Relaxation Dynamics in Integrable Field Theories

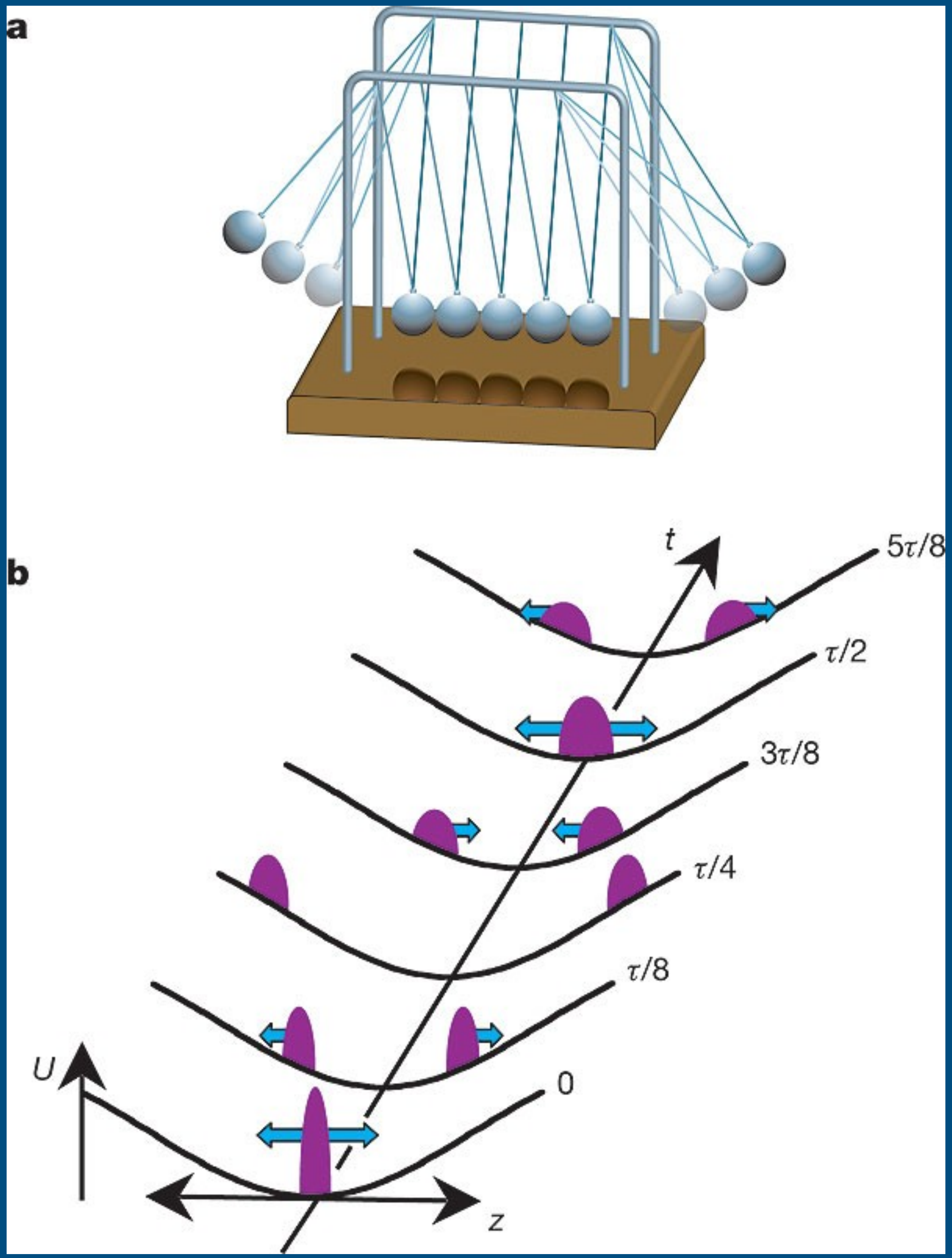
Manuscript(s) in preparation, under the supervision of Dr. D. Schuricht

Emanuele Di Salvo (Utrecht University) - Torino, 21/12/2023

A quantum Newton's cradle

Or integrability at work

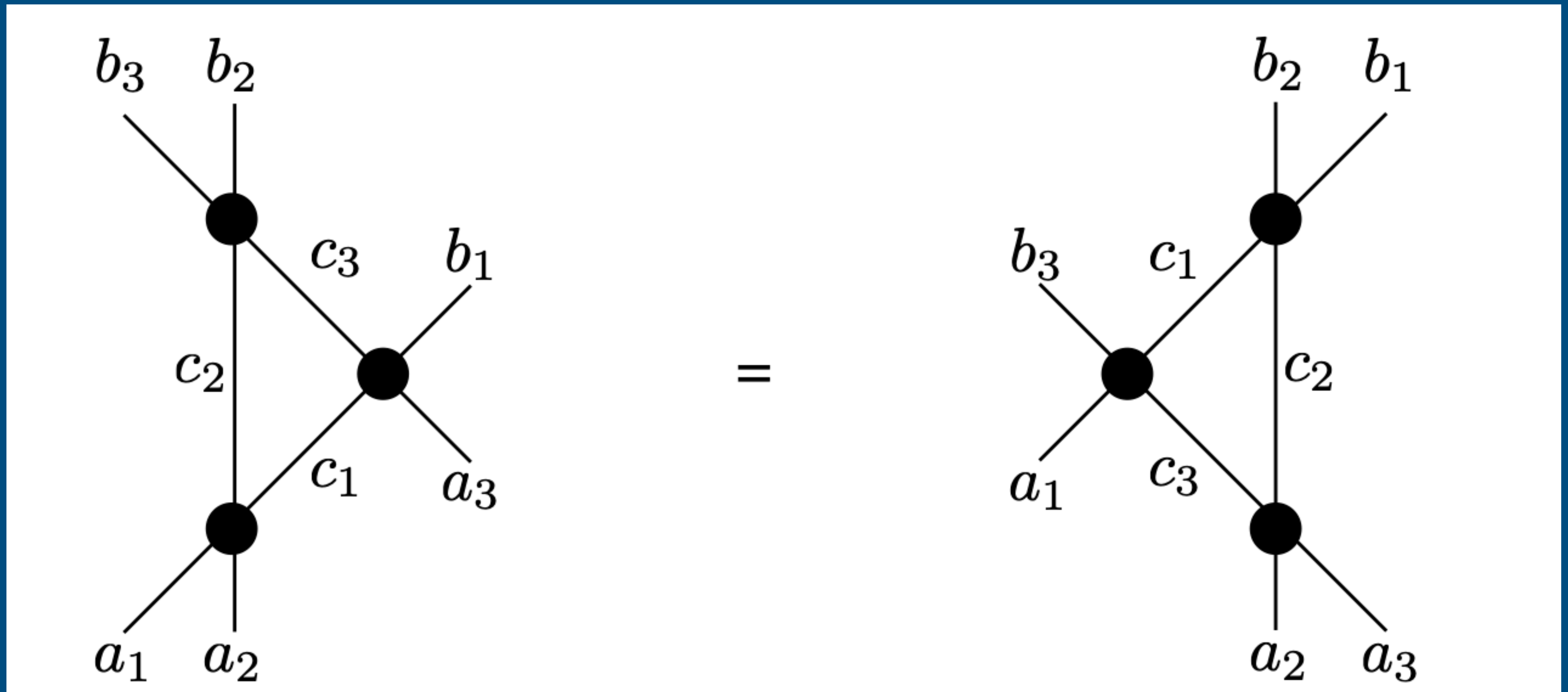
- Clouds of cold atoms interacting and oscillating for long times
- Substantial difference between 1D and higher dimensions
- Initial state memory
- First experimental realisation of an integrable model



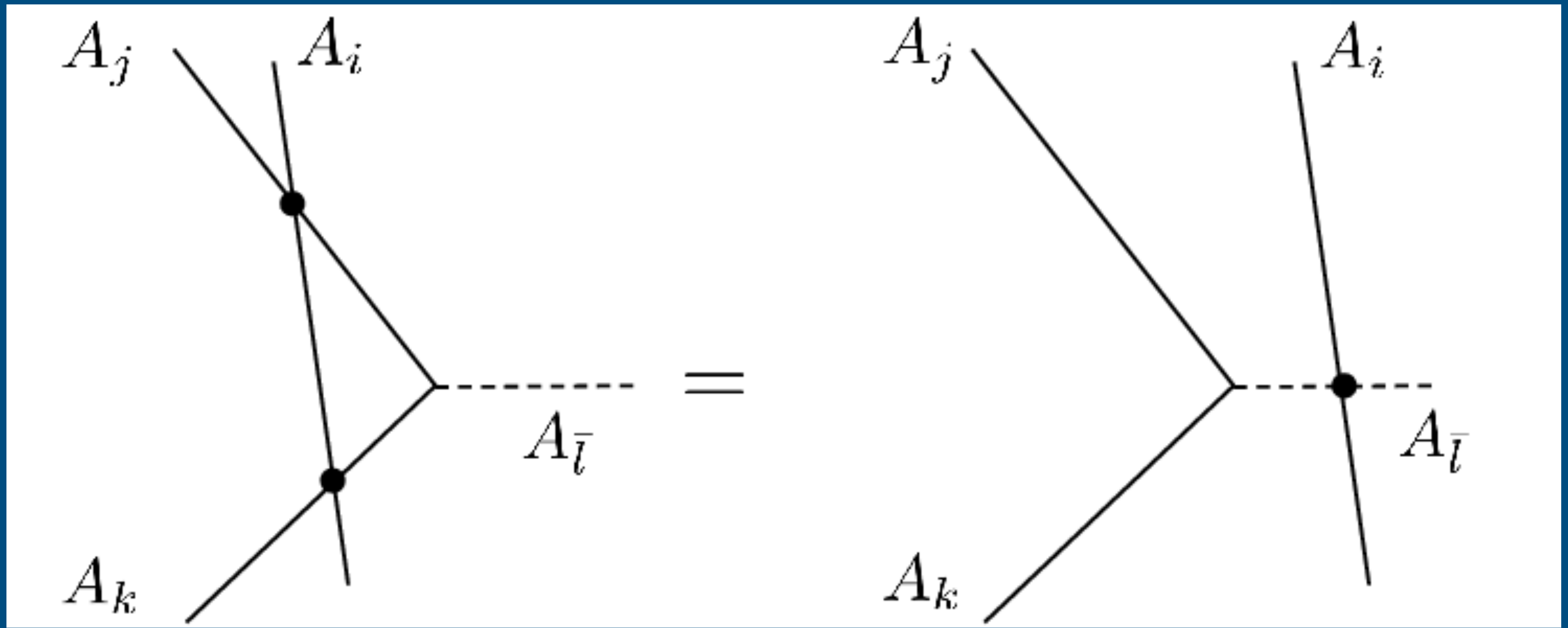
T. Kinoshita, T. Wenger and D. Weiss, "A quantum Newton's cradle", *Nature* (2006)

Integrable Field Theories

- Infinite number of conserved quantities constraints the S-matrix
- Elasticity and factorisability in scattering processes
- Bootstrap equation from unitarity, crossing symmetry and equivalence between bound states and asymptotic ones
- Stable particle content
- Simplest examples: Ising, Sinh-Gordon and Sine-Gordon models



Yang-Baxter equations



Bootstrap equation

Quantum Quenches in Integrable Systems

Local quench on a spin chain

Global quench

$$H(g_0) \to H(g)$$

Initial state's unitary evolution

Global quench

$$H(g_0) \to H(g)$$

- Extensive initial state $|\psi\rangle$
- Extensive in the norm $\langle\psi|\psi\rangle$, i.e. diverging in the thermodynamic limit (macroscopic state)
- Non-adiabaticity of the quenching
- Infinite number of emerging pairs of entangled particles with bosons exchange statistics ("Cooper pairs")

Relaxation towards the steady state

After a global quench

$$\langle O(t) \rangle = Z^{-1} \langle \psi | O(t) | \psi \rangle$$

Initial state decomposition: $$| \psi \rangle = \sum_{\lambda} e^{\delta s_\lambda} | \lambda \rangle$$

$$\langle O(t) \rangle = \sum_{\lambda} \sum_{\mu} e^{-\delta s_\lambda - \delta s_\mu} e^{i(\omega_\lambda - \omega_\mu)t} \langle \lambda | \mathcal{O} | \mu \rangle$$

Quench Action method

J.S. Caux, F.H.L. Essler - “Time evolution of local observables after quenching to an integrable model” *(Phys. Rev. Lett. 110, 257203 (2013))*

Quench Action method

$$\langle O(t) \rangle = \frac{\langle \psi | O(t) | \rho \rangle}{2\langle \psi | \rho \rangle} + h.c.$$

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Representative eigenstate

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Quench Action method

$$\langle O(t) \rangle = \frac{\langle \psi | O(t) | \rho \rangle}{2\langle \psi | \rho \rangle} + h.c.$$

Representative eigenstate

- Representative eigenstate's occupation density found by the means of gTBA
- More economic computational costs

J.S. Caux, F.H.L. Essler - "Time evolution of local observables after quenching to an integrable model" *(Phys. Rev. Lett. 110, 257203 (2013))*

Relaxation towards the steady state

After a global quench

$$\langle O(t) \rangle = e^{-\Gamma t} f(t) + g(t) + \langle O \rangle_\rho$$

Relaxation towards the steady state

After a global quench

$$\langle O(t)\rangle = e^{-\Gamma t} f(t) + g(t) + \langle O\rangle_\rho$$

O local*: $\quad f(t) \sim g(t) \sim t^{-\alpha} e^{2iMt}$

O semi-local*: $f(t) \sim t^{-\alpha} e^{2iMt}$, $g(t) = 0$

*with respect to fundamental fields, cf. Jordan-Wigner strings

Decomposing on the basis of eigenstates $\{\lambda\}$ again:

$$\langle O(t) \rangle = \sum_{\lambda} e^{\delta s_\lambda - \delta s_\rho + i(\omega_\lambda - \omega_\rho)t} \langle \lambda | O | \rho \rangle$$

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$$\mathcal{F}^O(\lambda | \rho)$$

Decomposing on the basis of eigenstates $\{\lambda\}$ again:

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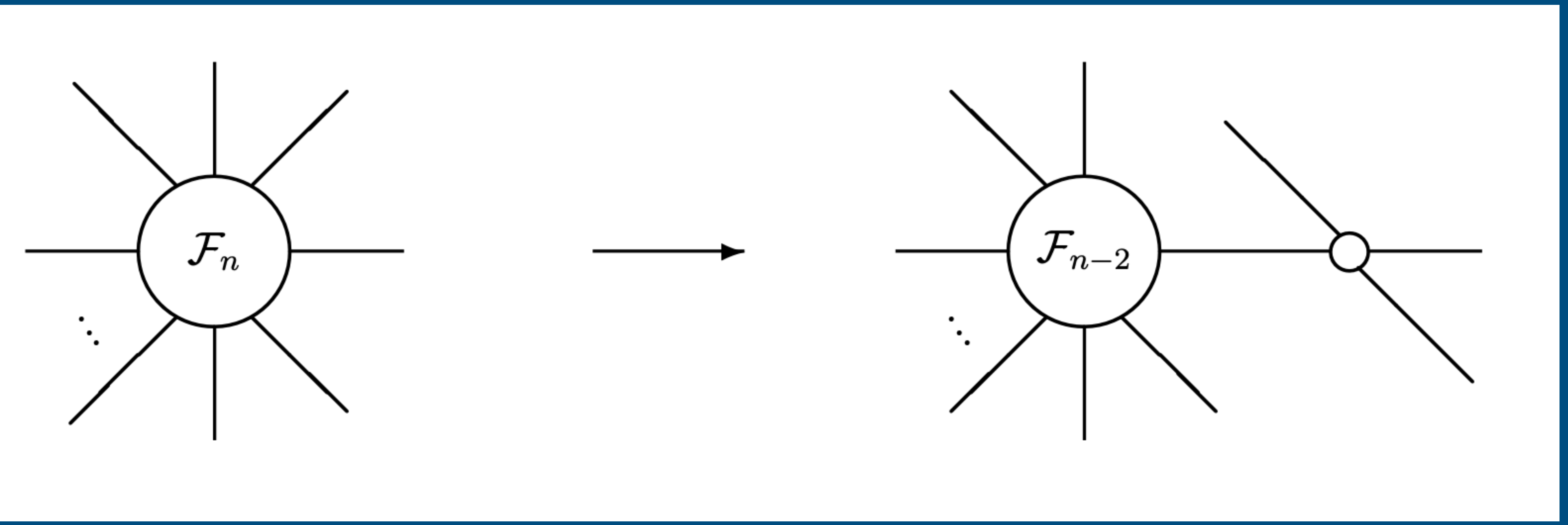
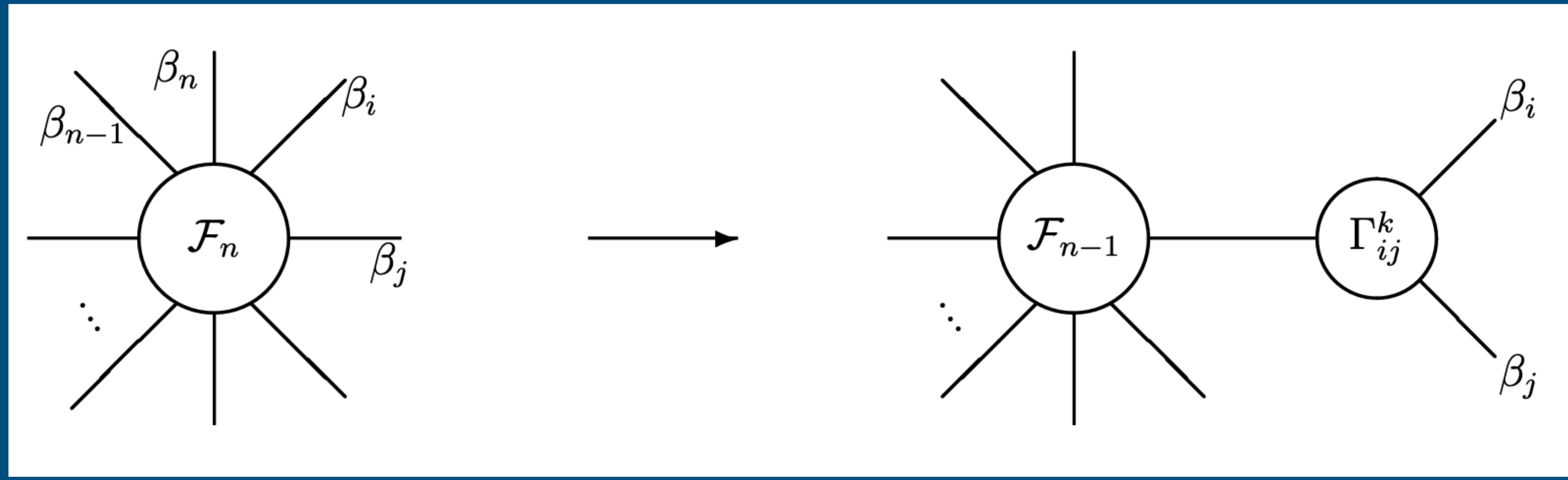


Annihilation pole axiom



Bound state axiom

Form Factors $\mathscr{F}$ of operator O

Axioms:

1. Scattering properties
2. Crossing symmetry
3. Locality
4. Annihilation pole
5. Bound state

Decomposing on the basis of eigenstates $\{\lambda\}$ again:

$$\langle O(t)\rangle = \sum_{\lambda} e^{\delta s_\lambda - \delta s_\rho + i(\omega_\lambda - \omega_\rho)t} \langle \lambda | O | \rho \rangle$$

Smirnov's decomposition formula:

$$\langle \lambda | O | \mu \rangle = \langle \lambda | \mu \rangle \mathscr{F}_0^O + \ldots + \mathscr{F}_{N+M}^O(\lambda_1 + i\pi, \ldots, \mu_M)$$

Splitting the matrix element into singular and regular parts as $|\lambda\rangle \to |\rho\rangle$:

$$\langle \lambda | O | \rho \rangle = \langle \lambda | O | \rho \rangle_{sing} + \langle \lambda | O | \rho \rangle_{reg}$$

Contour integration

Pole contribution $\sim e^{-\Gamma t} f(t)$

B. Bertini, D. Schuricht, F.H.L. Essler - "Quantum quench in the sine-Gordon model", *J. Stat. Mech. (2014) P10035*

Saddle point approximation

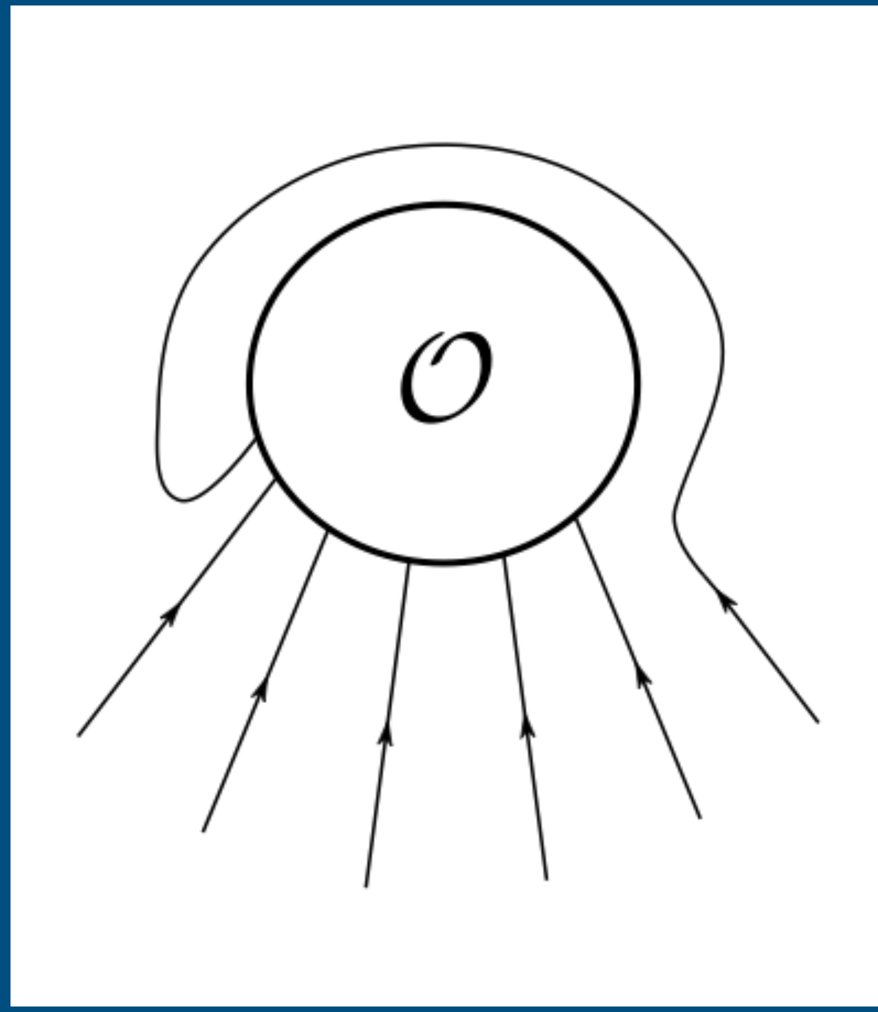
$\sim g(t)$

J.S. Caux, F.H.L. Essler - "Time evolution of local observables after quenching to an integrable model" *(Phys. Rev. Lett. 110, 257203 (2013)*

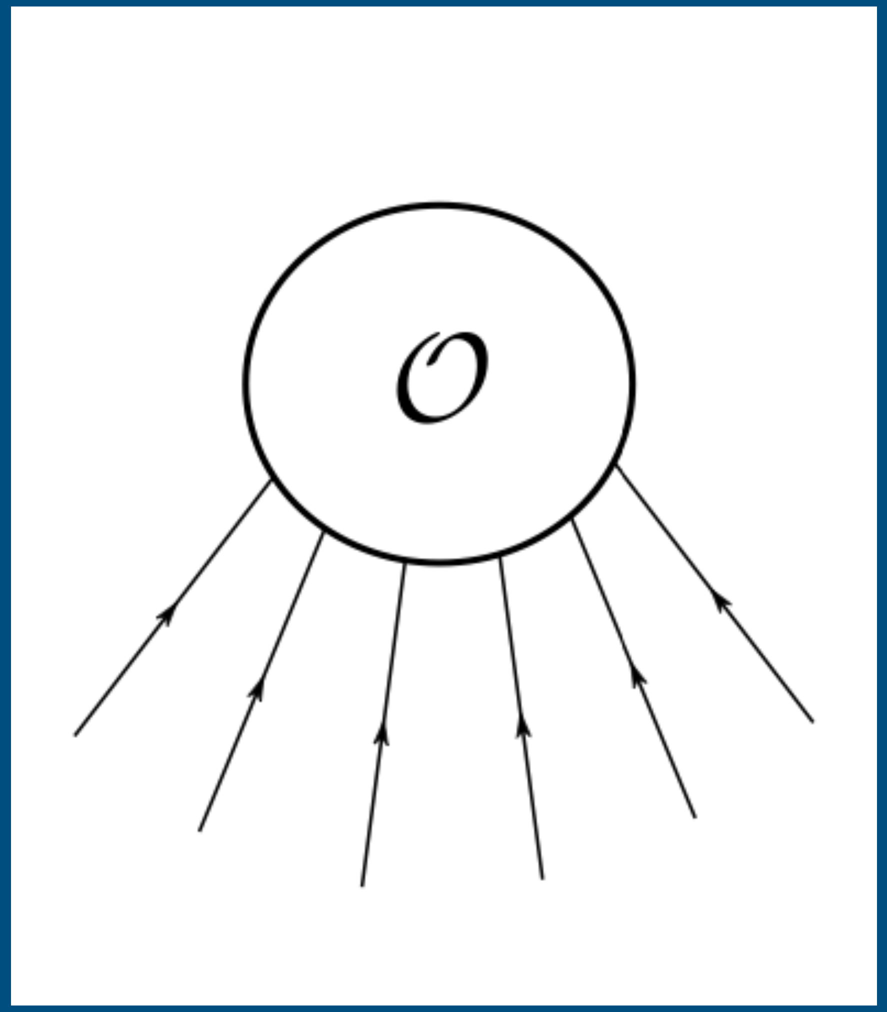
Role of locality

And semi-locality

$l(\mathcal{O}) \neq 0$



$$= e^{2i\pi l(\mathcal{O})}$$



Vanishing regular parts of matrix elements between representative state and other eigenstates λ

- $l(\mathcal{O})$ = semi-locality index
- Derived from OPE of the UV-CFT theory: $\mathcal{O}(z)\Phi(w) \sim (z-w)^{-l(\mathcal{O})}\Psi(z)$
- For example, l=1/2 for the spin field in Ising due to the Jordan-Wigner string

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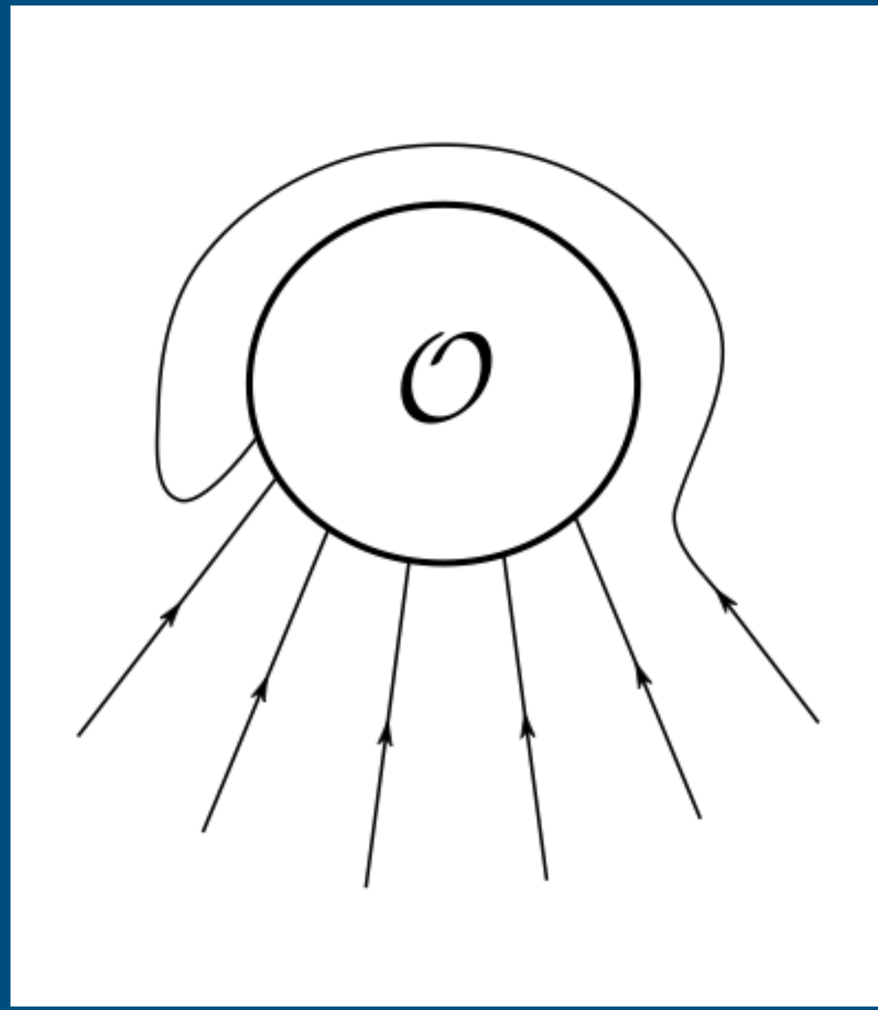

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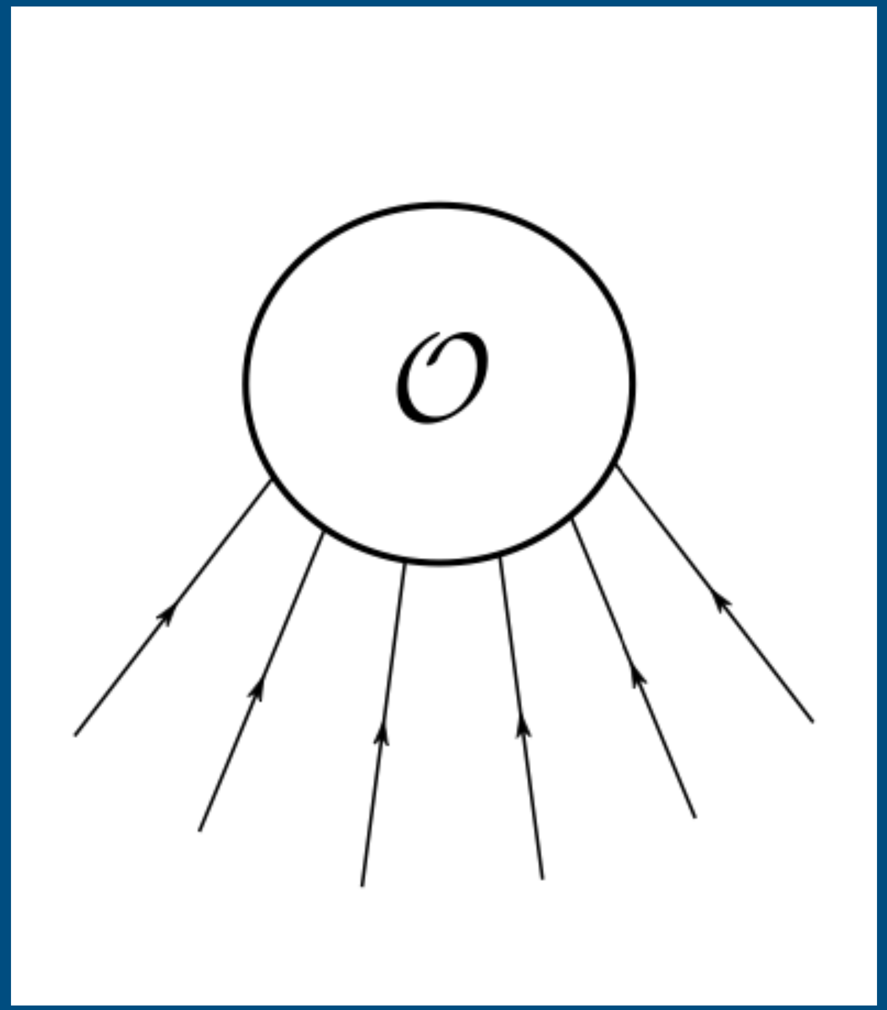

$$\Rightarrow \quad g(t) = 0$$

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Semi-local operators relax exponentially faster compared to local ones.

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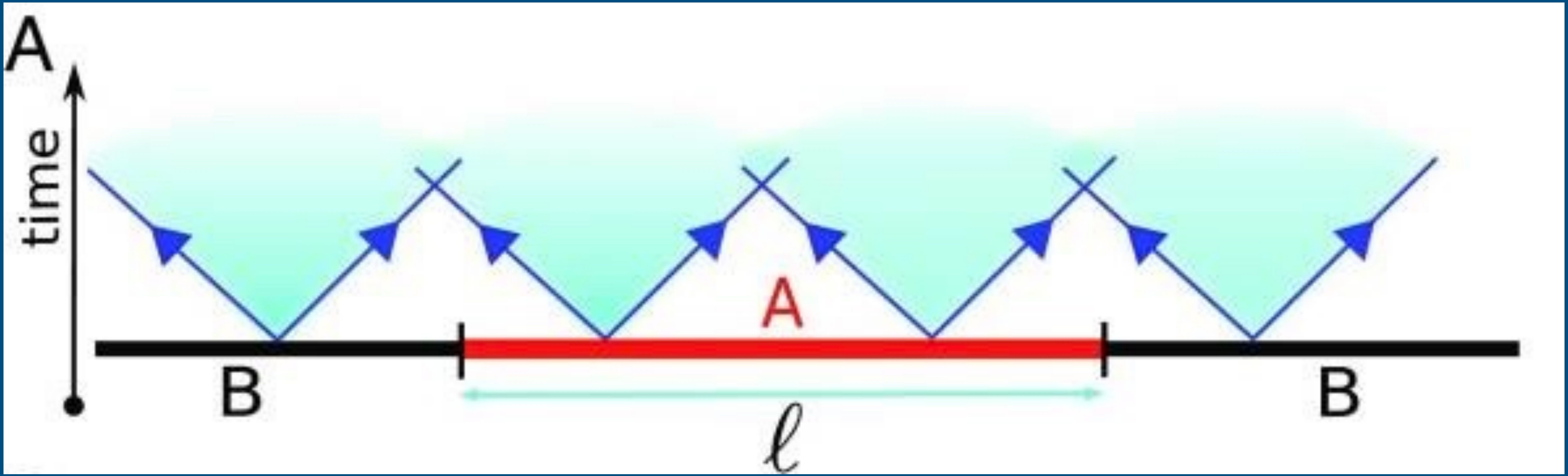
But why?

Entanglement entropy

In 1D models



Entanglement entropy of finite segment A

Entanglement entropy

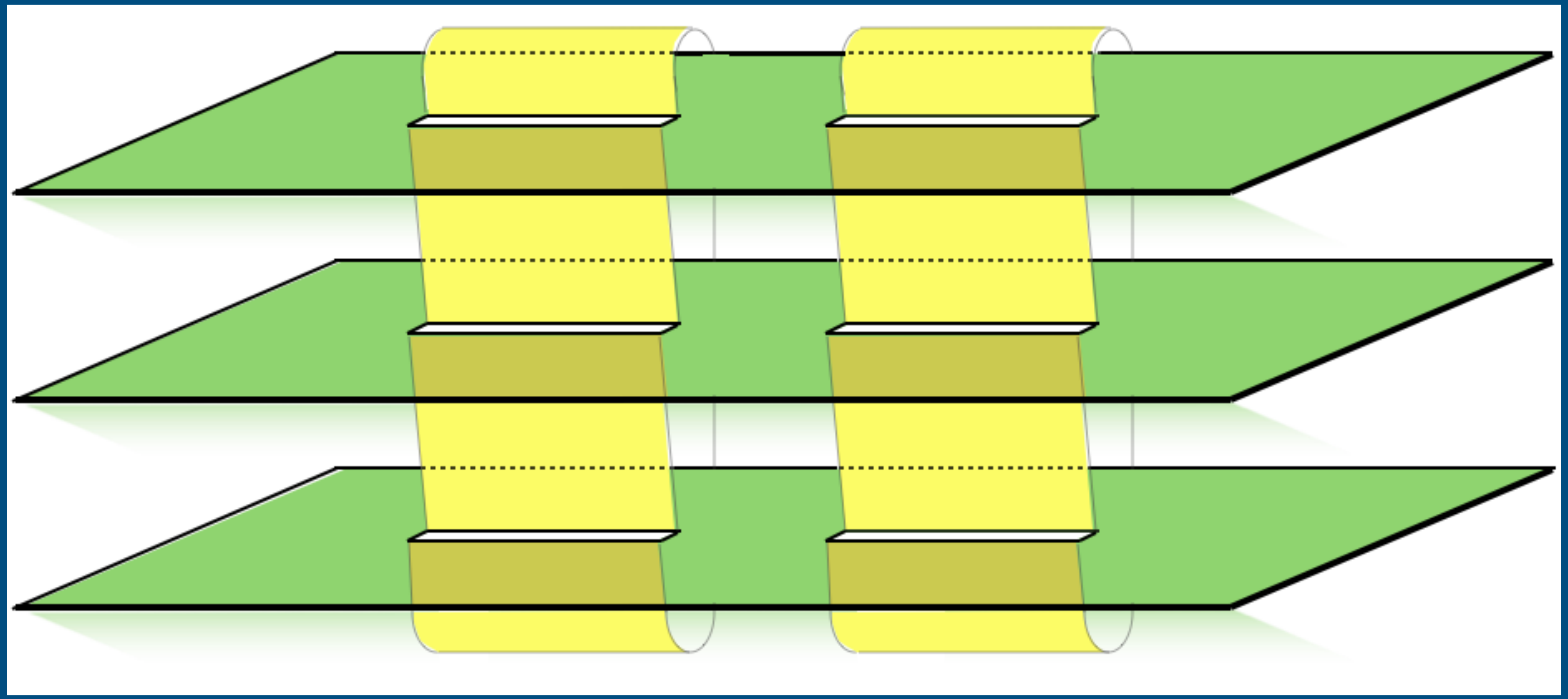
In 1D models

- Renyì entropies S_n
- Replica trick:

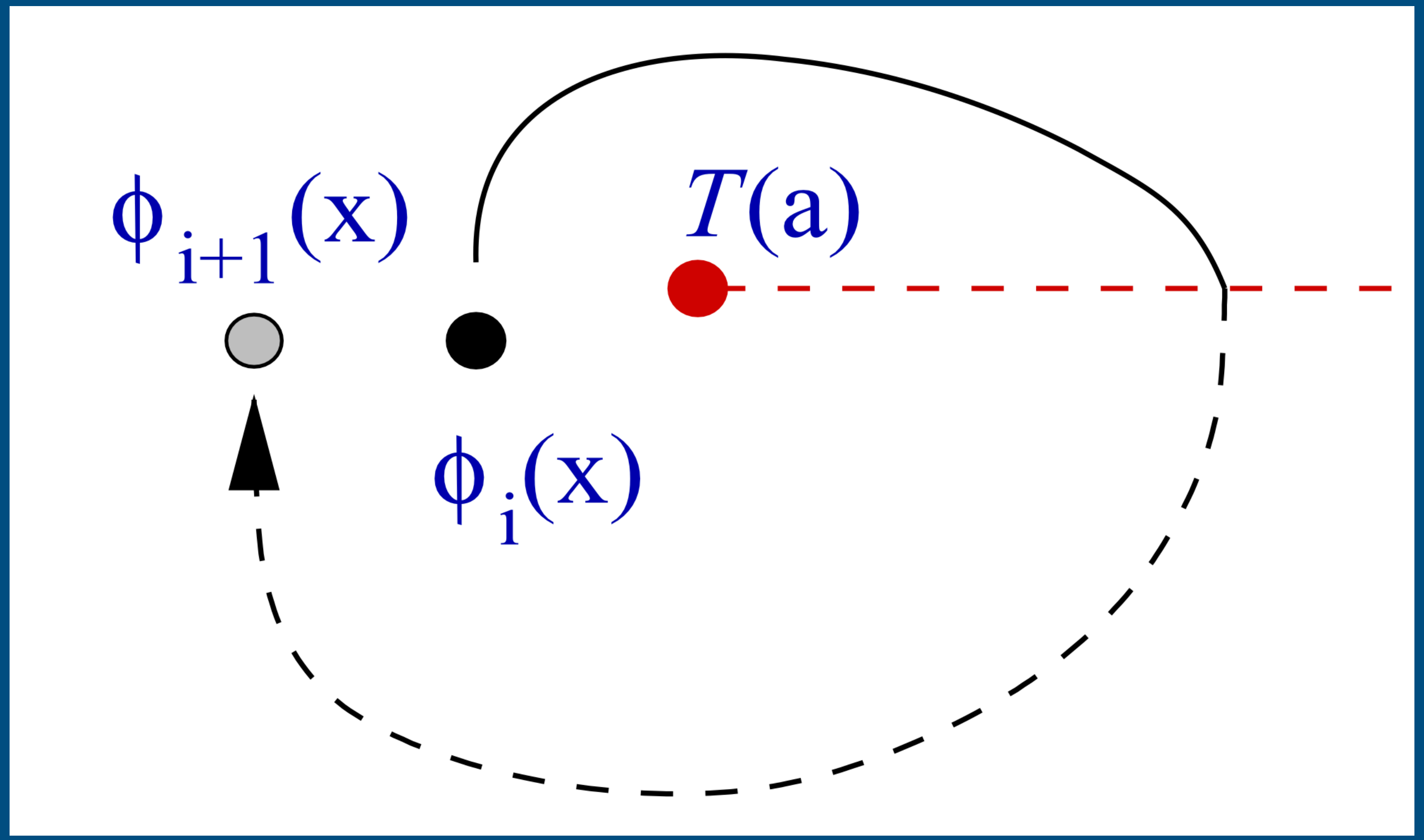
$$S_n(l) = -\, tr_A(\rho_{A,n} \log \rho_{A,n})$$

$$S(l) = \lim_{n \to 1} S_n(l)$$

- Equal-time correlator of branch twist operators $T_n(a_1, t)$ and $T_n^{\dagger}(a_2, t)$



3 copies of the same QFT on the Euclidean space



Action of branch twist operator on local fields

Entanglement growth after a quench

Semi-infinite line

- Subsystem A is now infinite
- Evaluating two-point function turns into just studying the one-point function of the operator $T_n(0,t)$ — clustering property of matrix elements
- Entanglement entropy growth:

$$S(t)/t = -\Gamma + \mathcal{O}\left(\frac{\log 2mt}{2mt}\right), \quad mt \gg 1$$

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$$S(t)/t = -\Gamma + \mathcal{O}\left(\frac{\log 2mt}{2mt}\right), \quad mt \gg 1$$

- Same value of damping exponent!

Entanglement growth after a quench

Segment of length l

- Evaluating two-point function is easier on the Euclidean plane
- Entanglement entropy grows and saturates:

$$S(l,t) \simeq -\Gamma t \ , \quad t < l/2$$

$$S(l,t) \simeq -\Gamma l/2 \ , \quad t > l/2$$

- Calabrese-Cardy quasiparticle picture retrieved as a first approximation
- Natural cutoffs

Conclusions

- General conclusion for integrable field theories
- Mainly related to the analytic structure of operator's Form Factors (semi-locality mainly)
- Local memory vs non-local
- Entanglement promotes relaxation of semi-local observables towards their equilibrium values
- All these results can be mapped into their respective non-relativistic system (Lieb-Liniger related system) by the means of out of equilibrium non-relativistic limit

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Thank you!

Linked Cluster expansion

And boundary initial state

- We choose the following boundary initial state:

$$|\psi\rangle = \exp\left\{ \int_0^{+\infty} d\theta K(\theta) Z^\dagger(-\theta) Z^\dagger(\theta) \right\} |0\rangle$$

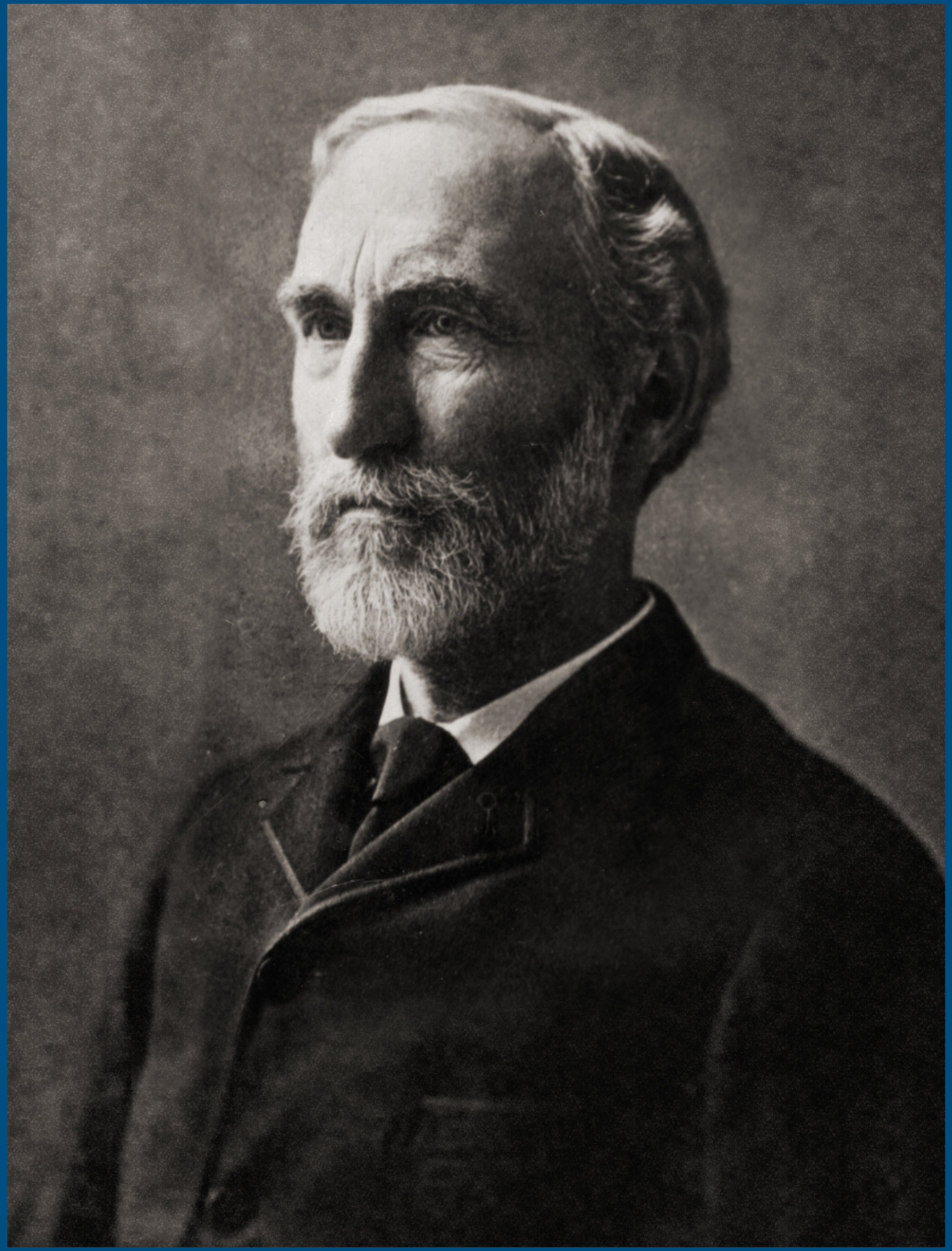
- Expand $\langle \mathcal{O}(t) \rangle$ in terms of K-functions and cancel the divergent contributions with finite volume regularisation

- Extra care in determining the non-exponentiating factors ($g(t)$)

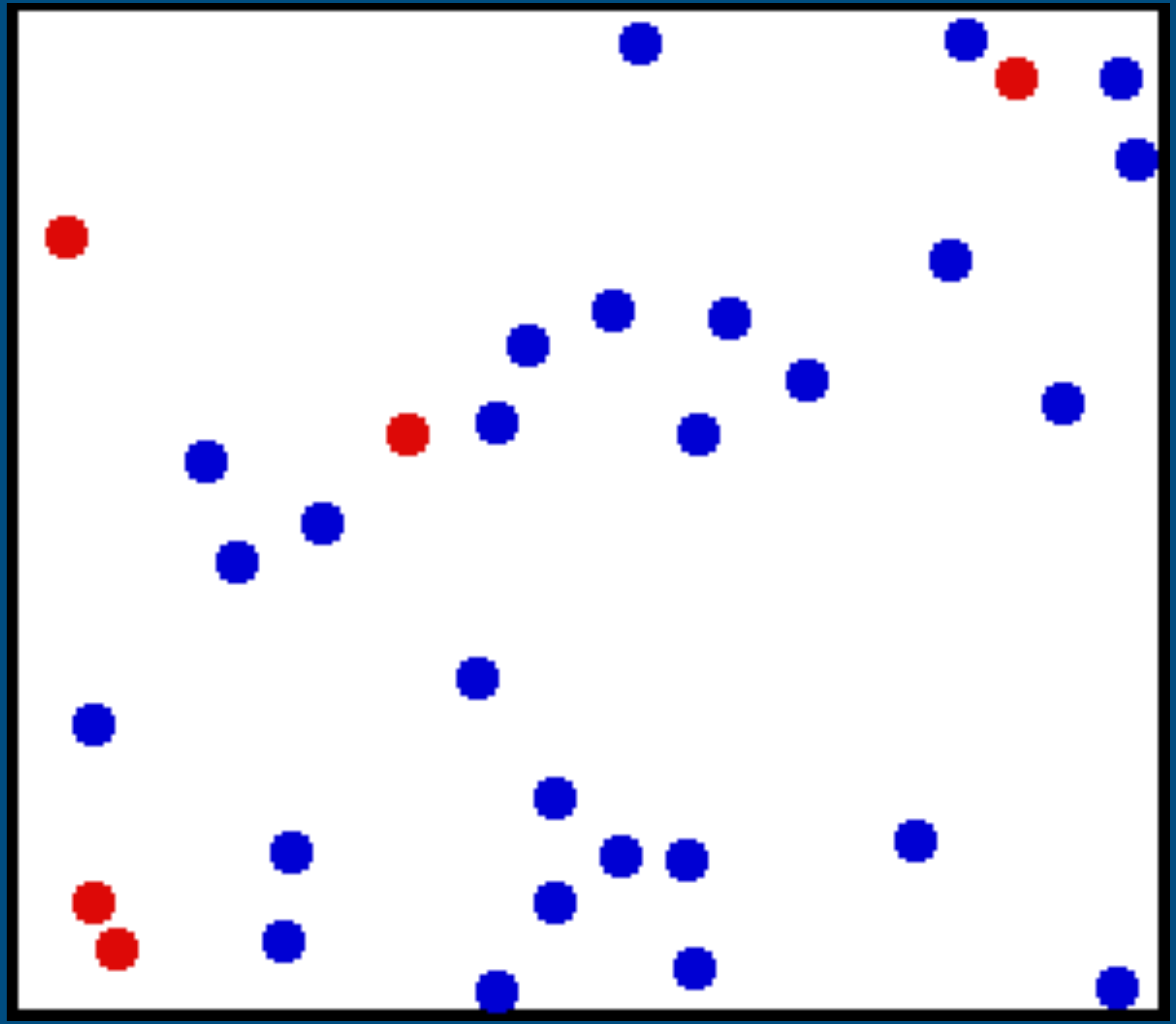
- Resum the expansion to get the full result

Before it's too late

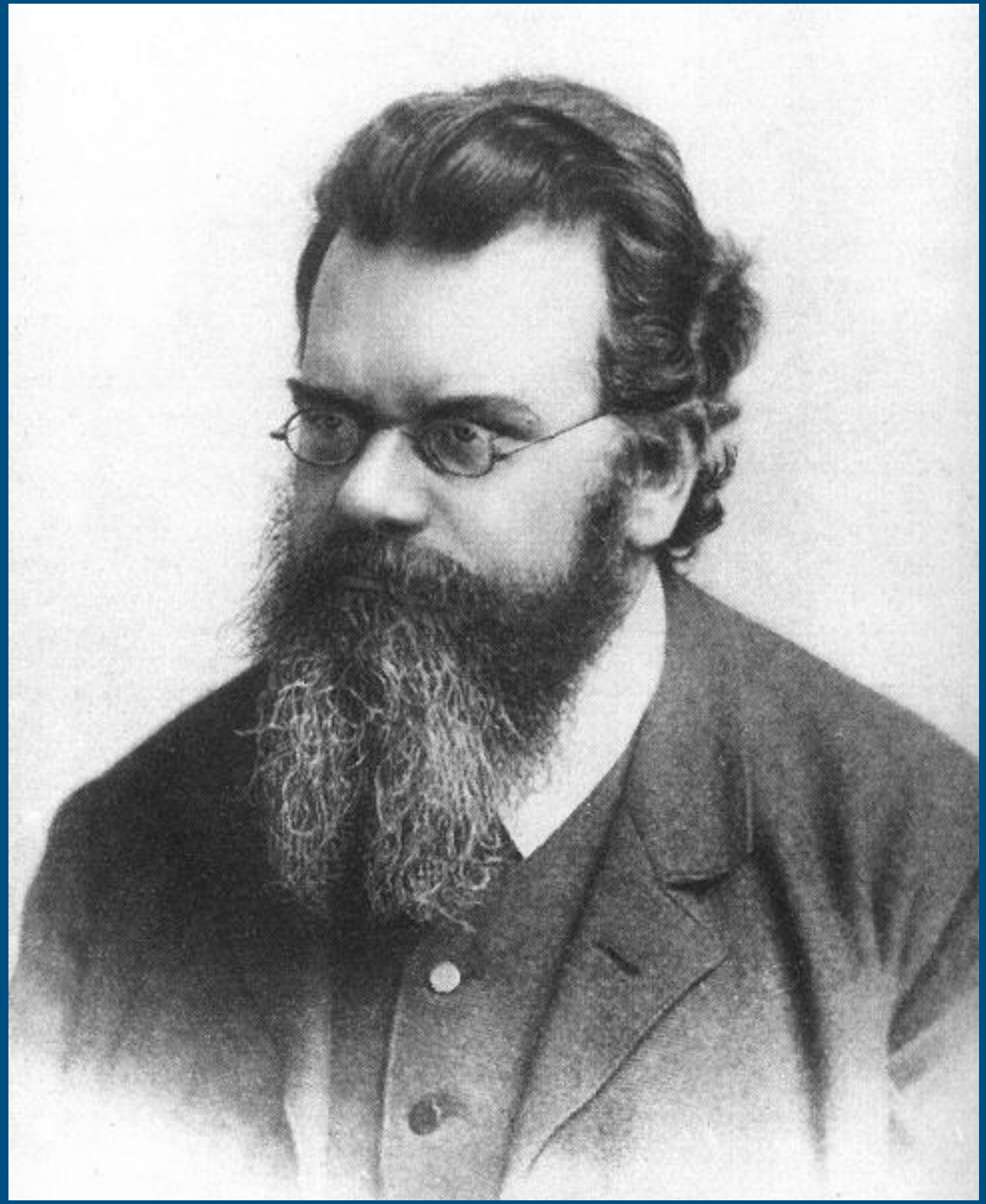
Ergodicity, thermalisation and relaxation



J. W. Gibbs



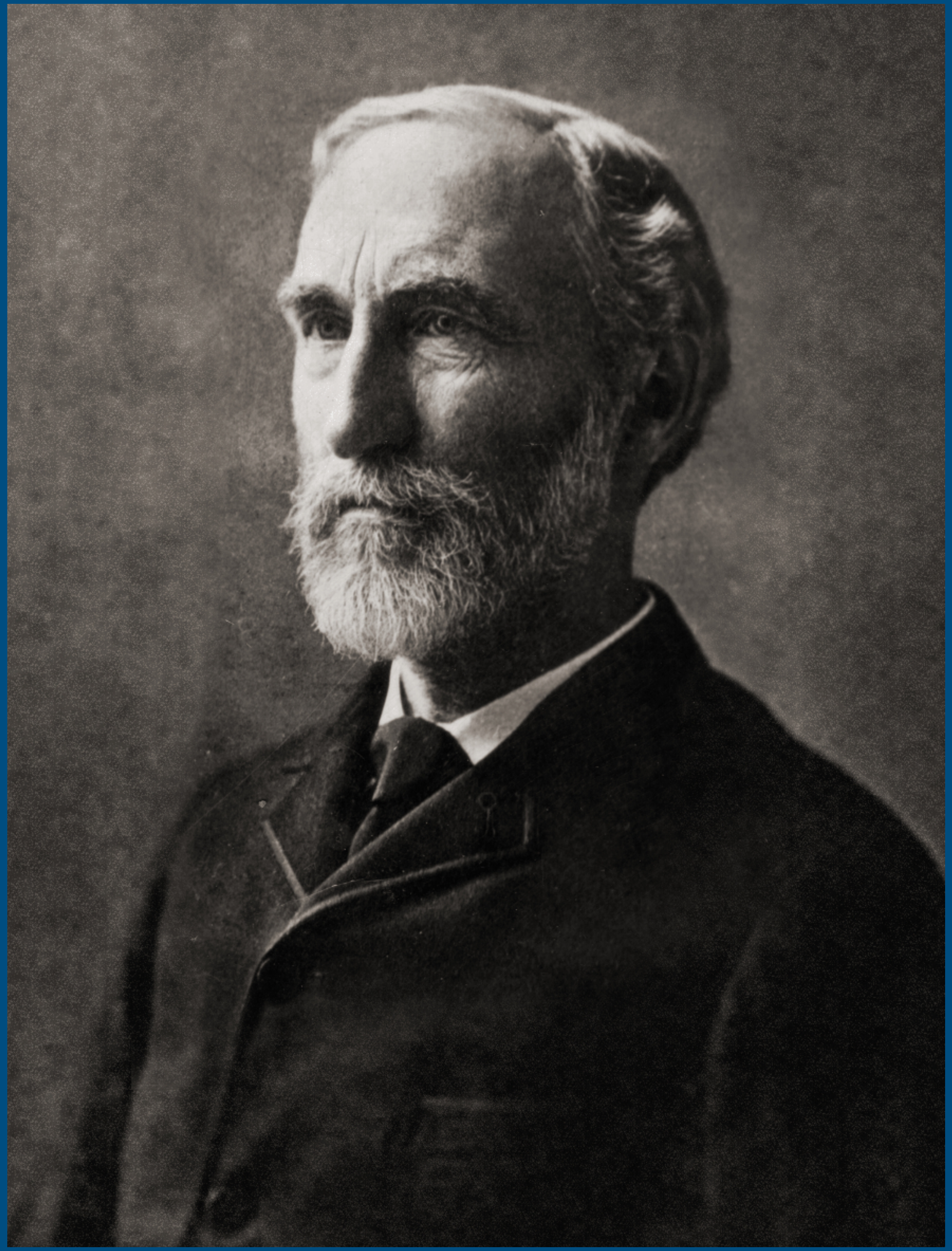
Ideal gas



L. Boltzmann

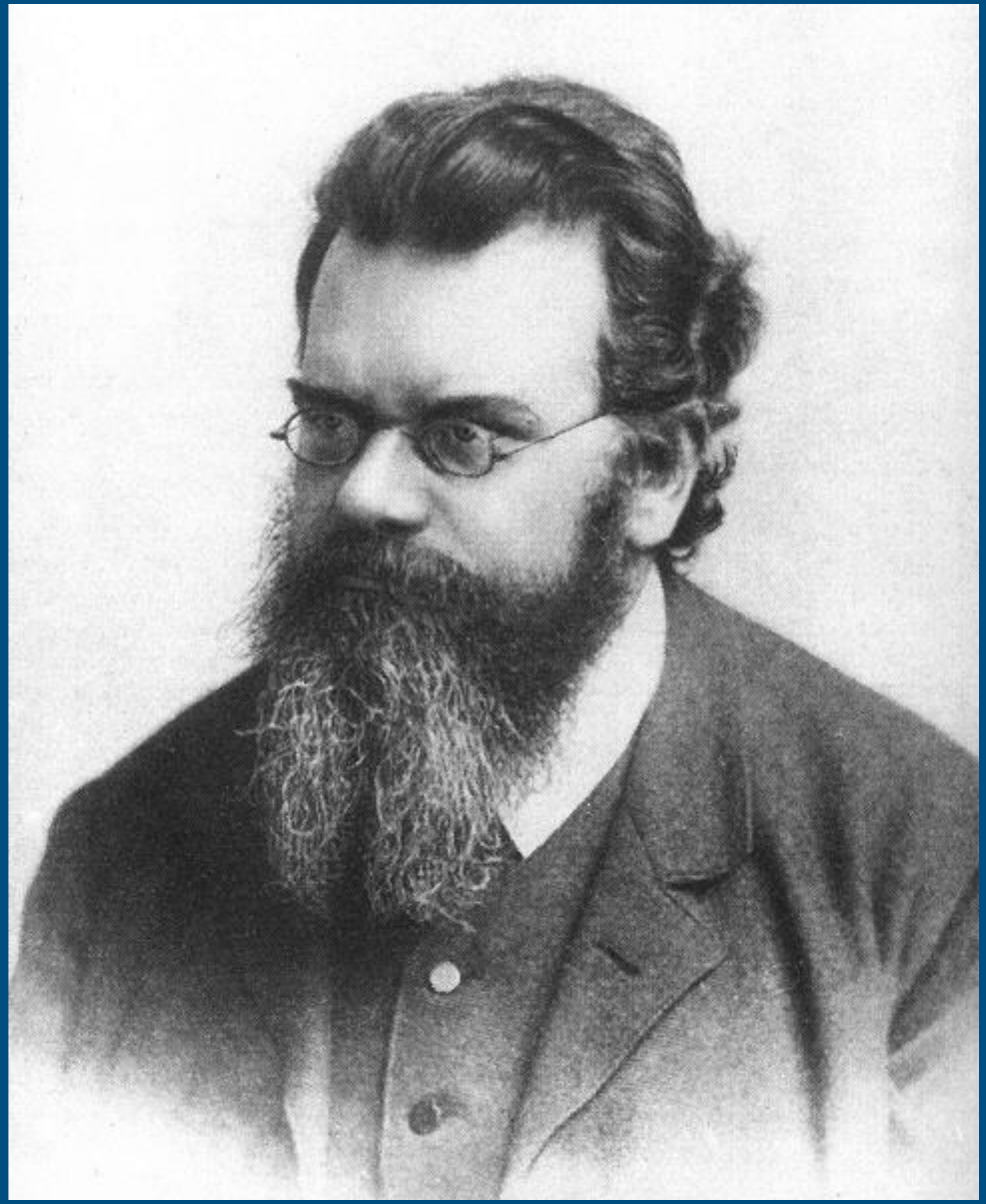
Before it's too late

Ergodicity, thermalisation and relaxation



J. W. Gibbs

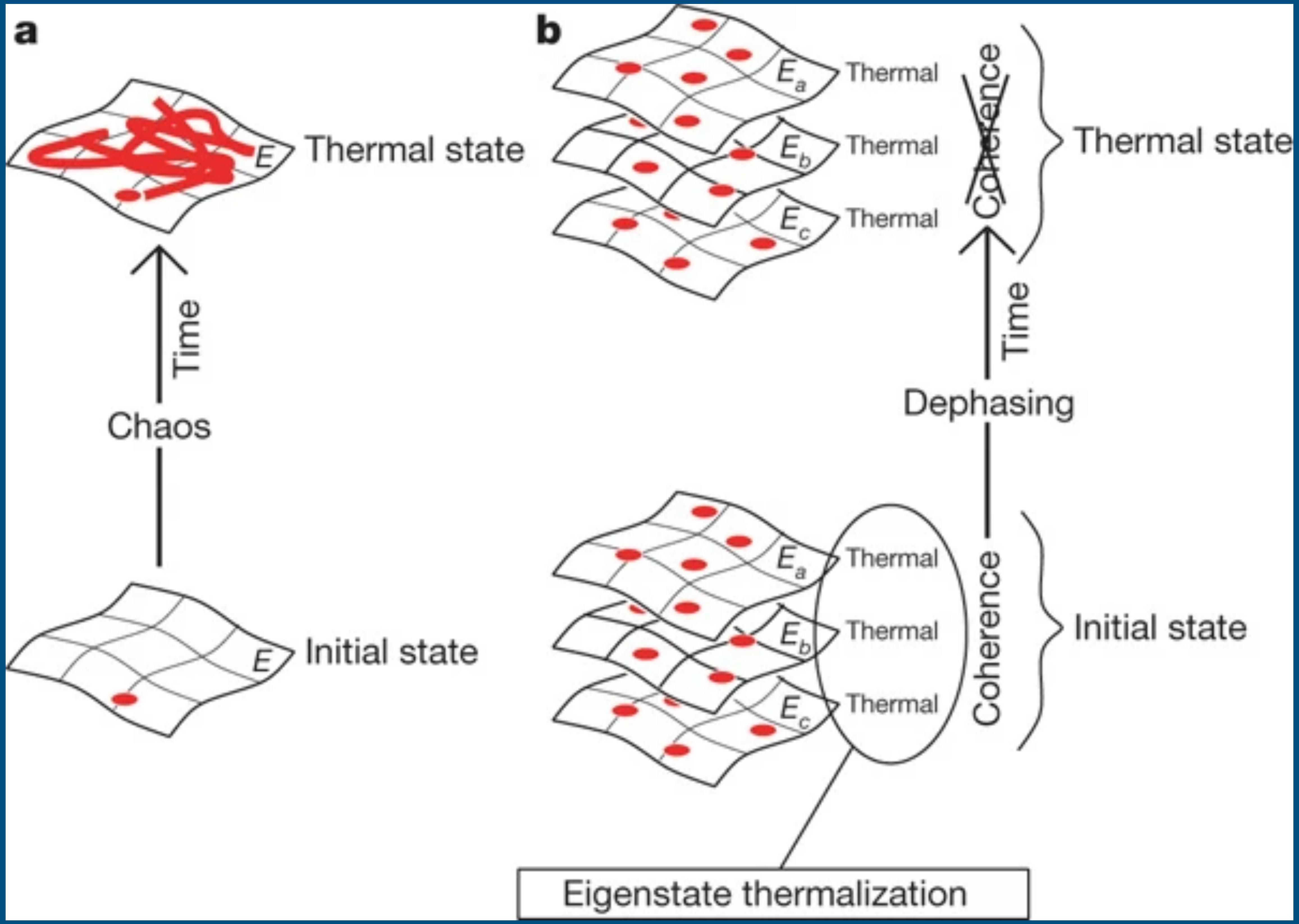
- System explores the whole space of energetically equivalent configurations
- After a transient period, the system is fully described by an equilibrium distribution function with temperature T



L. Boltzmann

What if it is quantum?

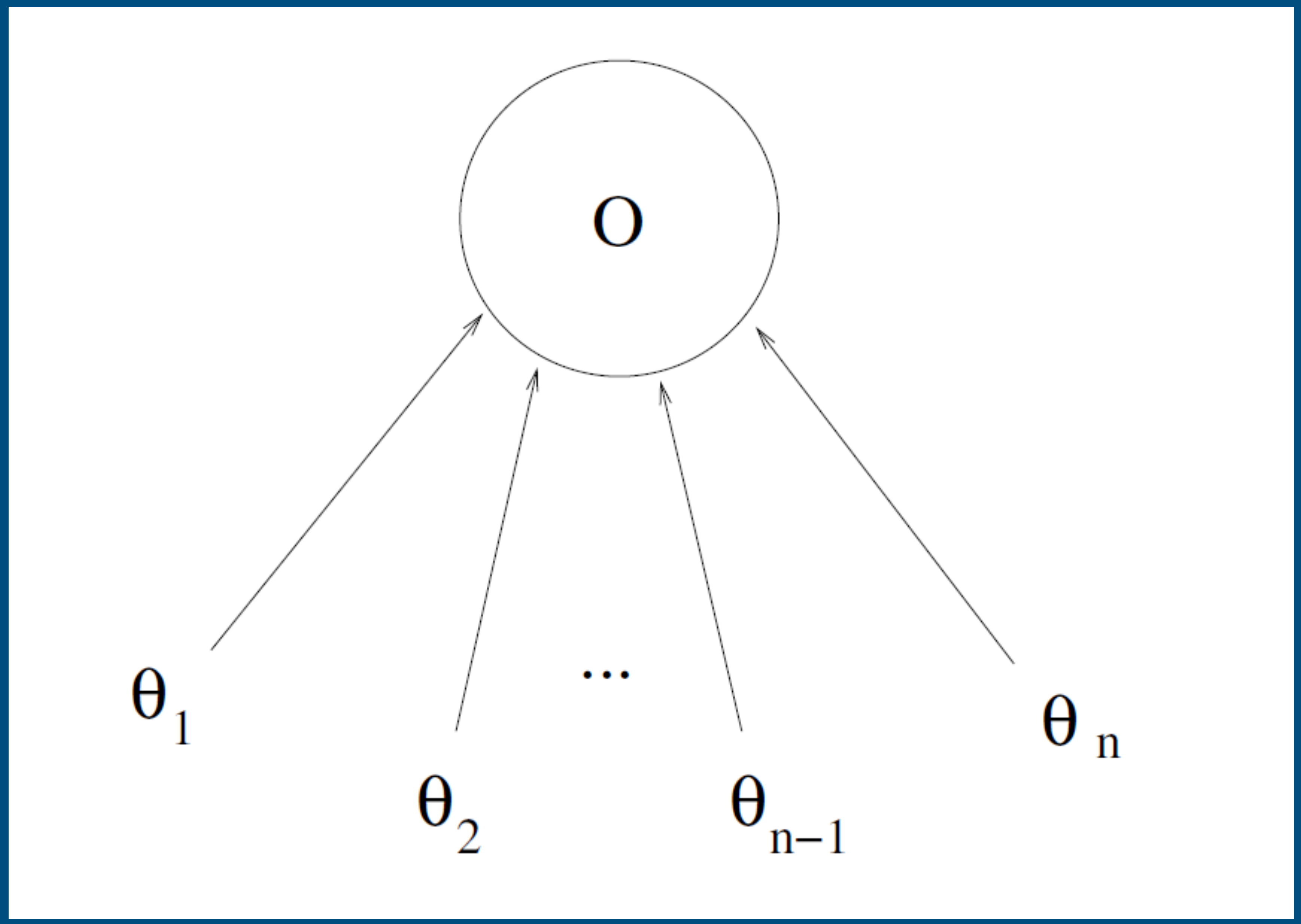
Or Eigenstate Thermalisation Hypothesis



Form Factors

Definition

$$\mathcal{F}_N^{O}(\theta_1, \ldots, \theta_N) = \langle 0 \,|\, O \,|\, \theta_1, \ldots, \theta_N \rangle$$



A n-particle form factor of the operator O

Form Factors

Recovering full matrix elements

$$\mathcal{F}^{O}_{N}(\theta_1, \ldots, \theta_N) = \langle 0 | O | \theta_1, \ldots, \theta_N \rangle$$

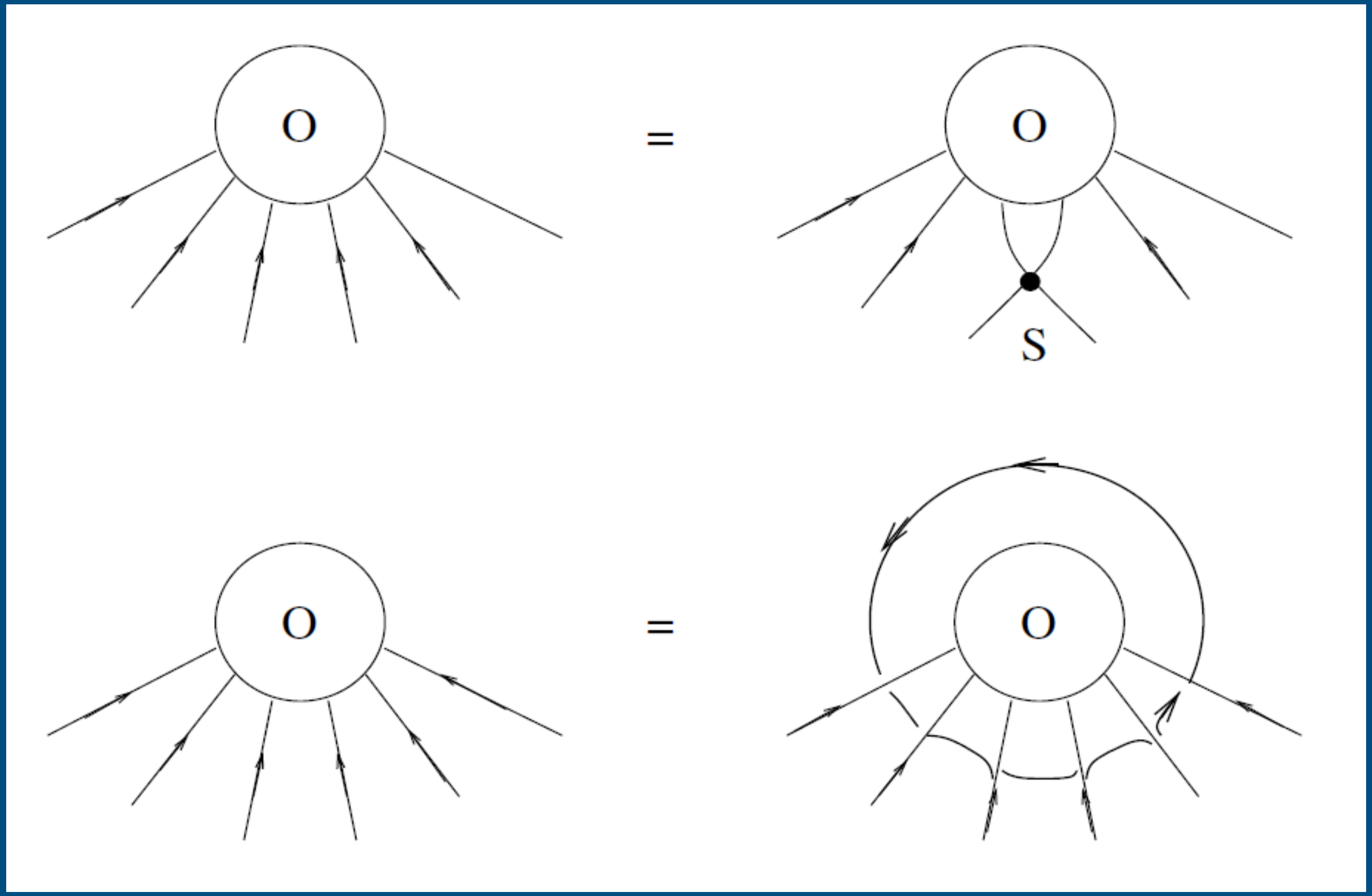
Crossing symmetry

$$\mathcal{F}^{O}_{N+1}(\xi_1 + i\pi, \theta_1, \ldots, \theta_N) = \langle \xi_1 | O | \theta_1, \ldots, \theta_N \rangle_{conn}$$

Form Factors

Watson's equations

$$\mathcal{F}_N^{O}(\theta_1, \ldots, \theta_N) = \langle 0 \,|\, O \,|\, \theta_1, \ldots, \theta_N \rangle$$



Form Factors

Watson's equations

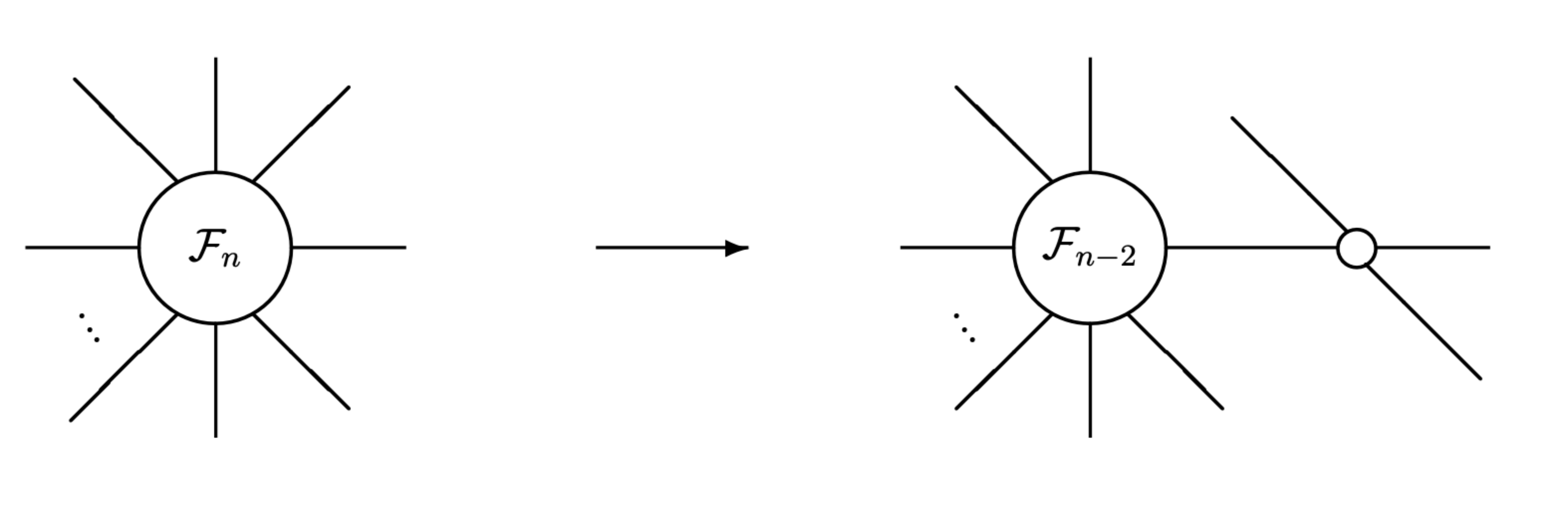
$$\mathcal{F}_N^O(\theta_1, \ldots, \theta_N) = \langle 0 | O | \theta_1, \ldots, \theta_N \rangle$$

$$\mathcal{F}_N^O(\theta_1, \ldots, \theta_{i+1}, \theta_i, \ldots \theta_N) = S(\theta_i - \theta_{i+1}) \mathcal{F}_N^O(\theta_1, \ldots, \theta_i, \theta_{i+1}, \ldots \theta_N)$$

$$\mathcal{F}_N^O(\theta_1 + 2i\pi, \ldots, \theta_N) = e^{2i\pi l(O)} \mathcal{F}_N^O(\theta_2, \ldots, \theta_N, \theta_1) = \prod_{i=2}^{N} S(\theta_i - \theta_1) \mathcal{F}_N^O(\theta_1, \ldots, \theta_N)$$

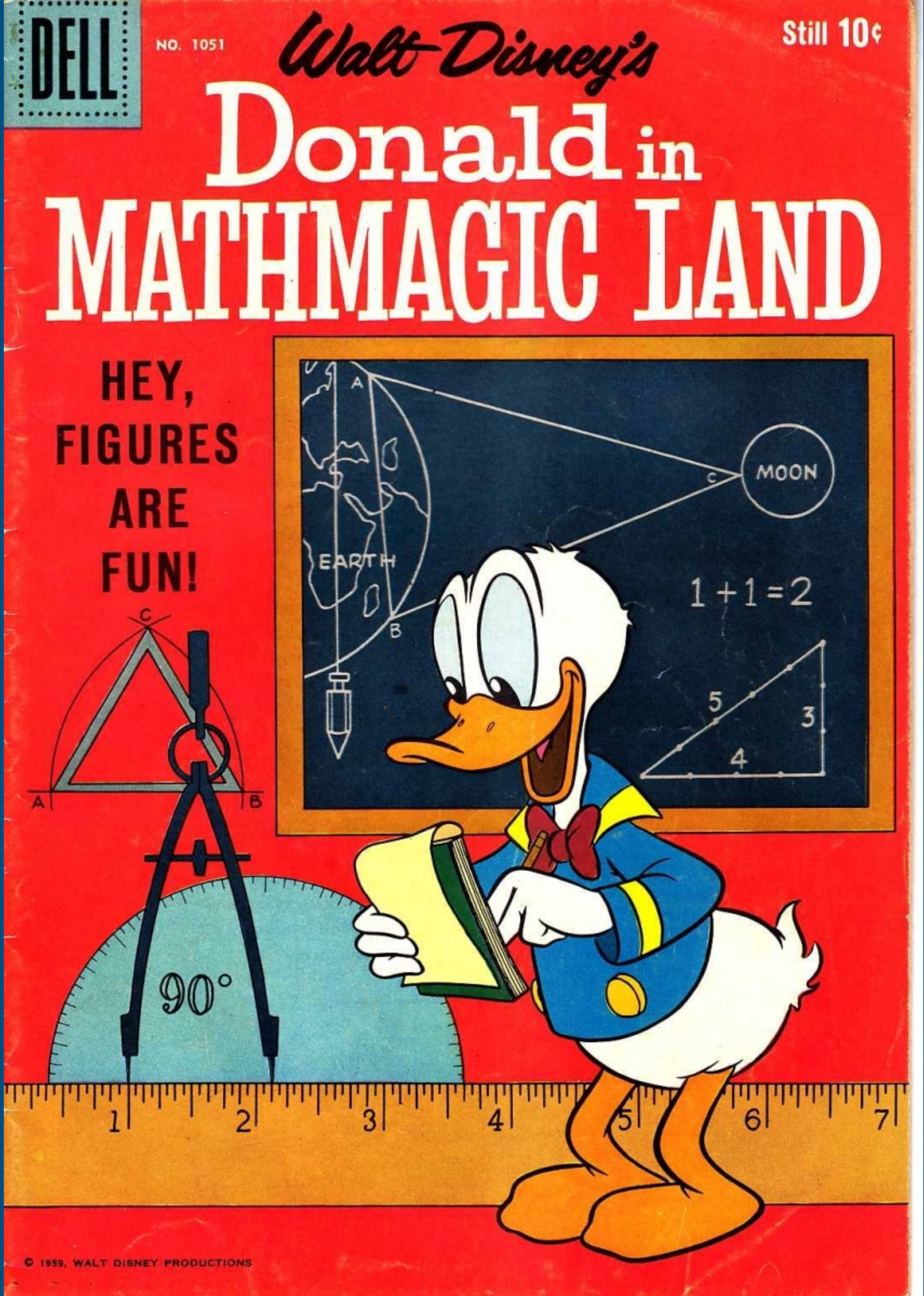
Form Factors

Annihilation pole axiom



Annihilation pole axiom

$$-iRes_{\bar{\theta}=\theta}\mathcal{F}_N^O(\bar{\theta}+i\pi,\theta,\theta_1,\ldots,\theta_{N-2}) = \left[1 - e^{2i\pi l(O)}\prod_{i=1}^{N-2} S(\theta_i-\theta)\right]\mathcal{F}_{N-2}^O(\theta_1,\ldots,\theta_{N-2})$$


DELL
NO. 1051
Walt Disney's
Still 10¢
Donald in
MATHMAGIC LAND
HEY, FIGURES ARE FUN!
MOON
EARTH
A
B
C
1+1=2
5
3
4
A
B
C
90°
1
2
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4
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6
7
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Ising model

ε and σ operators

- Massive fermionic theory, quench in the mass in the disordered phase $m_0 \to m$
- Free fermions S-matrix $S = -1$
- ε local vs σ semi-local wrt the fundamental field μ:

$$\sigma(z,\bar{z})\mu(0,0) \sim \frac{1}{\sqrt{2}\,|z|^{\frac{1}{4}}}\left(e^{i\frac{\pi}{4}}\sqrt{z}\psi(0) + e^{-i\frac{\pi}{4}}\sqrt{\bar{z}}\bar{\psi}(0)\right),\ l(\mathcal{O}) = \frac{1}{2}$$

Transverse field Ising model

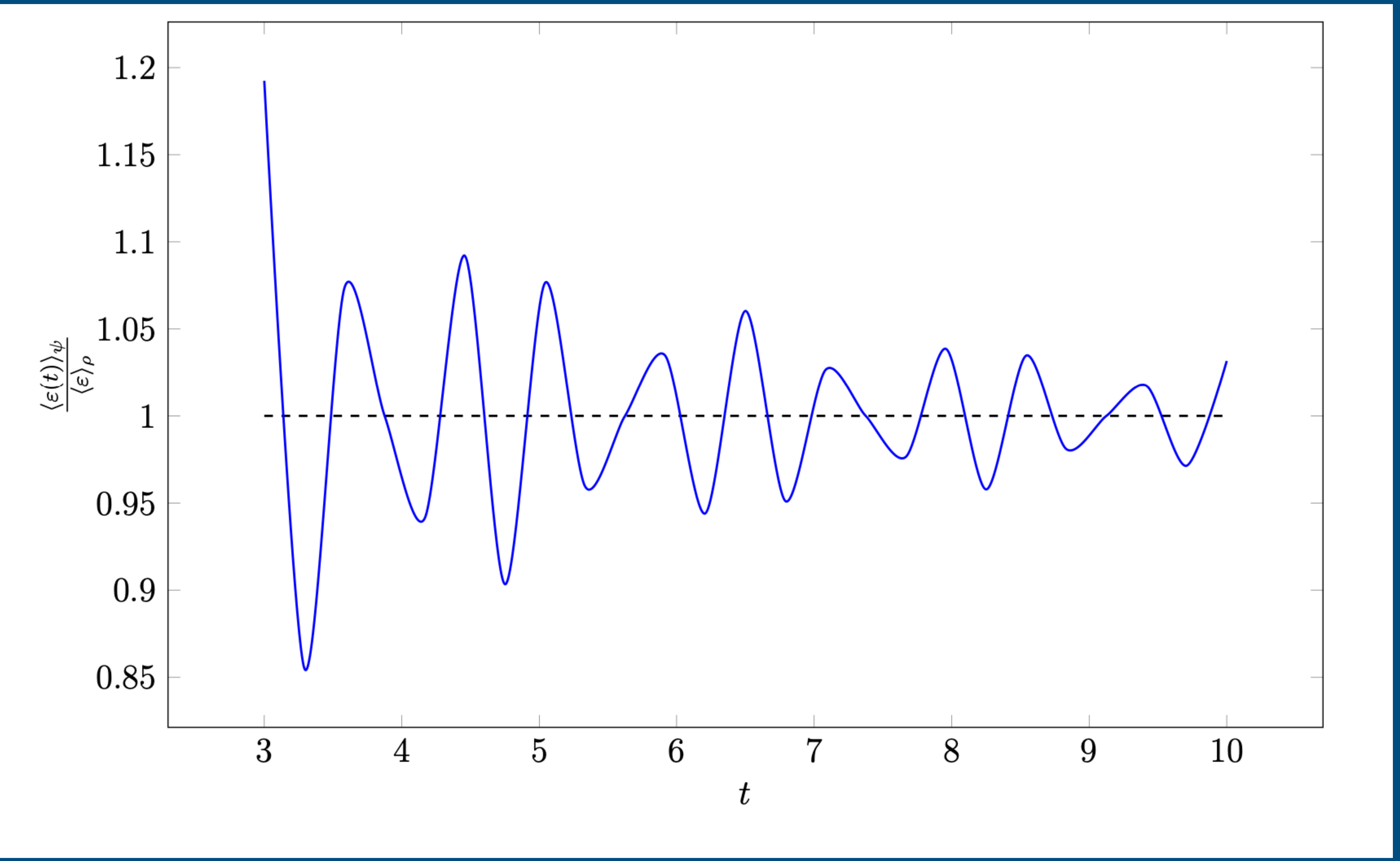
$\hat{\sigma}_i^z$ and $\hat{\sigma}_i^x$ operators

- Lattice theory:

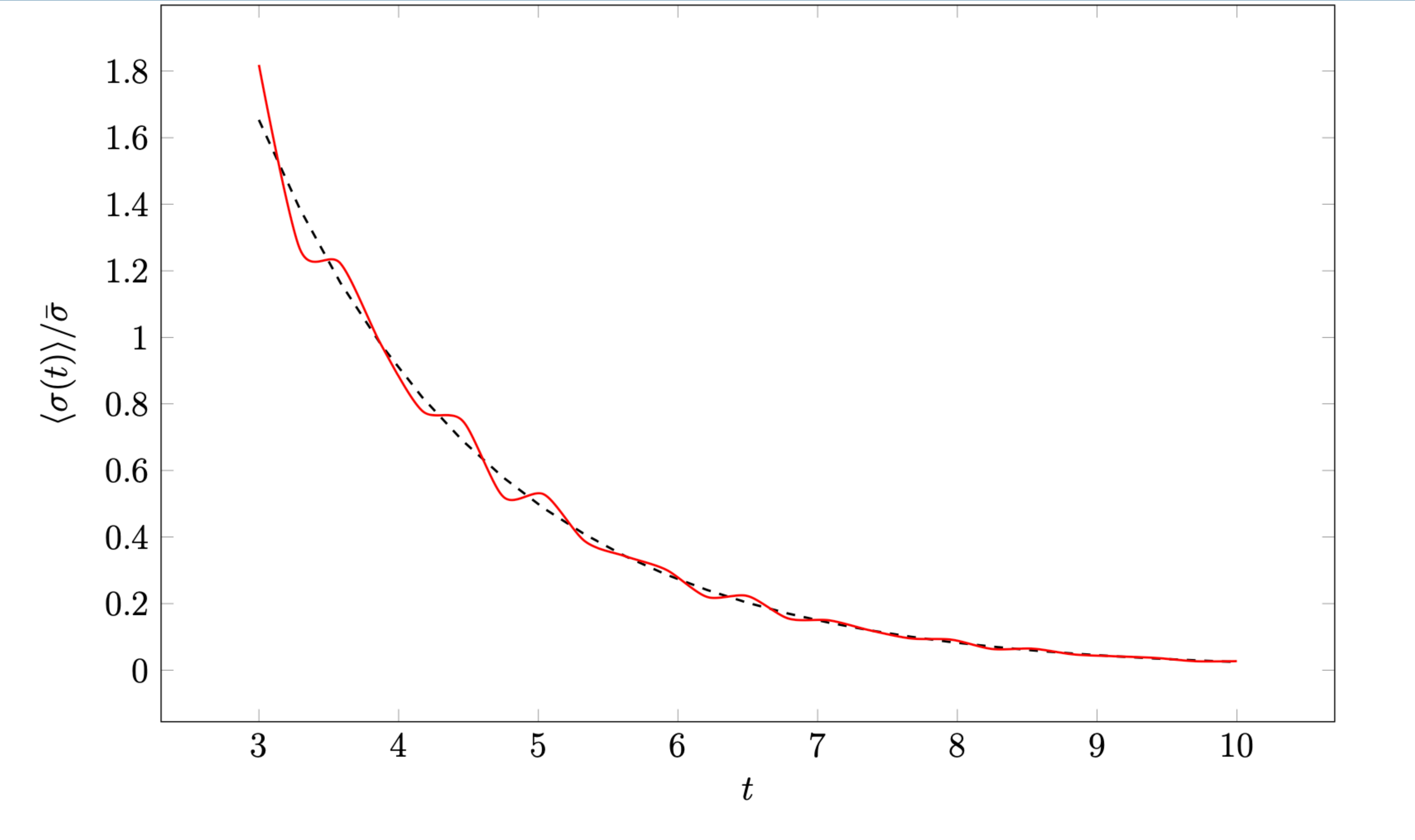
$$H = \sum_{i=1}^{L-1} \left[J\hat{\sigma}_i^x \hat{\sigma}_{i+1}^x + h\hat{\sigma}_i^z \right]$$

- Quench in the coupling $h_0 \to h$
- After Jordan-Wigner and Bogoliubov transformations we have free fermions
- $\hat{\sigma}_i^z$ local vs $\hat{\sigma}_i^x$ semi-local wrt the fermionic field $\hat{c}_i$

Energy/transverse spin operator



Spin/longitudinal spin operator



D. Fioretto, G. Mussardo -“Quantum Quenches in Integrable Field Theories”, *New J.Phys.12:055015,2010*
P. Calabrese, M. Fagotti, F.H.L. Essler - “Quantum Quench in the Transverse Field Ising chain I”, *J. Stat. Mech. (2012) P07016*

D. Schuricht, F.H.L. Essler - “Dynamics in the Ising field theory after a quantum quench*”, J. Stat. Mech. (2012) P04017*
P. Calabrese, M. Fagotti, F.H.L. Essler - “Quantum Quench in the Transverse Field Ising chain II”, *J. Stat. Mech. (2012) P07022*

Sinh-Gordon model

Vertex operator and twist operator τ

- Lagrangian field theory:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\mu^2}{g^2}(\cosh g\phi - 1)$$

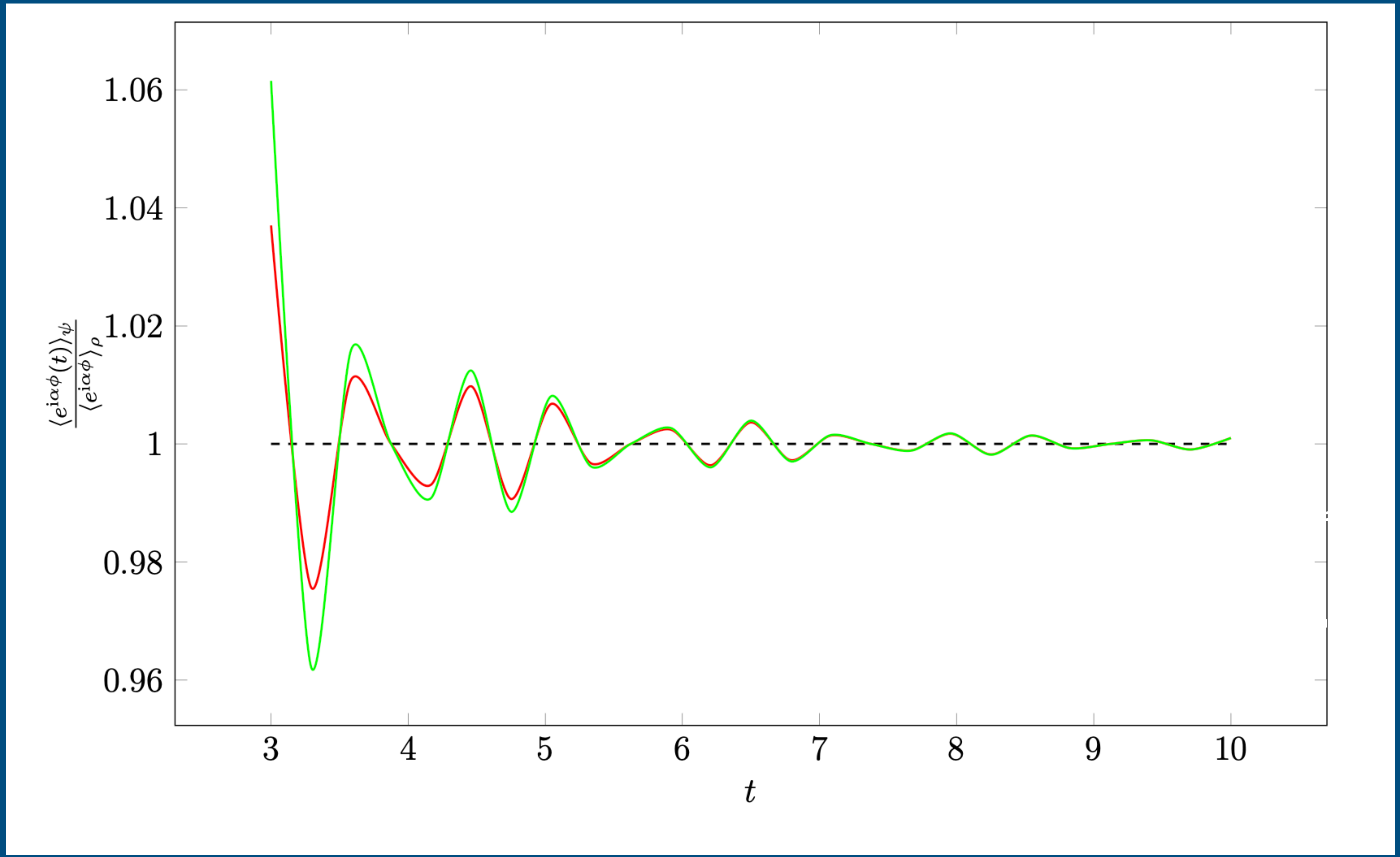
- Interacting theory with no bound states and non-trivial S-matrix:

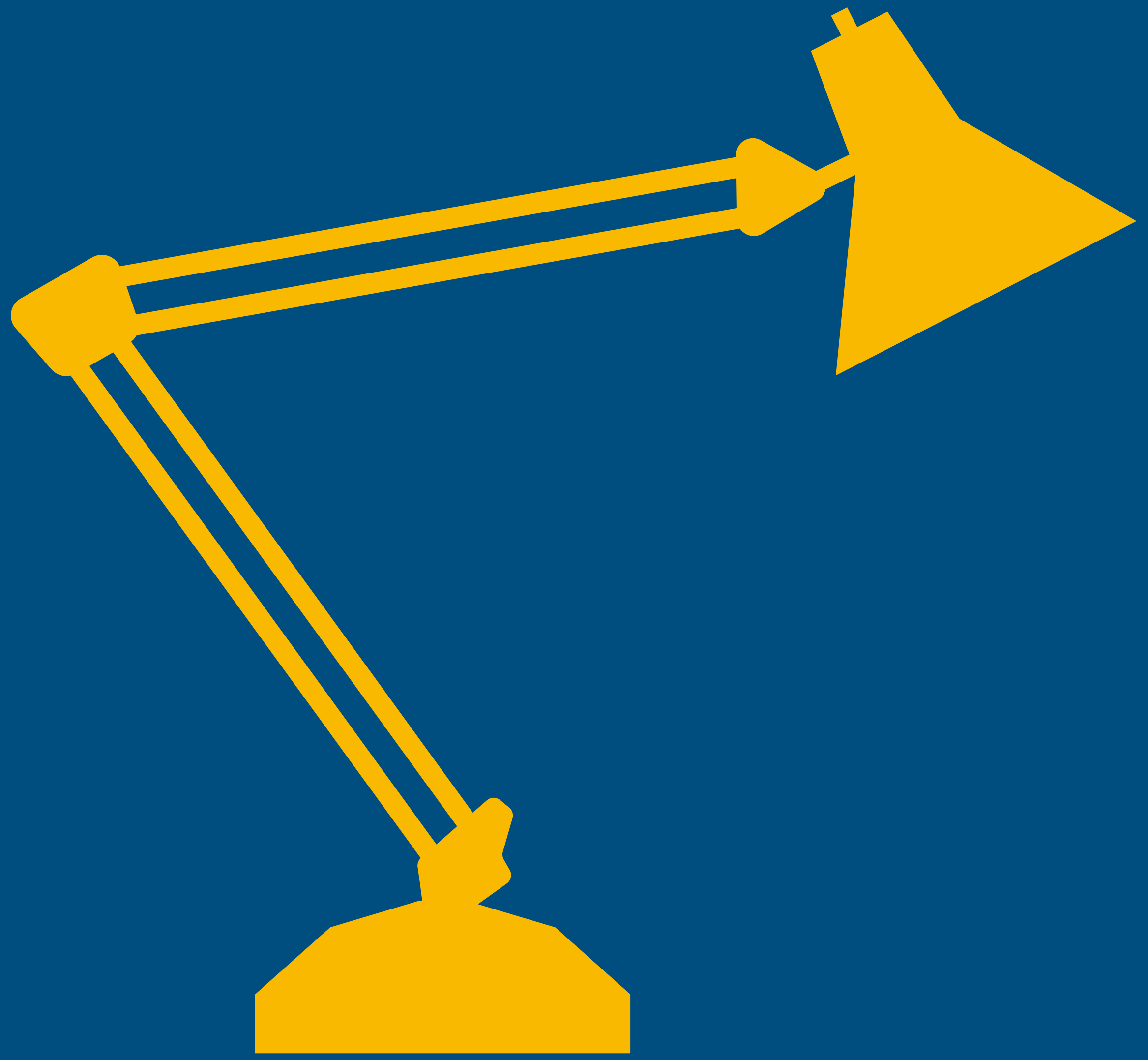
$$S(\theta) = \frac{\sinh\theta - i\sin B\pi}{\sinh\theta + i\sin B\pi}$$

- Local vertex operator $e^{i\alpha\phi}$ vs semi-local CFT twist operator τ

Vertex operator

CFT Twist operator



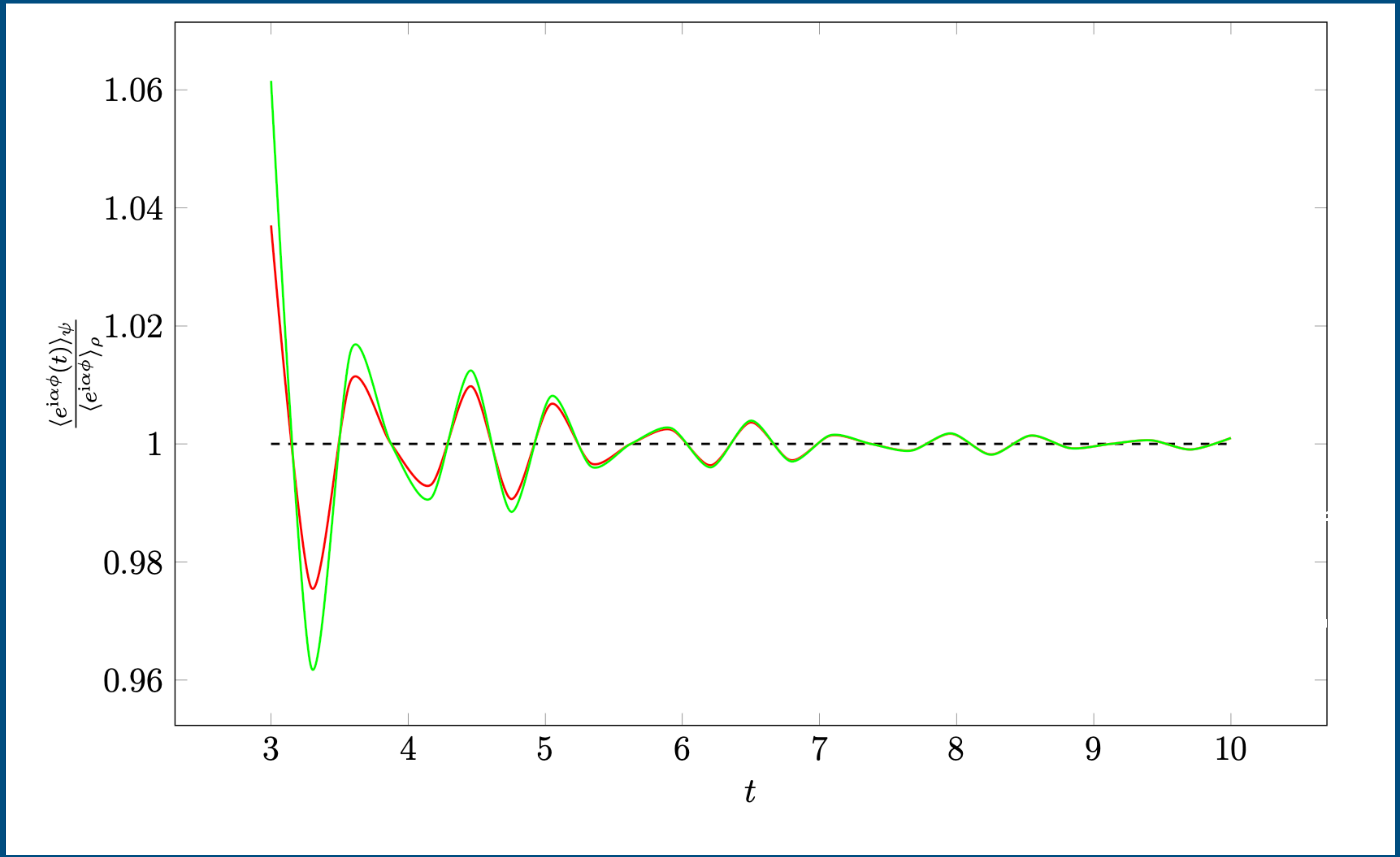


$= g(t)$

$= e^{-\Gamma t} f(t)$

E. Di Salvo, D. Schuricht - "Quantum quenches in the sinh-Gordon and Lieb-Liniger models", *J. Stat. Mech. (2023) 053107*

Vertex operator



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CFT Twist operator

Same behaviour of Ising’s spin operator $\sigma(t)$

Sine-Gordon model

Vertex operators

- Lagrangian field theory:

$$\mathscr{L} = \frac{1}{2}(\partial\phi)^2 - \frac{\mu^2}{\beta^2}(\cos\beta\phi - 1)$$

- Interacting theory of solitons with bound states (breathers)

- Local vertex operator $e^{i\beta\phi}$ (V. Gritsev *et al.*, PRL 2007) vs semi-local one $e^{i\beta\phi/2}$ (A. Cortes Cubero *et al.*, J. Stat. Mech. 2017)

Sine-Gordon model

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Branch twist operator's form factors

- Extension of S-matrix to particles in different replicated spaces
- Annihilation pole from adjacent sheets μ and $(\mu + 1)$
- Semi-locality index is now a matrix mixing different sectors of the theory