Entanglement entropy in the gauge Ising model: a Monte Carlo study

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Motivation

- Entanglement is a useful tool to investigate the properties of the ground state of quantum field theories in many areas of physics.
- Entanglement measures are used to
  - detect and characterize quantum phase transitions,
  - determine the number of effective degrees of freedom of a theory,
  - study the emergence of spacetime in quantum gravity,
  - ...
- Analytic calculations are typically performed for highly symmetric models, two dimensional systems or holographic theories.
- For general strongly interacting theories numerical calculations with tensor networks or (Quantum) Monte Carlo are necessary.
- However even numerical methods are limited by the highly non-locality of such observables, especially in high dimensions ($D = d + 1 > 2$).

Motivation

- Monte Carlo methods have been implemented in statistical models to study the entanglement entropy [Caraglio, Gliozzi; 0808.4094], [Alba, Tagliacozzo, Calabrese; 0910.0706, 1103.3166] as well as other entanglement measures such as entanglement negativity [Alba; 1302.1110].

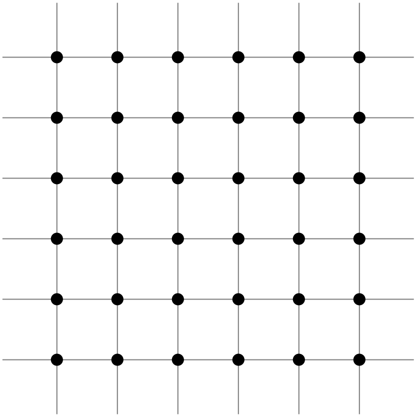
- $SU(N)$ gauge theories have also been studied [Buividovich, Polikarpov; 0802.4247] [Itou *et. al.*; 1512.01334] [Rabenstein *et. al.*; 1812.04279] [Jokela *et. al.*; 2304.08949], with the aim of understanding if entanglement can probe of confinement [Klebanov *et. al.*; 0709.2140].

Motivation

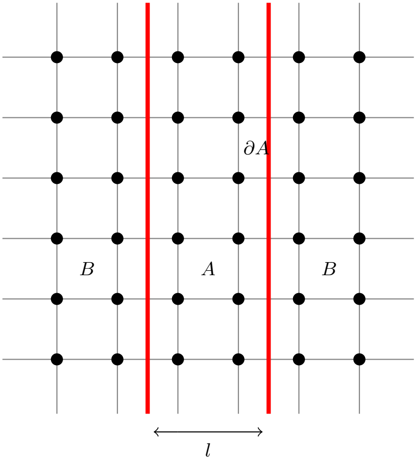
Monte Carlo calculations of entanglement measures for gauge theories present two main challenges:

- A significant **computational effort**, due to the high non-locality of the observables.
- The **ambiguity** in the definition of entanglement measures for gauge theories.

Entanglement in QFT



Entanglement in QFT



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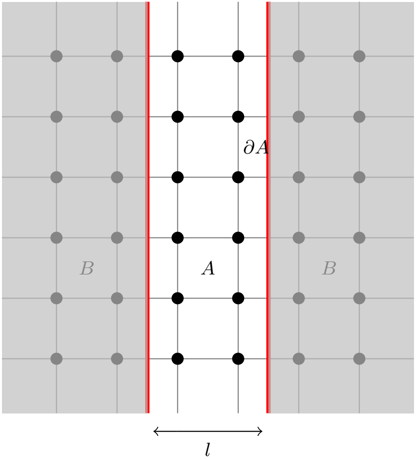

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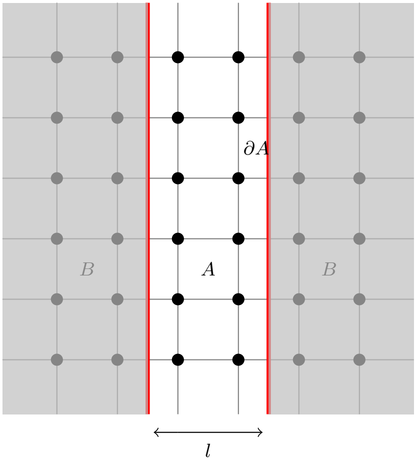


$$S_A(l) = -\operatorname{Tr}\{\rho_A \log \rho_A\} \qquad S_n(l) = \frac{1}{1-n} \log \operatorname{Tr} \rho_A^n$$

Area law and universal terms

- In $D = d + 1$ spacetime dimensions the entanglement entropy for pure states exhibits a UV divergent area law

$$S_A(l) = \alpha \frac{|\partial A|}{a^{D-2}} + \dots$$

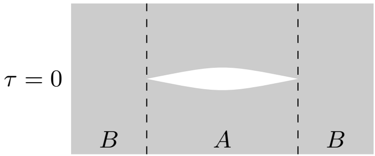
- Relevant physical information is expected to be encoded in the universal terms beyond the area law.
- In the slab geometry a common choice to regularize the UV divergence is to define the entropic c-function [Casini, Huerta; hep-th/0405111] [Nishioka, Takayanagi; hep-th/0611035]

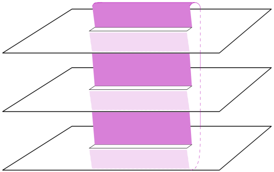
$$C(l) = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_A}{\partial l}$$

- The entropic c-function is expected to be monotonic under the RG flow, therefore it provides a "measure" of the number of degrees of freedom [Zamolodchikov; 1986][Casini, Huerta; cond-mat/0610375].

Replica trick

- A common way to calculate Rényi entropies (*i.e.* $\mathrm{Tr}\,\rho_A^n$) and other entanglement measurements is to exploit the replica trick [Calabrese, Cardy; hep-th/0405152].
- Consider for simplicity a $(1+1)$-dimensional system (such as a spin chain).





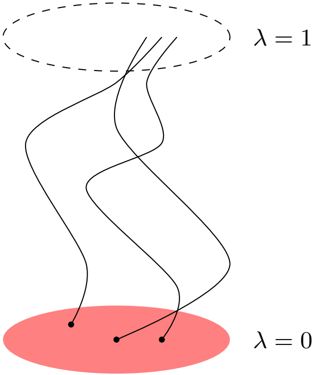
From [Cardy *et. al.*; 0706.3384].

$$S_n = \frac{1}{1-n} \log \frac{Z_n}{Z^n} \qquad C_n = \frac{l^{D-1}}{|\partial A|} \frac{\partial S_n}{\partial l} \sim \frac{1}{a} \log \frac{Z_n(l+a)}{Z_n(l)}$$

Jarzinski's theorem

- Jarzynski's theorem [Jarzynski; cond-mat/9610209] is an exact result that connects averages of out-of-equilibrium trajectories of a statistical system to equilibrium free energies.
- Consider the one parameter evolution $H_{\lambda=0} \rightarrow H_{\lambda=1}$. Jarzynki's theorem states that

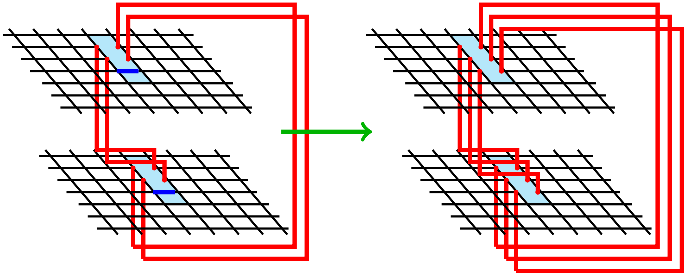
$$\left\langle \exp\left(-\int \beta\delta W\right)\right\rangle = \frac{Z_{\lambda=1}}{Z_{\lambda=0}}$$



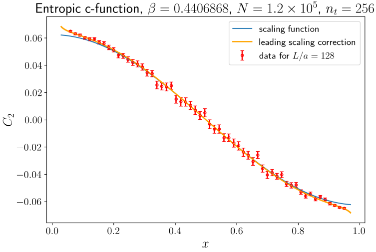
- In a Monte Carlo simulation the evolution is implemented by updating the configurations with a Hamiltonian which changes in Monte Carlo time.

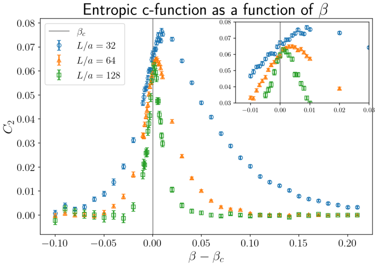
Our algorithm

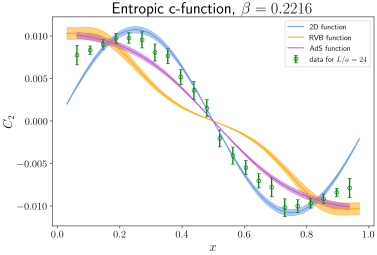
$$\frac{\partial S_n}{\partial l} \simeq \frac{1}{1-n}\frac{1}{a}\log\frac{Z_n(l+a)}{Z_n(l)}$$



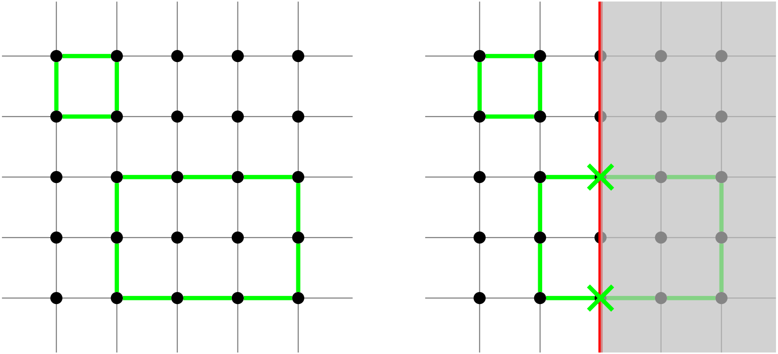
Some results







Entanglement in gauge theories



$$\mathcal{H}_{phys} \neq \mathcal{H}_{phys,A} \otimes \mathcal{H}_{phys,B}$$

The gauge Ising model in $D = 3$

- To avoid ambiguities in the definition of the replica space for a gauge theory we choose to study a theory that is dual to a spin model for which the Rényi entropy is well defined.
- The duality of the gauge Ising model in $D = 3$ implies that

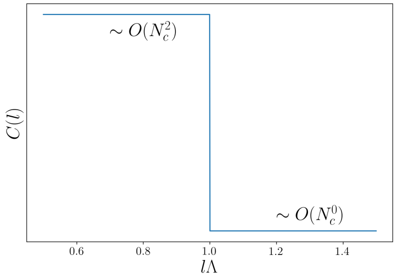
$$Z_{\text{gauge}} \propto Z_{\text{Ising}}$$

$$\langle O \rangle_{\text{gauge}} = \frac{\tilde{Z}_{\text{Ising}}}{Z_{\text{Ising}}}$$

- Therefore $\left(\frac{Z_n}{Z^n}\right)_{\text{gauge}} = \left(\frac{Z_n}{Z^n}\right)_{\text{Ising}}$ and we can perform the calculation of the Rényi entropy in the spin model.
- Despite its simplicity the gauge Ising model exhibits many phenomena that are present also in $SU(N)$ gauge theories, among which confinement.

Entanglement entropy in confining theories

- In confining theories that admit a dual holographic description it was shown [Klebanov *et. al.*; 0709.2140] that the entropic c-function has a sharp (first-order) transition.
- The scale of this transition is set by a scale which is of the same order of the hadronic scale Λ_{QCD}, the critical temperature T_c or the mass of the lightest glueball m_{glueball}.



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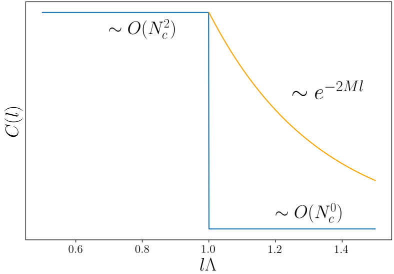
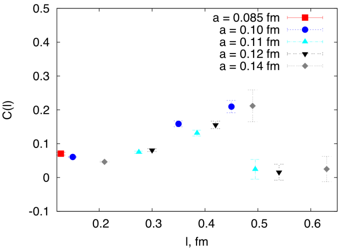
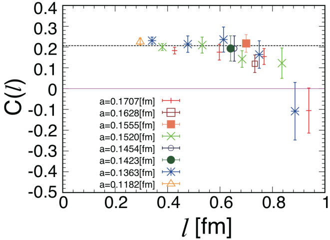


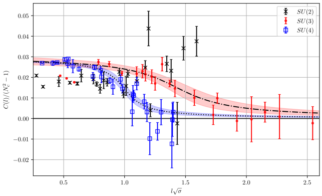
Monte Carlo simulations of $SU(N)$ gauge theories



From [Buividovich, Polikarpov; 0802.4247]

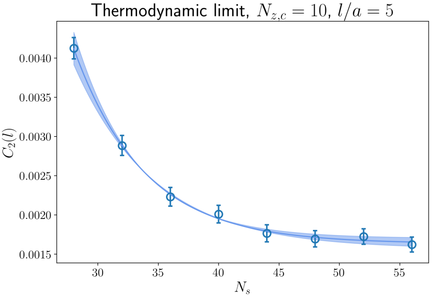


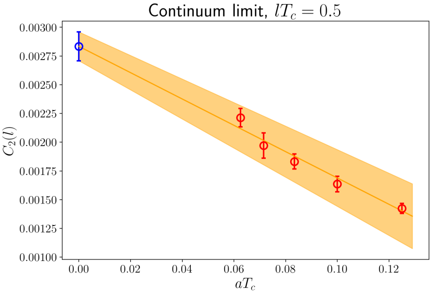
From [Itou *et. al.*; 1512.01334]



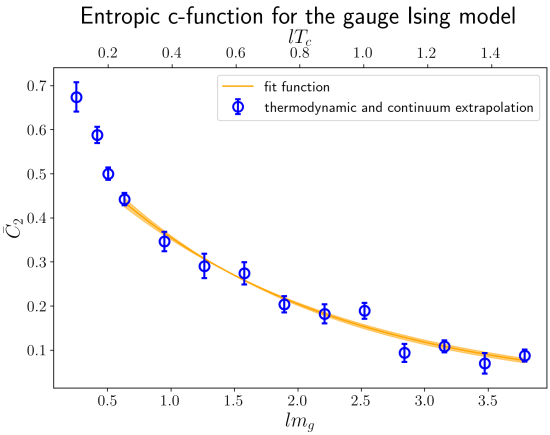
From [Rabenstein *et. al.*; 1812.04279]

Preliminary results





- $\bar{C}_2(l) \equiv \frac{C_2(l)}{C_2^{CFT}}$, C_2^{CFT} was found in [Kulchytskyy *et. al.*; 1904.08955].



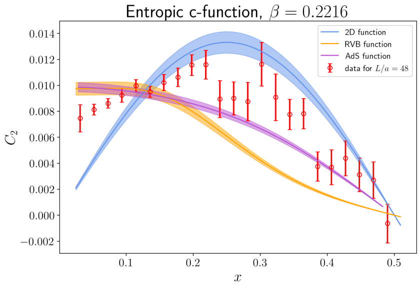
- Fit function $f(x) = a\exp(-2Mx)$, $M = 0.276(14)m_{\text{glueball}}$.

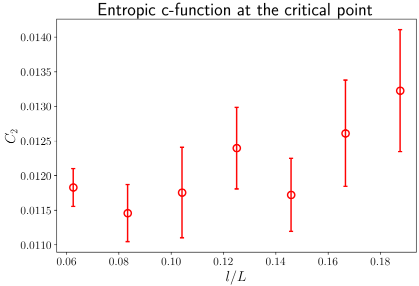
Conclusions and future prospects

- The aim of this work is to perform a **high-precision**, **non-ambiguous** study of the entanglement content of the ground state of a confining gauge theory.
- Thanks to an efficient algorithm based on a non-equilibrium theorem we are able to study the entropic c-function in the thermodynamic and continuum limit.
- We still need to better investigate the small lm_g regime to understand if a transition does occur.
- A complementary approach we are pursuing is to explicitly build the replica space of the gauge theory dual to the Ising model, by performing the duality in the replica space.
- It would be important to make contact with the Hamiltonian formalism.

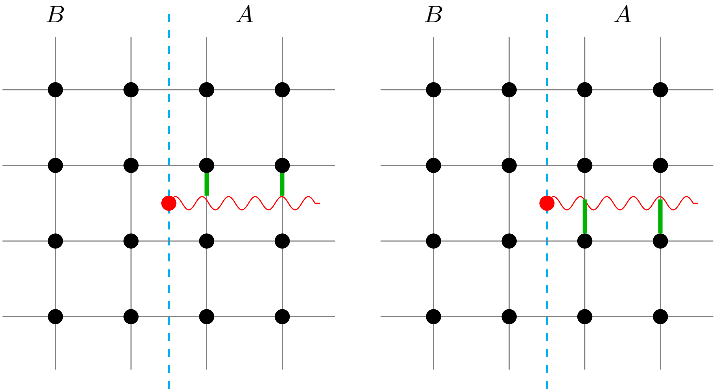
Backup slides

Entropic c-function at the critical point of the Ising model





Direct lattice



Dual lattice