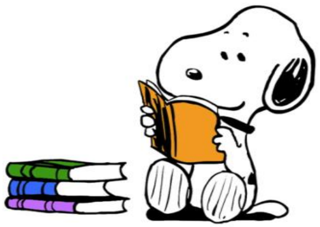
Turin Lattice Meeting 2023
UniTO, 21–22 December 2023

Mass of quantum topological excitations and order parameter finite size dependence

Marianna Sorba
SISSA, Trieste

Based on:

G. Delfino, M. Sorba, arXiv:2309.06206



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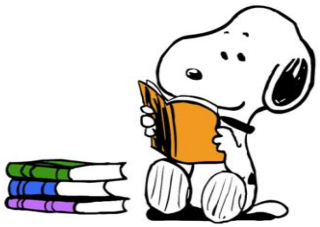

[1] G. Delfino, W. Selke and A. Squarcini, PRL 122 (2019) 050602
[2] M. Panero and A. Smecca, JHEP 03 (2021) 231

Vortex mass in the three-dimensional $O(2)$ scalar theory

Gesualdo Delfino[1,2], Walter Selke[3] and Alessio Squarcini[4,5]
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Universität Stuttgart, Pfaffenw
D-70569 Stuttgart, Ger

[1]

Topological excitations in statistical field theory at the upper critical dimension

Marco Panero and Antonio Smecca
Department of Physics, University of Turin and INFN, Turin,
Via Pietro Giuria 1, I-10125 Turin, Italy

[2]

The $O(n)$ vector model

$$\mathcal{H} = -\frac{1}{T}\sum_{\langle i,j\rangle} \mathbf{s}_i \cdot \mathbf{s}_j \,, \qquad \mathbf{s}_i = n\text{-component unit vector}$$

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- continuum description: $\mathbf{s}(\mathbf{x}, y) =$ order parameter field
  $\mathbf{x} = (x_1, ..., x_n)$, $y =$ imaginary time
- the **ground state manifold** and the **space boundary** both correspond to the sphere S^{n-1}, so different points on the space boundary can be mapped onto different ground states.

Topological particles

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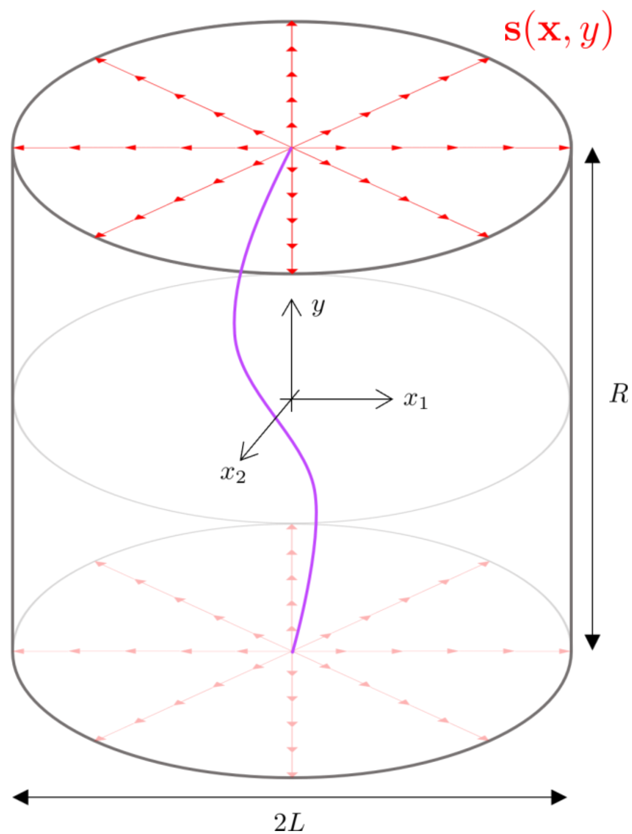
The propagation of these particles in imaginary time generates **topological defect lines** for the Euclidean system:

Hypercylinder $|\mathbf{x}| < L$, $|y| < R/2$
with $L \to \infty$, R large but finite.



One configuration of the particle trajectory is shown.

Order parameter

Once fixed the boundary conditions, the order parameter $\langle \mathbf{s}(\mathbf{x},0)\rangle_{\mathcal{B}}$ can be analytically determined from the **large $\boldsymbol{R}$ asymptotics** of the boundary states:

$$|B(\pm R/2)\rangle = \int \frac{d\mathbf{p}}{(2\pi)^n E_{\mathbf{p}}}\, a_{\mathbf{p}}\, e^{\pm \frac{R}{2} E_{\mathbf{p}}}\, |\tau(\mathbf{p})\rangle + \ldots$$

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Then:

$$\langle \mathbf{s}(\mathbf{x},0)\rangle_{\mathcal{B}} = \frac{\langle B(R/2)|\mathbf{s}(\mathbf{x},0)|B(-R/2)\rangle}{\langle B(R/2)|B(-R/2)\rangle}$$

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Then:

$$\langle \mathbf{s}(\mathbf{x}, 0) \rangle_{\mathcal{B}} \sim v \, \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(1 + \frac{n}{2}\right)} \, {}_1F_1\left(\frac{1}{2}, 1 + \frac{n}{2}; -z^2\right) z \, \hat{\mathbf{x}} \,, \qquad v = |\langle \mathbf{s}(\mathbf{x}, y) \rangle|$$

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- odd function of x
- interpolates between zero at $x=0$ and $\lim_{|\mathbf{x}|\to\infty} \langle \mathbf{s}(\mathbf{x},0)\rangle_{\mathcal{B}} \sim v\,\hat{\mathbf{x}}$
- depends on R.

Comparison with numerics

This analytical result was confirmed by Monte Carlo simulations performed on the lattice. In particular, for different values of R :

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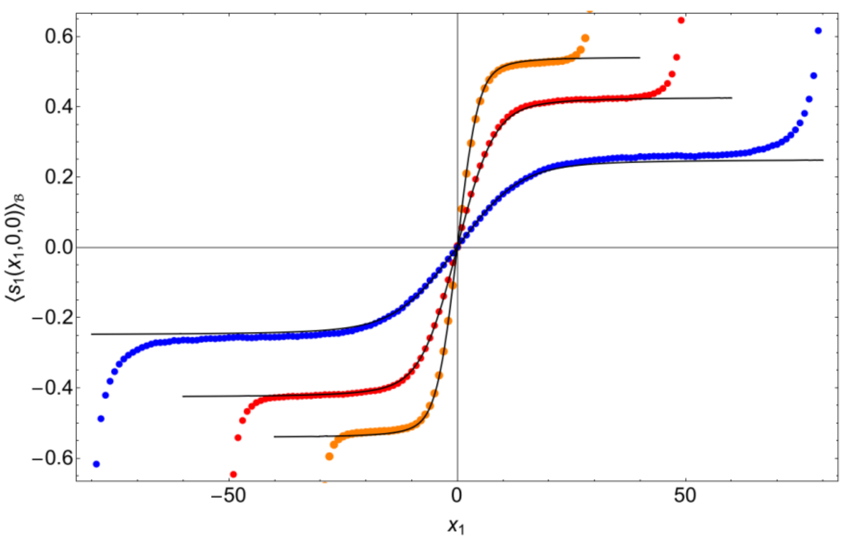

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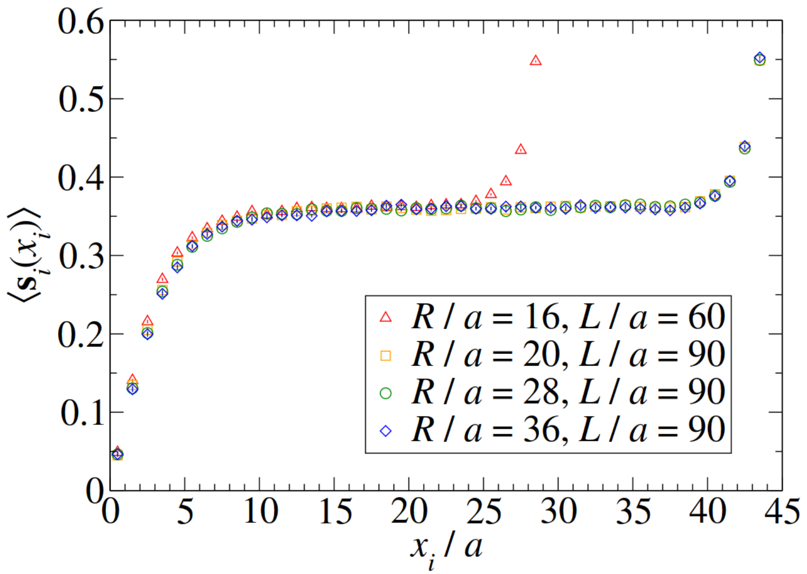

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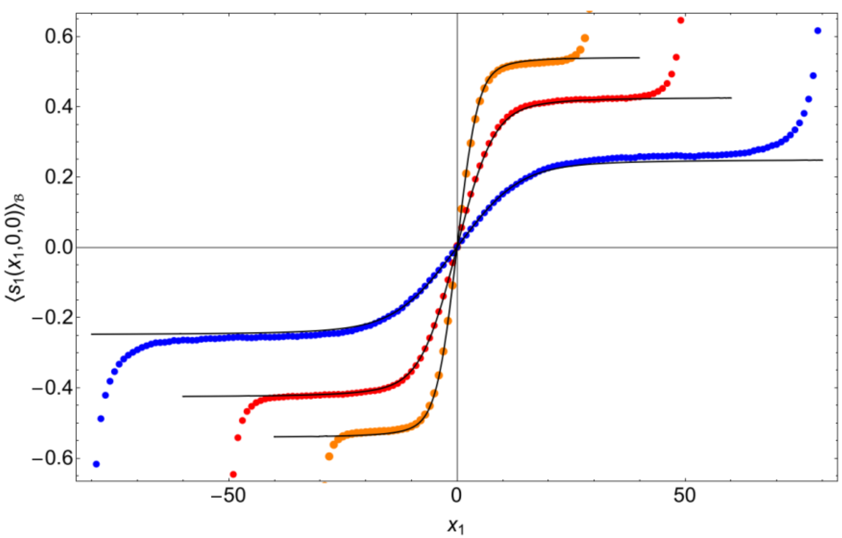

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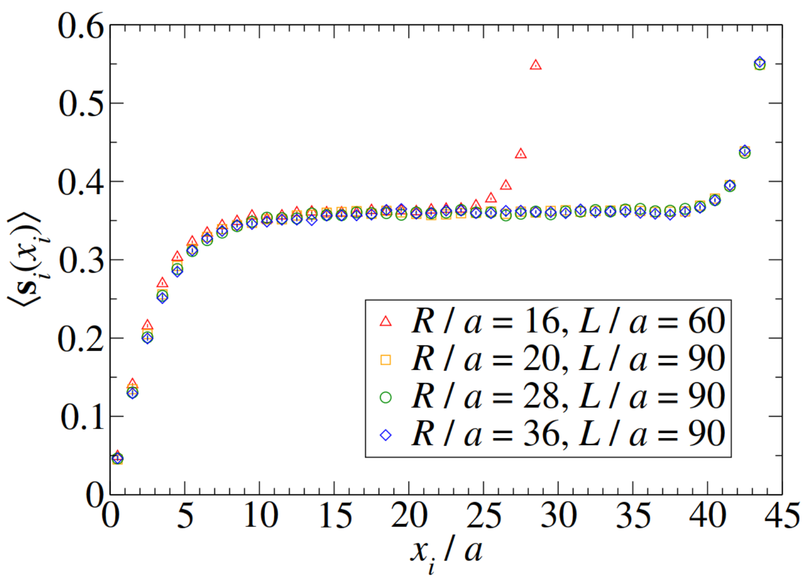

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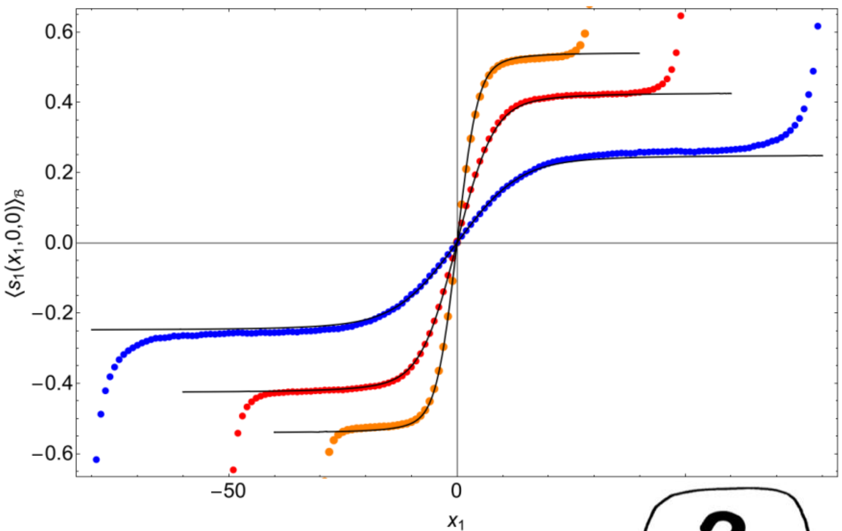

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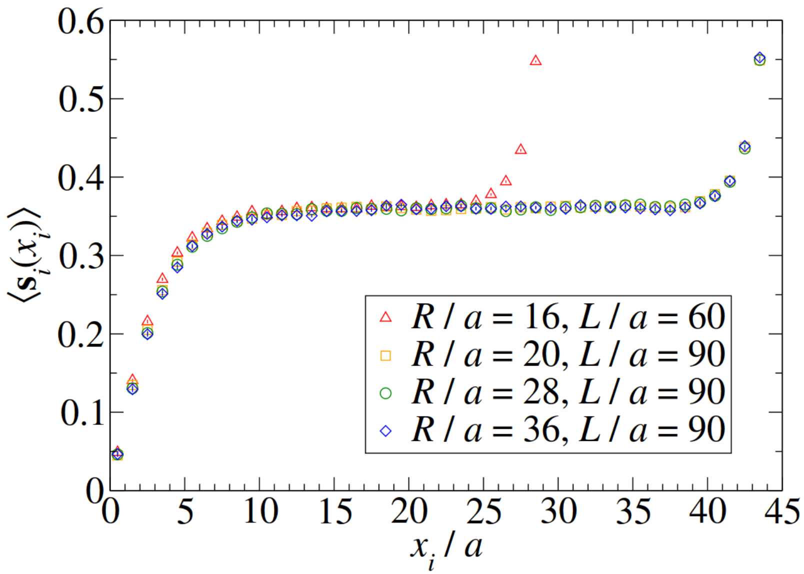

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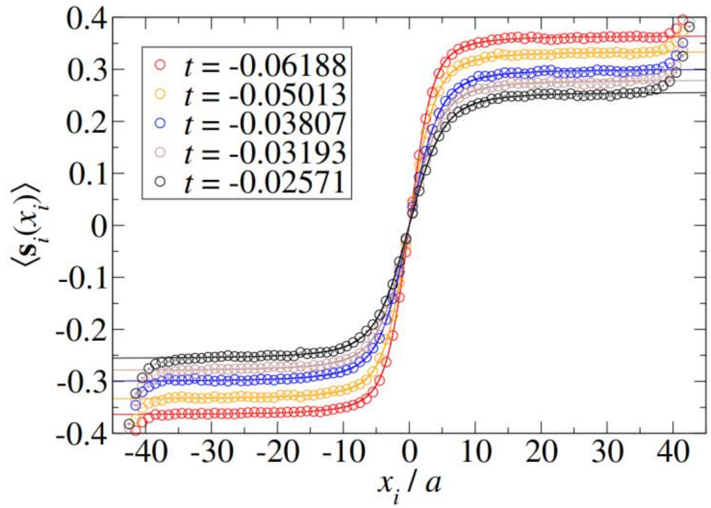
Residual fluctuations

Despite the R-independence, the overall $\mathbf{x}$-dependence of $\langle \mathbf{s}(\mathbf{x}, 0) \rangle_{\mathcal{B}}$ in [2] was found to be quite analogous to that observed in [1]:



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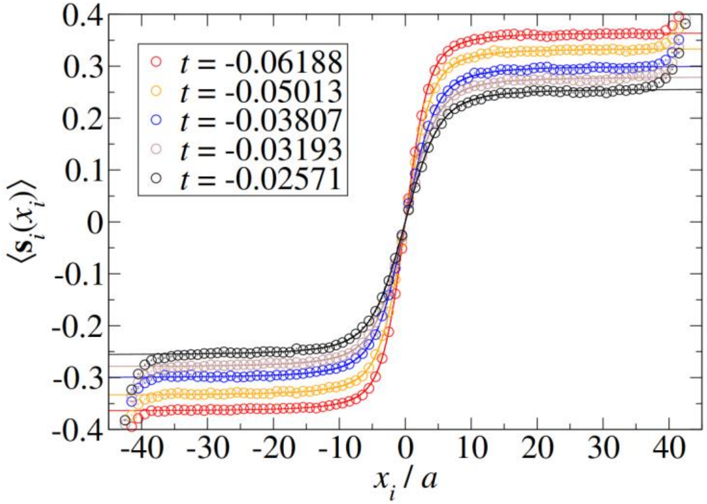

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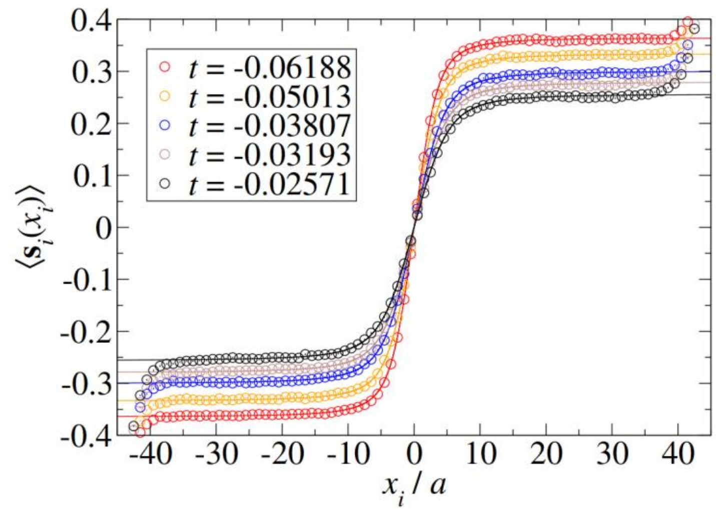

We should consider residual contributions not considered so far:

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These states are $\tau +$ **Goldstone bosons** associated to the spontaneous breaking of the continuous $O(n)$ symmetry.

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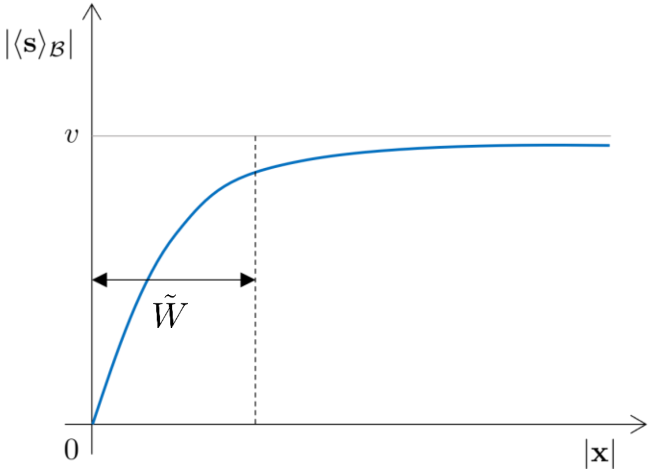
- remain appreciable within a width $\tilde{W}$
- cannot depend on m_τ , R.



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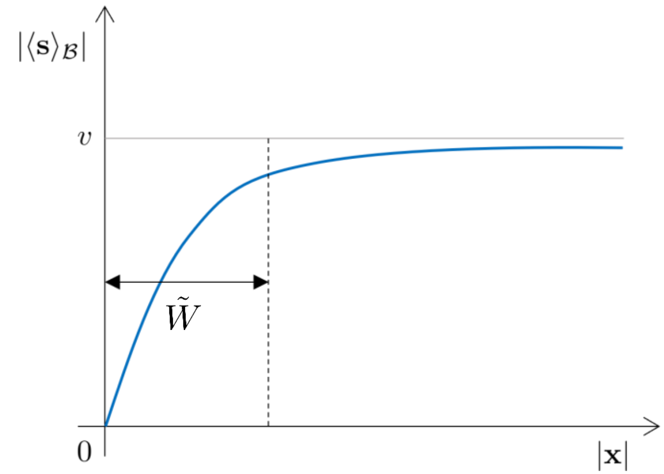

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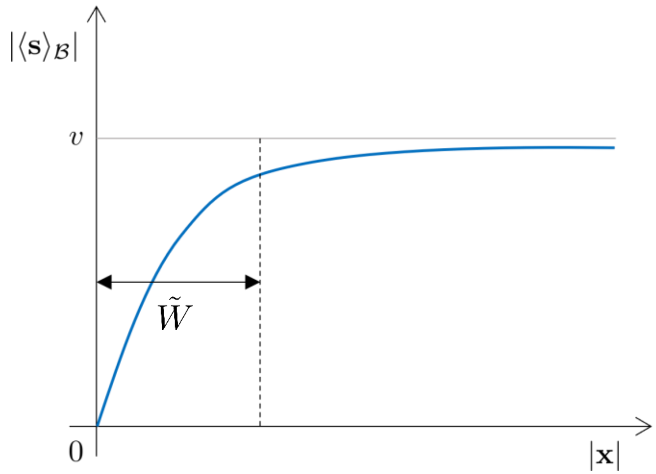

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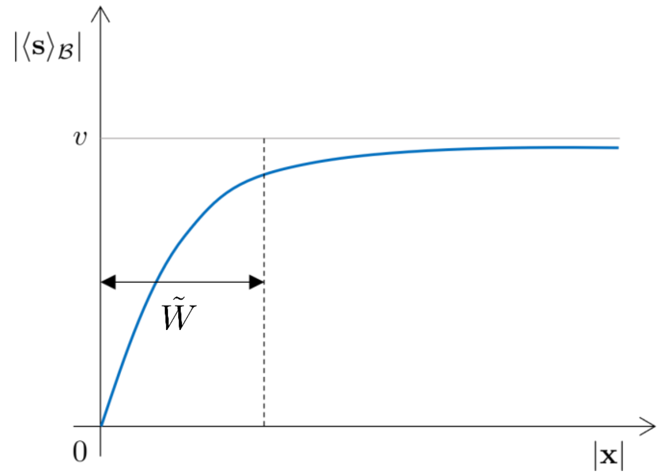

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The correspondent width in [1] is $W \propto \sqrt{R/m_\tau}$.

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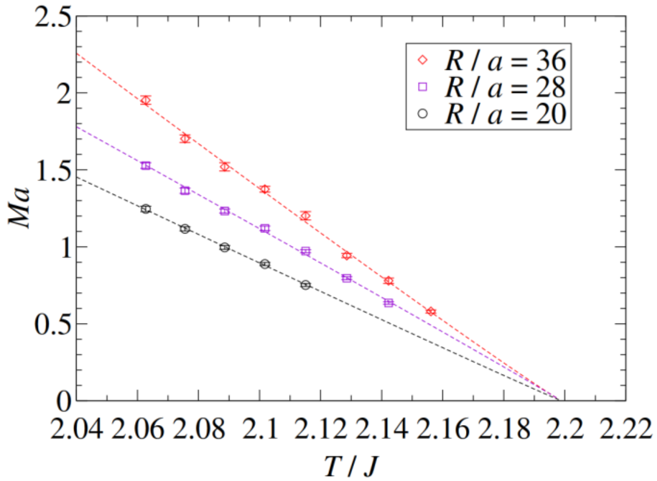

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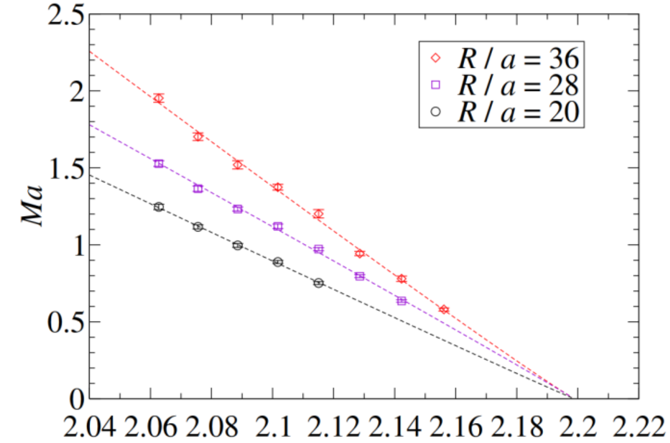

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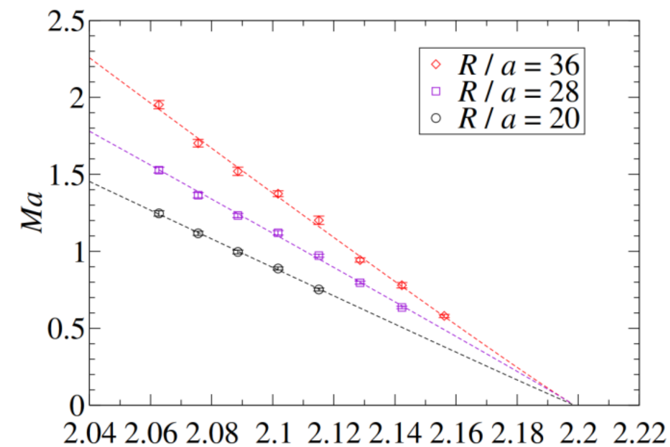

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This confirms that for $d \geq 4$ $m_\tau = \infty$ (R-independent) and also the order parameter is size-independent.

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Thank You!