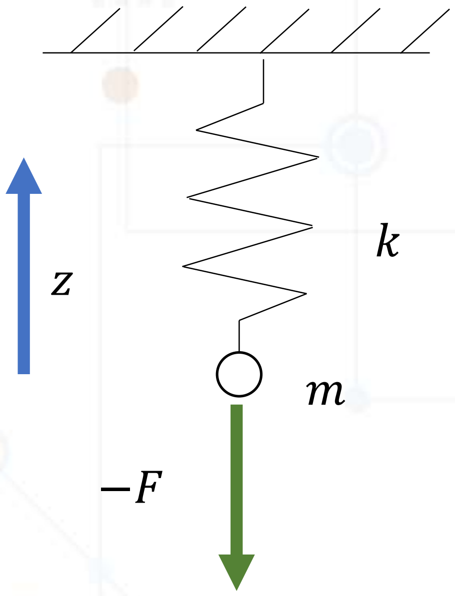
SIGRAV lecture 3: Thermal noise and inelastic effects

2/20/25

Vietri, Italy, 2/16/2024
S. Schlamminger
National Institute of Standards and Technology

Measure things with oscillators



$$m\,\ddot{z} + m\,\gamma\dot{z} + kz = F$$

$$\ddot{z} + \gamma\,\dot{z} + \omega_o^2\,z = \frac{F}{m} \qquad \omega_o = \sqrt{\frac{k}{m}}$$

Homogeneous solution:

$$z = C\ \exp\left(-\frac{\gamma}{2}t\right)\cos(\omega_D\,t + \phi)$$

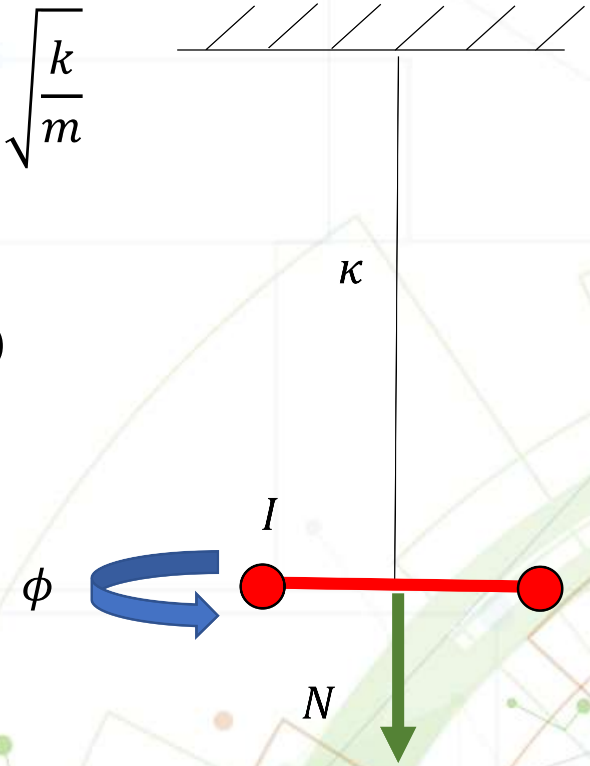
$$\omega_D = \sqrt{\omega_o^2 - \frac{\gamma^2}{4}}$$

Decay time: $\tau = \frac{2}{\gamma}$

Quality factor: $Q = \frac{\omega_o}{\gamma}$

$$I\,\ddot{\phi} + I\,\gamma\dot{\phi} + \kappa\phi = N$$

$$\omega_o = \sqrt{\frac{\kappa}{I}}$$



Response

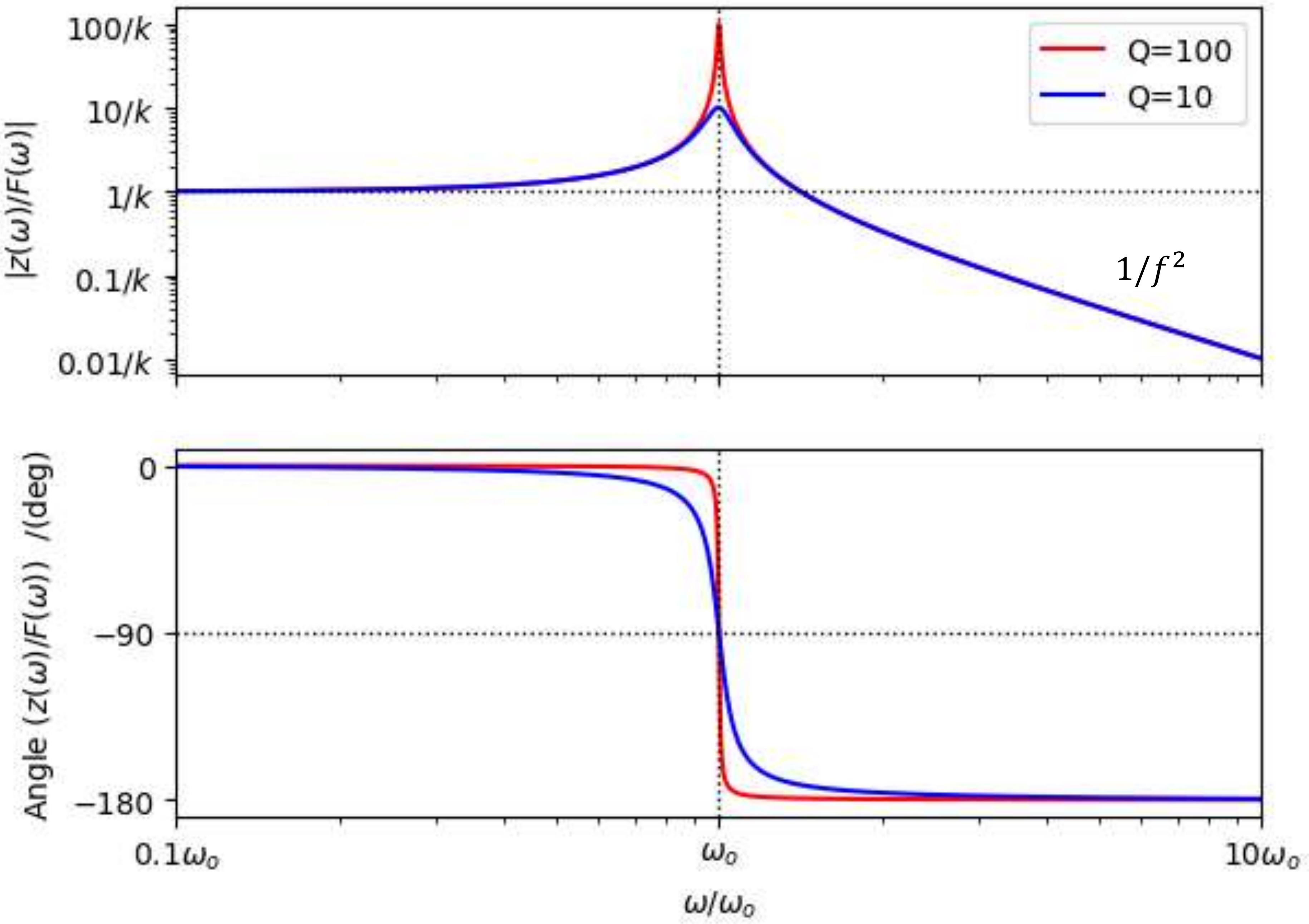
$$\ddot{z} + \gamma\,\dot{z} + \omega_o^2\,z = \frac{F}{m}$$

$$\omega_o = \sqrt{\frac{k}{m}}$$

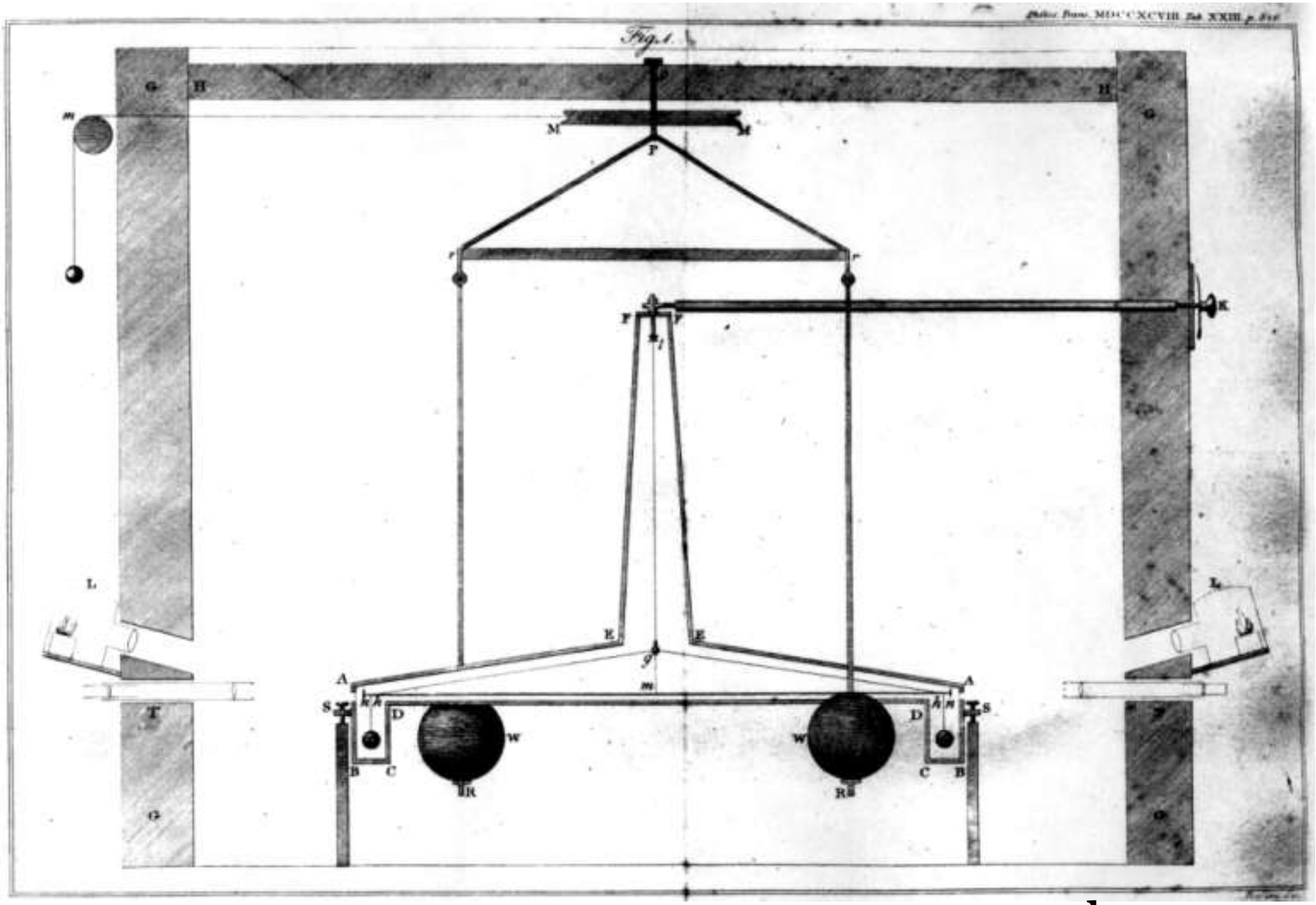
$$\omega_D = \sqrt{\omega_o^2 - \frac{\gamma^2}{4}}$$

$$Q = \frac{\omega_o}{\gamma}$$

$$\frac{z(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_o^2 - \omega^2 + i\gamma\omega}$$



The first experiment in the lab

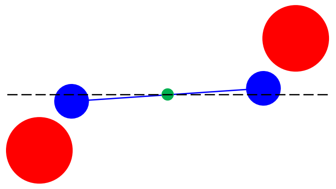


Cavendish, 1793-1798

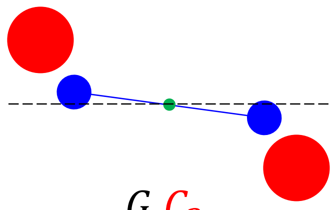
$$\rho_E = 5{,}448(33)\,\frac{\text{kg}}{\text{m}^3}$$

Cavendish method

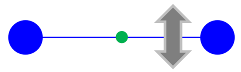
Static deflection



$$\theta_1 = \frac{G\,C_1}{\kappa}$$



$$\theta_2 = \frac{G\,C_2}{\kappa}$$



$$\kappa = I\omega^2$$

$$G = \frac{(\theta_1 - \theta_2)\omega^2}{C_1 - C_2} I$$

1798

DIE

GRAVITATIONS-CONSTANTE,

DIE

MASSE UND MITTLERE DICHTE DER ERDE

NACH EINER NEUEN EXPERIMENTELLEN BESTIMMUNG

VON

DR. PHIL. ET THEOL. CARL BRAUN, S. J.
IN MARIASCHEIN.

(Mit 3 Tafeln und 8 Textfiguren.)

BESONDERS ABGEDRUCKT AUS DEM LXIV. BANDE DER DENKSCHRIFTEN DER MATHEMATISCH-NATURWISSENSCHAFTLICHEN CLASSE DER KAISERLICHEN AKADEMIE DER WISSENSCHAFTEN.

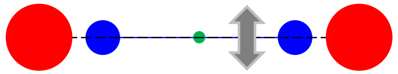
WIEN 1896.
AUS DER KAISERLICH-KÖNIGLICHEN HOF- UND STAATSDRUCKEREI.

IN COMMISSION BEI CARL GEROLD'S SOHN,
BUCHHÄNDLER DER KAISERLICHEN AKADEMIE DER WISSENSCHAFTEN.

- Inventor of the time-of-swing method.

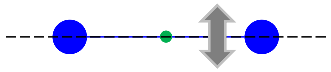
Time-of-swing method

Time-of-swing-method



$$\omega_n{}^2 = \frac{\kappa + \kappa_n}{I}$$

$$\kappa_n = G\, C_n$$



$$\omega_f{}^2 = \frac{\kappa + \kappa_f}{I}$$

$$\kappa_f = G\, C_f$$

$$G = \frac{\omega_n{}^2 - \omega_f{}^2}{(C_n - C_f)} I$$

1896

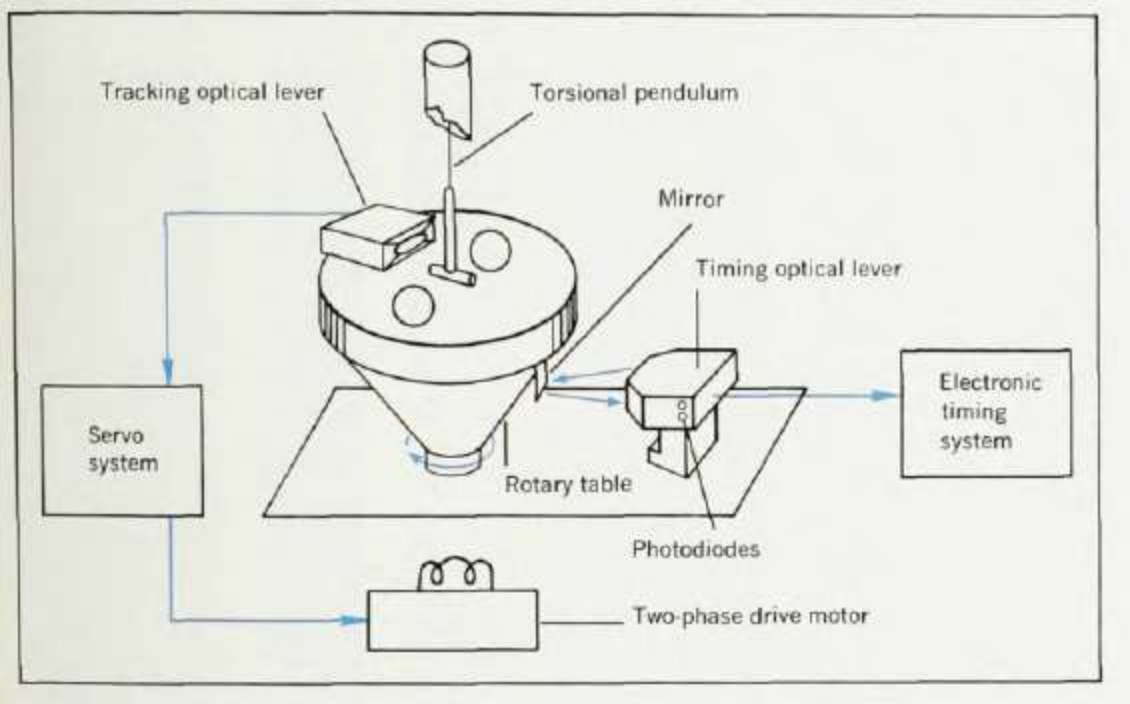
Yet another method

Finding a better value for G

Surprisingly, we know the gravitational constant only to within about half a percent. With this elegant method, we hope for a precision of at least one part in ten thousand.

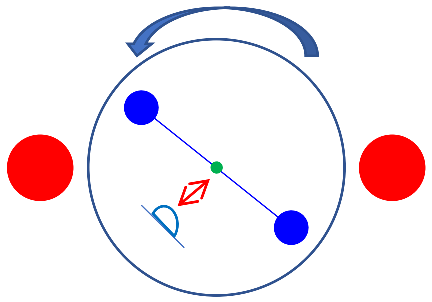
Jesse W. Beams

J.W. Beams,
Phys. Today **24** (5) 34 (1971)



Angular acceleration method

Angular acceleration method



$$\ddot{\theta} = \frac{16\pi}{5}\, G\, \frac{q_{22}Q_{22}}{I} \sin 2\theta$$

$$G = \frac{5}{16\pi} \frac{I}{q_{22}Q_{22}} \ddot{\theta}_0$$

1971

Electrostatic servo

Metrologia 24, 171 - 174 (1987)

A New Experiment for the Determination of the Newtonian Gravitational Constant

H. de Boer, H. Haars, and W. Michaelis

Physikalisch-Technische Bundesanstalt (PTB), D-3300 Braunschweig, Federal Republic of Germany

Received: March 17, 1987

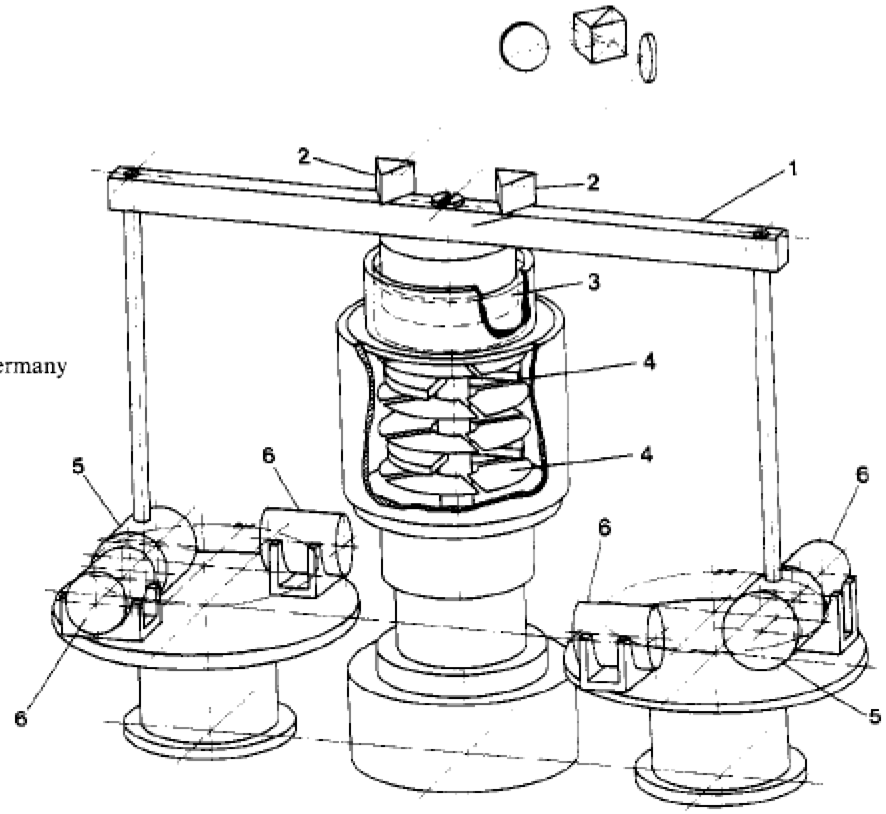
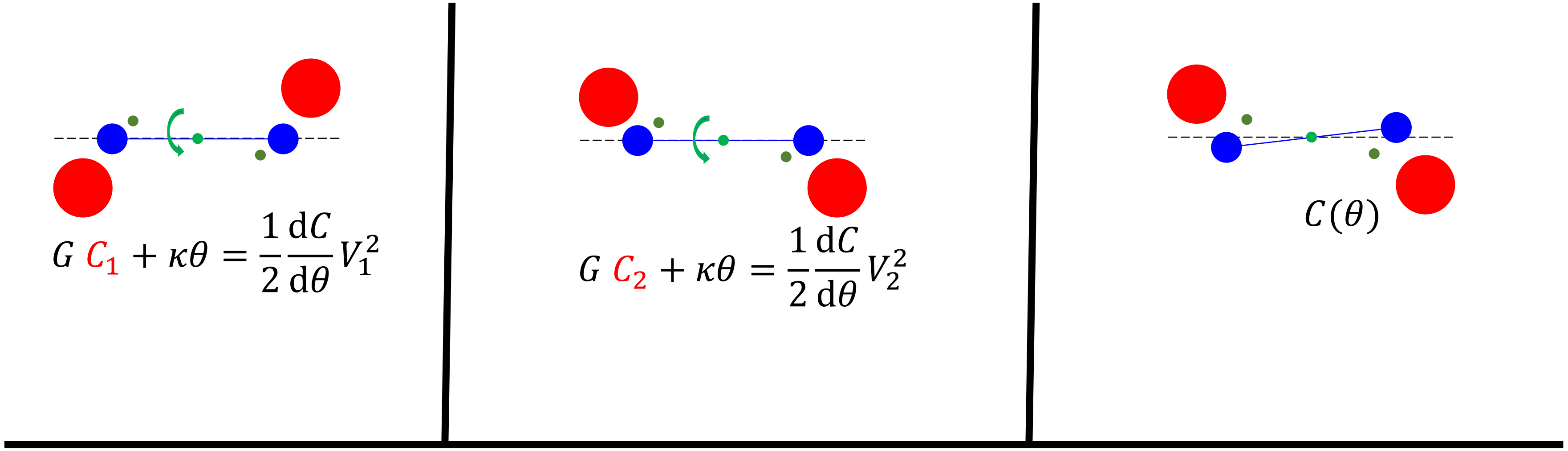


Fig. 1. Schematic drawing of the main parts of the experiment: 1 Torsion balance; 2 Interferometer; 3 Mercury bearing; 4 Electrometers; 5 Balance masses; 6 Masses mounted to the rotating tables

Servo method

Servo method



$$G\,C_1 + \kappa\theta = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta}V_1^2$$

$$G\,C_2 + \kappa\theta = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta}V_2^2$$

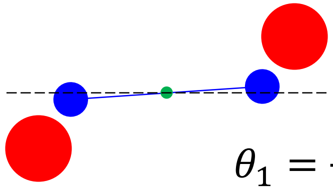
$$C(\theta)$$

$$G = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta}\frac{(V_1^2 - V_2^2)}{C_1 - C_2}$$

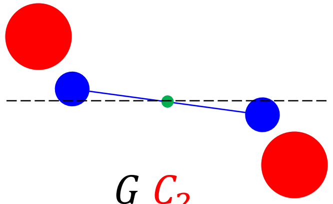
1987

TB physics

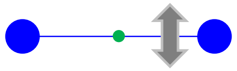
1798
Static deflection



$$\theta_1 = \frac{G\,C_1}{\kappa}$$



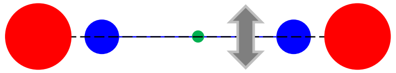
$$\theta_2 = \frac{G\,C_2}{\kappa}$$



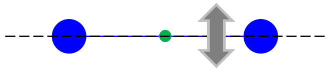
$$\kappa = I\omega^2$$

$$G = \frac{(\theta_1 - \theta_2)\omega^2}{C_1 - C_2} I$$

1896
Time of swing



$$\omega_n{}^2 = \frac{\kappa + \kappa_n}{I}$$

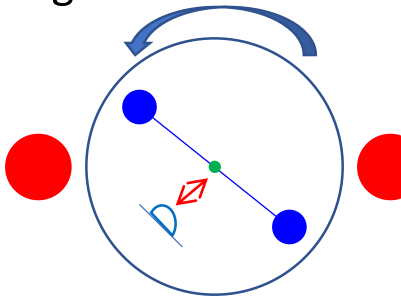


$$\omega_f{}^2 = \frac{\kappa + \kappa_f}{I}$$

$$\kappa_n = G\,C_n \qquad \kappa_f = G\,C_f$$

$$G = \frac{\omega_n{}^2 - \omega_f{}^2}{(C_n - C_f)} I$$

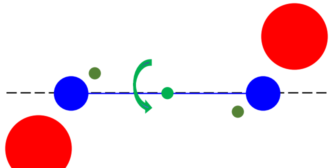
1971
Angular acc. servo



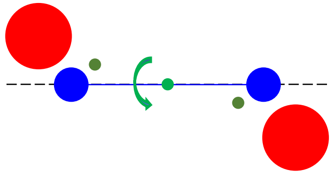
$$\ddot{\theta} = \frac{16\pi}{5}\, G\, \frac{q_{22}Q_{22}}{I} \sin 2\theta$$

$$G = \frac{5}{16\pi} \frac{I}{q_{22}Q_{22}} \ddot{\theta}_0$$

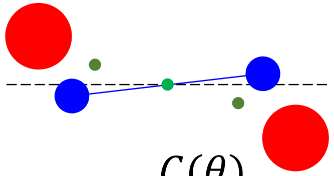
1987
Electrostatic servo



$$G\,C_1 + \kappa\theta = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta} V_1^2$$

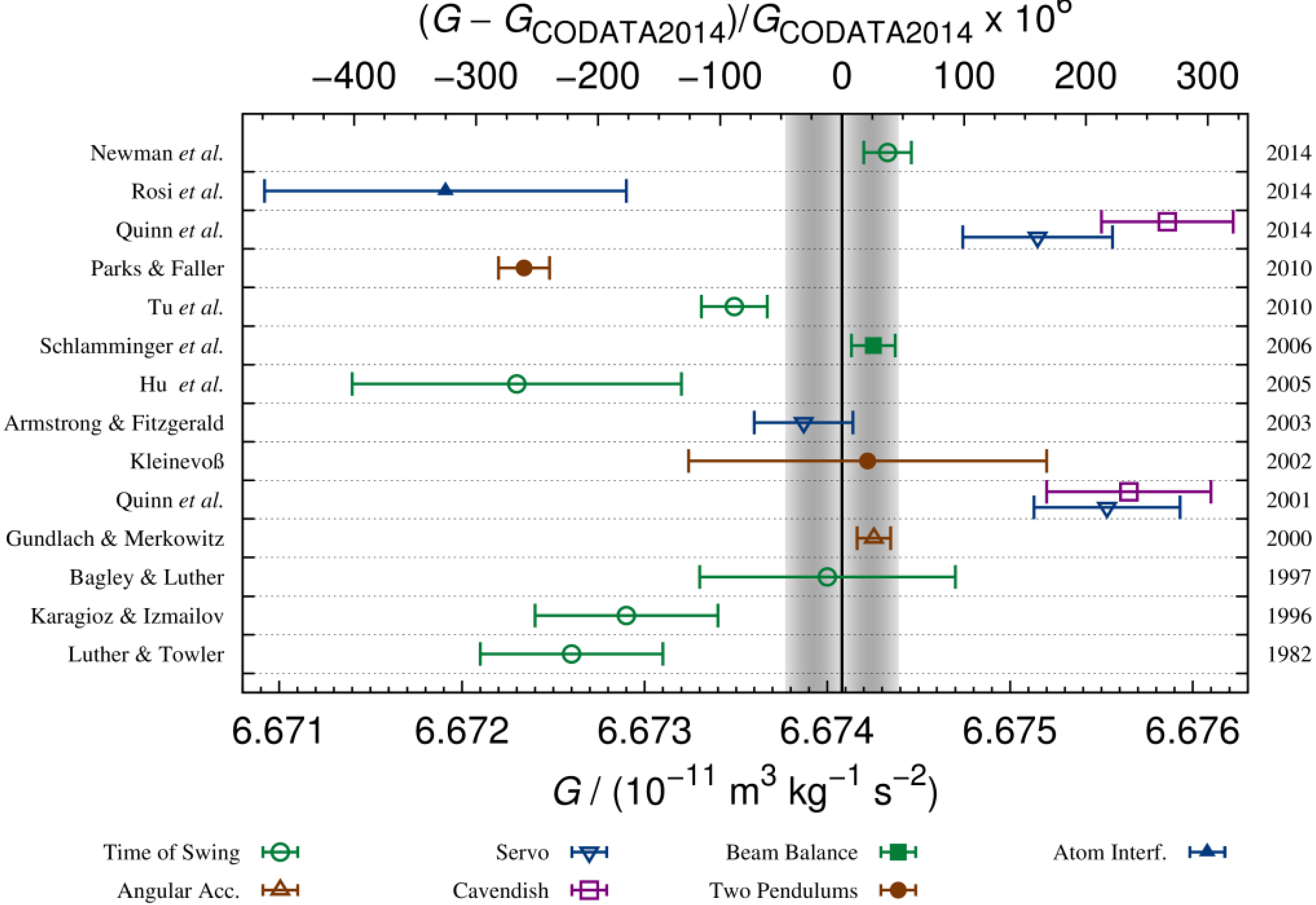


$$G\,C_2 + \kappa\theta = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta} V_2^2$$



$$C(\theta)$$

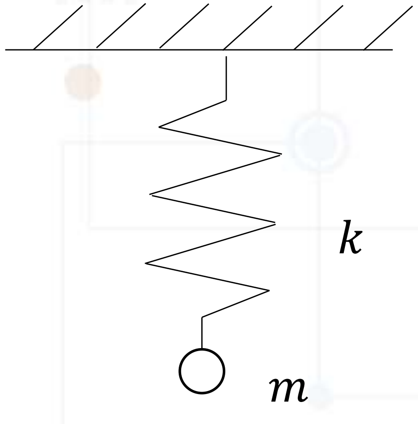
$$G = \frac{1}{2}\frac{\mathrm{d}C}{\mathrm{d}\theta} \frac{(V_1^2 - V_2^2)}{C_1 - C_2}$$



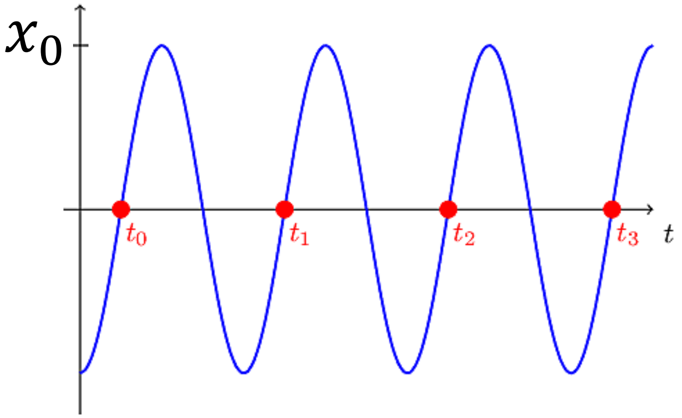
$(G - G_{CODATA2014})/G_{CODATA2014}$ x 10^6
−400
−300
−200
−100
0
100
200
300
Newman et al.
Rosi et al.
Quinn et al.
Parks & Faller
Tu et al.
Schlamminger et al.
Hu et al.
Armstrong & Fitzgerald
Kleinevoß
Quinn et al.
Gundlach & Merkowitz
Bagley & Luther
Karagioz & Izmailov
Luther & Towler
2014
2014
2014
2010
2010
2006
2005
2003
2002
2001
2000
1997
1996
1982
6.671
6.672
6.673
6.674
6.675
6.676
G / (10^{-11} m^3 kg^{-1} s^{-2})
Time of Swing
Servo
Beam Balance
Atom Interf.
Angular Acc.
Cavendish
Two Pendulums

approx 2016

Oscillators as probes



$$\omega_o = \sqrt{\frac{k}{m}}$$

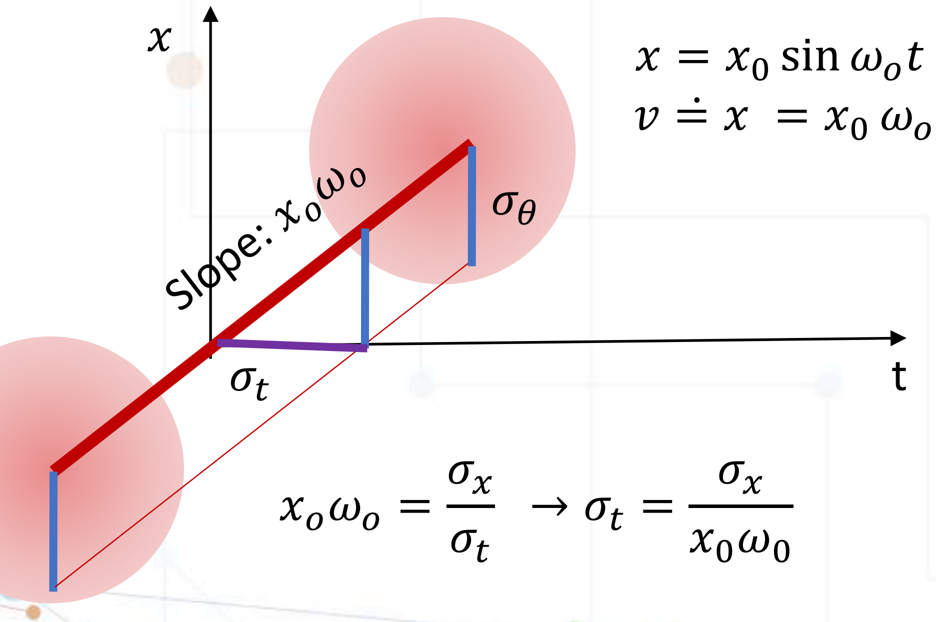


Imagine only thermal agitation…

What is the uncertainty in measuring these zero crossings?
And, what ultimately is the uncertainty of the period Determination.

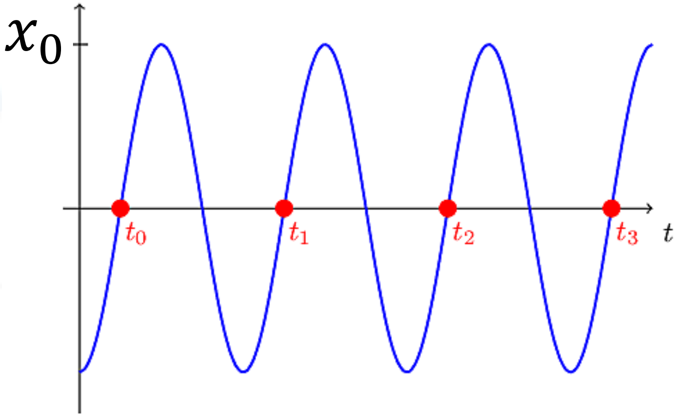
Oscillators as probes

Let’s zoom in at one zero crossing.



$$x = x_0 \sin \omega_o t$$
$$v \doteq x \ = x_0\, \omega_o\, cos\, \omega_o t$$

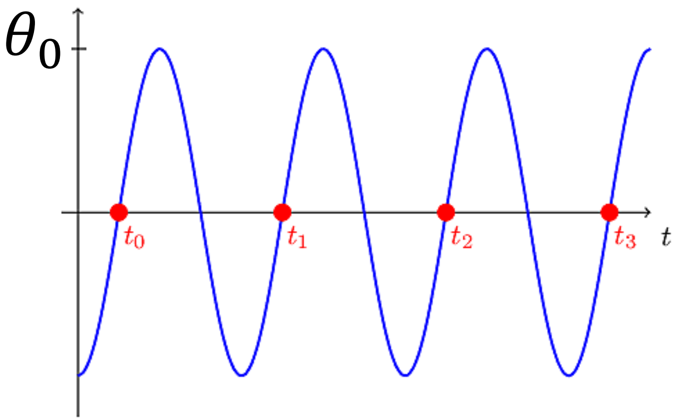
$$x_o\omega_o = \frac{\sigma_x}{\sigma_t} \ \rightarrow \sigma_t = \frac{\sigma_x}{x_0\omega_0}$$



Uncertainty in the orange balls:

$$\sigma_t = \frac{\sigma_x}{x_0\omega_0}$$

Oscillators as probes



$$E_N(T_o) = \frac{t_N - t_0}{N}$$

$$\sigma_{\mathrm{N}}^2 = \left(\frac{\partial E_{\mathrm{N}}}{\partial t_N}\right)^2 \sigma_t^2 + \left(\frac{\partial E_{\mathrm{N}}}{\partial t_0}\right)^2 \sigma_t^2 + 2\left(\frac{\partial E_{\mathrm{N}}}{\partial t_N}\right)\left(\frac{\partial E_{\mathrm{N}}}{\partial t_0}\right) r_{N,0}\sigma_t^2$$

$$\frac{\partial E_N}{\partial t_N} = \frac{1}{N}$$

$$\frac{\partial E_N}{\partial t_o} = -\frac{1}{N}$$

$$\sigma_N^2 = \underbrace{\left(\frac{\partial E_{\mathrm{N}}}{\partial t_N}\right)^2 \sigma_t^2 + \left(\frac{\partial E_{\mathrm{N}}}{\partial t_0}\right)^2 \sigma_t^2}_{\frac{2}{N^2}\sigma_t^2} + 2\underbrace{\left(\frac{\partial E_{\mathrm{N}}}{\partial t_N}\right)\left(\frac{\partial E_{\mathrm{N}}}{\partial t_0}\right)}_{\frac{-2}{N^2}\sigma_t^2} r_{N,0}\sigma_t^2$$

$$\sigma_N^2 = \frac{2}{N^2}\sigma_t^2(1 - r_{N,0})$$

Let's find the thermal noise amplitude

$$m\,\ddot{x} + m\,\gamma\,\dot{x} + \omega_0^2\,x = f \qquad -m\,\omega^2 X + i\,m\,\gamma\,\omega\,X + \omega_o^2\,X = F$$

Lorentzian approximation

$$R = \frac{X}{F} = \frac{1/m}{\omega^2 - \omega_o^2 + i\gamma\omega} = \frac{1/m}{(\omega - \omega_o)(\omega + \omega_o) + i\gamma\omega} \approx \frac{1/(m\omega_o)}{2(\omega - \omega_o) + i\gamma}$$

$$S_x^{th} = |R^2| S_F^{th}$$

$$S_F^{th} = 4\,k_b T\,m\,\gamma$$

Single sided power spectrum in the frequency domain!

$$< x^2 > = \int_0^\infty S_x^{th} \frac{d\omega}{2\pi} = \int_0^\infty 4\,k_b T\,m\,\gamma \frac{1/m^2}{(\omega^2 - \omega_o^2)^2 + \gamma^2\omega^2} \frac{d\omega}{2\pi}$$

$$< x^2 > \approx \int_0^\infty \frac{\frac{4\,k_b\,T\,\gamma}{m\omega_o^2}}{4(\omega - \omega_o)^2 + \gamma^2} \frac{d\omega}{2\pi} = \frac{k_b T}{m\,\omega_o^2} \int_0^\infty \frac{1}{1 + 4(\omega^2 - \omega_o^2)/\gamma^2} \frac{2}{\pi\gamma}\,d\omega$$

≈ 1 for $\gamma << \omega_o$

So, we find

$$< x^2 > = \frac{k_b T}{m\omega_o^2}$$

Note: $\omega_o^2 = \frac{k}{m}$

Hence: $< x^2 > = \frac{k_b T}{k}$

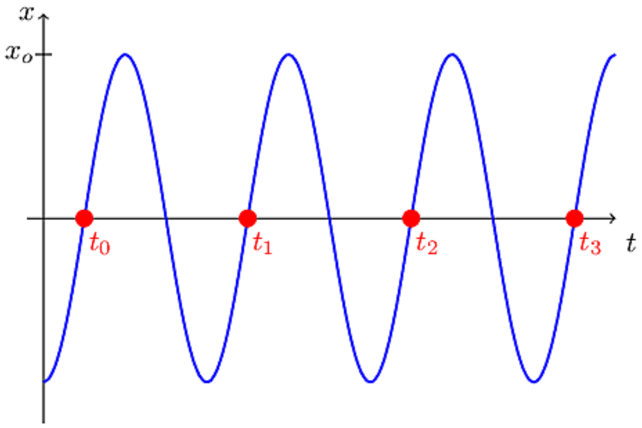
For a spring with k, we find $< E_{pot} > = \frac{1}{2}\, k < x^2 > = \frac{1}{2} k_b T$

Note :$< v^2 > = \omega_o^2 < x^2 > = \frac{k_b T}{m}$

And hence: $< E_{kin} > = \frac{1}{2} m < v^2 > = \frac{1}{2} k_b T$

$$< E_{tot} > = < E_{pot} > + < E_{kin} > = k_B T$$

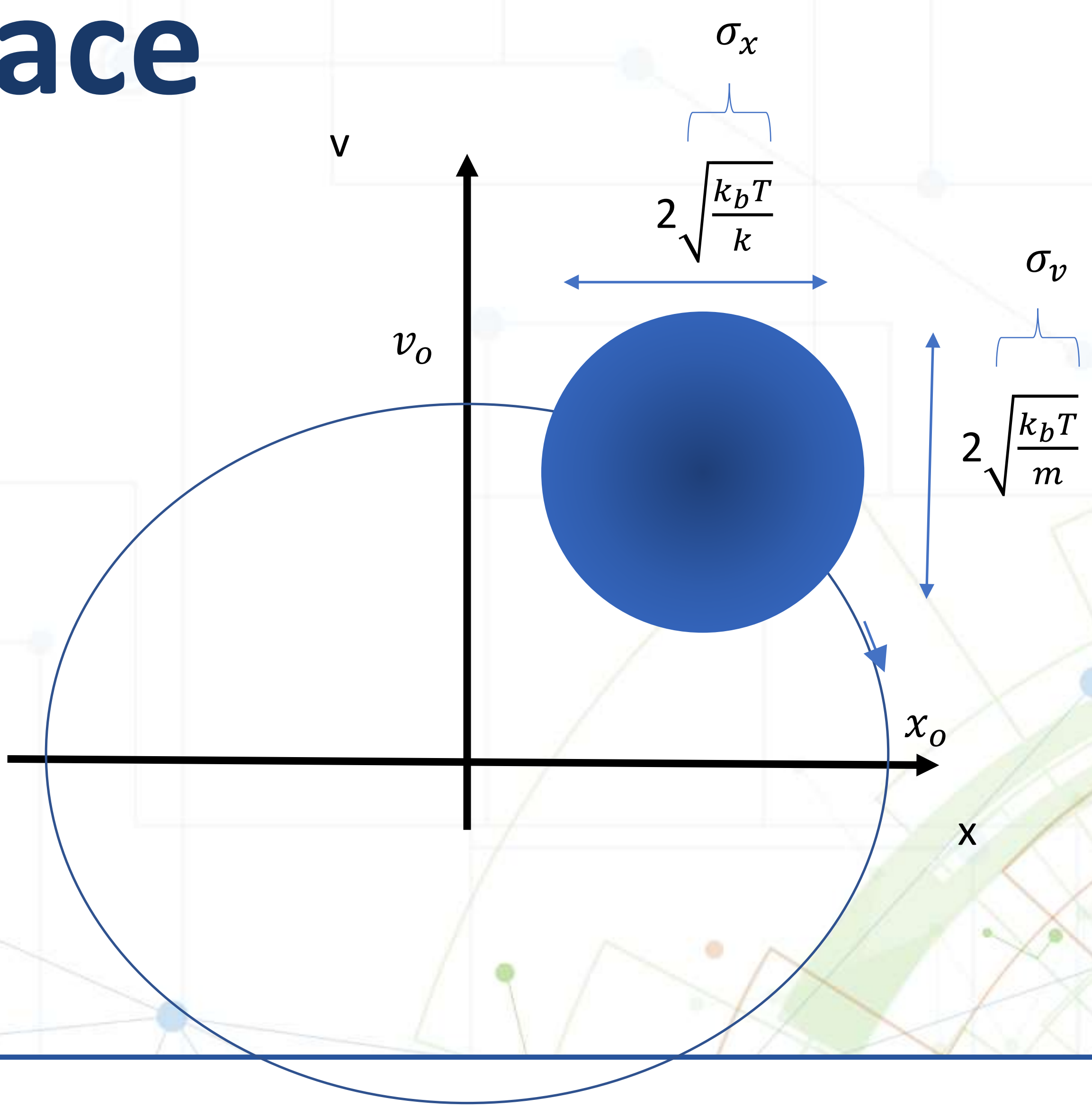
In phase space



Uncertainty in the orange balls:

$$\sigma_t = \frac{\sigma_x}{x_0 \omega_0}$$

$$\sigma_t = \frac{x_{th}}{x_o \omega_o}$$



Let's introduce:

$$x_{th} = \sqrt{\frac{k_b T}{k}}$$

$$\sigma_N^2 = \frac{2}{N^2}\sigma_t^2(1 - r_{N,0})$$

Let's find that correlation

Wiener-Khinchin

$$< x(\tau)x(0) > = R_{xx}(\tau) = \int_0^\infty S_{xx}(f)\exp(2\,\pi\,i\,f\;\tau)\frac{d\omega}{2\pi}$$

$$\int_0^\infty 4\,k_bT\,m\,\gamma\,\frac{1/m^2}{(\omega^2-\omega_o^2)^2+\gamma^2\omega^2}\exp(\,i\omega\,\tau)\frac{d\omega}{2\pi} \approx \int_0^\infty 2\,k_bT\,\frac{\frac{1}{m\omega_o^2}\;(\frac{\gamma}{2})}{(\omega-\omega_o)^2+\left(\frac{\gamma}{2}\right)^2}\exp(\,i\omega\,\tau)\frac{d\omega}{2\pi}$$

$$\approx \frac{k_bT}{m\,\omega_o^2}\int_0^\infty \frac{1}{\pi}\,\frac{\frac{\gamma}{2}}{(\omega-\omega_o)^2+\left(\frac{\gamma}{2}\right)^2}\exp(\,i\omega\,\tau)\,d\,\omega = \frac{k_bT}{m\,\omega_o^2}\exp\left(-\frac{\gamma}{2}|\tau|\right)\,\exp(-i\omega_o\tau)$$

For $\gamma << \omega_o$

Let's find that correlation (continued)

$$< x(\tau)x(0) >= \frac{k_b T}{m\ \omega_o^2} \exp\left(-\frac{\gamma}{2}|\tau|\right) \exp(-i\omega_o \tau)$$

We are interested, because we would like to find the zero crossings in $\tau = N\ T_0 = N \frac{2\pi}{\omega_o}$

In this case

$$< x(N\ T_o)x(0) >= \frac{k_b T}{m\ \omega_o^2} \exp(-\frac{\gamma}{2}\ N\ T_o)$$

We need to find

$$r = \frac{< x(NT_o)x(0) >}{\sqrt{< x(NT_o)x(NT_o) >< x(0)x(0) >}} = \exp\left(-\frac{\gamma}{2} N\ T_o\right)$$

Note the correlation in time for the zero crossings is the same, since $< t(N\ T_o)t(0) > = \frac{1}{x_o^2 \omega_o^2} < x(N\ T_o)x(0) >$

National Institute of Standards and Technology
U.S. Department of Commerce

Let's find that correlation (continued)

$$< x(\tau)x(0) >= \frac{k_b T}{m\,\omega_o^2} \exp\left(-\frac{\gamma}{2}|\tau|\right)\, \exp(-i\omega_o\tau)$$

We are interested in $\tau = N\,T_0 = N\frac{2\pi}{\omega_o}$, because we would like to find the zero crossings

In this case

$$< x(N\,T_o)x(0) >= \frac{k_b T}{m\,\omega_o^2} \exp(-\frac{\gamma}{2}\,N\,T_o)$$

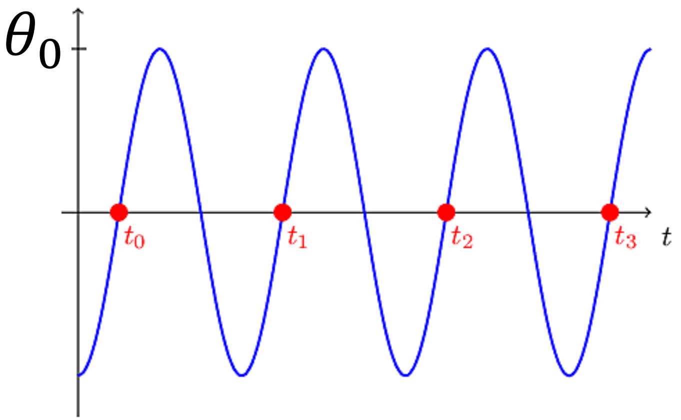
Use: $\frac{\gamma}{2} = \frac{\pi}{Q\,T_0}$

We need to find

$$r_{N0} = \frac{< x(NT_o)x(0) >}{\sqrt{< x(NT_o)x(NT_o) >< x(0)x(0) >}} = \exp\left(-\frac{\gamma}{2} N\,T_o\right) = \exp(-N\ \pi/Q)$$

Note the correlation in time for the zero crossings is the same, since $< t(N\,T_o)t(0) > = \frac{1}{x_o^2\omega_o^2} < x(N\,T_o)x(0) >$

Oscillators as probes



$$E_N(T_o) = \frac{t_N - t_0}{N}$$

$$\sigma_N^2 = \left(\frac{\partial E_N}{\partial t_N}\right)^2 \sigma_t^2 + \left(\frac{\partial E_N}{\partial t_0}\right)^2 \sigma_t^2 + 2\left(\frac{\partial E_N}{\partial t_N}\right)\left(\frac{\partial E_N}{\partial t_0}\right) r_{N,0}\sigma_t^2$$

$$\sigma_N^2 = \frac{2}{N^2}\sigma_t^2(1 - r_{N,0})$$

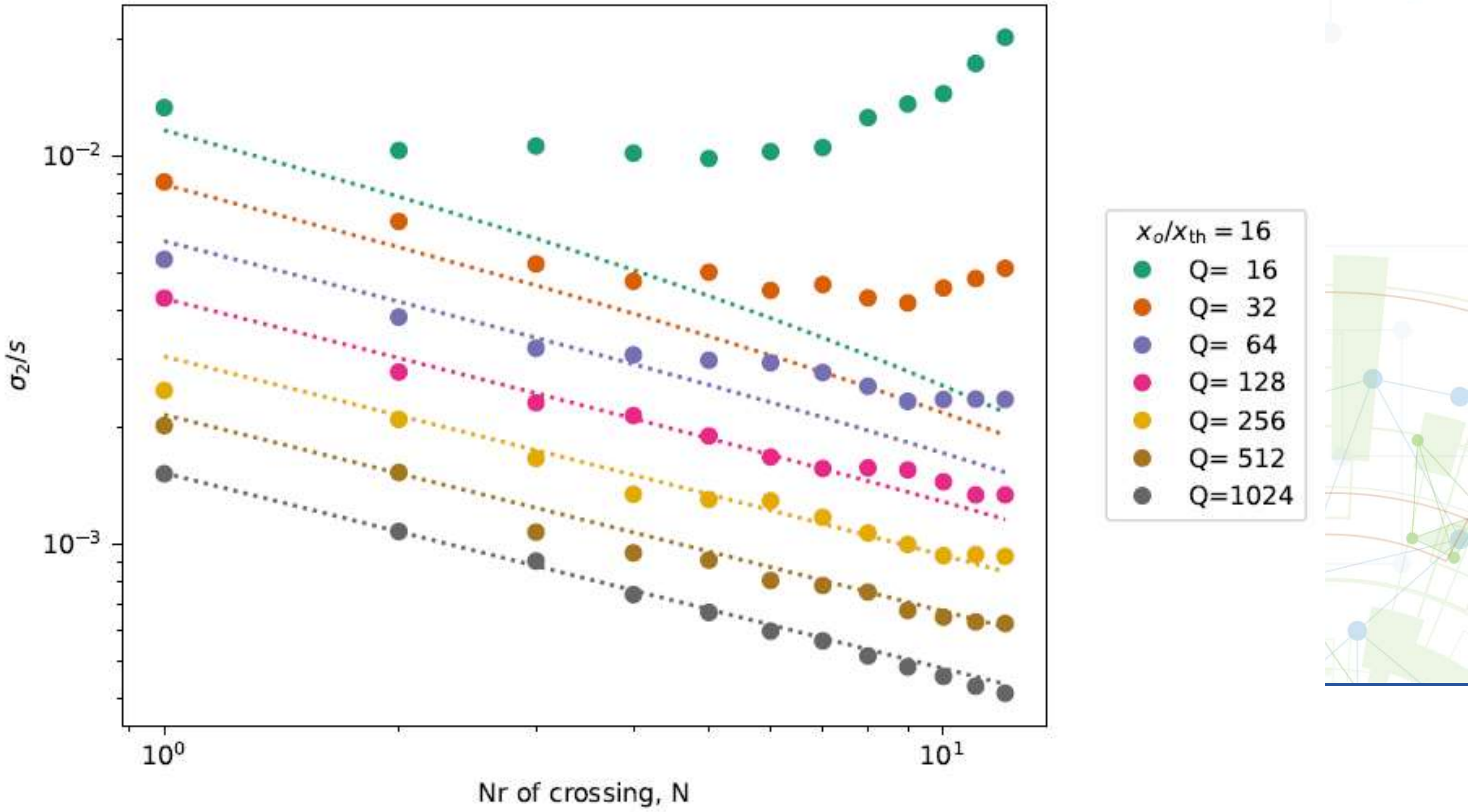
$$\sigma_t = \frac{x_{th}}{x_o \omega_o}$$

$$x_{th} = \sqrt{\frac{k_b T}{k}}$$

$$r_{N,0} = \exp\left(-\frac{N\,\pi}{Q}\right)$$

$$\sigma_n = \frac{\frac{\sqrt{2}}{N}\sqrt{\frac{k_b T}{k}}}{x_0 \omega_0}\sqrt{1 - \exp\left(-\frac{N\pi}{Q}\right)} \approx \frac{\sqrt{\frac{2\,k_b T}{k}}}{N\,x_o \omega_0}\sqrt{\frac{N\pi}{Q}}$$

$$\sigma_n \approx \frac{x_{th}}{x_0}\frac{1}{\omega_0}\sqrt{\frac{2\pi}{N\,Q}}$$



10^{-2}
10^{-3}
σ_2/s
10^0
10^1
Nr of crossing, N
$x_o/x_{\mathrm{th}} = 16$
Q= 16
Q= 32
Q= 64
Q= 128
Q= 256
Q= 512
Q=1024

Power spectral density – watch the definition

Single sided in f-domain

$$\int_{0}^{\infty} S_x^{ssf}\, df = \int_{0}^{\infty} S_x^{ssf} \frac{1}{2\pi} d\omega = < x^2 >$$

Double sided in f-domain

$$\int_{-\infty}^{\infty} S_x^{dsf}\, df = \int_{-\infty}^{\infty} S_x^{ssf} \frac{1}{2\pi} d\omega = < x^2 >$$

Single sided in ω-domain

$$\int_{0}^{\infty} S_x^{ss\omega} d\omega = < x^2 >$$

Double sided in ω-domain

$$\int_{-\infty}^{\infty} S_x^{ss\omega} d\omega = < x^2 >$$

$$S_x^{ssf} = 2 S_x^{dsf} = 2\pi S_x^{ss\omega} = 4\pi\, S_x^{ds\omega}$$

Power spectral density – watch the definition

$$S_x^{ssf} = 2S_x^{dsf} = 2\pi S_x^{ss\omega} = 4\pi\, S_x^{ds\omega}$$

Sometimes, people use double indices to distinguish single sided from double sided, e.g S_F and S_{FF}

$$S_F^{th}\, df\ = 4\, k_b T\, m\, \gamma\, df$$

$$S_{FF}^{th} df = 2\, k_b T\, m\, \gamma\, df$$

$$S_F^{th} d\omega = \frac{2}{\pi}\, k_b T\, m\, \gamma\, d\omega$$

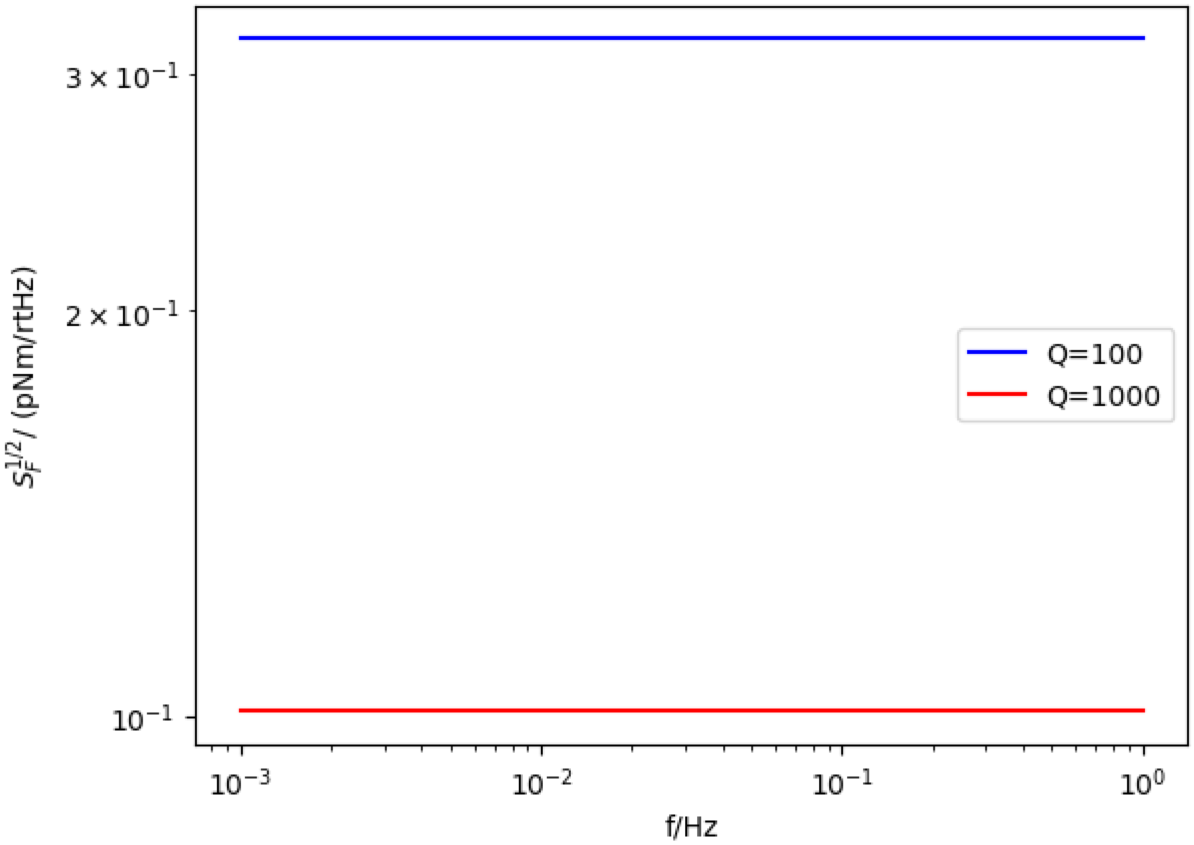
$$S_{FF}^{th} d\omega = \frac{1}{\pi}\, k_b T\, m\, \gamma\, d\omega$$

Power spectral densities of force & excursion

$$m\,\ddot{x} + m\,\gamma\,\dot{x} + \omega_o^2\,x = f$$

$$\gamma = \frac{\omega_0}{Q}$$

$$S_F^{th}\,df \;= 4\,k_bT\,m\frac{\omega_0}{Q}\,df$$



m =10 g

Power spectral densities of force & excursion

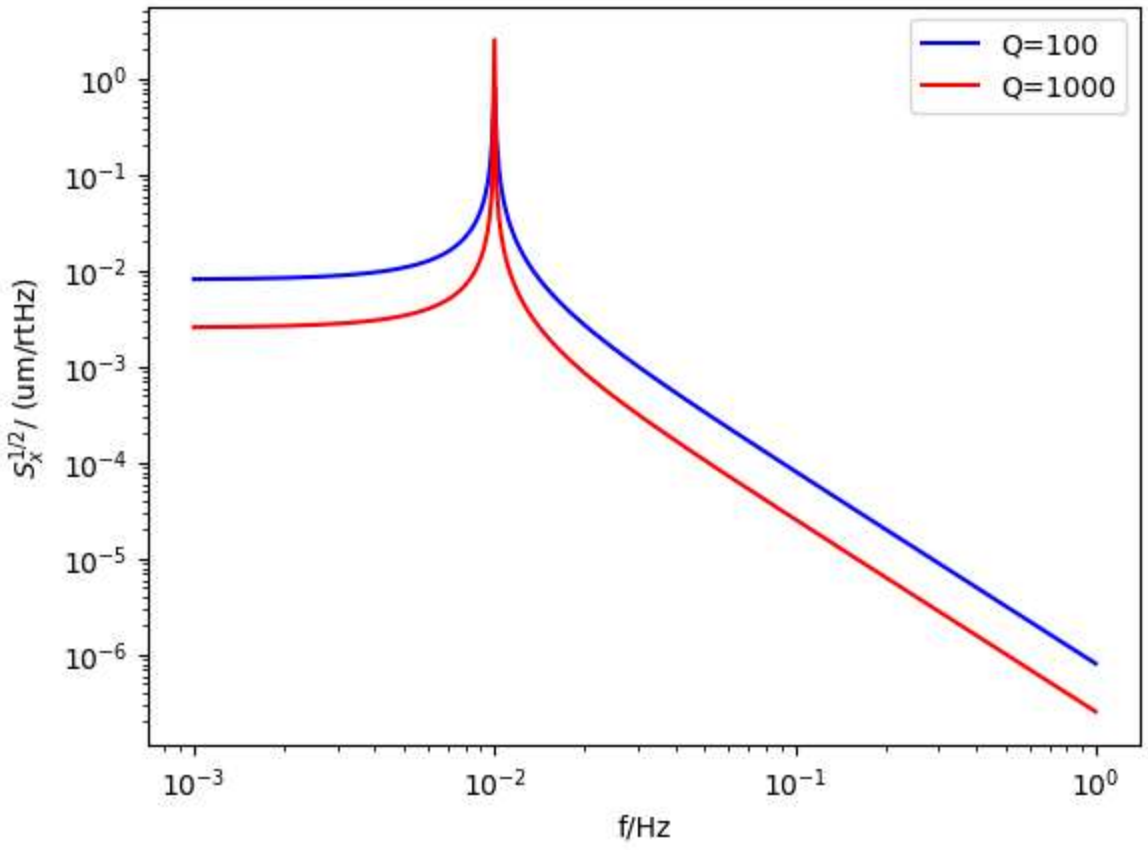
$$m\,\ddot{x} + m\,\gamma\,\dot{x} + \omega_o^2\,x = f$$

$$\gamma = \frac{\omega_0}{Q}$$

$$S_F^{th}\,df \;= 4\,k_bT\,m\frac{\omega_0}{Q}df$$

$$S_x^{th} = |R^2|S_F^{th}$$

$$S_x^{th} = 4\,k_bT\,m\frac{\omega_0}{Q}\,\frac{1/m^2}{(\omega^2-\omega_o^2)^2+\frac{1}{Q^2}\,\omega_0^2\,\omega^2}$$



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U.S. Department of Commerce

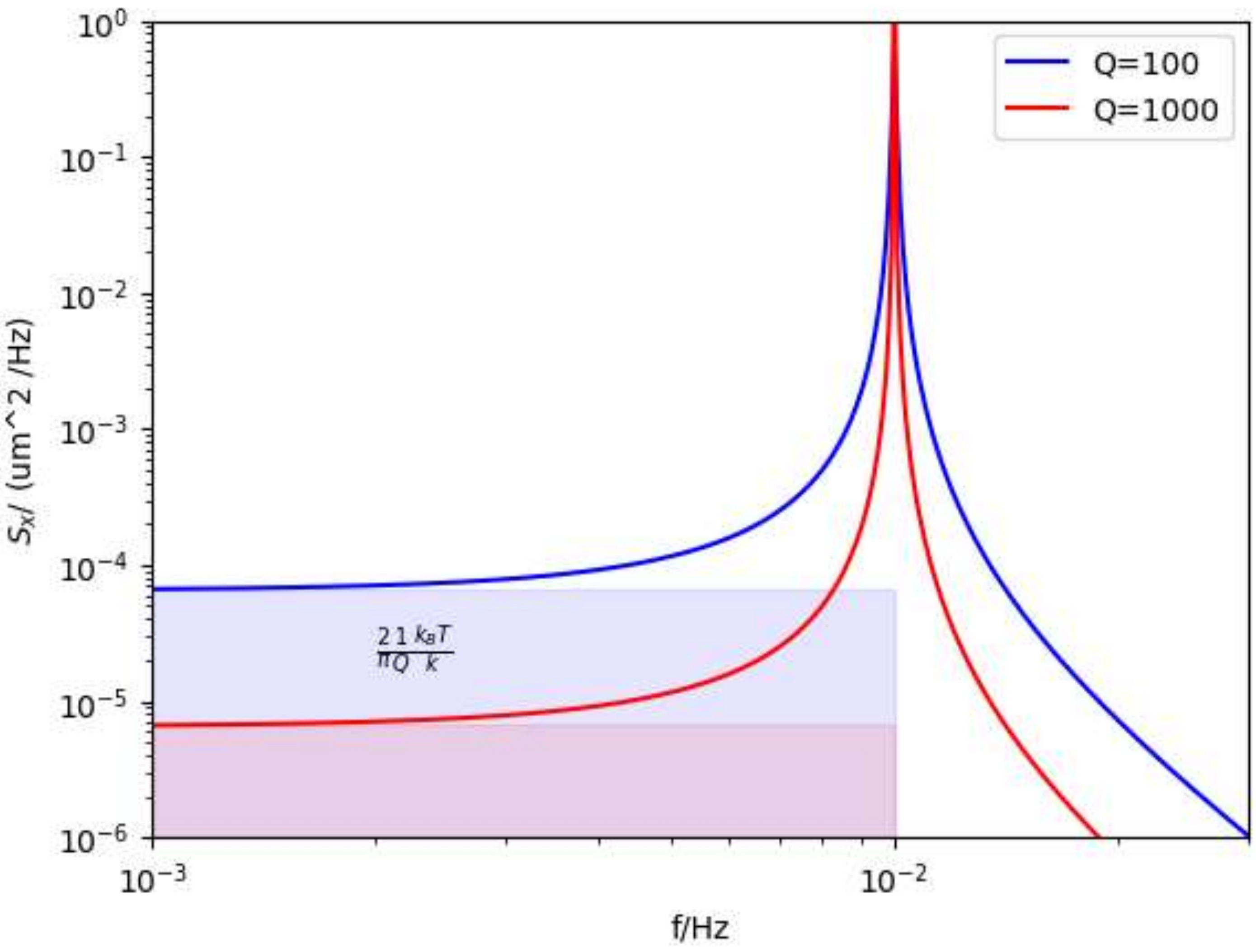
Where is the power

$$S_x^{th}(\omega) = 4\, k_b T\, m \frac{\omega_0}{Q} \frac{1/m^2}{(\omega^2 - \omega_o^2)^2 + \frac{1}{Q^2}\, \omega_0^2\, \omega^2}$$

$$S_x^{th}(0) = 4\, k_b T\, \frac{\omega_0}{Q} \frac{\frac{1}{m}}{\omega_o^4} = 4\, k_b T \frac{1}{Qk\, \omega_0}$$

$$\int_0^{\omega_o} 4\, k_b T \frac{1}{Qk\, \omega_0} \frac{d\omega}{2\pi} = \frac{2}{\pi} \frac{1}{Q} \frac{k_b T}{k}$$

$$< x^2 > = \int_0^{\infty} S_x^{th} \frac{d\omega}{2\pi} = \frac{k_b T}{k}$$



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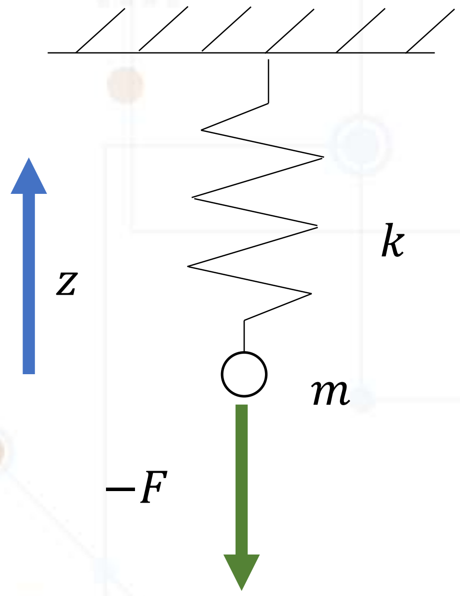
Velocity damping vs. internal damping



$$m\,\ddot{z} + m\,\gamma\dot{z} + kz = F$$

$$m\,\ddot{z} + k(1 + i\delta)z = F$$

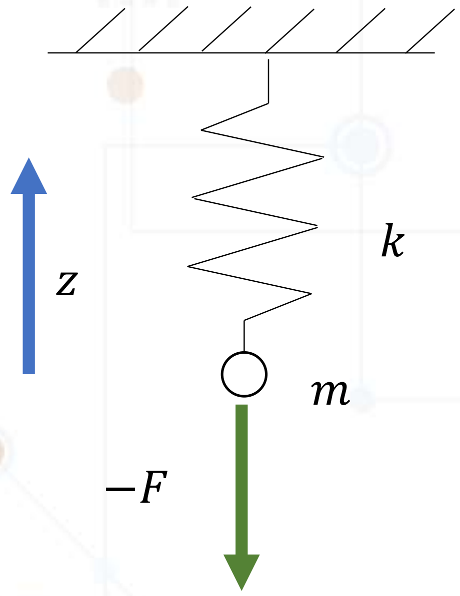
Internal damping

Examples of velocity damping:

- Viscous gas damping
- Eddy current damping
- Electrical damping/ patch field driven currents

Velocity damping can be eliminated by experimental design

Velocity damping vs. internal damping



$$m\,\ddot{z} + m\,\gamma\dot{z} + kz = F$$

$$\frac{z(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_o^2 - \omega^2 + \frac{i\omega_o\omega}{Q}}$$

$$m\,\ddot{z} + k(1 + i\delta)z = F$$

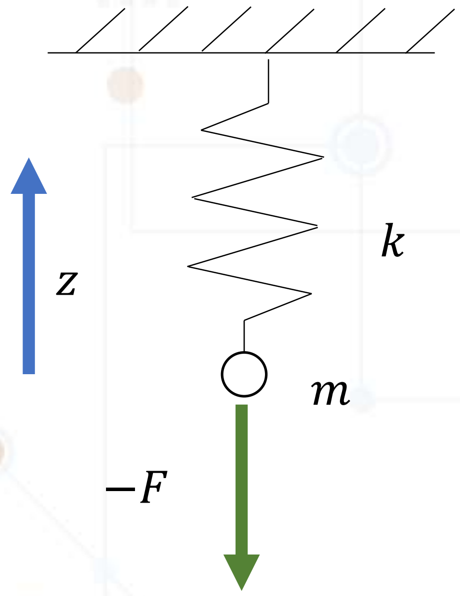
$$\frac{z(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_o^2 - \omega^2 + i\,\delta\,\omega_0^2}$$

How much difference can that make?

Real part of the mechanical impedance

For the velocity damping: $S_F^{th}\,df \;= 4\,k_bT\,m\,\gamma\,df = 4\,k_bT\,Re(Z)df$

Mechanical impedance



$$Z(\omega) = \frac{F}{v} = \frac{F}{i\,\omega\,z} = -\frac{i}{\omega}\,\frac{F}{z} = -\frac{i}{\omega}\,r^{-1}$$

Multiplicative inverse of the response function

$$r = \frac{z(\omega)}{F(\omega)} = \frac{1}{m}\,\frac{1}{\omega_o^2 - \omega^2 + \frac{i\omega_o\omega}{Q}}$$

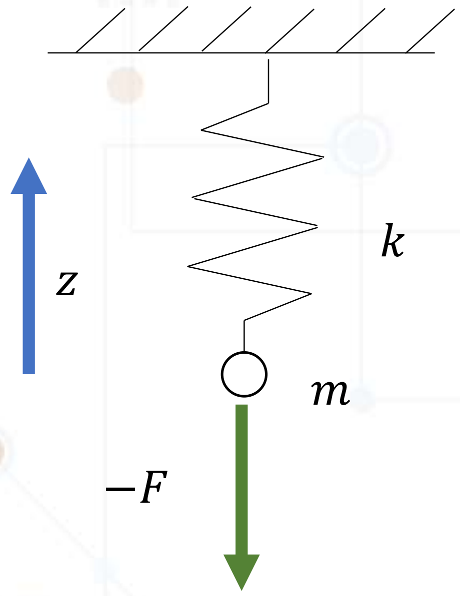
Previous result: $S_F^{th}\,df \;= 4\,k_bT\,m\,\gamma\,df$

$$r^{-1} = \frac{z(\omega)}{F(\omega)} = m\left(\omega_o^2 - \omega^2 + \frac{i\omega_o\omega}{Q}\right)$$

$$\frac{-i}{\omega}r^{-1} = -i\,\frac{\omega_o^2}{\omega}m + i\,\omega\,m + \frac{\omega_o}{Q}m$$

$$S_F^{th}\,df \;= 4\,k_bT\,Re(Z)df = 4\,k_bT\,\frac{\omega_o m}{Q}\,df$$

Velocity damping vs. internal damping



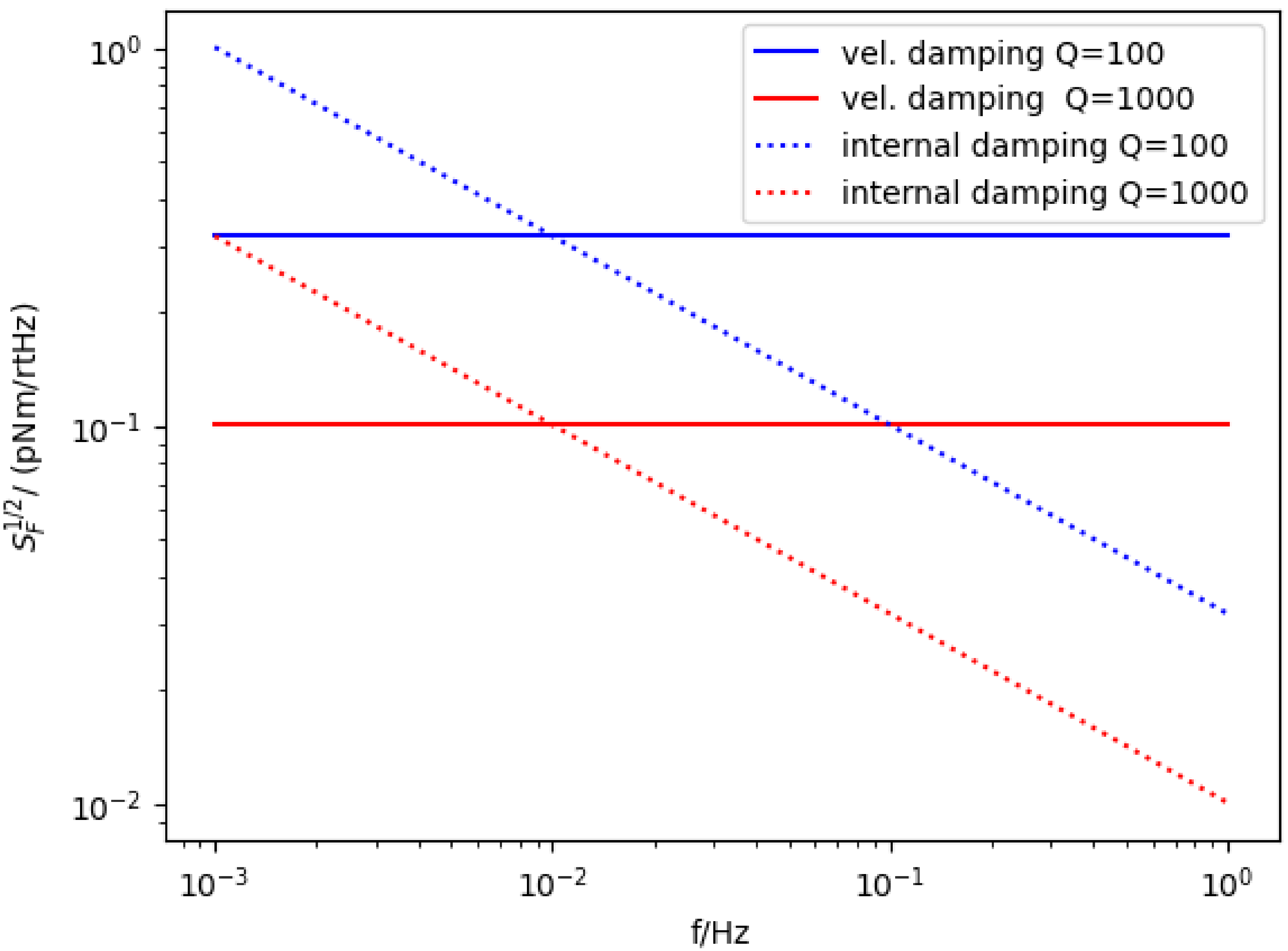
$$\frac{z(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_o^2 - \omega^2 + \frac{i\omega_o\omega}{Q}}$$

$$S_F^{th}\,df = 4\,k_bT\frac{\omega_o m}{Q}\,df = 4\,k_bT\,\kappa\,Q^{-1}\,\omega_o^{-1}$$

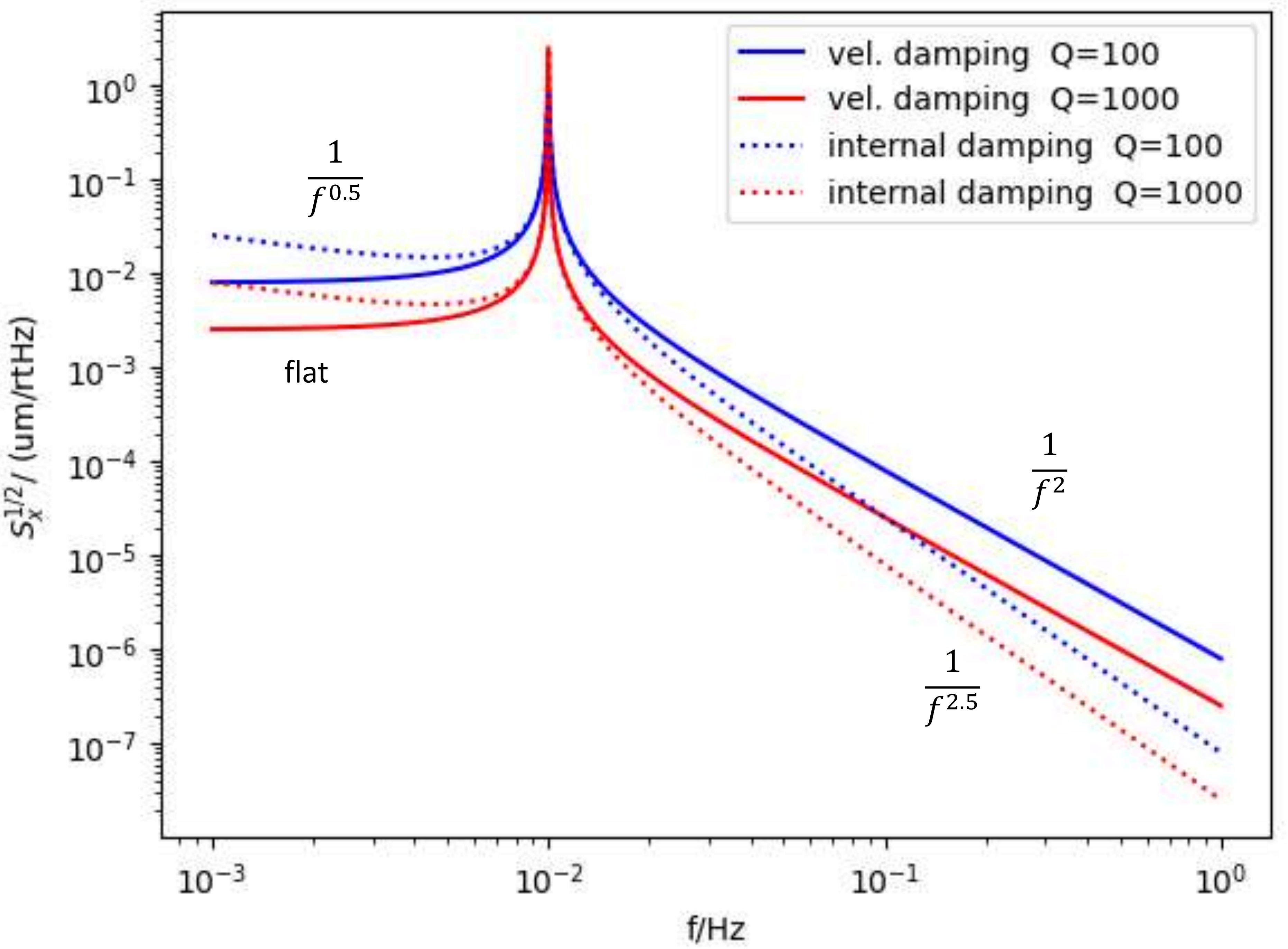
$$\frac{z(\omega)}{F(\omega)} = \frac{1}{m}\frac{1}{\omega_o^2 - \omega^2 + i\,\delta\,\omega_0^2}$$

$$S_F^{th}\,df = 4\,k_bT\frac{\omega_o^2\,\delta\,m}{\omega}\,df = 4\,k_bT\,\kappa\delta\;\omega^{-1}$$

$$S_F^{th}\,df = 4\,k_bT\,Re(Z)df = 4k_bT\,Re\left(-\frac{i}{\omega}r^{-1}\right)$$

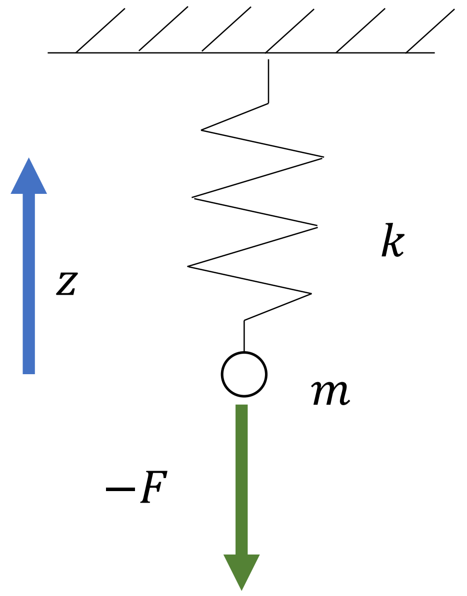

vel. damping Q=100
vel. damping Q=1000
internal damping Q=100
internal damping Q=1000
10^0
10^{-1}
10^{-2}
10^{-3}
$S_F^{1/2}$ / (pNm/rtHz)
f/Hz

1/f noise



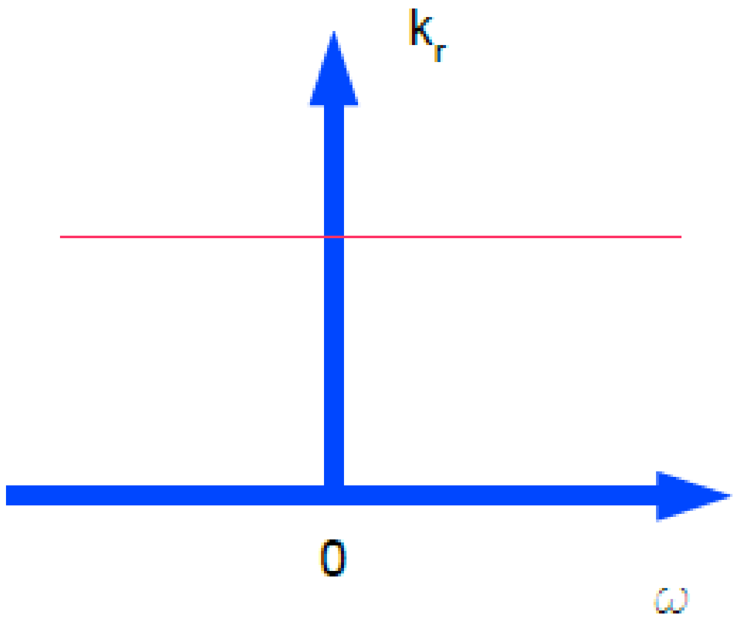
vel. damping Q=100
vel. damping Q=1000
internal damping Q=100
internal damping Q=1000
$\frac{1}{f^{0.5}}$
flat
$\frac{1}{f^{2}}$
$\frac{1}{f^{2.5}}$
$S_x^{1/2}$ / (um/rtHz)
f/Hz

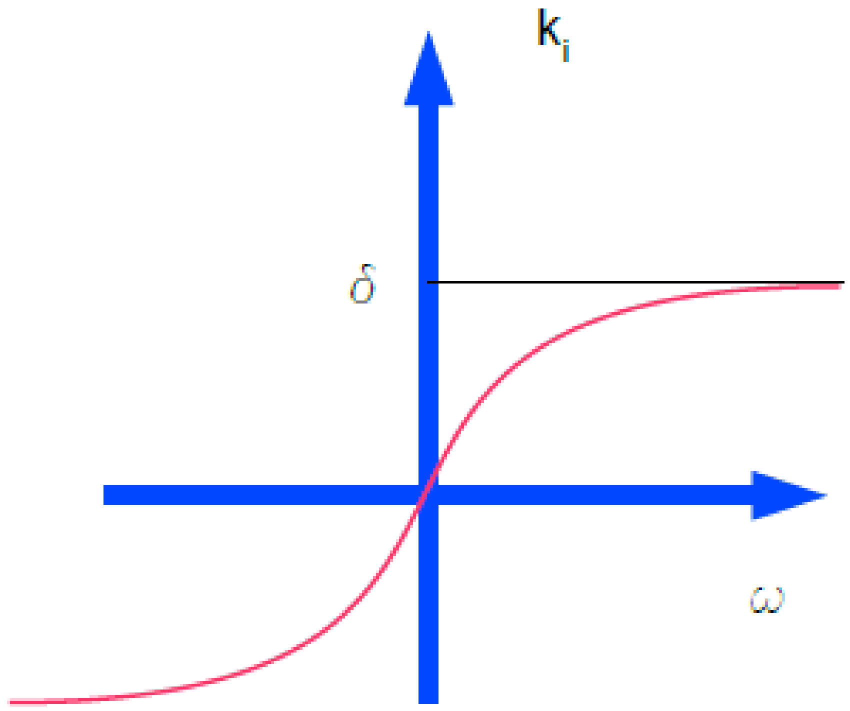
Where does the internal damping come from

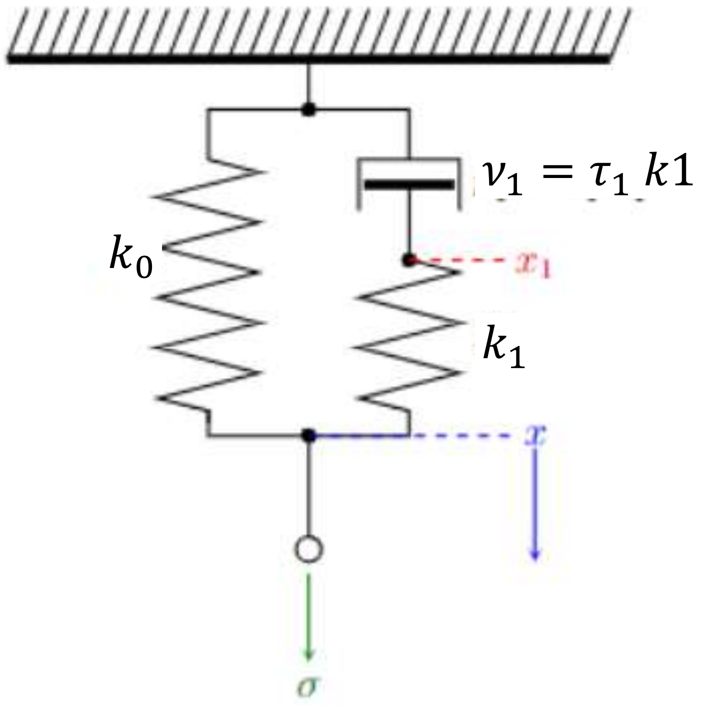


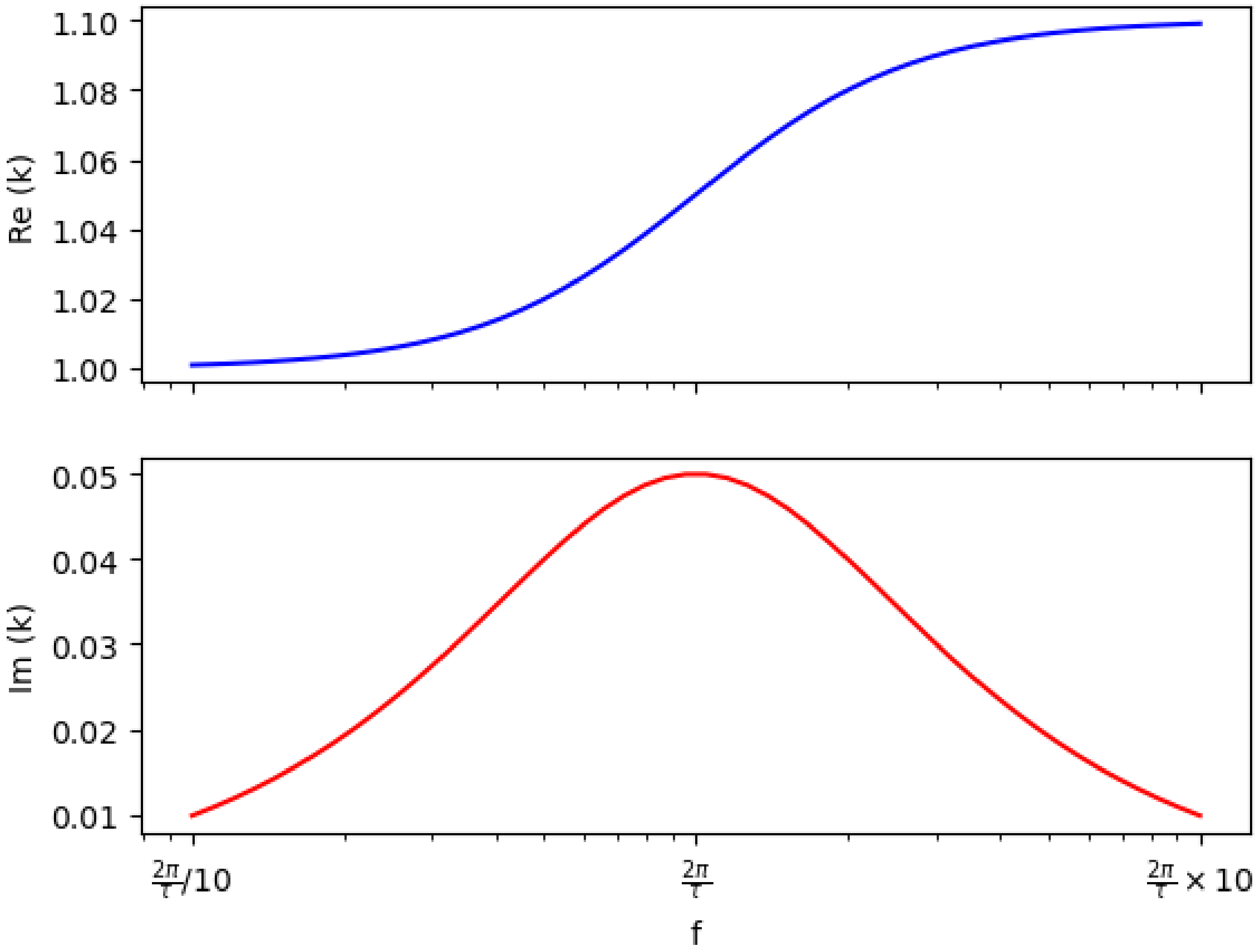
$$m\,\ddot{z} + k(1 + i\delta)z = F$$

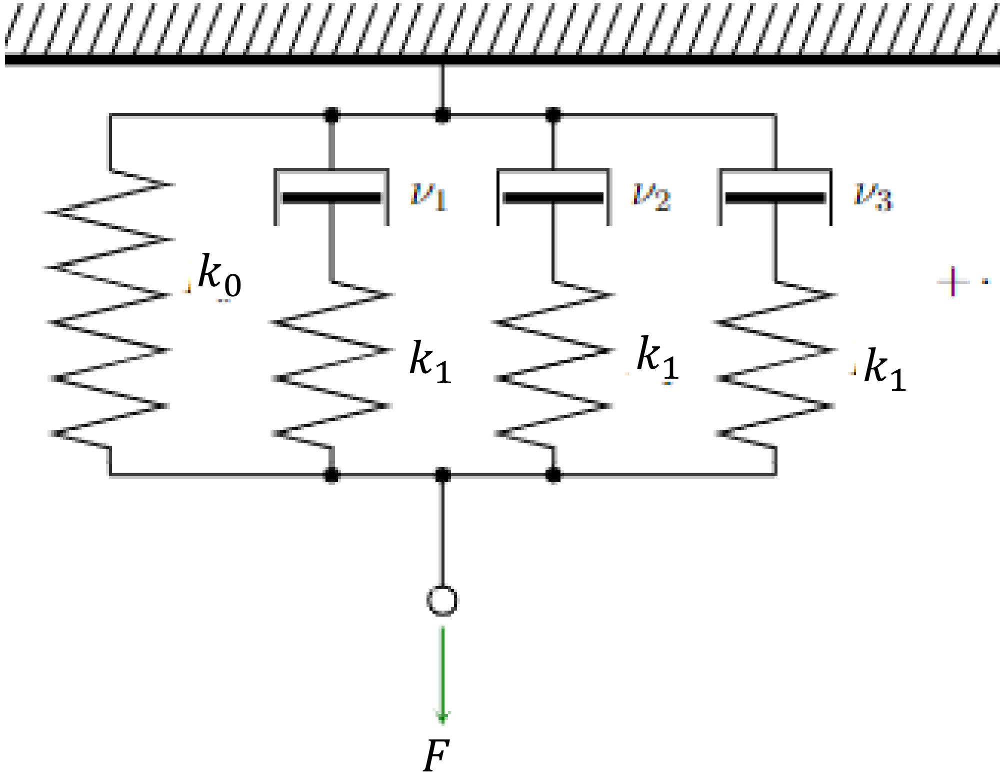
Kramer’s Kronig relationship relates the real part to the imaginary part and vice versa.

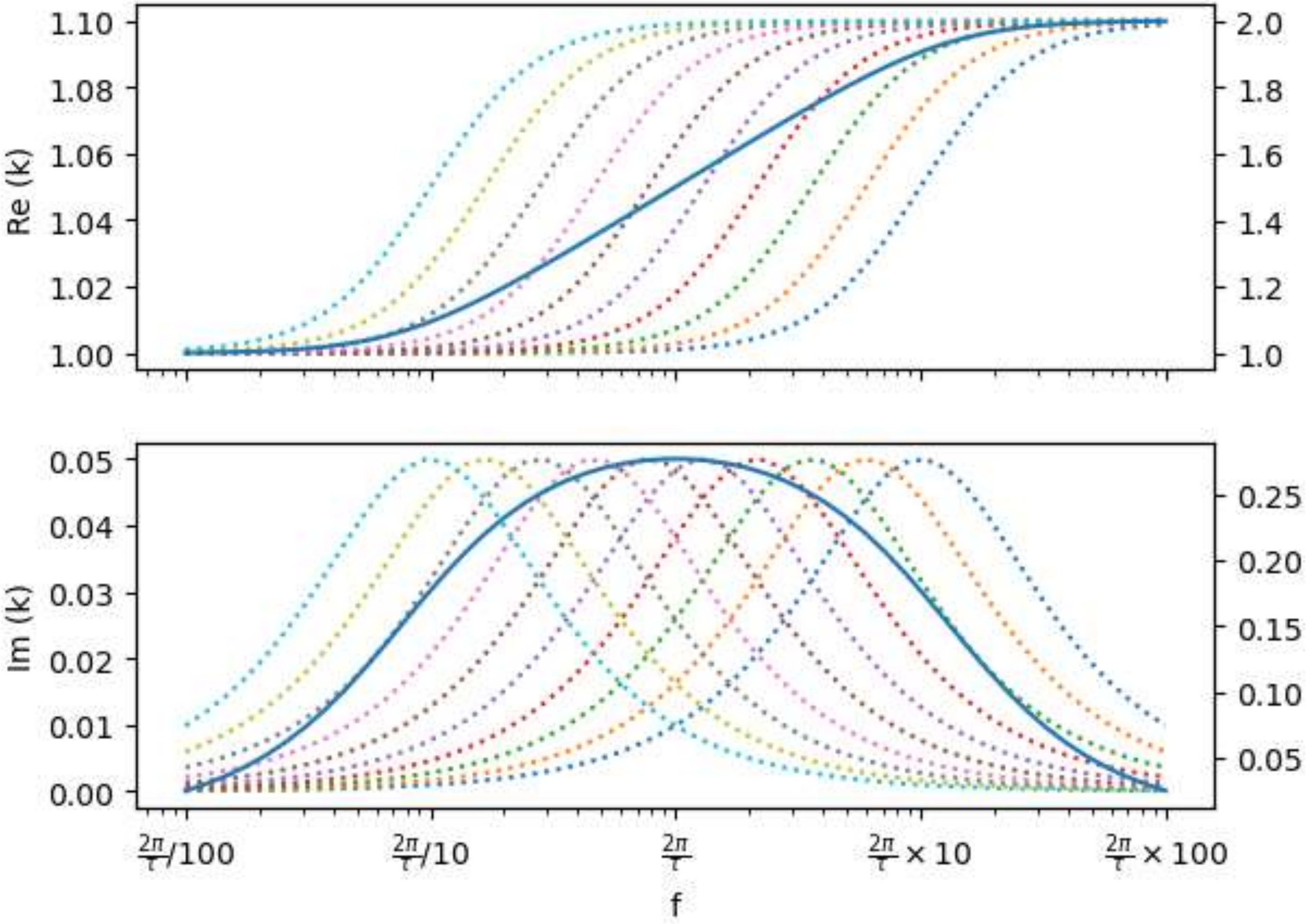





$\nu_1 = \tau_1\, k1$
k_0
x_1
k_1
x
σ


Re (k)
1.10
1.08
1.06
1.04
1.02
1.00
Im (k)
0.05
0.04
0.03
0.02
0.01
$\frac{2\pi}{\tau}/10$
$\frac{2\pi}{\tau}$
$\frac{2\pi}{\tau}\times 10$
f





Extended Maxwell model:

- A distribution of Maxwell units
- All have the same spring constant
- But a distribution of time constants

$$k(\omega) = k_0 + k_1 \int_{\tau_l}^{\tau_u} f(\tau)\, \frac{i\omega\tau}{i\omega\tau + 1} d\omega$$

Does the Time-of-Swing Method Give a Correct Value of the Newtonian Gravitational Constant?

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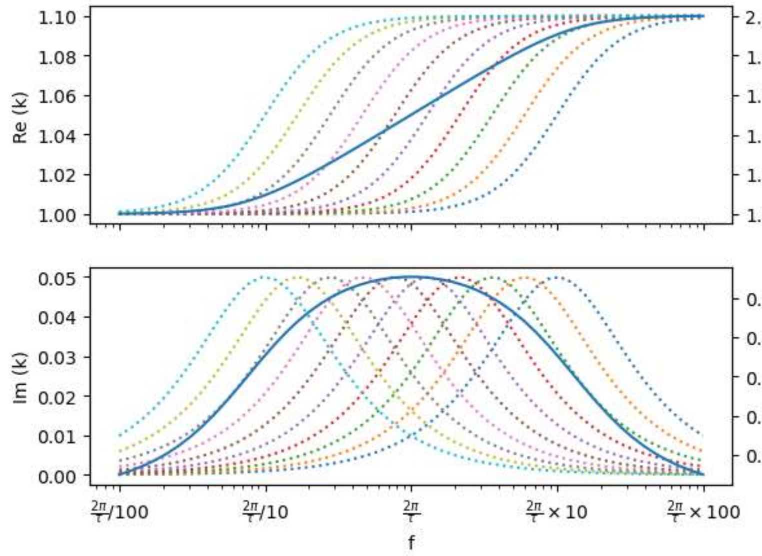
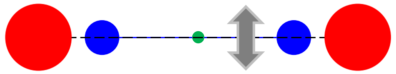
(Received 12 June 1995)

A standard way of measuring the Newtonian gravitational constant has been the time-of-swing method using a torsion pendulum. A key assumption is that the spring constant of the torsion fiber is independent of frequency. This is likely to be true to a good approximation if any damping present

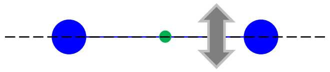


Time-of-swing method



$$\omega_n{}^2 = \frac{\kappa_1 + \kappa_n}{I}$$

$$\kappa_n = G\, C_n$$



$$\omega_f{}^2 = \frac{\kappa_2 + \kappa_f}{I}$$

$$\kappa_f = G\, C_f$$

$$\kappa_1 > \kappa_2$$

$$G_{reported} = \frac{\omega_n{}^2 - \omega_f{}^2}{(C_n - C_f)} I = G + \frac{\kappa_1 - \kappa_2}{(C_n - C_f)} I$$

$$G_{reported} > G$$

$$G_{reported} - G < \frac{1}{Q\pi}$$

Thanks

Thanks for listening to the thermal noise lecture!

Questions?