Helicity and OAM at Low-x and Large-N_c: an Exact Solution for Revised Helicity Evolution and the Small-x Asymptoics of OAM Distributions

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Based on recent works with Jeremy Borden and Brandon Manley

Quark and Gluon Helicity at Small x: a Brief Overview

YK, D. Pitonyak, M. Sievert, arXiv:1511.06737 [hep-ph], arXiv:1610.06197 [hep-ph] (KPS);
YK, M. Sievert, arXiv:1505.01176 [hep-ph], arXiv:1808.09010 [hep-ph];
F. Cougoulic, YK, A. Tarasov, and Y. Tawabutr, arXiv:2204.11898 [hep-ph] (KPS-CTT).

Sub-Eikonal Operators and Beyond

Dipole picture of DIS

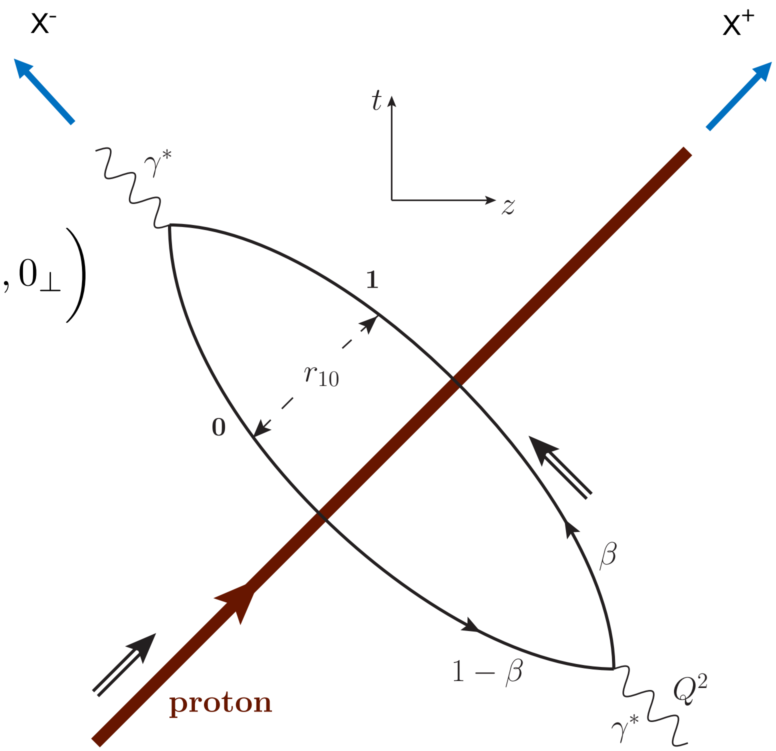
$$W^{\mu\nu} = \frac{1}{4\pi M_p} \int d^4x\, e^{iq\cdot x} \langle P| j^\mu(x)\, j^\nu(0) |P\rangle$$

$$q^\mu = \left(\frac{Q^2}{2q^-}, q^-, 0_\perp \right)$$

Large q^- → large x^- separation

$$e^{iq\cdot x} = e^{i\frac{Q^2}{2q^-}x^- + iq^- x^+}$$

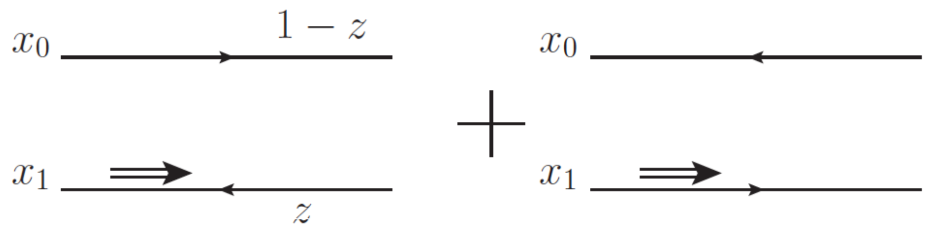
$$x^\pm = \frac{t \pm z}{\sqrt{2}}$$



aka the "shock wave"

Polarized Dipole: non-eikonal small-x physics

- All flavor-singlet small-x helicity observables depend on “polarized dipole amplitudes”:



$$G_{10}(z) \equiv \frac{1}{2N_c} \, \mathrm{Re} \left\langle\!\!\left\langle \mathrm{T} \, \mathrm{tr} \left[V_{\underline{0}} V_{\underline{1}}^{pol \, \dagger} \right] + \mathrm{T} \, \mathrm{tr} \left[V_{\underline{1}}^{pol} V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle (z)$$

unpolarized quark

polarized quark: eikonal propagation, non-eikonal spin-dependent interaction

$$V_{\underline{x}} = \mathcal{P} \exp \left[i g \int\limits_{-\infty}^{\infty} d x^{-} A^{+}(0^{+}, x^{-}, \underline{x}) \right]$$

- Double brackets denote an object with energy suppression scaled out:

$$\left\langle\!\!\left\langle \mathcal{O} \right\rangle\!\!\right\rangle (z) \equiv z s \left\langle \mathcal{O} \right\rangle (z)$$

Notation

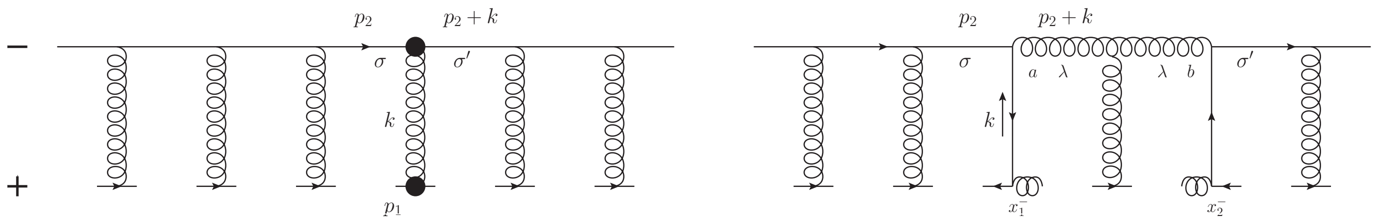
- Fundamental light-cone Wilson line:

$$V_{\underline{x}}[b^-, a^-] = \mathrm{P} \exp \left\{ ig \int\limits_{a^-}^{b^-} dx^- \, A^+(x^-, \underline{x}) \right\}$$

- Adjoint light-cone Wilson line:

$$U_{\underline{x}}[b^-, a^-] = \mathcal{P} \exp \left[ig \int\limits_{a^-}^{b^-} dx^- \, \mathcal{A}^+(x^+ = 0, x^-, \underline{x}) \right]$$

Sub-eikonal quark S-matrix in background gluon and quark fields



- The full sub-eikonal S-matrix for massless quarks is (Balitsky&Tarasov '15; KPS '17; YK, Sievert, '18; Chirilli '18; Altinoluk et al, '20; YK, Santiago '21)

$$V_{\underline{x},\underline{y};\sigma',\sigma} = V_{\underline{x}}\,\delta^2(\underline{x}-\underline{y})\,\delta_{\sigma,\sigma'}$$

"helicity independent" "helicity dependent"

$$-\vec{\mu}\cdot\vec{B} = -\mu_z\,B_z = \mu_z\,F^{12}$$

$$+\frac{i\,P^+}{s}\int\limits_{-\infty}^{\infty} dz^- d^2 z\; V_{\underline{x}}[\infty,z^-]\,\delta^2(\underline{x}-\underline{z})\,\left[-\delta_{\sigma,\sigma'}\,\overleftarrow{D}^i\; D^i + g\,\sigma\,\delta_{\sigma,\sigma'}\,F^{12}\right](z^-,\underline{z})\,V_{\underline{y}}[z^-,-\infty]\,\delta^2(\underline{y}-\underline{z})$$

$$-\frac{g^2 P^+}{2s}\delta^2(\underline{x}-\underline{y})\int\limits_{-\infty}^{\infty} dz_1^-\int\limits_{z_1^-}^{\infty} dz_2^- V_{\underline{x}}[\infty,z_2^-]t^b\psi_\beta(z_2^-,\underline{x})U_{\underline{x}}^{ba}[z_2^-,z_1^-]\left[\delta_{\sigma,\sigma'}\gamma^+ - \sigma\delta_{\sigma,\sigma'}\gamma^+\gamma^5\right]_{\alpha\beta}\bar{\psi}_\alpha(z_1^-,\underline{x})t^a V_{\underline{x}}[z_1^-,-\infty]$$

"helicity independent" "helicity dependent"

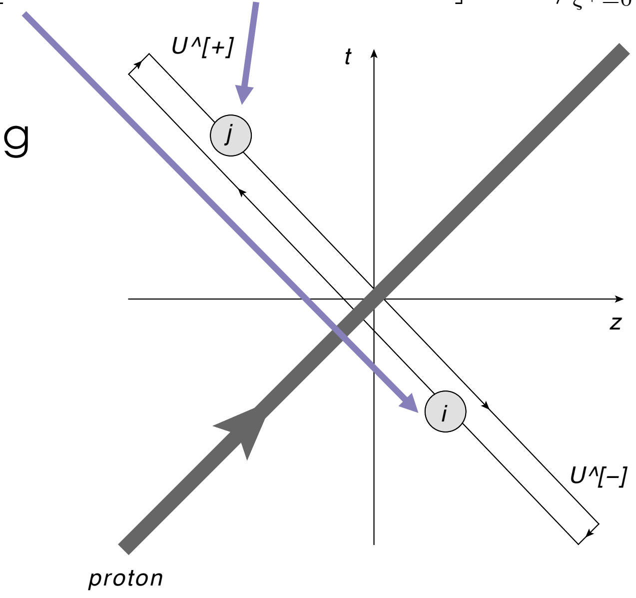
Helicity Distributions and Observables at Small x

Dipole Gluon Helicity TMD

- We start with the definition of the gluon dipole helicity TMD, corresponding to the Jaffe-Manohar PDF ΔG,

$$g_1^G(x, k_T^2) = \frac{-2i\, S_L}{x\, P^+} \int \frac{d\xi^- \, d^2\xi}{(2\pi)^3} \, e^{ixP^+ \xi^- - i\underline{k}\cdot\underline{\xi}} \left\langle P, S_L | \epsilon_T^{ij} \, \mathrm{tr} \left[F^{+i}(0) \, \mathcal{U}^{[+]\dagger}[0,\xi] \, F^{+j}(\xi) \, \mathcal{U}^{[-]}[\xi, 0] \right] | P, S_L \right\rangle_{\xi^+=0}$$

- Here $U^{[+]}$ and $U^{[-]}$ are future and past-pointing Wilson line staples (hence the name 'dipole' TMD, F. Dominguez et al '11 – looks like a dipole scattering on a proton):



Gluon Helicity

- A calculation gives

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \left[\left(1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right) G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right) \right]_{x_{10}^2 = \frac{1}{Q^2}}$$

$$g_{1L}^{G\,dip}(x, k_T^2) = \frac{N_c}{\alpha_s 2\pi^4} \int d^2 x_{10}\, e^{-i \underline{k} \cdot \underline{x}_{10}} \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] G_2 \left(x_{10}^2, zs = \frac{Q^2}{x} \right)$$

- Here we defined a new dipole amplitude G_2 (cf. Hatta et al, 2016; KPS 2017)

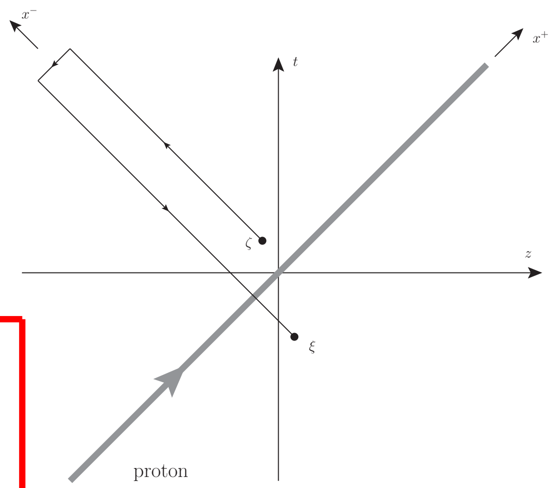
$$\int d^2 \left(\frac{x_1 + x_0}{2} \right) G_{10}^i(zs) = (x_{10})_\perp^i \, G_1(x_{10}^2, zs) + \epsilon^{ij} \, (x_{10})_\perp^j \, G_2(x_{10}^2, zs)$$

$$G_{10}^j(zs) \equiv \frac{1}{2N_c} \left\langle\!\!\left\langle \mathrm{tr} \left[V_{\underline{0}}^\dagger \, V_{\underline{1}}^{j\,\mathrm{G}[2]} + \left(V_{\underline{1}}^{j\,\mathrm{G}[2]} \right)^\dagger V_{\underline{0}} \right] \right\rangle\!\!\right\rangle$$

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} \equiv \frac{P^+}{2s} \int\limits_{-\infty}^{\infty} dz^- \, V_{\underline{z}}[\infty, z^-] \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i (z^-, \underline{z}) \right] V_{\underline{z}}[z^-, -\infty]$$

What is this D-D operator? Turns out it is related to the DD operator from before.

Quark Helicity PDF and TMD



- The flavor-singlet quark helicity PDF and SIDIS TMD are

$$\Delta\Sigma(x, Q^2) = -\frac{N_c\, N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

$$g_{1L}^S(x, k_T^2) = \frac{8\, i\, N_c\, N_f}{(2\pi)^5} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int d^2 x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\, \frac{\underline{x}_{10}}{x_{10}^2} \cdot \frac{\underline{k}}{\underline{k}^2} \left[Q(x_{10}^2, zs) + 2\, G_2(x_{10}^2, zs)\right]$$

- G_2 was defined before. This is the gluon admixture to quark helicity distributions.
- The dipole amplitude Q is due to F^{12} & axial current.
- The contribution of G_2 comes from the DD operator in the quark S-matrix.
- Hence, the DD operator is related to the Jaffe-Manohar distribution.

Amplitude Q

$$Q(x_{10}^2, zs) \equiv \int d^2 \left(\frac{x_0 + x_1}{2} \right) Q_{10}(zs)$$

- The amplitude Q is defined by

$$Q_{10}(zs) \equiv \frac{1}{2N_c} \, \text{Re} \left\langle\!\!\left\langle \text{T tr} \left[V_{\underline{0}} \, V_{\underline{1}}^{\text{pol}[1] \, \dagger} \right] + \text{T tr} \left[V_{\underline{1}}^{\text{pol}[1]} \, V_{\underline{0}}^{\dagger} \right] \right\rangle\!\!\right\rangle$$

with $V_{\underline{x}}^{\text{pol}[1]} = V_{\underline{x}}^{\text{G}[1]} + V_{\underline{x}}^{\text{q}[1]}$, where

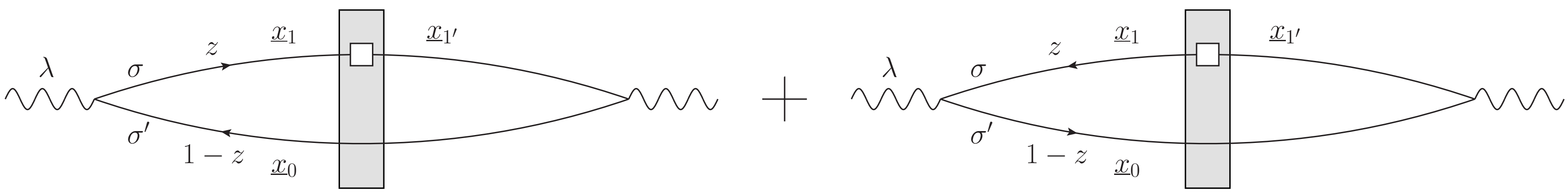
$$V_{\underline{x}}^{\text{G}[1]} = \frac{i \, g \, P^+}{s} \int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-] \, F^{12}(x^-, \underline{x}) \; V_{\underline{x}}[x^-, -\infty]$$

$$V_{\underline{x}}^{\text{q}[1]} = \frac{g^2 P^+}{2 \, s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \; \left[\gamma^+ \gamma^5 \right]_{\alpha\beta} \; \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty]$$

- This is the 'old' KPS polarized dipole amplitude. U = adjoint light-cone Wilson line.

g_1 structure function

- g_1 structure function is obtained similarly, using DIS in the dipole picture:



- One gets

$$g_1(x,Q^2) = -\sum_f \frac{N_c\, Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \left[Q(x_{10}^2, zs) + 2\,G_2(x_{10}^2, zs)\right]$$

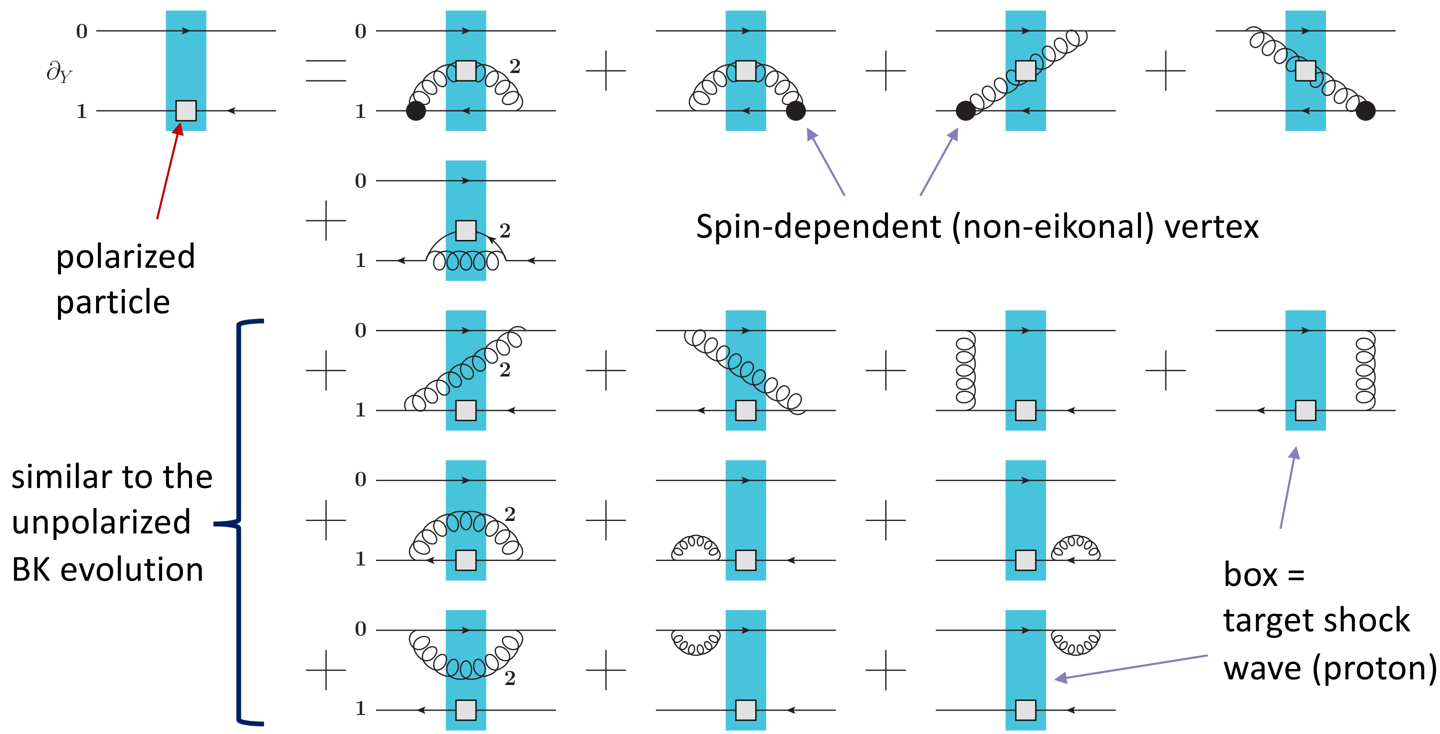
such that one reproduces the standard result $g_1(x,Q^2) = \frac{1}{2}\sum_f Z_f^2\, \Delta q_f^+(x,Q^2)$

$$\Delta q_f^+ = \Delta q_f + \Delta \bar{q}_f$$

Helicity Evolution

Evolution for Polarized Quark Dipole

One can construct an evolution equation for the polarized dipole:



Large N_c

- At large-N_c the equations close (Q → G).
- Everything with 2 in the subscript (e.g., G_2 and Γ_2) is new compared to the KPS ('15-'18) papers.

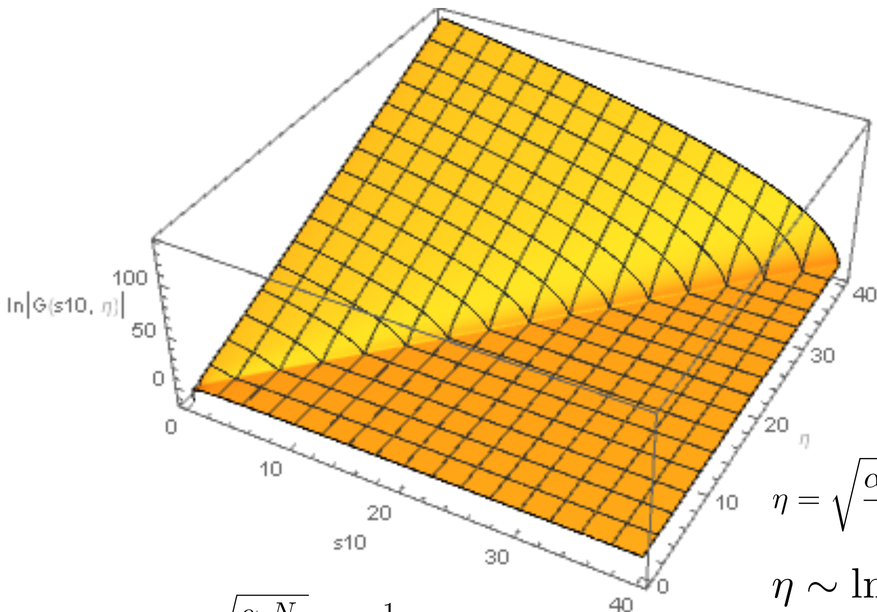
$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big],$$

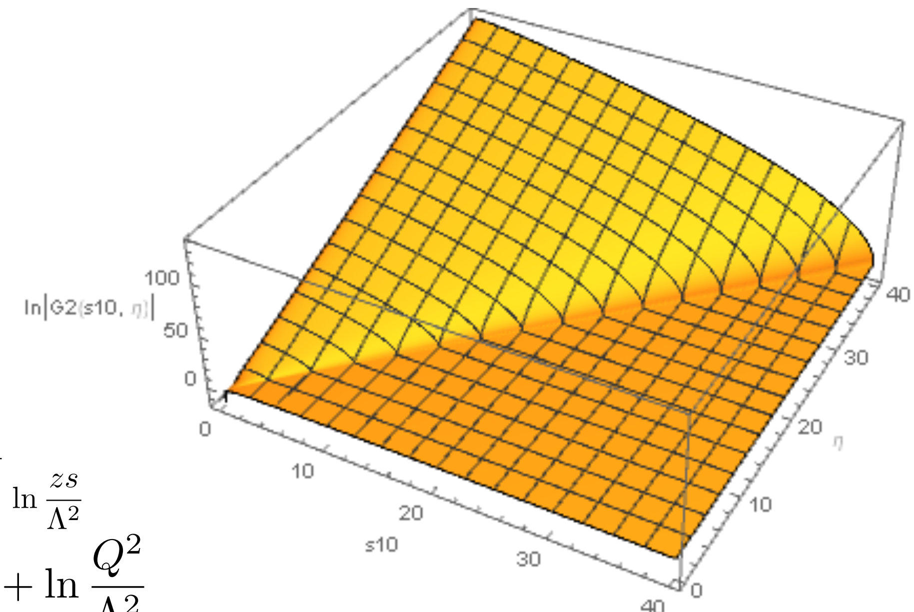
$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big],$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]$$

Solution of the Large-N_c Equations





$$\eta = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{zs}{\Lambda^2}$$

$$\eta \sim \ln \frac{1}{x} + \ln \frac{Q^2}{\Lambda^2}$$

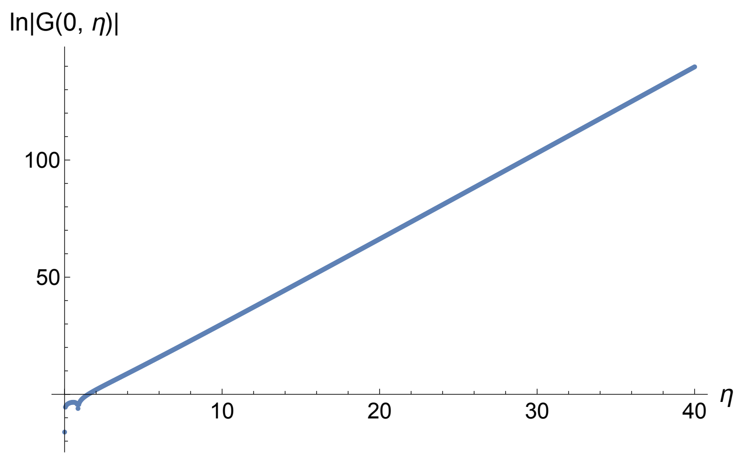
$$s_{10} = \sqrt{\frac{\alpha_s N_c}{2\pi}}\, \ln \frac{1}{x_{10}^2\, \Lambda^2}$$

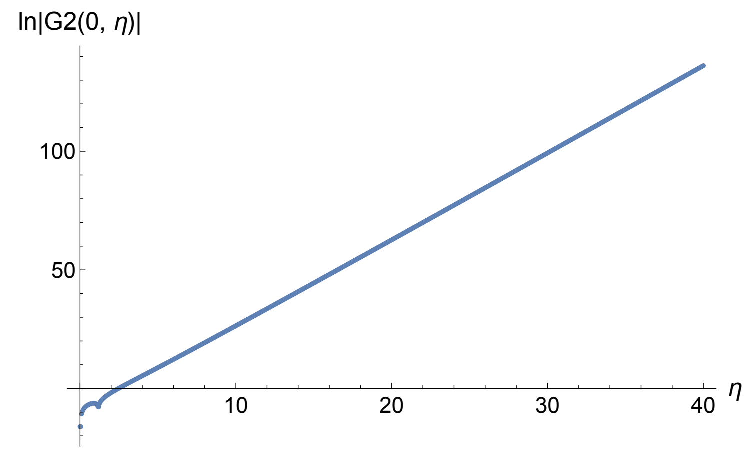
$$s_{10} \sim \ln \frac{Q^2}{\Lambda^2}$$

The large-N_c equations for G and G_2 can be solved numerically (and, possibly, analytically).

Small-x Asymptotics

- Fitting the slope of the log plots of G and G_2 vs e we can read off the small-x intercept (the power of x):





F. Cougoulic, YK, A. Tarasov, Y. Tawabutr, 2022

Small-x Asymptotics for Helicity Distributions

- The resulting small-x asymptotics for helicity PDFs and the g_1 structure function at large N_c is

$$\Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim g_1(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}}$$

- This power (aka the intercept) is in complete agreement with the work by J. Bartels, B. Ermolaev, and M. Ryskin (BER, 1996) using infrared evolution equations (with the analytic intercept constructed by KPS in 2016):

$$\alpha_h = \sqrt{\frac{17+\sqrt{97}}{2}}\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}} \approx 3.664\,\sqrt{\frac{\alpha_s\,N_c}{2\pi}}$$

- "Peace in the valley."
- **Right?**

Quark Helicity at Small x and Large-N_c: Analytic Solution

J. Borden, YK, 2304.06161 [hep-ph]

Analytic Solution of the Large-N_c Equations

- We want to solve these equations:

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \Big[\Gamma(x_{10}^2, x_{21}^2, z's) + 3\, G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) + 2\, \Gamma_2(x_{10}^2, x_{21}^2, z's) \Big],$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min\left[x_{10}^2, x_{21}^2 \frac{z'}{z''}\right]} \frac{dx_{32}^2}{x_{32}^2} \Big[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3\, G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) + 2\, \Gamma_2(x_{10}^2, x_{32}^2, z''s) \Big],$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max\left[x_{10}^2, \frac{1}{z's}\right]}^{\min\left[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2\, G_2(x_{21}^2, z's) \right],$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max\left[x_{10}^2, \frac{1}{z''s}\right]}^{\min\left[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}\right]} \frac{dx_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2\, G_2(x_{32}^2, z''s) \right]$$

Analytic Solution of the Large-N_c Equations

- The strategy is to use the double Laplace transform,

$$\bar{\alpha}_s \equiv \frac{\alpha_s N_c}{2\pi}$$

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \, e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \, G_{2\omega\gamma}$$

- One gets the expressions for all the other dipole amplitudes this way, for instance

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \, e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{\omega\gamma}{2\,\bar{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) - 2\, G_{2\omega\gamma} \right]$$

- Neighbor amplitudes involve several different double Laplace transforms:

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} \left[e^{\omega \ln(z'sx_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)} \right) + e^{\omega \ln(z'sx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \, G_{2\omega\gamma}^{(0)} \right]$$

Analytic Solution of the Large-N$_c$ Equations

- In the end, all the amplitudes in the double-Laplace space can be expressed in terms of the initial conditions/inhomogeneous terms, e.g.,

$$G_{2\omega\gamma} = G^{(0)}_{2\omega\gamma} + \frac{\bar{\alpha}_s}{\omega\,(\gamma - \gamma^-_\omega)\,(\gamma - \gamma^+_\omega)} \Bigg[2\,(\gamma - \delta^+_\omega)\left(G^{(0)}_{\delta^+_\omega \gamma} + 2\,G^{(0)}_{2\,\delta^+_\omega \gamma} \right) - 2\,(\gamma^+_\omega - \delta^+_\omega)\left(G^{(0)}_{\delta^+_\omega \gamma^+_\omega} + 2\,G^{(0)}_{2\,\delta^+_\omega \gamma^+_\omega} \right) + 8\,\delta^-_\omega \left(G^{(0)}_{2\omega\gamma} - G^{(0)}_{2\omega\gamma^+_\omega} \right) \Bigg]$$

with

$$\delta^\pm_\omega = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}\right] \qquad \gamma^\pm_\omega = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{16\,\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- More details in J. Borden, YK, 2304.06161 [hep-ph].

Small-x Asymptotics for Helicity Distributions

- Let's take a closer look at the anomalous dimension:

$$\Delta G(x, Q^2) = \int \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} \left(\frac{Q^2}{\Lambda^2}\right)^{\Delta\gamma_{GG}(\omega)} \Delta G_\omega(\Lambda^2)$$

- In the pure-glue case, Bartels, Ermolaev and Ryskin's (BER) anomalous dimension can be found analytically. It reads (KPS '16)

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1 - \frac{3\,\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \bar{\alpha}_s = \frac{\alpha_s\, N_c}{2\pi}$$

- Our evolution's anomalous dimension can be found analytically at large-N_c (J. Borden, YK, 2304.06161 [hep-ph]):

$$\Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

A Tale of Two Anomalous Dimensions

- The two anomalous dimensions look similar but are not the same function.

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1 - \frac{3\,\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- Their expansions in α_S start out the same, then **differ at four (!) loops** (the first 3 terms agree with the existing finite-order calculations, the four-loop result is unknown):

$$\Delta\gamma_{GG}^{BER}(\omega) = \underbrace{\frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5}} + \underbrace{\frac{504\,\bar{\alpha}_s^4}{\omega^7}} + \ldots$$

$$\Delta\gamma_{GG}^{us}(\omega) = \underbrace{\frac{4\,\bar{\alpha}_s}{\omega} + \frac{8\,\bar{\alpha}_s^2}{\omega^3} + \frac{56\,\bar{\alpha}_s^3}{\omega^5}} + \underbrace{\frac{496\,\bar{\alpha}_s^4}{\omega^7}} + \ldots$$

A Tale of Two Intercepts

$$\Delta G(x, Q^2) = \int \frac{d\omega}{2\pi i} \left(\frac{1}{x}\right)^{\omega} \left(\frac{Q^2}{\Lambda^2}\right)^{\Delta\gamma_{GG}(\omega)} \Delta G_{\omega}(\Lambda^2)$$

$$\Delta\gamma_{GG}^{BER}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \frac{1 - \frac{3\,\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] \qquad \Delta\gamma_{GG}^{us}(\omega) = \frac{1}{2}\left[\omega - \sqrt{\omega^2 - 16\,\bar{\alpha}_s \sqrt{1 - \frac{4\,\bar{\alpha}_s}{\omega^2}}}\right]$$

- The intercept (largest power Re[ω]) is given by the right-most singularity (branch point) of the anomalous dimension.
- For BER this gives $\alpha_h = \sqrt{\frac{17 + \sqrt{97}}{2}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.664 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$
- For us $\alpha_h = \frac{4}{3^{1/3}} \sqrt{\mathrm{Re}\left[(-9 + i\,\sqrt{111})^{1/3}\right]} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.661 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$

A Tale of Two Intercepts

$$\Delta\Sigma(x, Q^2)\Big|_{x \ll 1} \sim \Delta G(x, Q^2)\Big|_{x \ll 1} \sim \left(\frac{1}{x}\right)^{\alpha_h}$$

- The power α_h is known as the intercept.
- BER:

$$\alpha_h = \sqrt{\frac{17 + \sqrt{97}}{2}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.664 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

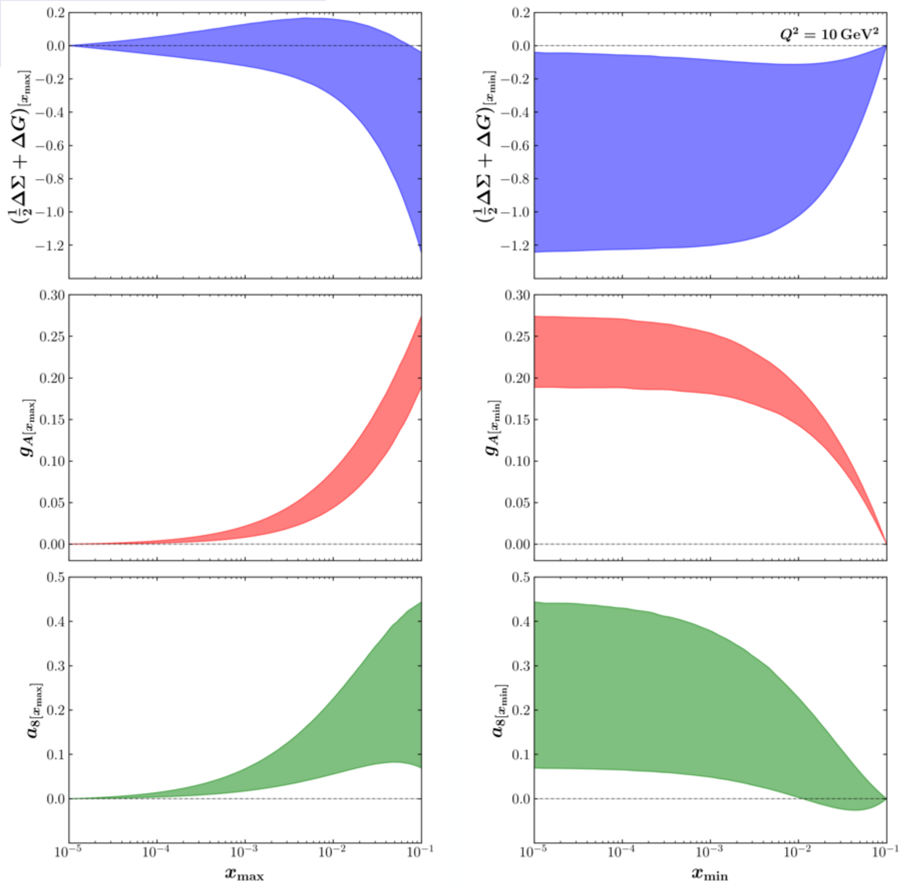
- Us:

$$\alpha_h = \frac{4}{3^{1/3}} \sqrt{\mathrm{Re}\left[(-9 + i\,\sqrt{111})^{1/3}\right]} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.661 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

- Our numerical solution also gave the intercept of 3.660 or 3.661, but we believed we had larger error bars.
- We (still) disagree with BER. Albeit in the 3rd decimal point…

JAMsmallx: Adamiak, Baldonado, YK, Melnitchouk, Pitonyak, Sato, Sievert, Tarasov, Tawabutr 2023

How much spin is there at small x?



$$\left(\frac{1}{2}\Delta\Sigma + \Delta G\right)_{[x_{\min}]}(Q^2) \equiv \int_{x_{\min}}^{x_0} dx \left(\frac{1}{2}\Delta\Sigma + \Delta G\right)(x, Q^2),$$

$$g_{A[x_{\min}]}(Q^2) \equiv \int_{x_{\min}}^{x_0} dx \left[\Delta u^+(x,Q^2) - \Delta d^+(x,Q^2)\right],$$

$$a_{8[x_{\min}]}(Q^2) \equiv \int_{x_{\min}}^{x_0} dx \left[\Delta u^+(x,Q^2) + \Delta d^+(x,Q^2) - 2\,\Delta s^+(x,Q^2)\right]$$

$$\left(\frac{1}{2}\Delta\Sigma + \Delta G\right)_{[x_{\max}]}(Q^2) \equiv \int_{10^{-5}}^{x_{\max}} dx \left(\frac{1}{2}\Delta\Sigma + \Delta G\right)(x, Q^2),$$

$$g_{A[x_{\max}]}(Q^2) \equiv \int_{10^{-5}}^{x_{\max}} dx \left[\Delta u^+(x,Q^2) - \Delta d^+(x,Q^2)\right],$$

$$a_{8[x_{\max}]}(Q^2) \equiv \int_{10^{-5}}^{x_{\max}} dx \left[\Delta u^+(x,Q^2) + \Delta d^+(x,Q^2) - 2\,\Delta s^+(x,Q^2)\right]$$

$$\int_{10^{-5}}^{0.1} \mathrm{d}x \left(\frac{1}{2}\Delta\Sigma + \Delta G\right)(x) = -0.64 \pm 0.60$$

Negative net spin at small x!

Potentially a lot of spin at small x. However, the uncertainties are large. Need a way to constrain the initial conditions.

Quark and Gluon OAM at Small x and Large N_c

YK, B. Manley, 2310.18404 [hep-ph]; B. Manley, 2401.05508 [hep-ph]

OAM Distributions

- We begin by writing the (Jaffe-Manohar) quark and gluon OAM in terms of the Wigner distribution as

$$L_z = \int \frac{d^2 b_\perp \, db^- \, d^2 k_\perp \, dk^+}{(2\pi)^3} \, (\underline{b} \times \underline{k})_z \, W(k,b)$$

- After much algebra, we arrive at the quark and gluon OAM distributions at small x :

$$L_{q+\bar{q}}(x,Q^2) = \frac{N_c N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \Big[Q(x_{10}^2, zs) - 3\, G_2(x_{10}^2, zs) - I_3(x_{10}^2, zs) - 2\, I_4(x_{10}^2, zs) + I_5(x_{10}^2, zs) + 3\, I_6(x_{10}^2, zs) \Big]$$

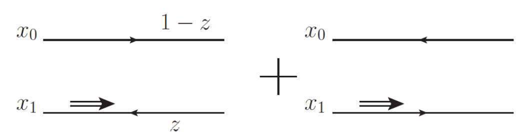
$$L_G(x,Q^2) = -\frac{2\, N_c}{\alpha_s \pi^2} \left\{ \left[2 + 6\, x_{10}^2 \frac{\partial}{\partial x_{10}^2} + 2\, x_{10}^4 \frac{\partial^2}{\partial (x_{10}^2)^2} \right] \left[I_4(x_{10}^2, zs) + I_5(x_{10}^2, zs) \right] + \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] \left[I_5(x_{10}^2, zs) + I_6(x_{10}^2, zs) \right] \right\}_{x_{10}^2 = 1/Q^2, \, zs = Q^2/x}$$

OAM Distributions and Moment Amplitudes

$$L_{q+\bar{q}}(x,Q^2) = \frac{N_c N_f}{2\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{dz}{z} \int\limits_{\frac{1}{zs}}^{\min\left\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\right\}} \frac{dx_{10}^2}{x_{10}^2} \Big[Q(x_{10}^2, zs) - 3\, G_2(x_{10}^2, zs) - I_3(x_{10}^2, zs) - 2\, I_4(x_{10}^2, zs) + I_5(x_{10}^2, zs) + 3\, I_6(x_{10}^2, zs) \Big]$$

$$L_G(x,Q^2) = -\frac{2\,N_c}{\alpha_s \pi^2} \left\{ \left[2 + 6\, x_{10}^2 \frac{\partial}{\partial x_{10}^2} + 2\, x_{10}^4 \frac{\partial^2}{\partial (x_{10}^2)^2} \right] \left[I_4(x_{10}^2, zs) + I_5(x_{10}^2, zs) \right] + \left[1 + x_{10}^2 \frac{\partial}{\partial x_{10}^2} \right] \left[I_5(x_{10}^2, zs) + I_6(x_{10}^2, zs) \right] \right\}_{x_{10}^2 = 1/Q^2,\ zs = Q^2/x}$$

- Q and G_2 are the same as above. However, we also now have the impact parameter **moments of dipole amplitudes**, labeled I_3, I_4, I_5 and I_6:



$$\int d^2 x_1\, x_1^i\, Q_{10}(zs) = x_{10}^i\, I_3(x_{10}^2, zs) + \ldots,$$

$$\int d^2 x_1\, x_1^i G_{10}^j(zs) = \epsilon^{ij}\, x_{10}^2\, I_4(x_{10}^2, zs) + \epsilon^{ik}\, x_{10}^k\, x_{10}^j\, I_5(x_{10}^2, zs) + \epsilon^{jk}\, x_{10}^k\, x_{10}^i\, I_6(x_{10}^2, zs) + \ldots$$

Evolution for Moment Dipole Amplitudes

Evolution equations for the moment amplitudes in DLA and at large N_c are derived in YK, B. Manley, 2310.18404 [hep-ph].

They can be solved numerically (same ref) and analytically (B. Manley, 2401.05508 [hep-ph])

$$\begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \end{pmatrix}(x_{10}^2, zs) = \begin{pmatrix} I_3^{(0)} \\ I_4^{(0)} \\ I_5^{(0)} \\ I_6^{(0)} \end{pmatrix}(x_{10}^2, zs) + \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z's}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 2\,\Gamma_3 - 4\,\Gamma_4 + 2\,\Gamma_5 + 6\,\Gamma_6 - 2\,\Gamma_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}(x_{10}^2, x_{21}^2, z's)$$

$$+ \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int\limits_{\max[x_{10}^2, \frac{1}{z's}]}^{\min[\frac{z}{z'} x_{10}^2, \frac{1}{\Lambda^2}]} \frac{dx_{21}^2}{x_{21}^2} \begin{pmatrix} 4 & -4 & 2 & 6 & -4 & -6 \\ 0 & 4 & 2 & -2 & 0 & 1 \\ -2 & 2 & -1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \\ G \\ G_2 \end{pmatrix}(x_{21}^2, z's)$$

$$\begin{pmatrix} \Gamma_3 \\ \Gamma_4 \\ \Gamma_5 \\ \Gamma_6 \end{pmatrix}(x_{10}^2, x_{21}^2, z's) = \begin{pmatrix} I_3^{(0)} \\ I_4^{(0)} \\ I_5^{(0)} \\ I_6^{(0)} \end{pmatrix}(x_{10}^2, z's)$$

$$+ \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{1}{s x_{10}^2}}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z''s}}^{\min[x_{10}^2, x_{21}^2 \frac{z'}{z''}]} \frac{dx_{32}^2}{x_{32}^2} \begin{pmatrix} 2\,\Gamma_3 - 4\,\Gamma_4 + 2\,\Gamma_5 + 6\,\Gamma_6 - 2\,\Gamma_2 \\ 0 \\ 0 \\ 0 \end{pmatrix}(x_{10}^2, x_{32}^2, z''s)$$

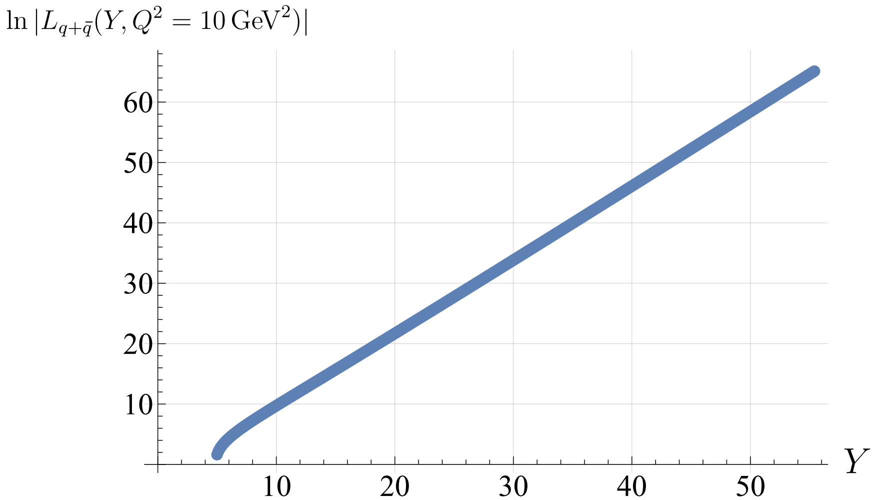
$$+ \frac{\alpha_s N_c}{4\pi} \int\limits_{\frac{\Lambda^2}{s}}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{dz''}{z''} \int\limits_{\max[x_{10}^2, \frac{1}{z''s}]}^{\min[\frac{z'}{z''} x_{21}^2, \frac{1}{\Lambda^2}]} \frac{dx_{32}^2}{x_{32}^2} \begin{pmatrix} 4 & -4 & 2 & 6 & -4 & -6 \\ 0 & 4 & 2 & -2 & 0 & 1 \\ -2 & 2 & -1 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 4 \end{pmatrix} \begin{pmatrix} I_3 \\ I_4 \\ I_5 \\ I_6 \\ G \\ G_2 \end{pmatrix}(x_{32}^2, z''s)$$

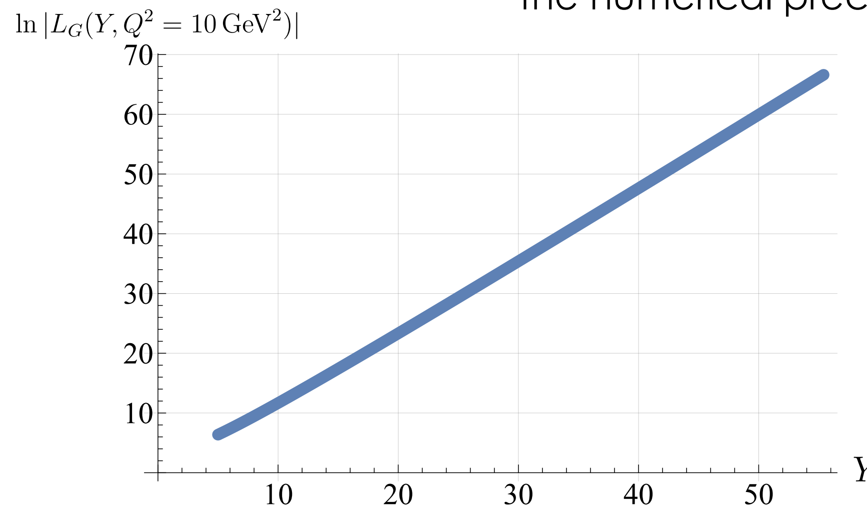
Small-x asymptotics of OAM distributions

- Solving the above evolution equations numerically, we arrive at

$$L_{q+\bar{q}}(x, Q^2) \sim L_G(x, Q^2) \sim \left(\frac{1}{x}\right)^{3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Consistent with Boussarie, Hatta, Yuan 2019 (based on BER IREE from 1996) within the numerical precision





Two intercepts, again

- The evolution equations for moment dipole amplitudes have been solved analytically by B. Manley in 2401.05508 [hep-ph]. The solution was constructed using the double Laplace transform, similar to the solution for the impact-parameter integrated amplitudes.
- The resulting small-x OAM asymptotics at large N_c is the same as for helicity PDFs,

$$L_{q+\bar{q}}(x, Q^2) \sim L_G(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$$

with the intercept

$$\alpha_h = \frac{4}{3^{1/3}} \sqrt{\mathrm{Re}\left[(-9 + i\,\sqrt{111})^{1/3}\right]} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.661 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

- This slightly disagrees with the work of Boussarie, Hatta, and Yuan (2019), which resulted in the same intercept as BER:

$$\alpha_h = \sqrt{\frac{17 + \sqrt{97}}{2}} \sqrt{\frac{\alpha_s\, N_c}{2\pi}} \approx 3.664 \sqrt{\frac{\alpha_s\, N_c}{2\pi}}$$

OAM Distribution to hPDF Ratios

- Following Boussarie *et al* (2019), we consider the ratios of OAM distributions to helicity PDFs at small x.
- For these ratios, Boussarie *et al*, predict, using the Wandzura-Wilczek approximation:

$$\frac{L_{q+\bar{q}}(x,Q^2)}{\Delta\Sigma(x,Q^2)} = -\frac{1}{1+\alpha_h} \qquad \frac{L_G(x,Q^2)}{\Delta G(x,Q^2)} = -\frac{2}{1+\alpha_h}$$

$$\Delta\Sigma(x,Q^2)\Big|_{x\ll 1} \sim \Delta G(x,Q^2)\Big|_{x\ll 1} \sim \left(\frac{1}{x}\right)^{\alpha_h} \qquad L_{q+\bar{q}}(x,Q^2) \sim L_G(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$$

OAM Distribution to hPDF Ratios

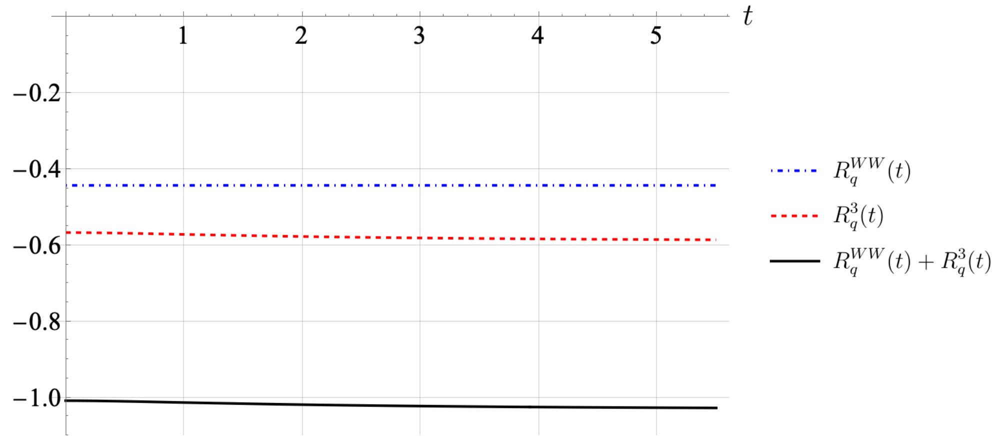
- Analytic solution from B. Manley, 2401.05508 [hep-ph], gives

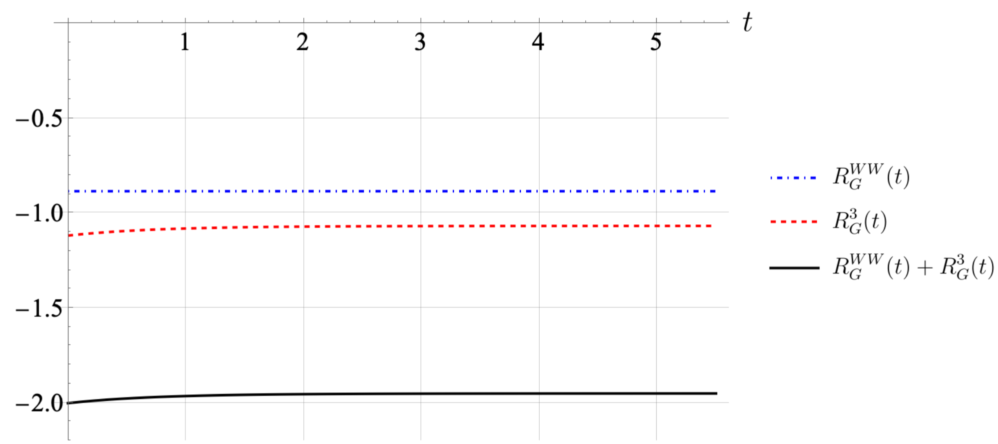
$$t = \sqrt{\frac{\alpha_s N_c}{2\pi} \ln \frac{Q^2}{\Lambda^2}}$$

$$\alpha_s = 0.25 \qquad \Lambda = 1 \text{ GeV}$$

$$\frac{L_{q+\bar{q}}(x,Q^2)}{\Delta\Sigma(x,Q^2)} = -\frac{1}{1+\alpha_h}$$

$$\frac{L_G(x,Q^2)}{\Delta G(x,Q^2)} = -\frac{2}{1+\alpha_h}$$





Conclusions

We have constructed an exact analytical solution of the large-N_c helicity evolution equations. It yielded small-x asymptotics of $\Delta\Sigma(x, Q^2)$ and $\Delta G(x, Q^2)$. The solution agrees with the results in the literature to the 3 known loops but slightly disagrees with BER starting from 4 loops and on the intercept.

We have obtained the small-x asymptotics of OAM distributions for quarks and gluons at large N_c. Again, the intercept slightly disagrees with the one in the literature. Our ratios of the OAM distributions to hPDFs also appear to deviate from the ones obtained using the WW approximation.

Stay tuned for the additional corrections to the large-N_c&N_f helicity evolution due to q→G and G→q transitions in the shock wave (J. Borden, M. Li, YK, in preparation; see also G. Chirilli, 2021) + the exact analytical solution of those modified equations.

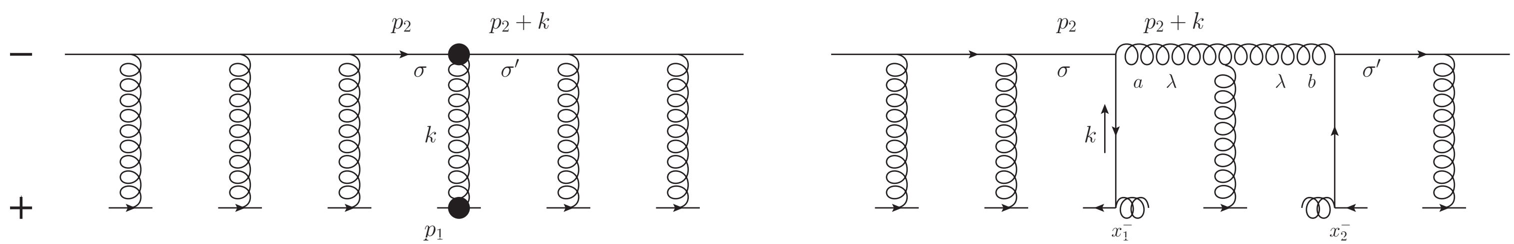
Small-x phenomenology of the polarized DIS and SIDIS data has been done, giving an estimate of the amount of quark and gluon spin at small x. Error bars are large though.

Gluon production in longitudinally polarized p+p collisions at small x has also been calculated recently (see talk by Ming Li on Wednesday): ready for phenomenology (N. Baldonado et al, in preparation).

Backup Slides

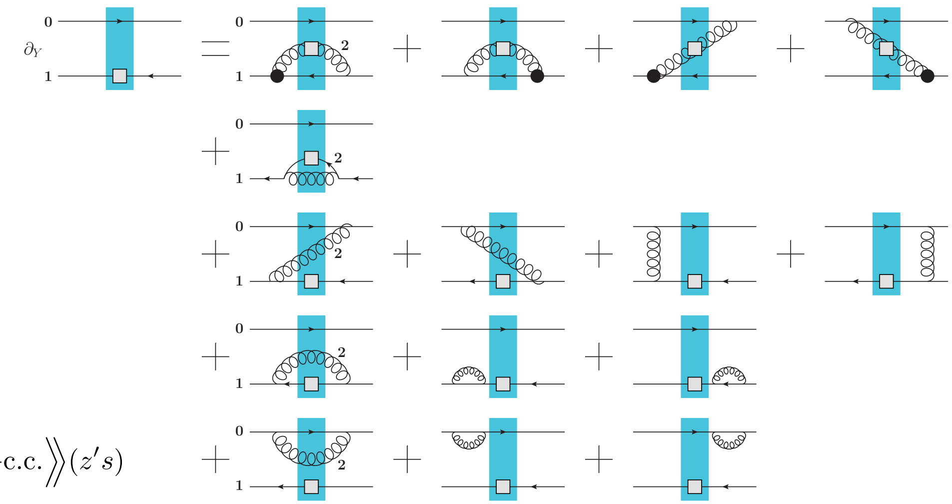
Polarized fundamental "Wilson line"

- To complete the definition of the polarized dipole amplitude, we need to construct the definition of the polarized "Wilson line" V^{pol}, which is the leading helicity-dependent contribution for the quark scattering amplitude on a longitudinally-polarized target proton.



- At the leading order we can either exchange one non-eikonal *t*-channel gluon (with quark-gluon vertices denoted by blobs above) to transfer polarization between the projectile and the target, or two *t*-channel quarks, as shown above.
- We employ a blend of Brodsky & Lepage's LCPT and background field method-inspired operator treatment. We refer to the latter as the **light-cone operator treatment (LCOT)**.

Evolution for Polarized Quark Dipole



$$\langle\langle \ldots \rangle\rangle = z\, s\, \langle \ldots \rangle$$

$$\begin{aligned}
&\frac{1}{2N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\right\rangle(zs) = \frac{1}{2N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] + \mathrm{c.c.}\right\rangle\!\right\rangle_0(zs) \\
&+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2 \Bigg\{ \left[\frac{1}{x_{21}^2} - \frac{\underline{x}_{21}}{x_{21}^2}\cdot\frac{\underline{x}_{20}}{x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba} + \mathrm{c.c.}\right\rangle\!\right\rangle(z's) \\
&+\left[2\frac{\epsilon^{ij}\, x_{21}^j}{x_{21}^4} - \frac{\epsilon^{ij}\,(x_{20}^j + x_{21}^j)}{x_{20}^2\, x_{21}^2} - \frac{2\,\underline{x}_{20}\times\underline{x}_{21}}{x_{20}^2\, x_{21}^2}\left(\frac{x_{21}^i}{x_{21}^2} - \frac{x_{20}^i}{x_{20}^2}\right)\right]\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b V_{\underline{0}} t^a V_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{i\,\mathrm{G}[2]}\right)^{ba}\right\rangle\!\right\rangle(z's)\Bigg\} \\
&+\frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int \frac{d^2x_2}{x_{21}^2}\Bigg\{\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{1}}^{ba}\right\rangle\!\right\rangle(z's) + 2\,\frac{\epsilon^{ij}\,\underline{x}_{21}^j}{x_{21}^2}\,\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{i\,\mathrm{G}[2]\,\dagger}\right] U_{\underline{1}}^{ba}\right\rangle\!\right\rangle(z's)\Bigg\} \\
&+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\, \frac{x_{10}^2}{x_{21}^2\, x_{20}^2}\Bigg\{\frac{1}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right] U_{\underline{2}}^{ba}\right\rangle\!\right\rangle(z's) - \frac{C_F}{N_c^2}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]\right\rangle\!\right\rangle(z's)\Bigg\} + \\
&\mathrm{c.c.}
\end{aligned}$$

Equation does not close!

OAM Distribution to hPDF Ratios

$$\frac{L_{q+\bar{q}}(x,Q^2)}{\Delta\Sigma(x,Q^2)} = -\frac{1}{1+\alpha_h}$$

- Our numerical solution gives the following ratios (α_s=0.25):

$$\frac{L_G(x,Q^2)}{\Delta G(x,Q^2)} = -\frac{2}{1+\alpha_h}$$

$$Y = \ln\frac{1}{x}$$