Role of the polFF for gluons …first hint
- ❑ Future work:

- open theo issues in hadron-in-jet production

- Transverse Λ polarization in SIDIS and Λ-jet in ep (@EIC)

- (pol)TMDFFs for light/heavy quarks into heavy hadrons, flavor-dependent widths…

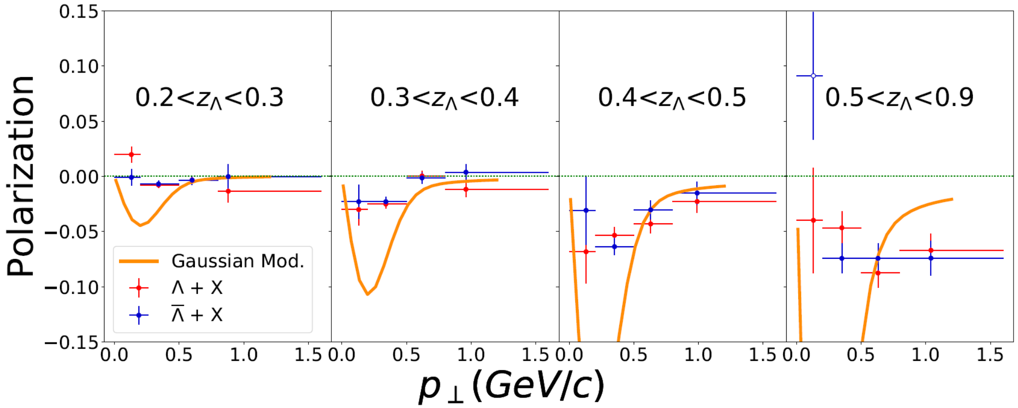
THANKS for the ATTENTION

BACK-UP SLIDES

OPEN ISSUES (FORMER ANALYSIS)

[UD, Gamberg, Murgia, Zaccheddu 2022]

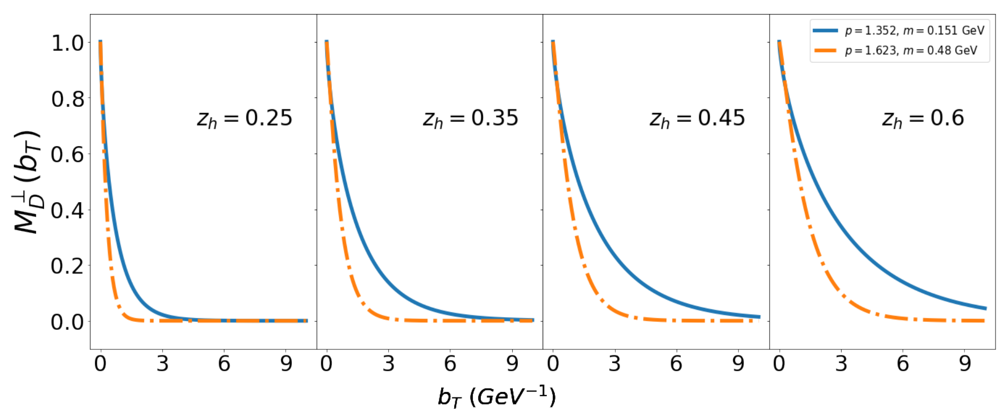
- Tension between *2h* & *1h* data description [e^+e^- collisions]



1h estimates from *2h*-fit

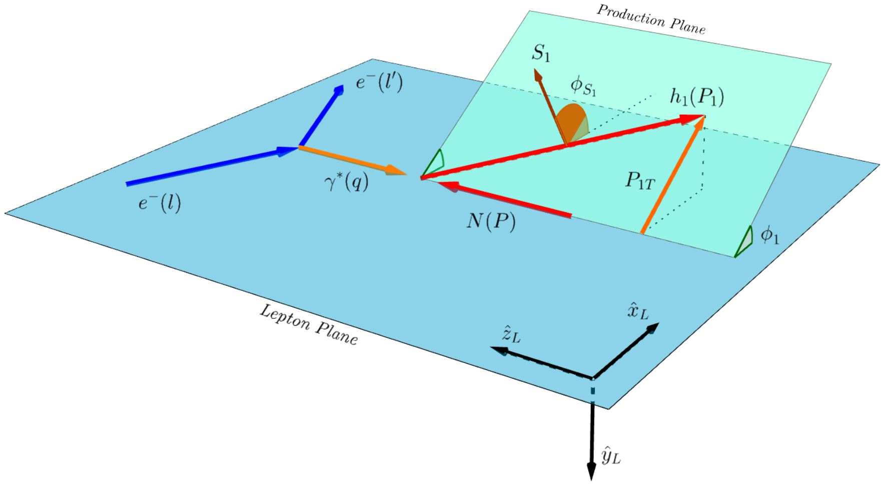
Fit of *2h* and *1h* data:

2 nonperturbative models



2 factorization schemes?

SIDIS



$$P_n^{h_1}(x_B, z_h) = \frac{M_1 \int dq_T\, q_T\, d\phi_1\, \mathcal{B}_1\left[\widetilde{f}_1 \widetilde{D}_{1T}^{\perp(1)}\right]}{\int dq_T\, q_T\, d\phi_1\, \mathcal{B}_0\left[\widetilde{f}_1 \widetilde{D}_1\right]}$$

$$M_{f_1}(b_T, x) = \frac{1}{2\pi} e^{-g_1 \frac{b_T^2}{4}} \left(1 - \frac{\lambda g_1^2}{1+g_1} \frac{b_T^2}{4}\right)$$

Bacchetta, Delcarro, Pisano, Radici, Signori 2017

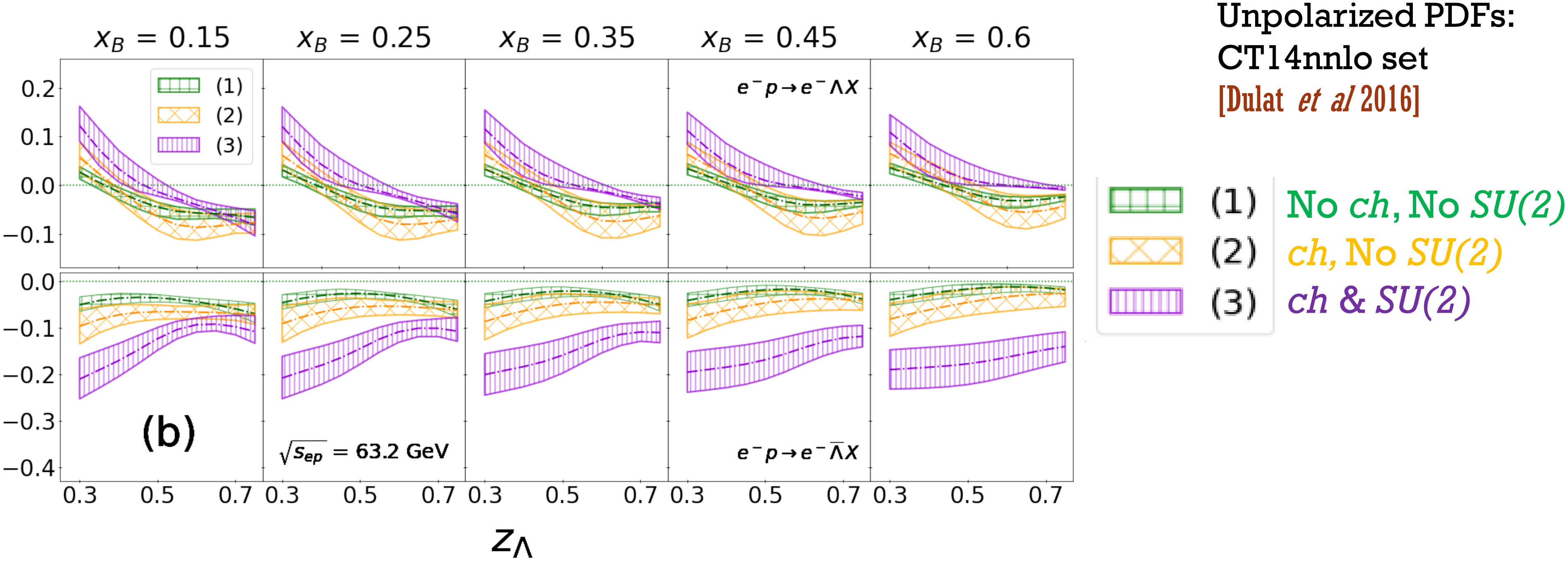
$$\mathcal{B}_0\left[\widetilde{f}_1 \widetilde{D}_1\right] = \frac{1}{z^2} \sum_q e_q^2 \int \frac{db_T}{(2\pi)} b_T J_0(b_T\, q_T) f_{q/N}(x; \bar{\mu}_b) d_{h/q}(z; \bar{\mu}_b)$$
$$\times M_{f_1}(b_c(b_T), x) M_{D_h}(b_c(b_T), z)\, e^{-g_K(b_c(b_T); b_{\max}) \ln\left(\frac{Q^2 z}{x M_P M_h}\right) - S_{\text{pert}}(b_*; \bar{\mu}_b)}$$

Coll PDF

$$\mathcal{B}_1\left[\widetilde{f}_1 \widetilde{D}_{1T}^{\perp(1)}\right] = \frac{1}{z^2} \sum_q e_q^2 \int \frac{db_T}{(2\pi)} b_T^2 J_1(b_T\, q_T) f_{q/N}(x; \bar{\mu}_b) D_{1T,q}^{\perp(1)}(z; \bar{\mu}_b)$$
$$\times M_{f_1}(b_c(b_T), x) M_{D_1}^{\perp}(b_c(b_T), z)\, e^{-g_K(b_c(b_T); b_{\max}) \ln\left(\frac{Q^2 z}{x M_P M_h}\right) - S_{\text{pert}}(b_*; \bar{\mu}_b)}$$

Nonpert funct.

PREDICTIONS FOR EIC



Unpolarized PDFs: CT14nnlo set [Dulat *et al* 2016]

- Sc. 1 & 2: no significant differences
- Λ pol. decreases and becomes negative
- $\bar{\Lambda}$ is always negative

- Sc. 3: similar behavior
- Λ pol. slightly larger
- $\bar{\Lambda}$ most significant difference

Impact study in SIDIS@EIC (Sc 1 at LO) Kang, Terry, Vossen, Xu, Zhang 2022

$$S_{\rm pert}(b_*;\bar{\mu}_b) = -\widetilde{K}(b_*;\bar{\mu}_b)\ln\frac{Q^2}{\bar{\mu}_b^2} - \int_{\bar{\mu}_b}^{Q}\frac{d\mu'}{\mu'}\left[2\gamma_D(g(\mu');1) - \gamma_K(g(\mu'))\ln\frac{Q^2}{\mu'^2}\right]$$

$$\bar{\mu}_b = \frac{C_1}{b_*(b_T)}$$

$$C_1 = 2e^{-\gamma_E}$$

Perturb. CS Kernel

RG anomalous dimensions

cusp

Nonperturbative functions

Bacchetta, Delcarro, Pisano, Radici, Signori 2017

$$M_D(b_T,z) = \frac{g_3\, e^{-b_T^2\frac{g_3}{4z^2}} + \frac{\lambda_F}{z^2}g_4^2\left(1 - g_4\frac{b_T^2}{4z^2}\right)e^{-b_T^2\frac{g_4}{4z^2}}}{g_3 + \frac{\lambda_F}{z^2}g_4^2}$$ light hadrons

$$M_D(b_T,z,p,m) = \frac{2^{2-p}}{\Gamma(p-1)}(b_T m/z_p)^{p-1}K_{p-1}(b_T m/z_p)$$ for Λ Boglione, Simonelli 2021

$$g_K(b_T;b_{\rm max}) = \frac{g_2 b_T^2}{2}\,; \quad g_2 = 0.13\,{\rm GeV}^2$$

$$b_c(b_T) = \sqrt{b_T^2 + b_{\rm min}^2}$$

$$b_*(b_T;b_{\rm min},b_{\rm max}) = b_{\rm max}\left(\frac{1 - e^{-b_T^4/b_{\rm max}^4}}{1 - e^{-b_T^4/b_{\rm min}^4}}\right)^{1/4}$$

$$b_*(b_c(b_T)) \to \begin{cases} b_{\rm min} & b_T \ll b_{\rm min} \\ b_T & b_{\rm min} \ll b_T \ll b_{\rm max} \\ b_{\rm max} & b_T \gg b_{\rm max} \end{cases}$$

Collinear FF sets: for π/K de Florian. Sassot, Stratmann 2007; for Λ Albino, Kniehl, Kramer 2008