Presenting Honeycomb/Snowflake

numerical implementation for twist-3 PDF evolution equations

Simone Rodini

Deutsches Elektronen-Synchrotron DESY, Notkestr. 85, 22607 Hamburg, Germany

In collaboration with Alexey Vladimirov and Lorenzo Rossi

May 29, 2024

Contents

Introduction

Why

Polarized processes

g_2 structure function
and Single-Spin Asymemtries

Sivers function $f_{1T}^{\perp}$
and polarized TMDs in general

Lattice quasi-distributions

[...]

Fundamentals

The operators

$$\bar{q}_i(z_1 n) g F_a^{\mu+}(z_2 n) q_j(z_3 n) \qquad F_a^{\mu+}(z_1 n) g F_b^{\nu+}(z_2 n) F_c^{\rho+}(z_3 n)$$

The color projectors

$$T^a_{ij} \qquad i f^{abc} \qquad d^{abc}$$

The spin projectors

for the quark-gluon-quark
$\gamma^+,\ \gamma^+\gamma_5,\ i\sigma^{\alpha+}\gamma_5$

for the 3-gluons
a bit more complex

Symmetries and independent distributions

From Parity, Time-Reversal and Charge Conjugation:

2 independent quark-gluon-quark distributions
and
2 independent 3-gluon distributions

$$\mathfrak{S}^{\pm}(x_1, x_2, x_3) = \pm\mathfrak{S}^{\pm}(-x_1, -x_2, -x_3)$$

$$\mathfrak{F}^{\pm}(x_1, x_2, x_3) = \mp\mathfrak{F}^{\pm}(-x_1, -x_2, -x_3) \quad \mathfrak{F}^{\pm}(x_1, x_2, x_3) = \mp\mathfrak{F}^{\pm}(x_1, x_3, x_2)$$

Support

From momentum conservation in intermediate state

$\boldsymbol{x_i \in [-1, 1]}$ $\quad i = 1, 2, 3$

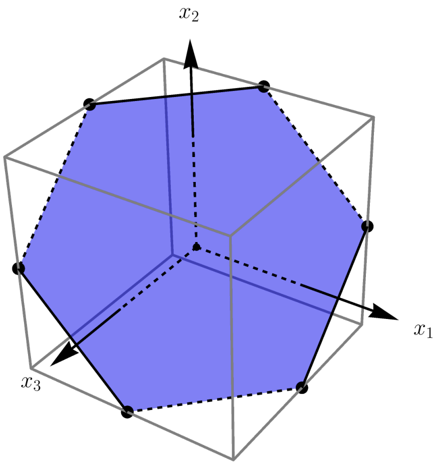
$\Rightarrow$ cube in 3D space

From total momentum conservation

$$\boldsymbol{x_1 + x_2 + x_3 = 0}$$

$\Rightarrow$ slice of the cube

hexagon

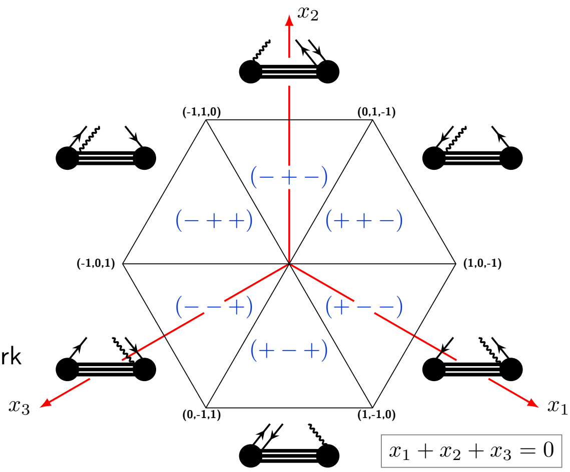


Partonic interpretation

$$\begin{cases} x_1 > 0 & \text{emission of anti-quark} \\ x_1 < 0 & \text{absorption of quark} \end{cases}$$

$$\begin{cases} x_2 > 0 & \text{emission of gluon} \\ x_2 < 0 & \text{absorption of gluon} \end{cases}$$

$$\begin{cases} x_3 > 0 & \text{emission of quark} \\ x_3 < 0 & \text{absorption of anti-quark} \end{cases}$$



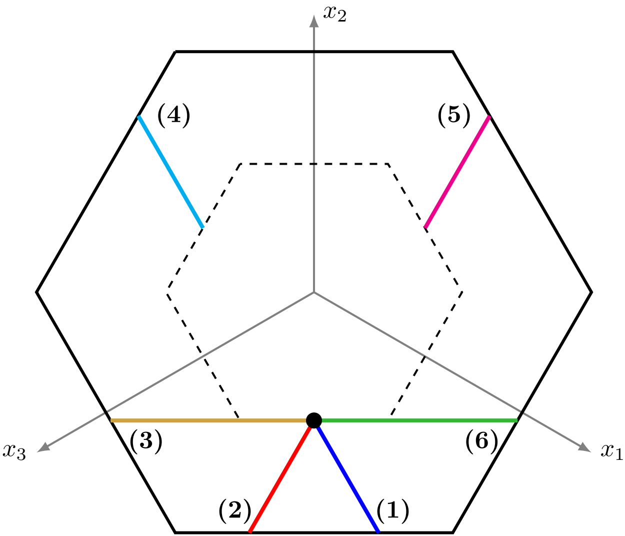
Evolution equations

Flavor separation, singlet $= \mathfrak{S}^{\pm}_{S} = u + d + s + \cdots$
non-singlet

$$\mathfrak{S}^{\pm}_{\mathsf{NS}_1} = \mathfrak{S}^{\pm}_{u} - \mathfrak{S}^{\pm}_{d}$$
$$\mathfrak{S}^{\pm}_{\mathsf{NS}_2} = \mathfrak{S}^{\pm}_{u} + \mathfrak{S}^{\pm}_{d} - 2\mathfrak{S}^{\pm}_{s}$$
$$\cdots$$

$$\mu^2 \frac{\partial}{\partial \mu^2} \mathfrak{S}^{\pm}_{\mathsf{NS}_i} = -\frac{\alpha_s(\mu)}{4\pi} \mathbb{H}_{\mathsf{NS}} \mathfrak{S}^{\pm}_{\mathsf{NS}_i}$$
$$\mu^2 \frac{\partial}{\partial \mu^2} \begin{pmatrix} \mathfrak{S}^{\pm}_{S} \\ \mathfrak{F}^{\pm} \end{pmatrix} = -\frac{\alpha_s(\mu)}{4\pi} \begin{pmatrix} \mathbb{H}^{\pm}_{qq} & \mathbb{H}^{\pm}_{qg} \\ \mathbb{H}^{\pm}_{gq} & \mathbb{H}^{\pm}_{gg} \end{pmatrix} \begin{pmatrix} \mathfrak{S}^{\pm}_{S} \\ \mathfrak{F}^{\pm} \end{pmatrix}$$
$$\mu^2 \frac{\partial}{\partial \mu^2} [E, H] = -\frac{\alpha_s(\mu)}{4\pi} \mathbb{H}_{\mathsf{CO}} [E, H]$$

Kernels action



Grids

The goal

$$\mathfrak{S}_{\mathsf{NS}}(\boldsymbol{x}) = \sum_{i,j} \mathcal{I}_{ij}(\boldsymbol{x}) \mathfrak{S}_{\mathsf{NS}}^{ij} \qquad \mathfrak{S}_{\mathsf{NS}}^{ij} = \mathfrak{S}_{\mathsf{NS}}(\boldsymbol{x}(i,j))$$

so that

$$\mu^2 \frac{d}{d\mu^2} \mathfrak{S}_{\mathsf{NS}}^{ij} = -a_s(\mu) \sum_{i'j'} \mathbb{H}_{\mathsf{NS}}{}_{i'j'}^{ij} \mathfrak{S}_{\mathsf{NS}}^{i'j'} \qquad \mathbb{H}_{\mathsf{NS}}{}_{i'j'}^{ij} = \mathbb{H}_{\mathsf{NS}} \mathcal{I}_{i'j'}(\boldsymbol{x}(i,j))$$

Two **independent** steps:

1) fix the discretization $\boldsymbol{x}(i,j)$

2) fix the weights $\mathcal{I}_{ij}(\boldsymbol{x})$

Discretization

Nice properties to have:

1) Denser grid towards $\boldsymbol{x} = 0$

2) Compatible with hexagonal symmetry

First guess: equispaced grid

$$\boldsymbol{x}(i,j) = \left(\frac{i}{N}, \frac{j}{N}, \frac{-i-j}{N} \right) \qquad i, j, i+j \in [-N, N]$$

~~Denser grid towards $\boldsymbol{x} = 0$~~

We can do better!

Discretization

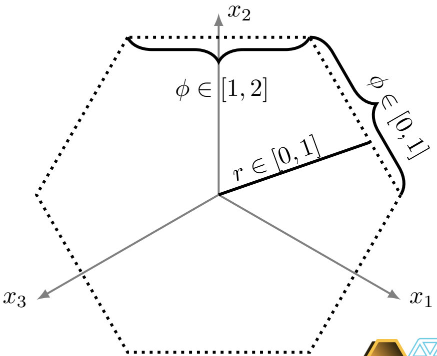
Step 1: change coordinate system

$$\boldsymbol{x} \to (\phi, r)$$

$$r = \|\boldsymbol{x}\|_\infty = \max(|x_1|, |x_2|, |x_3|) \in [0, 1]$$

$$r(\boldsymbol{x}) = \max(|x_1|, |x_2|, |x_3|)$$

$$\phi(\boldsymbol{x}) = \begin{cases} \frac{x_2}{r} & x_1 > 0, x_2 \geq 0, x_3 < 0 \\ \left(1 - \frac{x_1}{r}\right) & x_1 \leq 0, x_2 > 0, x_3 < 0 \\ \left(3 - \frac{x_2}{r}\right) & x_1 < 0, x_2 > 0, x_3 \geq 0 \\ \left(3 - \frac{x_2}{r}\right) & x_1 < 0, x_2 \leq 0, x_3 > 0 \\ \left(4 + \frac{x_1}{r}\right) & x_1 \geq 0, x_2 < 0, x_3 > 0 \\ \left(6 + \frac{x_2}{r}\right) & x_1 > 0, x_2 < 0, x_3 \leq 0 \end{cases}$$



Discretization

Now we can discretize angle/radius:

$$(i,j) \to (\phi_i, r_j) \qquad i \in [0, 6n) \quad j \in [0, M]$$

$$\phi_i = \frac{i \bmod n}{n} + \left[\frac{i}{n}\right] \qquad \Longrightarrow \qquad i = n\phi_i$$

$$r_j = \left[\cosh\left(\frac{j-M}{Mc}\right)\right]^{-\alpha} \qquad \Longrightarrow \qquad j = M\left[1 + c \text{ acosh}\left(\frac{1}{\sqrt[\alpha]{r_j}}\right)\right]$$

Generalizing to **continuous** variables in $(\mathfrak{f}, \mathfrak{r}) \in [0, 6n) \times [0, M]$

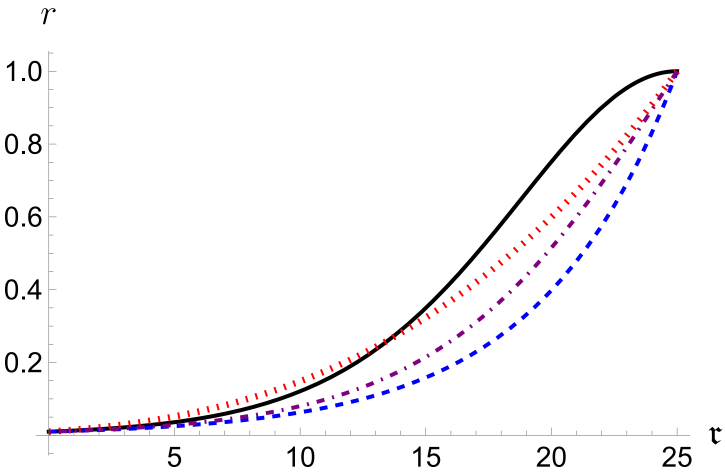
$$\mathfrak{f}(\boldsymbol{x}) = n\phi(\boldsymbol{x})$$

$$\mathfrak{r}(\boldsymbol{x}) = M\left[1 + c \text{ acosh}\left(\frac{1}{\sqrt[\alpha]{r(\boldsymbol{x})}}\right)\right]$$

$r_0 = x_{\text{min}}$

Discretization

Different choices for the radial map

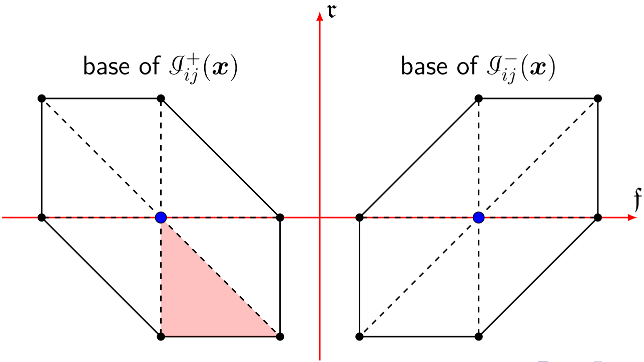


Weights

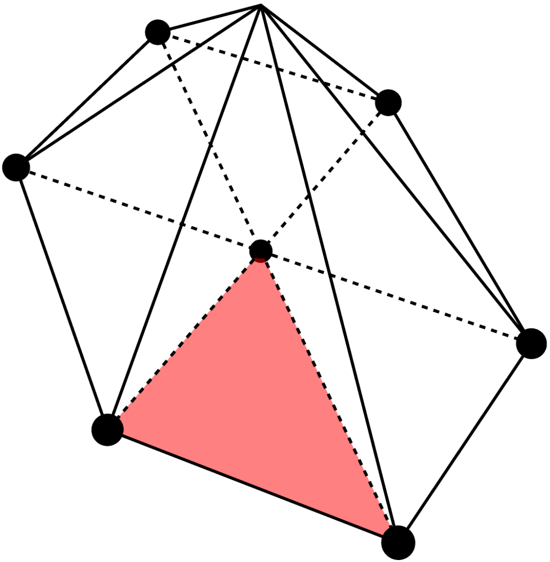
Two nearest-neighbor weights

$$\mathscr{I}^{\pm}_{ij}(\boldsymbol{x}) = \max\Big[0, 1 - \max\Big(|\mathfrak{f}(\boldsymbol{x}) - i| \ , \ |\mathfrak{r}(\boldsymbol{x}) - j| \ , \ |\mp \mathfrak{f}(\boldsymbol{x}) - \mathfrak{r}(\boldsymbol{x}) \pm i + j|\Big)\Big]$$

$$\mathscr{I}_{ij}(\boldsymbol{x}) = \frac{\mathscr{I}^{+}_{ij}(\boldsymbol{x}) + \mathscr{I}^{-}_{ij}(\boldsymbol{x})}{2}$$



Weights



One kernel

$$\widehat{\mathcal{H}}_{12}\mathcal{G}^{+}_{i'j'}(\boldsymbol{x}^{ij}) = \underbrace{\int_{-\infty}^{\infty}}\, dv\, \frac{x_1^{ij}\Theta(x_1^{ij}, -v)}{v(v - x_1^{ij})} \underbrace{\left[\mathcal{G}^{+}_{i'j'}(\boldsymbol{x}^{ij}) - \mathcal{G}^{+}_{i'j'}(x_1^{ij} - v, x_2^{ij} + v, x_3^{ij})\right]}$$

Only non-zero for $i' = i$, $j' = j$

Always present

Infinite integration range but...

Compact and ‘very small’ support

$$\boldsymbol{x}^{ij} = \boldsymbol{x}(\phi_i, r_j)$$

One kernel

After a bit of (very simple) algebra

$$\widehat{\mathscr{H}}_{12}\mathscr{G}^{+}_{i'j'}(\boldsymbol{x}^{ij}) = \delta_{i\neq i' \text{ or } j\neq j'}\int_{v^{i'j'}_{\text{min}}}^{v^{i'j'}_{\text{max}}} dv\,\frac{x_1^{ij}\Theta(x_1^{ij},-v)}{v(x_1^{ij}-v)}\mathscr{G}^{+}_{i'j'}(x_1^{ij}-v, x_2^{ij}+v, x_3^{ij})$$

$$+\delta_{ii'}\delta_{jj'}\mathscr{G}^{+}_{ij}(\boldsymbol{x}^{ij})\left[\theta(x_1^{ij})\log\left(1-\frac{x_1^{ij}}{v^{ij}_{\text{min}}}\right)+\theta(-x_1^{ij})\log\left(1-\frac{x_1^{ij}}{v^{ij}_{\text{max}}}\right)\right]$$

$$+\delta_{ii'}\delta_{jj'}\int_{v^{ij}_{\text{min}}}^{v^{ij}_{\text{max}}} dv\,\frac{x_1^{ij}\Theta(x_1^{ij},-v)}{v(v-x_1^{ij})}\left[\mathscr{G}^{+}_{ij}(\boldsymbol{x}^{ij})-\mathscr{G}^{+}_{ij}(x_1^{ij}-v, x_2^{ij}+v, x_3^{ij})\right]$$

Off-diagonal

Integral of plus-prescription outside the support

Plus-prescription, diagonal part

Few numbers

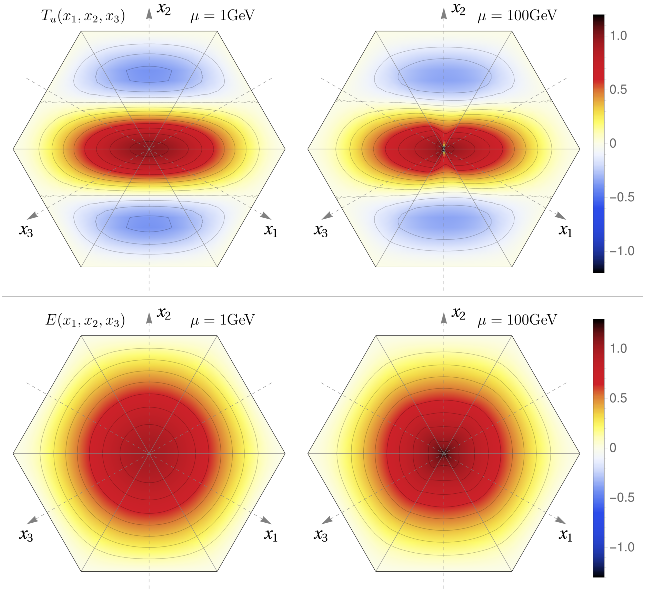
1. With our $\mathcal{I}_{ij}(\boldsymbol{x})$ only few $\%$ of entries $\mathbb{H}^{i'j'}_{\text{NS},ij}$ are non-zero
   $\Rightarrow$ **custom sparse-algebra implementation**

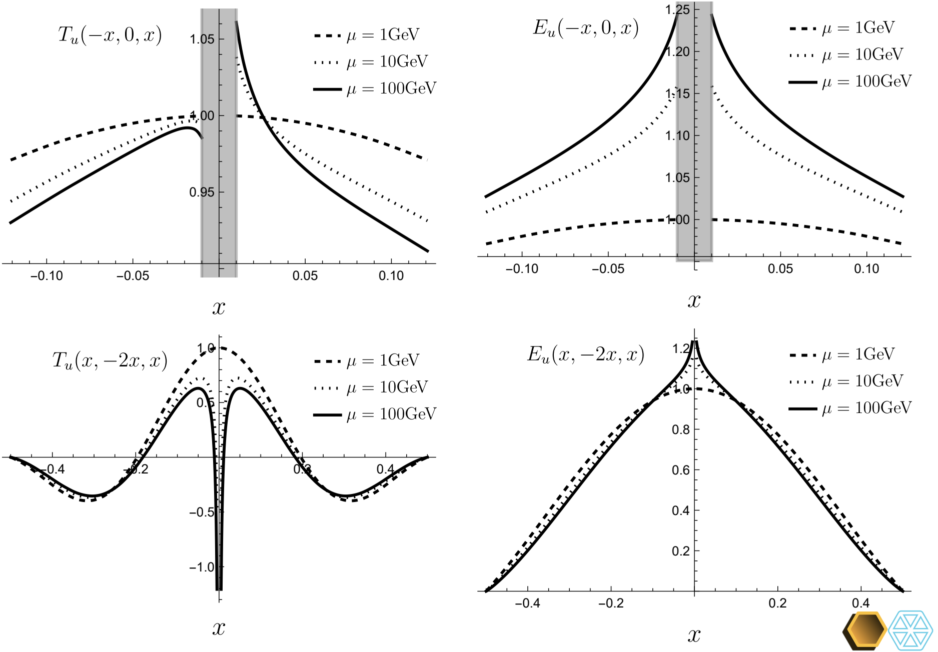
2. with 'just' 120×25 points, kernels sizes of $\mathcal{O}(\mathbf{10} \sim \mathbf{20}\text{MB})$

3. average interpolation accuracy $\sim 10^{-3}$
   Worse towards the edge of the physical region:
   for $r \leq 0.9$ accuracy improves to $\boldsymbol{\sim 10^{-4}}$

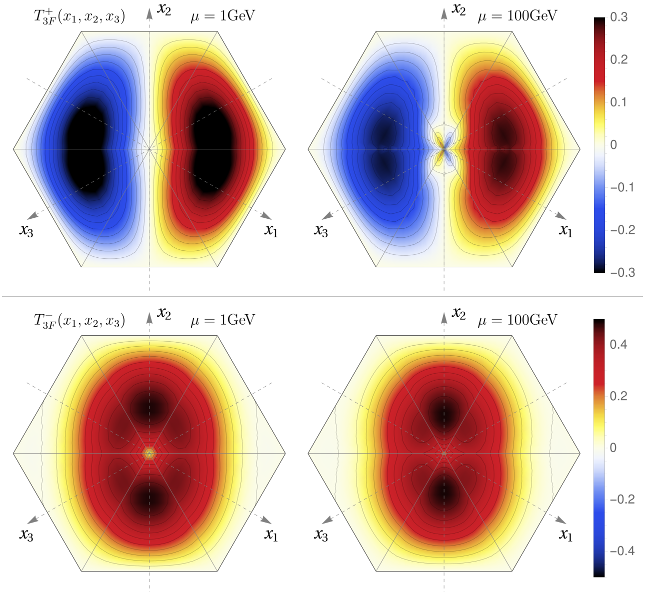
Evolution results and comments

T_u and E_u

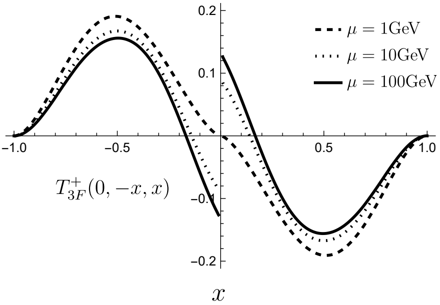

$T_u(x_1, x_2, x_3)$
$\mu = 1\text{GeV}$
$\mu = 100\text{GeV}$
$E(x_1, x_2, x_3)$
$\mu = 1\text{GeV}$
$\mu = 100\text{GeV}$
x_1
x_2
x_3
1.0
0.5
0
−0.5
−1.0

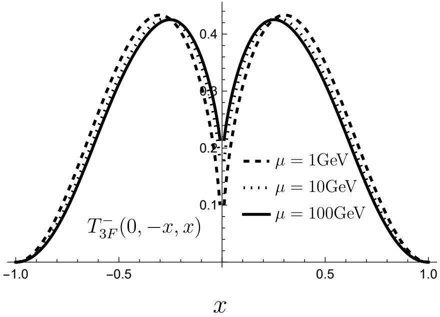

$T_u(-x,0,x)$
$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
1.05
1.00
0.95
−0.10
−0.05
0.05
0.10
x
$E_u(-x,0,x)$
$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
1.25
1.20
1.15
1.10
1.05
1.00
−0.10
−0.05
0.05
0.10
x
$T_u(x,-2x,x)$
$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
1.0
0.5
−0.5
−1.0
−0.4
−0.2
0.2
0.4
x
$E_u(x,-2x,x)$
$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
1.2
1.0
0.8
0.6
0.4
0.2
−0.4
−0.2
0.2
0.4
x

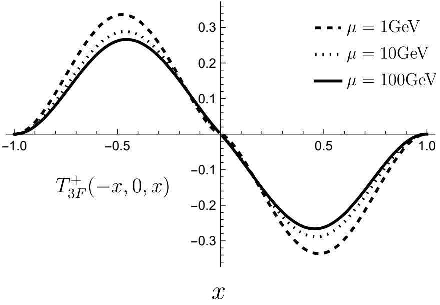
3-gluon

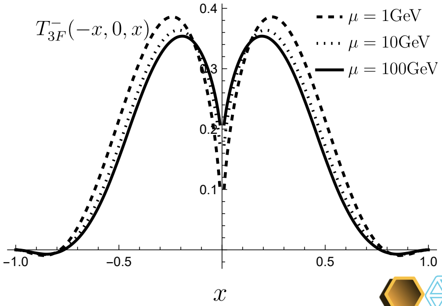


$T^{+}_{3F}(x_1,x_2,x_3)$
x_2
$\mu = 1\text{GeV}$
x_3
x_1
x_2
$\mu = 100\text{GeV}$
x_3
x_1
0.3
0.2
0.1
0
−0.1
−0.2
−0.3
$T^{-}_{3F}(x_1,x_2,x_3)$
x_2
$\mu = 1\text{GeV}$
x_3
x_1
x_2
$\mu = 100\text{GeV}$
x_3
x_1
0.4
0.2
0
−0.2
−0.4


$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
$T^+_{3F}(0,-x,x)$
0.2
0.1
-0.1
-0.2
-1.0
-0.5
0.5
1.0
x


$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
$T^-_{3F}(0,-x,x)$
0.4
0.3
0.2
0.1
-1.0
-0.5
0.5
1.0
x


$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
$T^+_{3F}(-x,0,x)$
0.3
0.2
0.1
-0.1
-0.2
-0.3
-1.0
-0.5
0.5
1.0
x


$T^-_{3F}(-x,0,x)$
$\mu = 1\text{GeV}$
$\mu = 10\text{GeV}$
$\mu = 100\text{GeV}$
0.4
0.3
0.2
0.1
-1.0
-0.5
0.5
1.0
x

Conclusions and outlooks

1. The two codes are available on github at VladimirovAlexey/Snowflake and at QCDatHT/honeycomb
2. Both should be easily integrable in existing programs
3. Evolution effects are not overwhelmingly strong, but also not negligible