Florence Theory Group Day

Algebraic Area of Lattice Random Walks and Exclusion Statistics

Joint work with Stéphane Ouvry (UPSaclay) and Alexios Polychronakos (CCNY)

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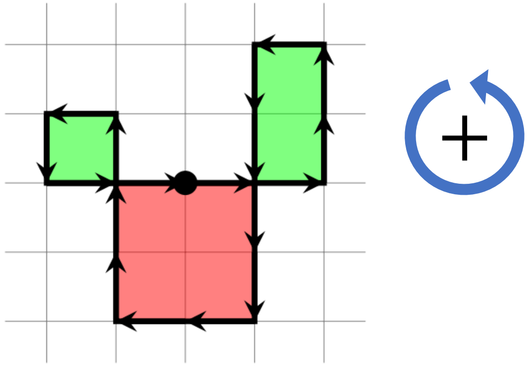
PhD at Université Paris-Saclay

March 25, 2024

Algebraic area A of closed random walks on a square lattice



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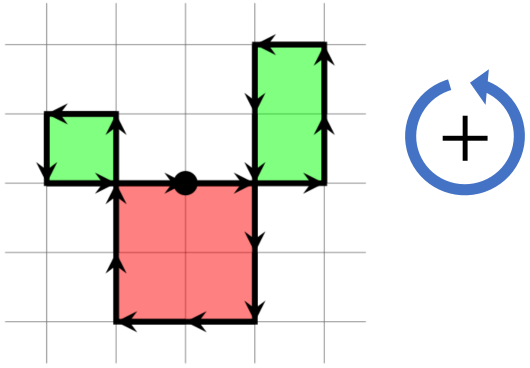
Algebraic area: area swept by the closed walk, weighted by the winding number in each winding sector

Algebraic area A of closed random walks on a square lattice



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Question: a formula for the number $C_{\mathbf{n}}(A)$ of closed **n**-step square lattice walks that enclose an algebraic area A? (**n** is necessarily even, **n**=2*n*)

e.g. $C_2(0) = 4$, $C_4(0) = 28$, $C_4(1) = C_4(-1) = 4$

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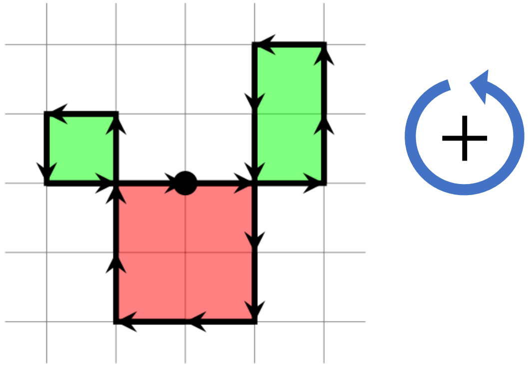

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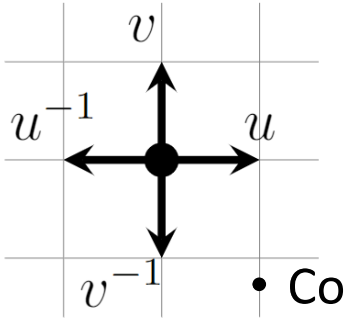
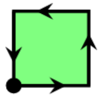


square lattice walks



$v^{-1}\,u^{-1} v\,u = \mathrm{Q}^1$

- Commutation relation: $v\,u = \mathrm{Q}\,u\,v$
- Generating function for algebraic area enumeration of **n**-step closed walks

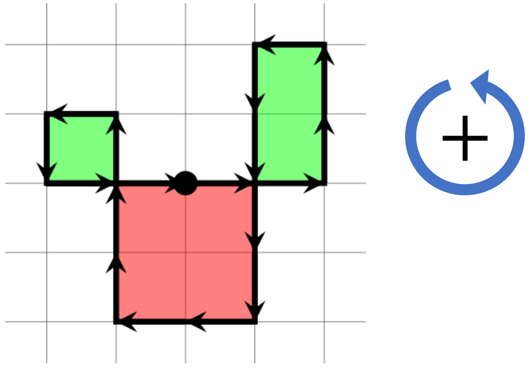
$$(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\,\mathrm{Q}^A + \ldots$$

$$\mathbf{Tr}\,(v^n u^m) = \delta_{n,0}\delta_{m,0}$$

$$\Rightarrow \mathbf{Tr}\,(u + u^{-1} + v + v^{-1})^{\mathbf{n}} = \sum_A C_{\mathbf{n}}(A)\,\mathrm{Q}^A$$

e.g., $\mathbf{Tr}\left(u + u^{-1} + v + v^{-1}\right)^4 = 28 + 4\mathrm{Q} + 4\mathrm{Q}^{-1}$

Algebraic area A of closed random walks on a square lattice



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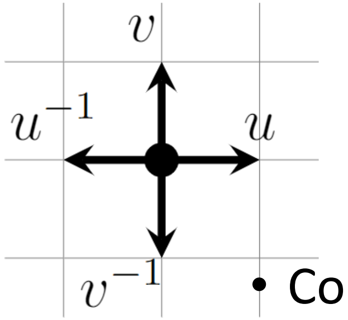

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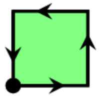

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physics

Hofstadter model: a charged particle hopping on a square lattice in a perpendicular magnetic field

- Hamiltonian $H = u + u^{-1} + v + v^{-1}$
  u, *v* are "magnetic translation operators"
- $\mathrm{Q} = \mathrm{e}^{2\pi \mathrm{i}\phi/\phi_0}$; ϕ/ϕ_0: magnetic flux per plaquette

  rational flux $\phi/\phi_0 = p/q$ with p, q coprime
- $\sum_A C_{\mathbf{n}}(A)\, \mathrm{Q}^A = \mathbf{Tr}\ H^{\mathbf{n}}$

aim: $\mathbf{Tr}\ H^{\mathbf{n}}$

Rational flux, i.e., $\mathrm{Q} = \mathrm{e}^{2\pi \mathrm{i}\, p/q}$ with p, q coprime

$$u = \mathrm{e}^{\mathrm{i}k_x} \begin{pmatrix} \mathrm{Q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Q}^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathrm{Q}^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Q}^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \quad v = \mathrm{e}^{\mathrm{i}k_y} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

u, v: $q \times q$ “clock” and “shift” matrices

Quantum trace

$$\mathbf{Tr}\, H^{\mathbf{n}} = \frac{1}{q} \int_0^{2\pi} \int_0^{2\pi} \frac{dk_x}{2\pi} \frac{dk_y}{2\pi} \, \mathrm{tr}\, H^{\mathbf{n}}$$

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$u \to -uv, \quad v \to v$

$k_x = k_y = 0, \ \mathbf{n} < q$

reduces to →

Usual trace

$$\mathbf{Tr}\, H^{\mathbf{n}} = \frac{1}{q}\mathrm{tr}\, H_2^{\mathbf{n}}$$

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}$$

$f_k = 1 - \mathrm{Q}^k, \ g_k = 1 - \mathrm{Q}^{-k}$

Aim: $\mathrm{tr}\ H_2^{\mathbf{n}}$

Rational flux, i.e., $\mathrm{Q} = \mathrm{e}^{2\pi \mathrm{i}\, p/q}$ with p, q coprime $u = \mathrm{e}^{\mathrm{i}k_x} \begin{pmatrix} \mathrm{Q} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \mathrm{Q}^2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mathrm{Q}^3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \mathrm{Q}^{q-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}, \; v = \mathrm{e}^{\mathrm{i}k_y} \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$

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Aim: tr $H_2^{\mathbf{n}}$

- Approach 1: **secular determinant** det $(I - z\, H_2)$ and its relation to ***exclusion statistics***

$$\ln \det(I - zH_2) = \mathrm{tr}\, \ln(I - zH_2) = -\sum_{\mathbf{n}=1}^{\infty} \frac{z^{\mathbf{n}}}{\mathbf{n}} \mathrm{tr}\, H_2^{\mathbf{n}}$$

- Approach 2: **direct computation** (combinatorics of periodic Dyck paths)

$$\mathrm{tr}\, H_2^{\mathbf{n}} = \sum_{k_1=1}^{q} \sum_{k_2=1}^{q} \cdots \sum_{k_{\mathbf{n}}=1}^{q} h_{k_1 k_2} h_{k_2 k_3} \cdots h_{k_{\mathbf{n}} k_1}$$

h_{ij}: matrix element of H_2

Approach 1: via secular determinant $\det(I - z\,H_2)$

$$H_2 = \begin{pmatrix} 0 & f_1 & 0 & \cdots & 0 & 0 \\ g_1 & 0 & f_2 & \cdots & 0 & 0 \\ 0 & g_2 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & f_{q-1} \\ 0 & 0 & 0 & \cdots & g_{q-1} & 0 \end{pmatrix}$$

Secular determinant $\det(I - zH_2) = \sum_{n=0}^{\lfloor q/2 \rfloor} (-1)^n Z_n z^{2n}$

Kreft coefficient $Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}$, $s_k := g_k f_k = 4\sin^2(k\pi p/q)$

[Kreft 1993]

"+2 shifts" $Z_0 = 1$

e.g., for $q = 7$, $Z_3 = s_1\, s_3\, s_5 + s_1\, s_3\, s_6 + s_1\, s_4\, s_6 + s_2\, s_4\, s_6$

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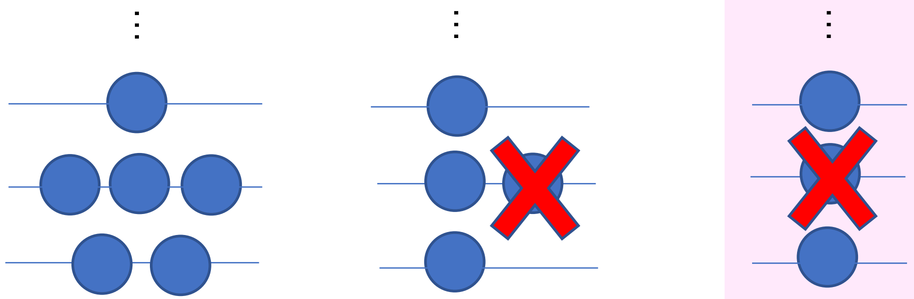
Interpretation in statistical mechanics



Exclusion parameter g bosons $(g = 0)$ fermions $(g = 1)$ $g = 2$

Z_n: partition function for n particles occupying $q - 1$ energy levels. These particles obey $g = 2$ ***exclusion statistics*** (no two particles can occupy adjacent quantum states)

stronger exclusion than fermions!

Closed random walks on a square lattice → ***Exclusion statistics*** with exclusion parameter $g = 2$

Use techniques from statistical mechanics to compute tr $H_2^{\mathbf{n}}$

Approach 1: via secular determinant $\det(I - z\,H_2)$

Kreft coefficient $$Z_n = \sum_{k_1=1}^{q-2n+1} \sum_{k_2=1}^{k_1} \cdots \sum_{k_n=1}^{k_{n-1}} s_{k_1+2n-2} s_{k_2+2n-4} \cdots s_{k_{n-1}+2} s_{k_n}\,,\quad s_k := g_k f_k = 4\sin^2(k\pi p/q)$$

Introduce cluster coefficient b_n via $\log\left(\sum_{n=0}^{\lfloor q/2 \rfloor} Z_n x^n\right) = \sum_{n=1}^{\infty} b_n x^n$, $\operatorname{tr} H_2^{\mathbf{n}=2n} = 2n(-1)^{n+1} b_n$

$$\operatorname{tr} H_2^{\mathbf{n}=2n} = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \ldots, l_j) \sum_{k=1}^{q-j} s_k^{l_1} s_{k+1}^{l_2} \cdots s_{k+j-1}^{l_j}$$

The composition is an ordered partition.

e.g. four compositions of n=3: (3), (2,1), (1,2), (1,1,1)

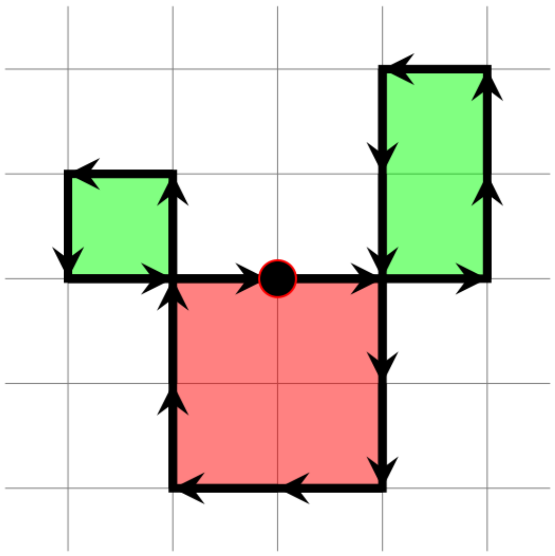
$$c_2(l_1, l_2, \ldots, l_j) = \frac{1}{l_1} \prod_{i=2}^{j} \binom{l_{i-1} + l_i - 1}{l_i}$$

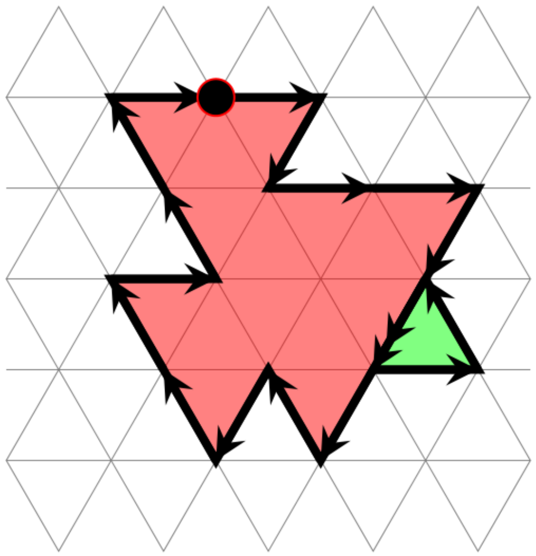
$$C_{\mathbf{n}}(A) = 2n \sum_{\substack{l_1, l_2, \ldots, l_j \\ \text{composition of } n}} c_2(l_1, l_2, \ldots, l_j) \sum_{k_3=-l_3}^{l_3} \sum_{k_4=-l_4}^{l_4} \cdots \sum_{k_j=-l_j}^{l_j} \binom{2l_1}{l_1 + A + \sum_{i=3}^{j}(i-2)k_i} \binom{2l_2}{l_2 - A - \sum_{i=3}^{j}(i-1)k_i} \prod_{i=3}^{j} \binom{2l_i}{l_i + k_i}$$

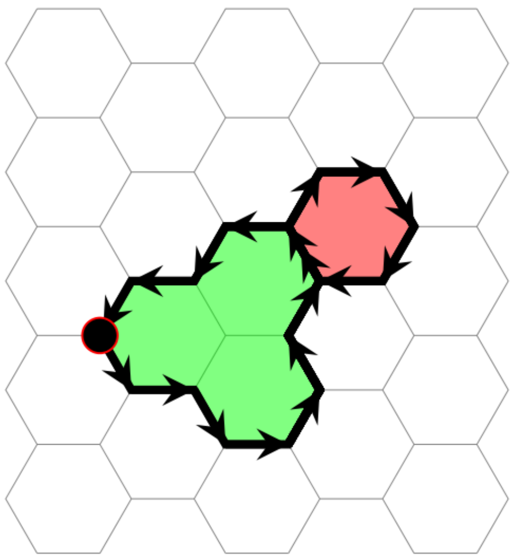
[Ouvry, Wu 2019]

$\det(I - zH_2)$ → Z_n → b_n → $\operatorname{tr} H_2^{\mathbf{n}=2n}$ → Generalize to the "+g shifts" (g-exclusion)

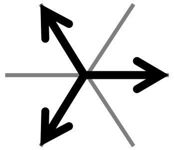
Closed random walks on various lattices



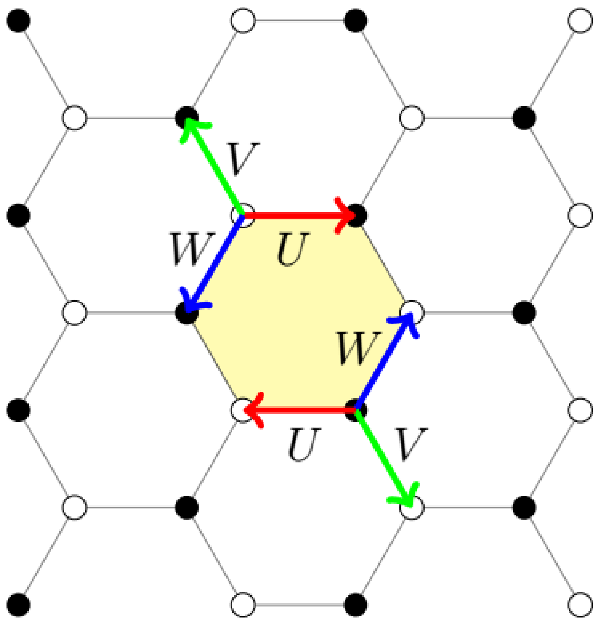




Square	Chiral* triangular	Honeycomb
$g = 2$	$g = 3$	mixture of $g = 1$ and $g = 2$
	[Ouvry, Polychronakos 2019]	[Gan, Ouvry, Polychronakos 2022]

* Only three out of six directions at each step are allowed, that is  .

Honeycomb lattice walks



Hamiltonian $H = U + V + W$

honeycomb algebra $U^2 = V^2 = W^2 = I, \quad (UVW)^2 = \mathrm{Q}$

$$U = \begin{pmatrix} 0 & u \\ u^{-1} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & v \\ v^{-1} & 0 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & \mathrm{Q}^{1/2} v u^{-1} \\ \mathrm{Q}^{-1/2} u v^{-1} & 0 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & u + v + \mathrm{Q}^{1/2} v u^{-1} \\ u^{-1} + v^{-1} + \mathrm{Q}^{-1/2} u v^{-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & A \\ A^\dagger & 0 \end{pmatrix}, \qquad H_{1,2} = AA^\dagger$$

set $\mathrm{e}^{-\mathrm{i}k_x} = -\mathrm{Q}^{1/2}$

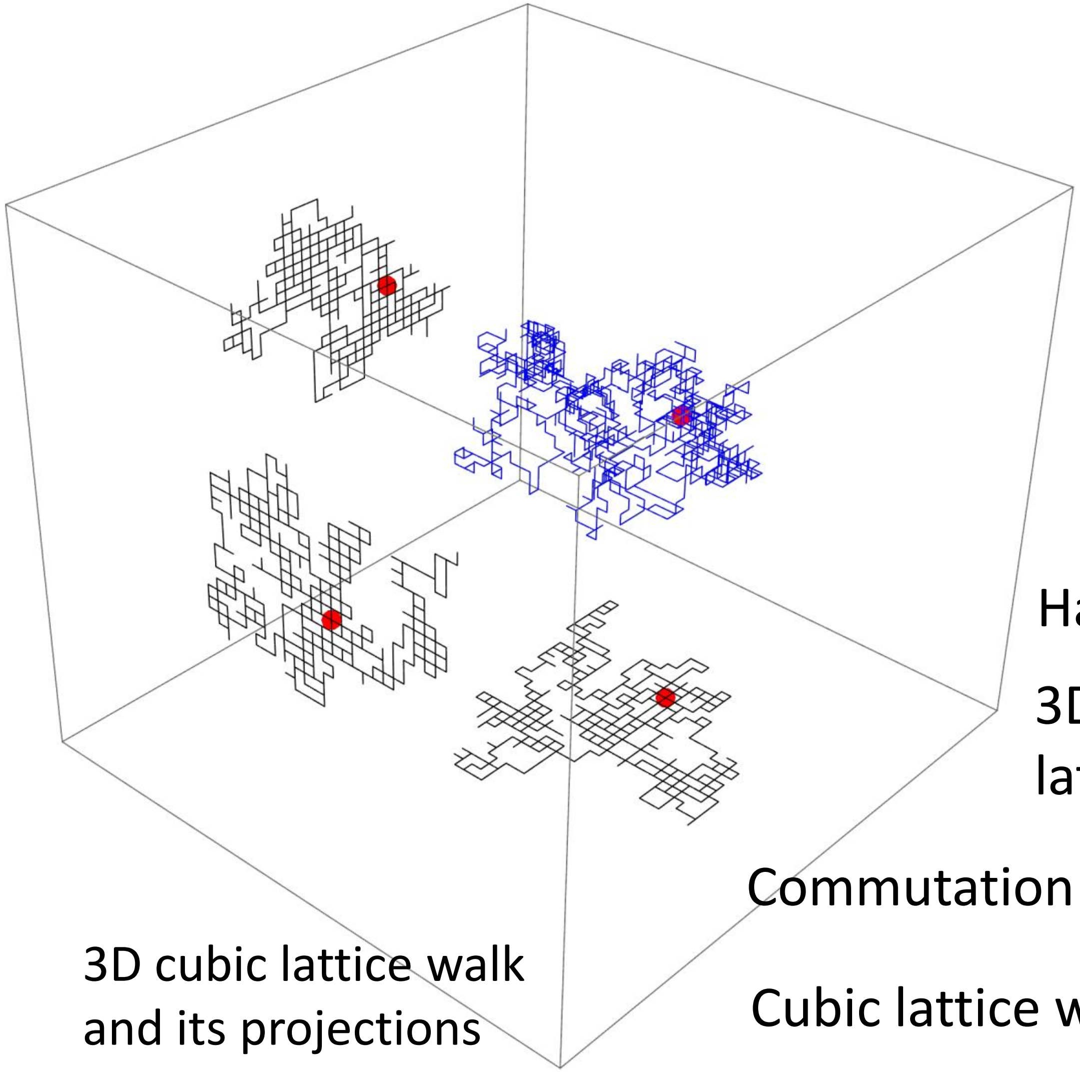
$k_y = 0, \mathbf{n} < q$

$$\det(I - zH) = \det(I - z^2 H_{1,2}) = \sum_{n=0}^{q} (-1)^n Z_n z^{2n}$$

$\det(I - z^2 H_{1,2})$ → Z_n → b_n → $\mathbf{Tr}\, H^{\mathbf{n}=2n} = \mathbf{Tr}\, H_{1,2}^n = \frac{1}{q} \mathrm{tr}\, H_{1,2}^n$

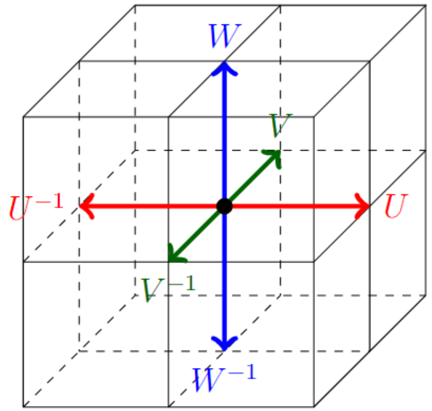
Cubic lattice walks [Gan 2023]

How to define the algebraic area?



3D cubic lattice walk
and its projections

Algebraic area of 3D walks: sum of algebraic areas obtained from the walk's projection onto the xy, yz, zx-planes along the $-z, -x, -y$ directions



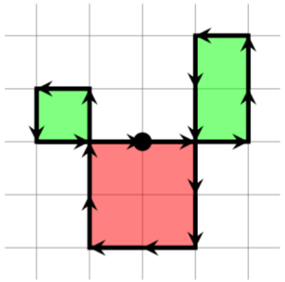
Hamiltonian $H = U + V + W + U^{-1} + V^{-1} + W^{-1}$

3D Hofstadter model: a charged particle hopping on a cubic lattice coupled to a magnetic field $\mathbf{B} = (1,1,1)$

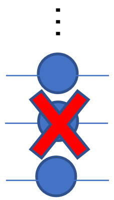
Commutation relations $VU = \mathrm{Q}UV,\ WV = \mathrm{Q}VW,\ UW = \mathrm{Q}WU$

Cubic lattice walks ➡ mixture of g=1, g=1, and $g = 2$ with equal numbers of g=1 exclusion particles of two types

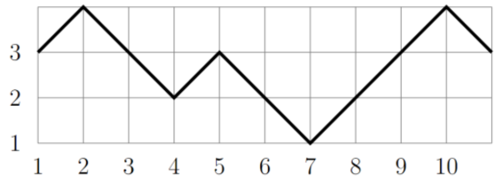
Take-home message



Algebraic area enumeration of closed lattice random walks



Quantum exclusion statistics — Combinatorics of Dyck/Motzkin paths



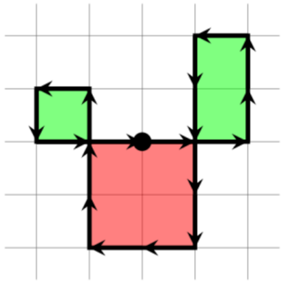
Outlook

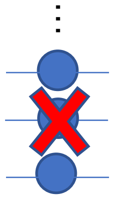
- Various lattice random walks, e.g., triangular lattice [thesis], hypercubic lattice [ongoing work]
- Classification of random walks based on the exclusion parameter g
- Connection to exactly solvable models

  e.g., open Ising spin-1/2 chain: *free-fermionic* spectrum $\pm\epsilon_1 \pm \epsilon_2 \pm \cdots$ with ϵ_k obtained from $g = 2$ exclusion matrix H_2 [Baxter 1989], closed Ising chain [ongoing work]
- Potential applications in high energy physics?

Take-home message

Algebraic area enumeration of closed lattice random walks





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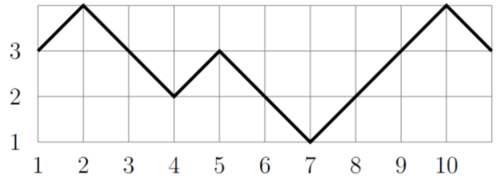

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Thank You!