Holographic thermal entropy from geodesic bit threads

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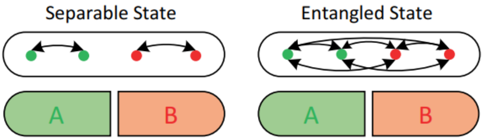
Based on: S. C, F. Gentile, D. Seminara and E. Tonni [arXiv:2403.03930]

Entanglement Entropy

Divide the total system in two subsystems A and B

$$S_A = -tr_A \rho_A \log \rho_A$$

$\rho_A = tr_B |\Psi\rangle\langle\Psi|$ = reduced density matrix of A $\rightarrow$ trace out B from the total density matrix $\rho_{\text{tot}} = |\Psi\rangle\langle\Psi|$



$|\psi\rangle_{AB} = |\psi\rangle_A \otimes |\psi\rangle_B$ $\qquad$ $|\psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\psi\rangle_B$

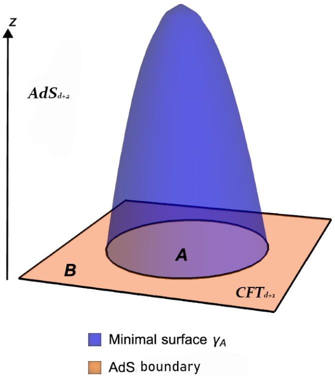
- If the total system is in a pure state, S_A is a good measure of the entanglement
- If the total system is in a mixed state, S_A is counting not only the entanglement, but also the initial entropy of the system

Holographic Entanglement Entropy

Ryu and Takayanagi (2006)

$$S_A = \frac{\text{Area of } m(A)}{4G_N}$$

$m(A) = d -$ dimensional static minimal surface in AdS$_{d+2}$ (“RT surface") anchored on A



Bit Threads

Freedman and Headrick (2017):

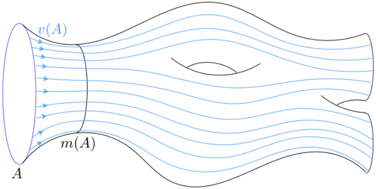
- How one should think about the minimal surface m(A)?
- Can we rewrite the HEE formula without involving m(A)?

They introduced a vector field V in the bulk:

$$\nabla_\mu \boldsymbol{V}^\mu = 0\,, \qquad |\boldsymbol{V}| \leq C\,, \qquad \text{Flow lines = Bit Threads}$$

- Flux of V through a surface γ: $\phi(\gamma) = \int_\gamma \sqrt{h} n_\mu V^\mu$
- They demonstrated: $\max_{\boldsymbol{V}} \phi(A) = C \min_{m \sim A} \text{Area}(m)$

There exists a vector field $\boldsymbol{V}$ whose flux through A equals the area of $m(A)$



Reformulation of the Ryu-Takayanagi formula

If we set $C = \dfrac{1}{4G_N}$

$$S(A) = \max_{\boldsymbol{V}} \phi(A)$$

→ Each flow line (bit thread) leaving the region A carries one bit of information about the microstate of A

→ The minimal surface acts as a bottleneck limitating the number of threads emanating from A

→ The minimal surface is unique, but there are infinite classes of bit threads that maximizes the flux

Sphere in S-AdS$_{d+2}$ black hole

- Constant time slice of Schwarzschild AdS$_{d+2}$ black hole:

$$ds^2_{|\text{t=const}} = \frac{1}{z^2}\left(dr^2 + \frac{dz^2}{f(z)} + r^2 d\Omega_{d-1}\right) \qquad \text{with } f(z) = 1 - \left(\frac{z}{z_h}\right)^{d+1}$$

- Minimal surface → solution of a second order differential equation

$$\frac{z''(r)}{f(z)} + \frac{(d-1)\; z'(r)\left(\frac{z'(r)^2}{f(z)}+1\right)}{rf(z)} + \frac{d\left(\frac{z'(r)^2}{f(z)}+1\right)}{z(r)} - \frac{z'(r)^2\;\frac{\partial f(z)}{\partial z(r)}}{2f(z)^2} = 0$$

No analytic solution known → we find a numerical solution $z_m(r_m)$

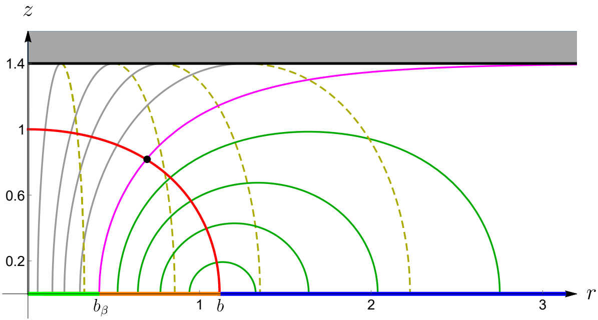
- Flow lines of the vector field V = **geodesics**

$$z'(r) = \pm\frac{1}{z}\sqrt{f(z)\,(C^2 - z^2)}$$

The integration constant C is determined by imposing that geodesics intersect orthogonally the RT surface at a point $z_m(r_m)$

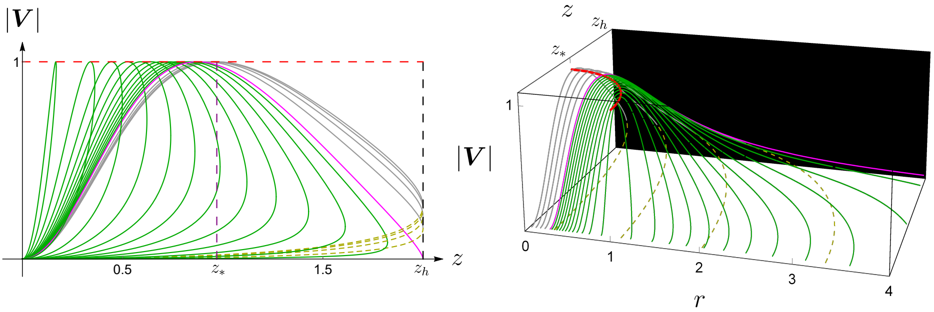
The two branches (ascending and descending) of each geodesic can be expressed as

$$r_{\gtrless}(z) = r_m(z_m) + \int_{z_m}^{\tilde{z}_*} \frac{v}{\sqrt{f(v)\left(C^2 - v^2\right)}}\, dv \pm \int_{z}^{\tilde{z}_*} \frac{v}{\sqrt{f(v)\left(C^2 - v^2\right)}}\, dv$$



Three types of geodesics:

1) Green: two endpoints on the boundary (maximum height $\tilde{z}_* = C$)
2) Grey: one endpoint on the boundary and one on the horizon (maximum height $\tilde{z}_* = z_h$)
3) Magenta: one endpoint on the boundary and its reaches the horizon only for (limiting case of the other two)

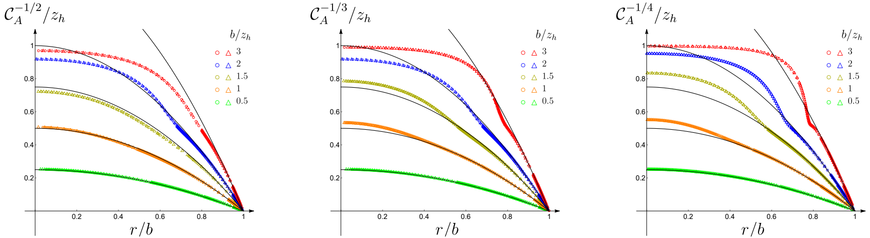


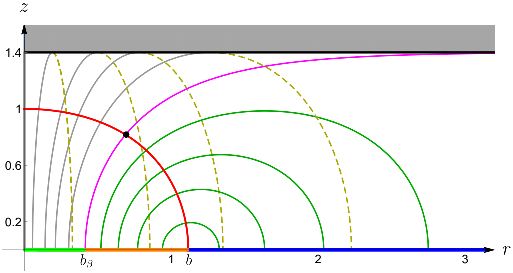
Magnitude of the vector field $\boldsymbol{V}$:

$$\left|\boldsymbol{V}_{\gtrless}\right| = \left(\frac{z}{z_m}\right)^d \frac{\sqrt{C^2-z_m^2}}{\sqrt{C^2-z^2}} \left(\frac{r_m}{r_\gtrless}\right)^{d-1} \frac{(\partial_{z_m} r_<)\big|_{z=z_m}}{\partial_{z_m} r_\gtrless}$$

The entanglement entropy S_A is computed as the flux of bit threads through the region A:

$$S_A = \phi(A) = \frac{L^d_{\text{AdS}}\Omega_{d-1}}{4G_N}\int_0^{b-\epsilon}(n_z \boldsymbol{V}^z \frac{1}{z^d})_{|z=0}\, r^{d-1}dr = \frac{L^d_{\text{AdS}}\Omega_{d-1}}{4G_N}\int_0^{b-\epsilon} C_A(r)\, r^{d-1}dr$$





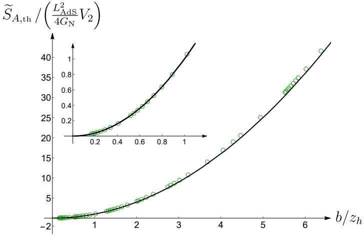
Does the flux through the green interval A_β have a meaning? → **Yes!**

If we rename as $\tilde{S}_{A,th}$ the flux of bit threads through the green interval

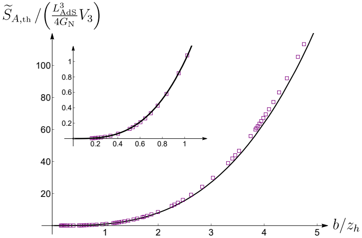
$$\tilde{S}_{A,th} = \frac{L_{AdS}^d \Omega_{d-1}}{4G_N} \int_0^{b_\beta} (n_z V^z \frac{1}{z})|_{z=0} \, r^{d-1} dr$$

We find, at least numerically, that the geodesic bit threads identify a subinterval A_β that contains the information regarding the thermal entropy

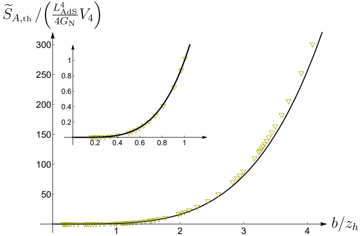
$$S_{A,th} = \frac{L_{\text{AdS}}^d}{4G_N} V_d \left(\frac{b}{z_h}\right)^d \quad \text{where} \quad V_d = \frac{\pi^{d/2}}{\Gamma(d/2+1)}$$



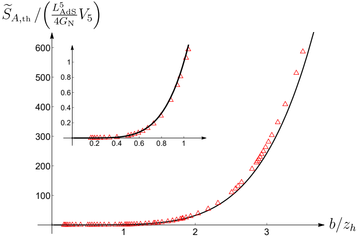
$\widetilde{S}_{A,\mathrm{th}} \Big/ \left(\frac{L^2_{\mathrm{AdS}}}{4G_{\mathrm{N}}} V_2\right)$
b/z_h
40 35 30 25 20 15 10 5 −2
1 2 3 4 5 6
1 0.8 0.6 0.4 0.2
0.2 0.4 0.6 0.8 1



$\widetilde{S}_{A,\mathrm{th}} \Big/ \left(\frac{L^3_{\mathrm{AdS}}}{4G_{\mathrm{N}}} V_3\right)$
b/z_h
100 80 60 40 20
1 2 3 4 5
1 0.8 0.6 0.4 0.2
0.2 0.4 0.6 0.8 1



$\widetilde{S}_{A,\mathrm{th}} \Big/ \left(\frac{L^4_{\mathrm{AdS}}}{4G_{\mathrm{N}}} V_4\right)$
b/z_h
300 250 200 150 100 50
1 2 3 4
1 0.8 0.6 0.4 0.2
0.2 0.4 0.6 0.8 1



$\widetilde{S}_{A,\mathrm{th}} \Big/ \left(\frac{L^5_{\mathrm{AdS}}}{4G_{\mathrm{N}}} V_5\right)$
b/z_h
600 500 400 300 200 100
1 2 3
1 0.8 0.6 0.4 0.2
0.2 0.4 0.6 0.8 1

Thank you for the attention!