Source masses: geometry, materials, and calculation

SIGRAV school 2024

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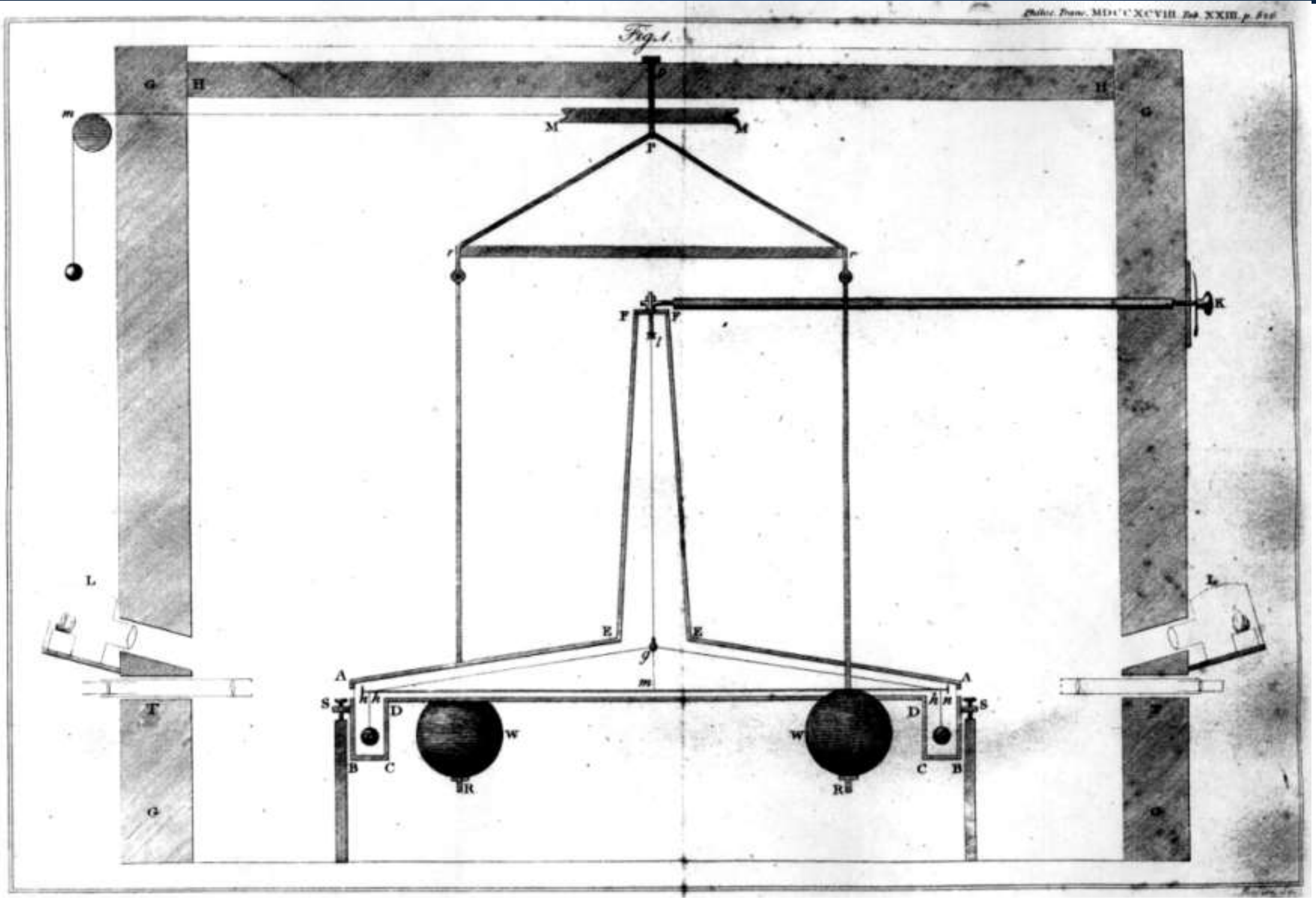
Source masses vs. test masses

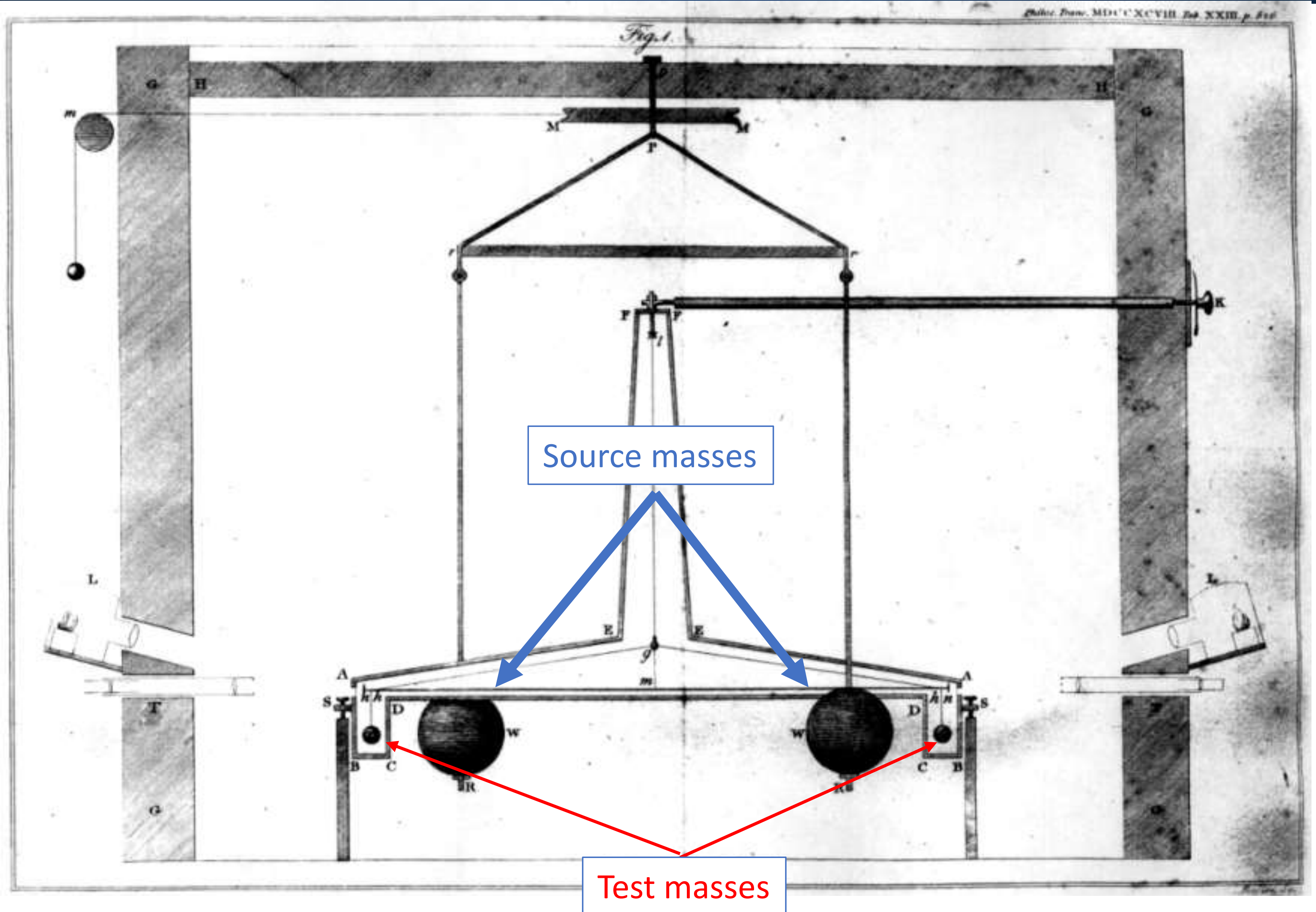


Cavendish Experiment

1793 – 1798

$$\rho_E = 5448(33)\,\frac{\text{kg}}{\text{m}^3}$$

Source masses vs. test masses

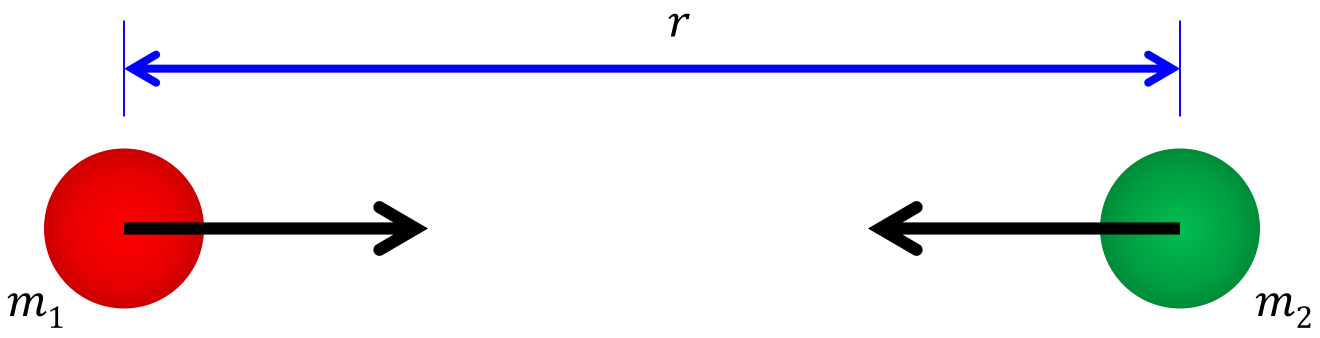


Cavendish Experiment

1793 – 1798

$$\rho_E = 5448(33)\,\frac{\text{kg}}{\text{m}^3}$$

Newton's law is symmetric



$$F = G\frac{m_1 m_2}{r^2}$$

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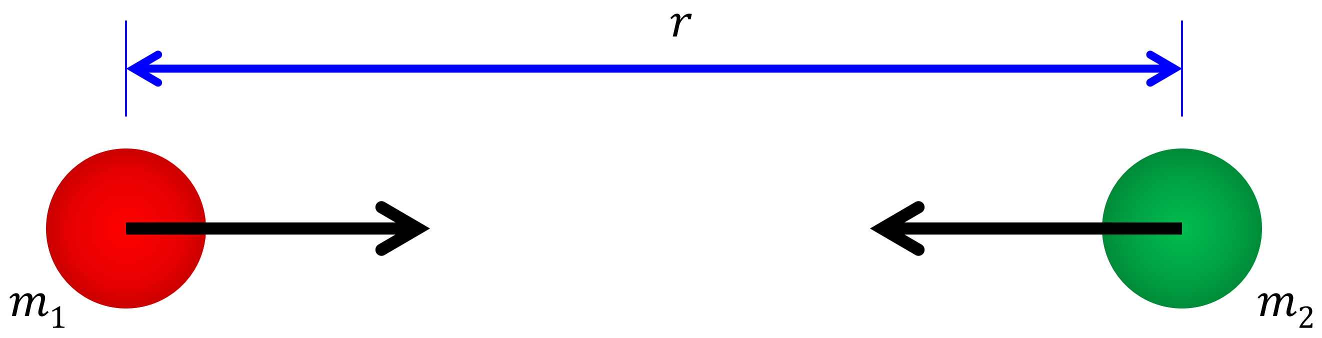


$$F = G\frac{m_1 m_2}{r^2} \qquad F = G\frac{m_1 m_2}{r^2}$$

Let's use one mass as a sensor. For example, suppose we measure the acceleration on mass m_1, we find:

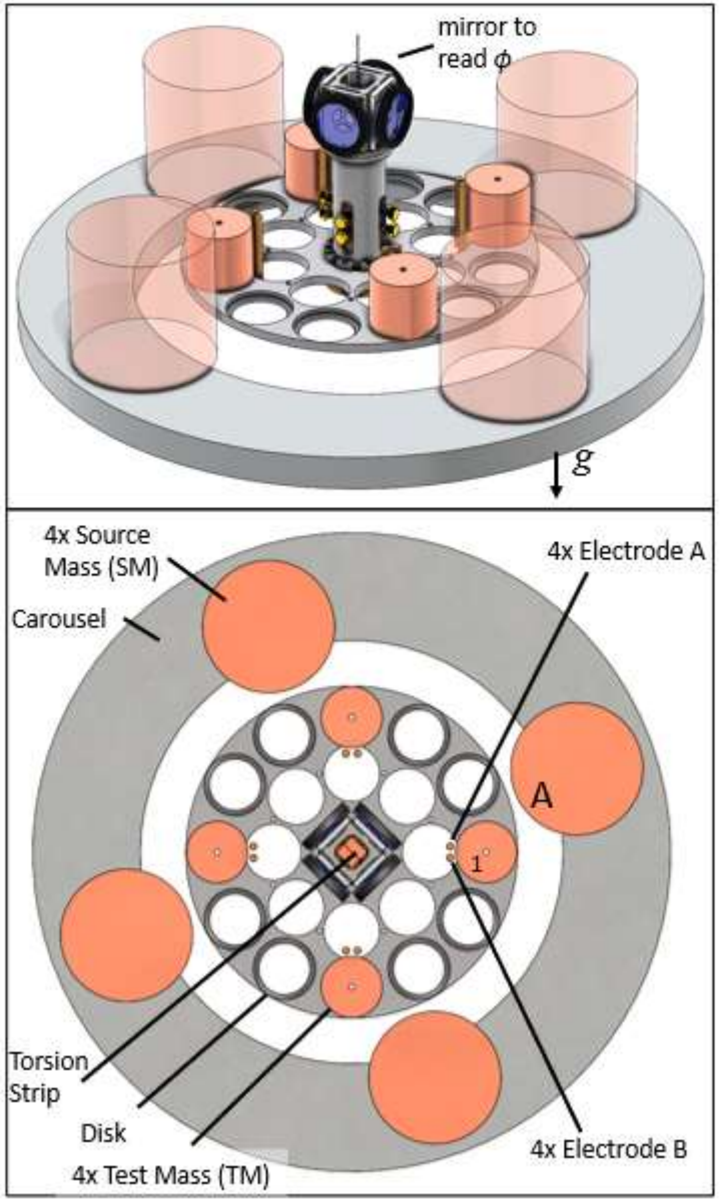
$$a_1 = \frac{F}{m_1} = G\frac{m_2}{r^2}$$

The measurement is independent of m_1.

$$G = \frac{a_1 r^2}{m_2}$$

A good way to memorize the units of *G*
m/s^2 × m^2/kg

IRL = In real Life



$$\phi_{ccw} = \phi_o + \frac{\Gamma G}{\kappa}$$

$$\phi_{cw} = \phi_o - \frac{\Gamma G}{\kappa}$$

$$\Delta\phi = 2\frac{\Gamma G}{\kappa} \qquad \Gamma = 35\, M\, m \frac{r^4}{R^5}$$

$$\omega^2 = \frac{\kappa}{I}$$

$$\Delta\phi = 70\frac{G\, Mm\, r^4}{\kappa\, R^5}$$

Depends on m

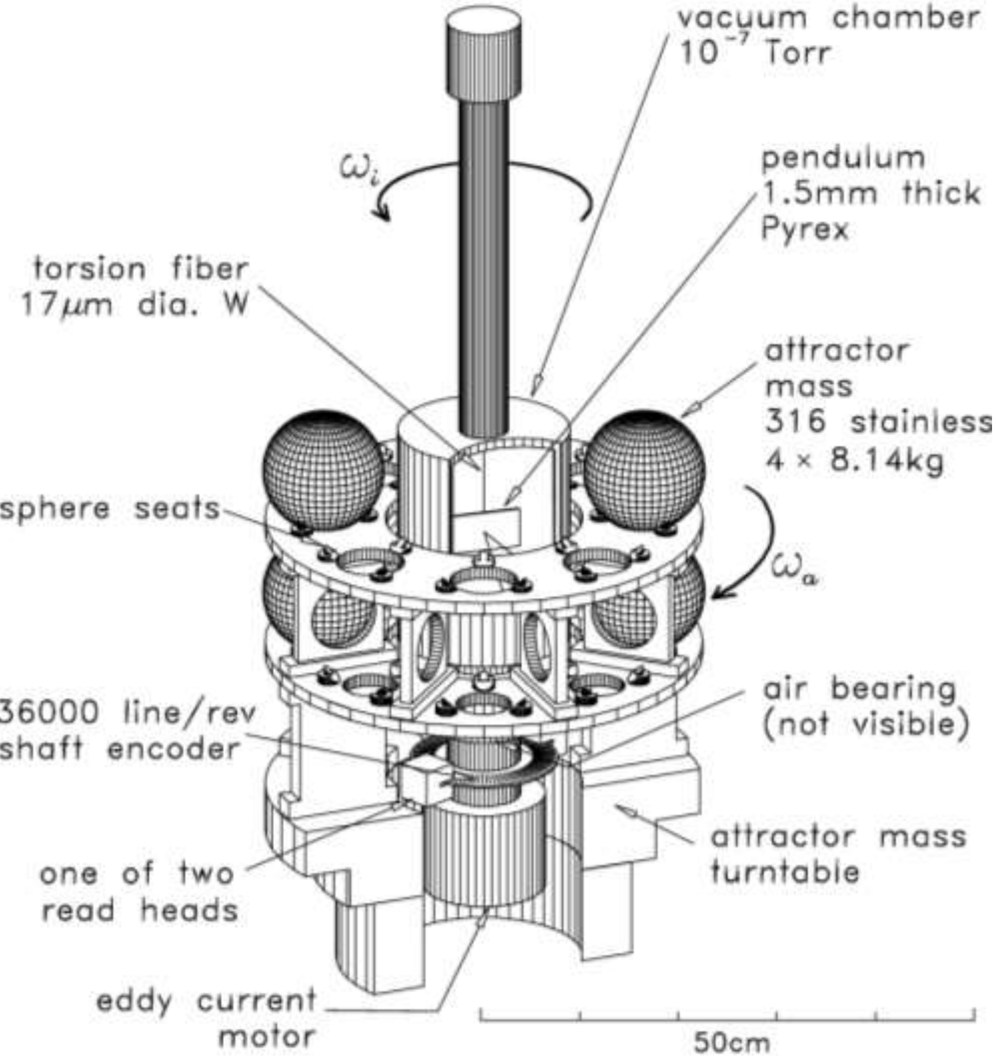
$$\kappa = \frac{4\pi^2}{T_0^2}\left(4\, m\, r^2 + I_{disk}\right)$$

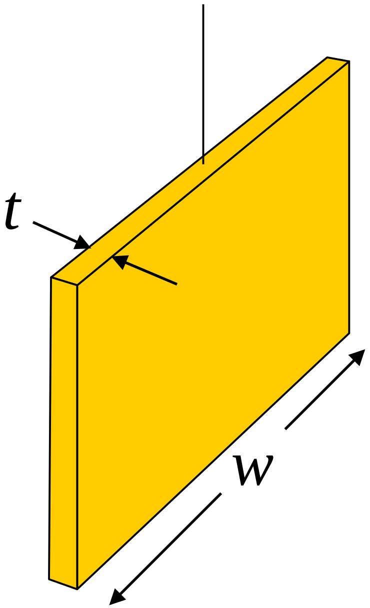
$$G = \frac{16\pi^2}{T_0^2}\frac{R^5}{70\, M\, r^2}\left(1 + \frac{I_{disk}}{4mr^2}\right)\Delta\phi$$

Weakly dependent on m

Here: $\frac{I_{disk}}{4mr^2} \approx 0.11$

Another example: Jens' Gundlach's exp.





J.H. Gundlach & S.M. Merkowitz
PRL 85, 2869 (2000).

He measured angular acceleration

$$\alpha(\phi) = \frac{4\pi G}{I} \sum_{l=2}^{\infty} \frac{1}{2l+1} \sum_{m=-l}^{l} m\, q_{lm} Q_{lm}\, e^{im\phi}$$

With:

I: moment of inertia of the pendulum
q_{lm}: Inner multipole moments – of the pendulum
Q_{lm}: Outer multipole moments – of the source mass assembly

$$q_{lm} = \int \rho_p(\vec{r}) r^l Y^*_{lm}(\hat{r})\ d^3r$$

$$Q_{lm} = \int \rho_p(\vec{r}) r^{-l-1}\, Y^*_{lm}\,(\hat{r})\, d^3r$$

Advantage of a thin plate

NIST

For a two-fold symmetry, the leading term is the quadrupole moment

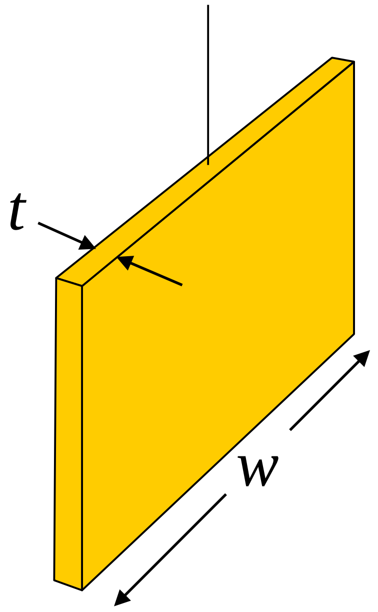
$$\alpha_{22}(\phi) = \frac{16\pi G}{5\,I}\frac{q_{22}}{I}\,Q_{22}\sin(2\phi)$$

Employing a thin vertical plate with homogenous density, we obtain

$$\frac{q_{22}}{I} = \frac{\int \rho_p \sqrt{\frac{15}{32\,\pi}}(x^2 - y^2)\;\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}{\int \rho_p (x^2 + y^2)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z}$$

If $y^2 \ll x^2$, i.e. $t \ll w$:

$$\frac{q_{22}}{I} = \sqrt{\frac{15}{32\,\pi}} \quad \Rightarrow \quad \alpha_{22}(\phi) \approx 4\sqrt{\frac{3}{10\,\pi}}\,G\,Q_{22}\sin(2\phi)$$



More precisely:

$$\frac{q_{22}}{I} = \sqrt{\frac{15}{32\,\pi}}\left(1 - 2\frac{t^2}{w^2}\right)$$

In JHG's experiment:

$t = 1.5$ mm, $w = 76$ mm

Hence the relative correction is

$$8 \times 10^{-4}$$

Let's look at the source mass

NIST

TABLE III. Field masses used in determinations of G. Adapted from Ref. 133.

Field mass (total) (kg)	Material	Geometry	Measurement principle	References
1.6	Stainless steel	Spheres	Torsion balance	57
21	Tungsten	Spheres	Torsion balance	50
33	Stainless steel	Spheres	Torsion balance	53
45	Cu 0.7% Te	Cylinders	Torsion balance	72
118	Copper	Rings	Torsion balance	59
480	Tungsten	Cylinders	Double pendulum	103
516	Tungsten	Cylinders	Atom gravimeter	132
521	Tungsten alloy	Cylinders	Free-fall gravimeter	118
1 152	Brass	Cylinders	Double pendulum	102
5 775	Lead	Sphere	Beam balance	106
13 520	Mercury	Cylinder tank	Beam balance	114
100 000	Lead	Rectangular block	Beam balance	109

C. Rothleitner and SS, Invited Review Article: Measurements of the Newtonian constant of gravitation, G, *Review of Scientific Instruments* **88**, 111101 (2017).

Three qualities of the source mass

In general, three decisions need to be made when it comes to the source mass:

- The material
  - Density
  - Homogeneity
  - Magnetic properties
- The size
- The shape

Density is important

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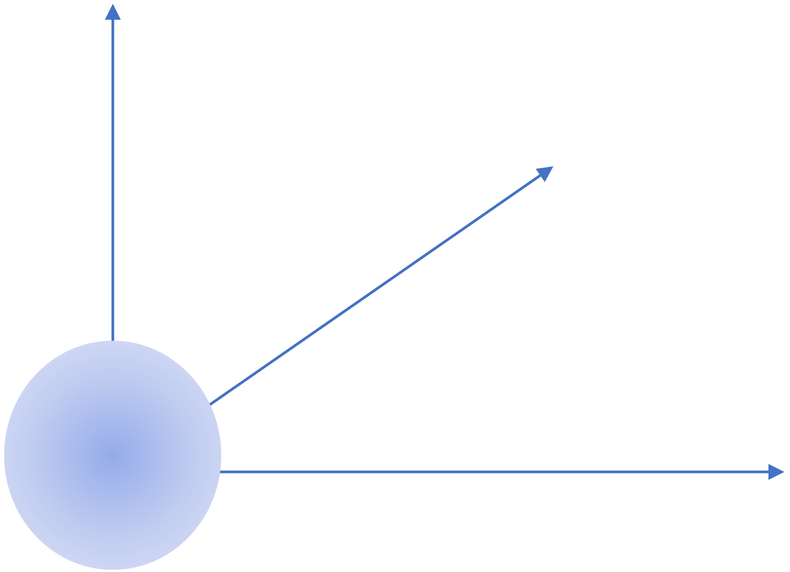


For a spherical mass, the field is

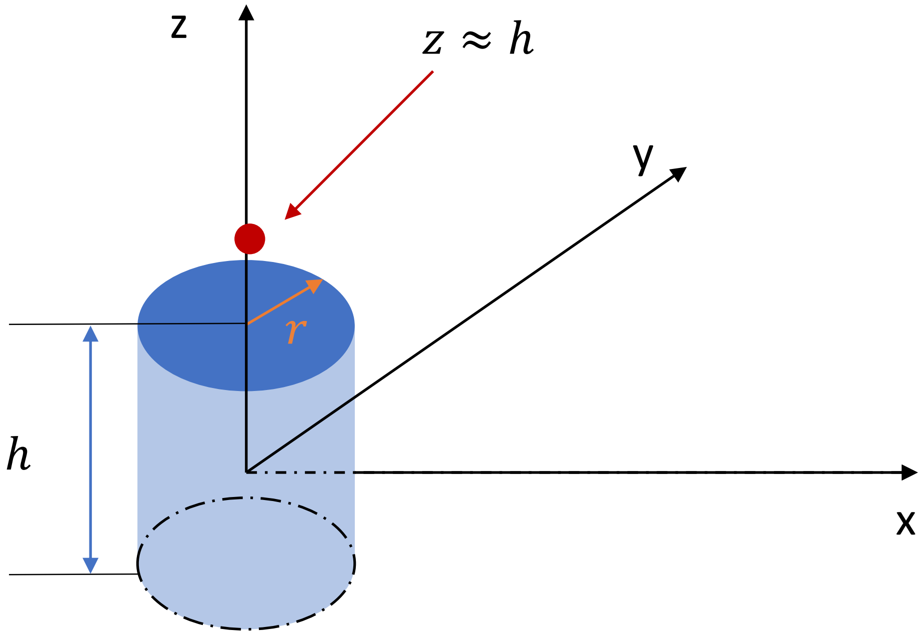
$$a_r = -G\frac{M}{r^2}$$

So why does the density matter?
Because you can get closer. Suppose, $r_{\max}$ is the largest distance from the CM that one is willing to go. Then the most signal one can generate is

$$a_r = -G\frac{\frac{\rho 4}{3}\, r_{max}^3}{r_{max}^2} = -\frac{4}{3}\rho G\, \mathrm{r}_{\max}$$

Note the dimensionality here:
Acceleration = G times density times radius

For a cylinder



$$a_z(0,0,z) = -2\pi\,\rho\,G\,\left(h + \sqrt{r^2 + (h/2 - z)^2} - \sqrt{r^2 + (h/2 + z)^2}\right)$$

From a previous experiment:

$\rho_a = 13\,600\ \mathrm{kg/m^3}$ mercury
$h_a = 64\ \mathrm{cm}$
$r_a = 49.5\ \mathrm{cm}$

Suppose we would change the material from mercury to water and keep
The same height.

$\rho_b = \rho_a/13.6$ water
$h_b = h_a$
$r_b \approx 13.6\,r_a$

Look how much mass we need to generate the same gravitational acceleration:

$$m = \rho\,r^2\pi\,h$$

Hence,
$m_b \approx 13.6\,m_a$ For mercury: 6.8 tons for water: 92 tons

We agreed on high density, but what else?

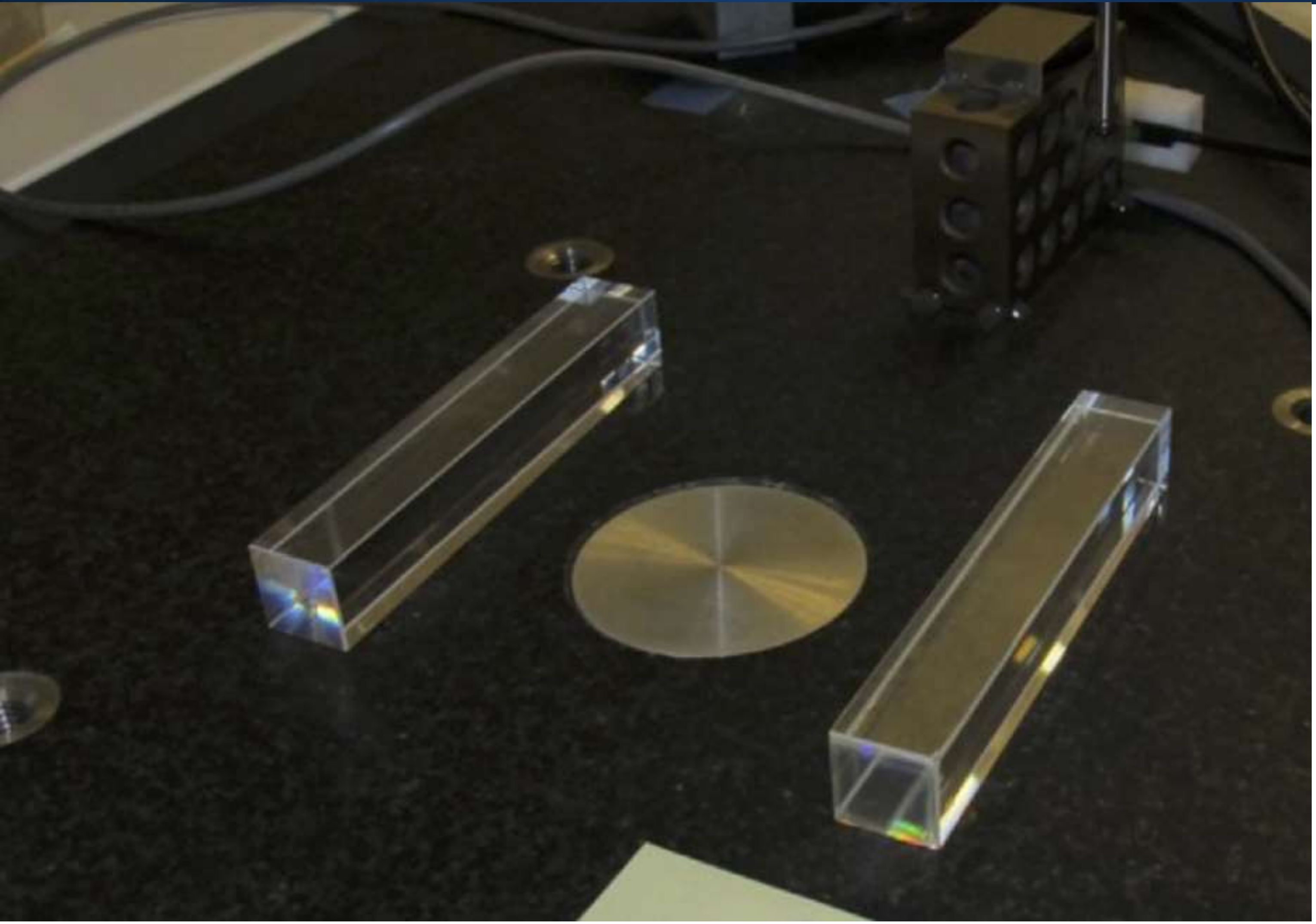
Material	Density kg/m^3
Osmium	22 570
Iridium	22 420
Platinum	21 450
Rhenium	21 020
Plutonium	*19 840*
Gold	19 320
Tungsten	19 300
Uranium	*19 100*
Tantalum	16 600
Mercury	*13 564*

Material	Density kg/m^3
Rhodium	12 410
Thorium	11 700
Lead	11 340
Silver	10 500
Molybdenum	10 220
Copper	8 940
Cobalt	8 900
Nickel	8 900
Cadmium	8 650
Brass	8 600

In red font: refractory metals => very high melting point, above 1850°C. The materials are not made via melting, but sintering. Hence, the density homogeneity is not that great.

In blue and italics: Not practical for other reasons (radioactive, poisonous).

$PbWO_4$ Lead Tungstate – a promising material



K T A Assumin-Gyimah *et al.* Neutron phase contrast imaging of $PbWO_4$ crystals for G experiment test masses using a Talbot-Lau neutron interferometer 2022 *Class. Quantum Grav.* **39** 245014

Crystals are made for scintillators in particle detectors. They are optically clear and have a density of 8 300 kg/m^3

Lead Tungstate

Table 1. List of non-hygroscopic, optically-transparent, high-density crystals and their physical properties.

Material	$PbWO_4$	$CdWO_4$	LSO	LYSO	BGO
Density ($g\,cm^{-3}$)	8.3	7.9	7.4	7.3	7.13
Atomic numbers	82, 74, 8	48, 74, 8	71, 32, 8	71, 39, 32, 8	83, 32, 8
Refractive index (light)	2.2	2.2–2.3	1.82	1.82	2.15
Thermal expansion	8.3 (para)	10.2	5	5	7
Coefficient(s) ($10^{-6}\,°C^{-1}$)	19.7 (perp)	—	—	—	—

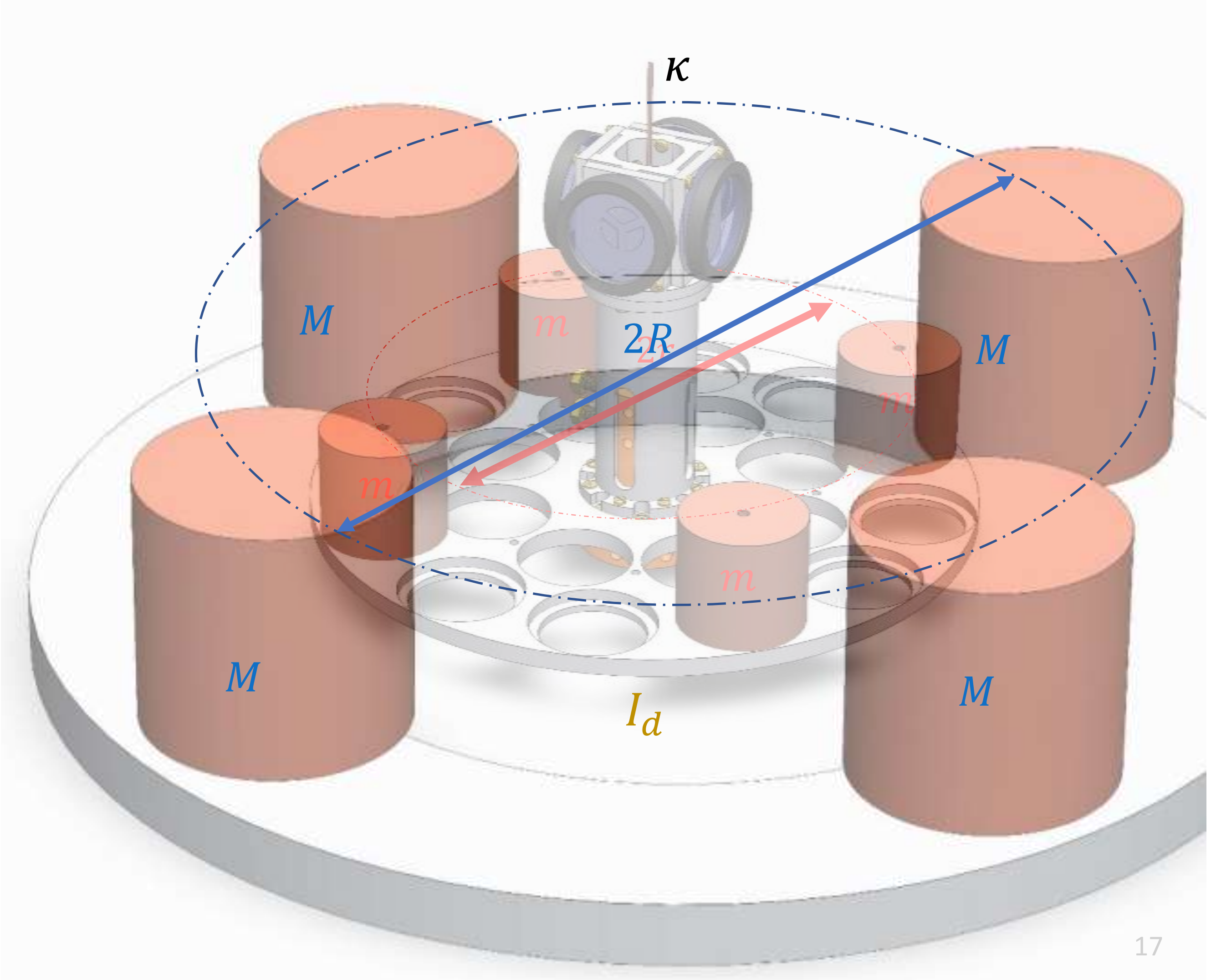
Result: $\frac{1}{N}\frac{dN}{dx} < 0.5 \times 10^{-6}\text{cm}^{-1}$

Fractional atomic density gradient.

K T A Assumin-Gyimah *et al.* Neutron phase contrast imaging of $PbWO_4$ crystals for G experiment test masses using a Talbot-Lau neutron interferometer 2022 *Class. Quantum Grav.* **39** 245014

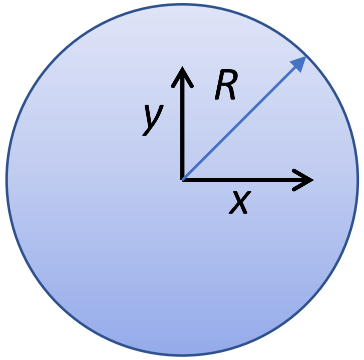
An example experiment

m	1.2	kg
r	120	mm
κ	204	μN m
I_d	745	g dm^2
$I_d/(4m\,r^2)$	0.11	
T_o	121	s



Other ways to reduce the effect of gradients

Measure it in an air bearing:



$$\Delta y = -\frac{R\Delta\rho}{4\,\rho_0}$$

gradient points down to make it stable in air bearing

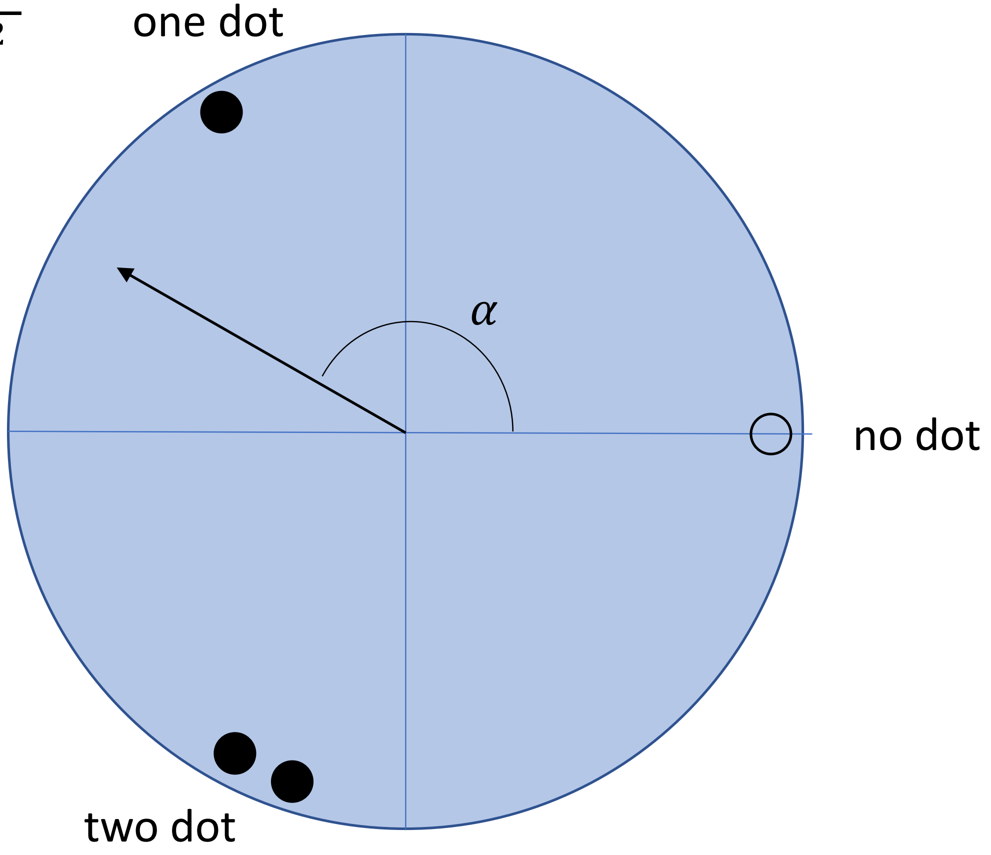
$$I\,\ddot{\theta} - Mg\Delta y\theta = 0 \qquad I = \frac{1}{2}MR^2 \quad \text{does not depend on } \delta\rho$$

$$\omega^2 = -\frac{Mg\Delta y}{I} = \frac{4\pi^2}{T^2}$$

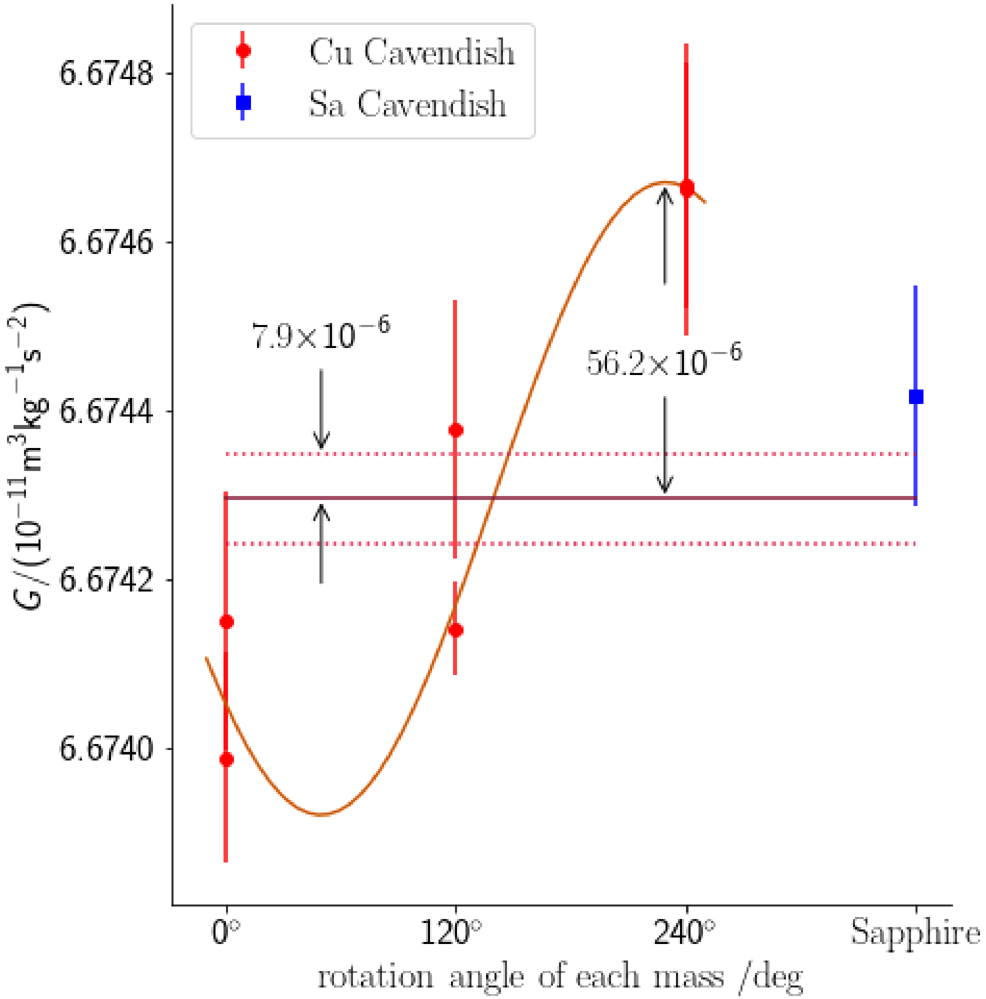
$$\frac{\Delta\rho}{\rho_0} = \frac{8\pi^2 R}{T^2 g}$$

#	T (s)	$\frac{\Delta\rho}{\rho_0} \times 10^6$	α
1	68.0	102.7	159
2	58.8	137.3	-15
3	54.5	159.9	159
4	44.7	237.6	140

angle between the line that connects the geometric center to the no dot mark and the direction of maximum density.



Average over different orientations



Here, the source masses were rotated. The measurements
Were done in three orientations.
The mean value is independent of the gradient.

Magnetic properties

The magnetic force on the test mass in the z direction is given by

$$F = -\frac{\mu_o}{2}\frac{\partial}{\partial z}\int \chi\,\vec{H}\cdot\vec{H}\,\mathrm{d}V - \mu_o\frac{\partial}{\partial z}\int \vec{M}\cdot\vec{H}\,\mathrm{d}V$$

So for the test mass it's important to have a small magnetic susceptibility χ. In the above equation the volume magnetic susceptibility is used, sometimes abbreviated as χ_V

$\chi > 0$ the material is paramagnetic => the force is towards larger $\vec{H}\cdot\vec{H}$.

$\chi < 0$ the material is diamagnetic => the force is towards smaller $\vec{H}\cdot\vec{H}$.

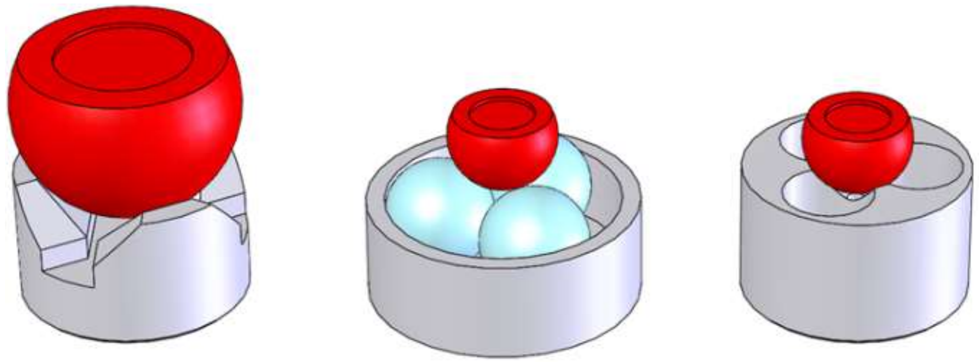
Material	χ_V
Aluminum	2.2×10^{-5}
Silver	-2.3×10^{-5}
Copper	-9.6×10^{-6}
Nickel	600
Iron	200 000

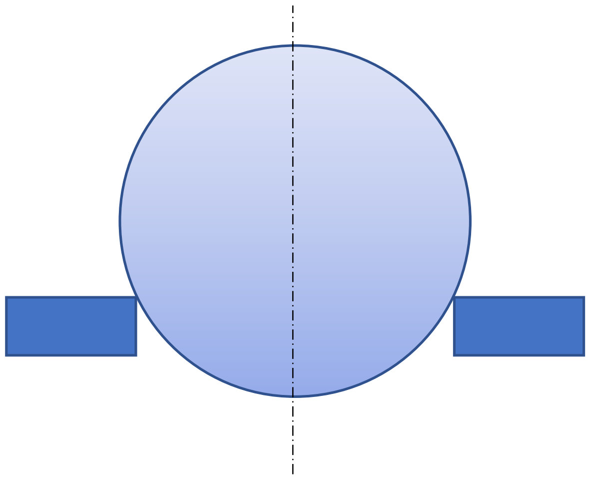
Source mass material properties

- Large density
- No permanent magnetic moment
- Low χ is desired but not required – as long as the ambient $\vec{H}$ field is small.
- Low thermal expansion coefficient => for position and size measurements.
- Low vapor pressure if inside vacuum.
- Conductive or possibly gold coated, absent any electrostatic shield
- Machineability and measurability (ground and polished, diamond turned)

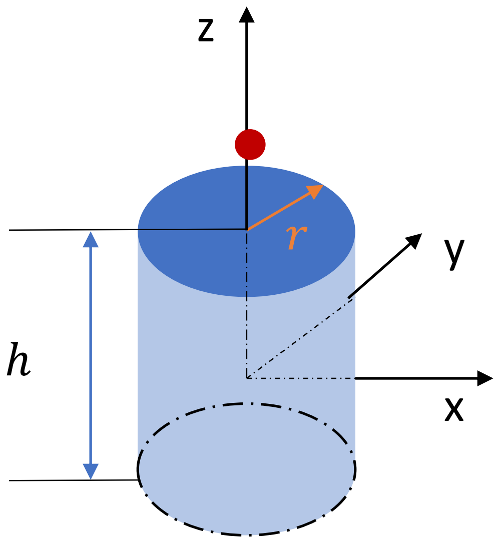
Shapes - sphere

- A single dimension radius R. No fillets.
- Easy to calculate $a_r = G\ M\ /r^2$ for $r > R$
- Easy to manufacture by random orbit grinding. See for example ball bearings.
- Supporting the sphere can be hard.





Shapes - cylinder



- Two dimensions, height h and radius r
- Fillets on two circles.
- Easy to machine (turned on a lathe).
- Stands on a surface.
- Field along the axis is easy to calculate, see below

$$a_z(0,0,z) = -2\pi\,\rho\,G\,\left(h + \sqrt{r^2 + (h/2 - z)^2} - \sqrt{r^2 + (h/2 + z)^2}\right)$$

The radial field

A.H. Cook and Y.T. Chen, On the significance of the radial Newtonian gravitational force of the finite cylinder, *J. Phys. A: Math. Gen.* **15**,1591 (1982).

Without loss in generality, they calculate the horizontal force on a point in the plane of the bottom of the cylinder.

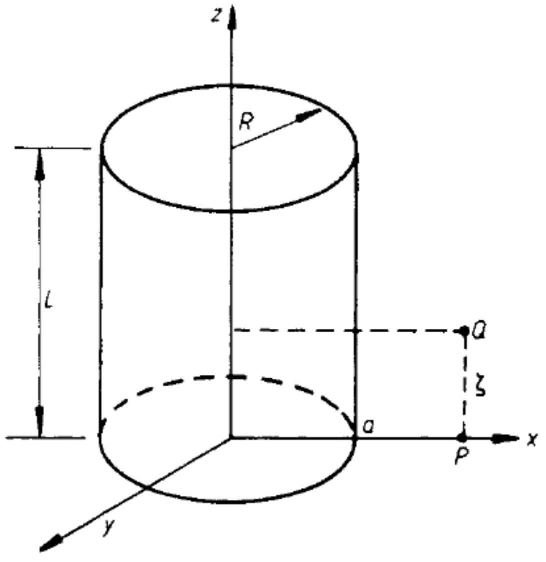


Figure 1.

Chen and Cook find
For the optimal shape:
$\frac{2R}{L}$=1.029282

Better gravitational efficiency than for a sphere

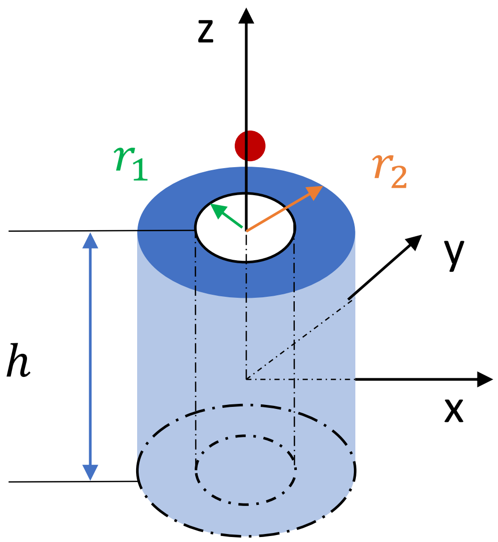
$$
\begin{aligned}
\frac{F_a^{(1)}}{2G\rho} &\equiv C_y(L, R, a) \\
&= \frac{L^2}{a\sqrt{1+[(R+a)/L]^2}+\sqrt{1+[(R-a)/L]^2}} \\
&\times\left\{\left[1+2\frac{R^2+a^2}{L^2}+\sqrt{1+[(R-a)/L]^2}\times\left(\sqrt{1+[(R+a)/L]^2}+\frac{R^2+a^2}{L^2}\right)\right.\right. \\
&\left.+\left(\frac{R^2-a^2}{L^2}\right)^2\frac{1}{1-\sqrt{1+[(R-a)/L]^2}}\right]K(k) \\
&-\frac{1}{2}[1+[(R+a)/L]^2+\sqrt{1+[(R-a)/L]^2}]^2E(k) \\
&-2\frac{R^2+a^2}{L^2}\sqrt{1+[(R-a)/L]^2}\prod(\tfrac{1}{2}\pi, k, k) \\
&\left.+2\frac{(R+a)^2}{L^2}\sqrt{1+[(R-a)/L]^2}\prod(\tfrac{1}{2}\pi, \alpha^2, k)\right\}+I_0
\end{aligned}
$$

Elliptical integral of the 1st kind

Elliptical integral of the 2nd kind

Elliptical integral of the 3rd kind

The hollow cylinder

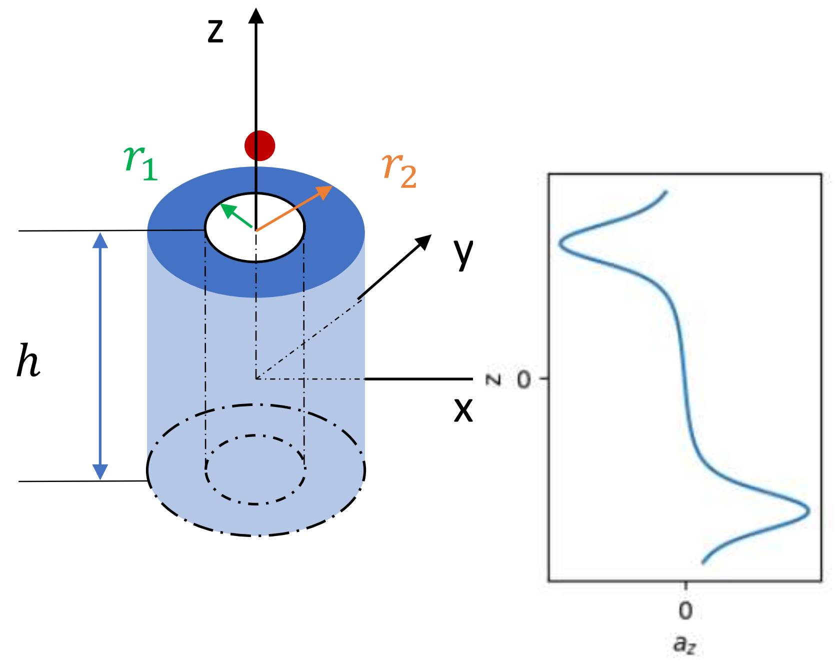


- Three dimensions, height h, inner, r_1 and outer radius r_2
- Fillets on four circles.
- Easy to machine (turned on a lathe).
- Stands on a surface.
- Fields can be calculated from two cylinders, i.e. subtracting a cylinder with density $-\rho$ from one with ρ

On axis:

$$a_z(0,0,z) = -2\pi\,\rho\,G\,(R_{2+} - R_{2-} + R_{1+} - R_{1-}) \text{ with } R_{1,2\pm} = \sqrt{r_{1,2}^2 + \left(z \pm \frac{h}{2}\right)^2}$$

The hollow cylinder

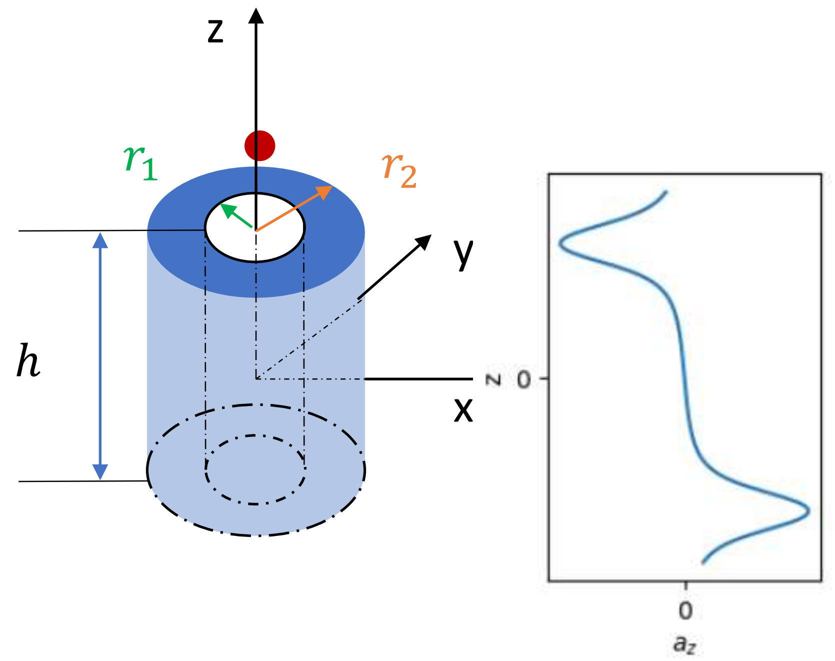


Maximum near the end of the cylinder.

On axis:

$$a_z(0,0,z) = -2\pi\, \rho\, G\, (R_{2+} - R_{2-} + R_{1+} - R_{1-}) \text{ with } R_{1,2\pm} = \sqrt{r_{1,2}^2 + \left(z \pm \frac{h}{2}\right)^2}$$

The hollow cylinder



gravitational potential $\Phi(r, \phi, z)$ with $a_z = \frac{\partial \Phi}{\partial z}$

We note: $\nabla^2 \Phi(r, \phi, z)$ =0
In a cylindrical coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \phi^2} + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

0 for cylindrical symmetry

Near the axis of symmetry, the potential can be developed

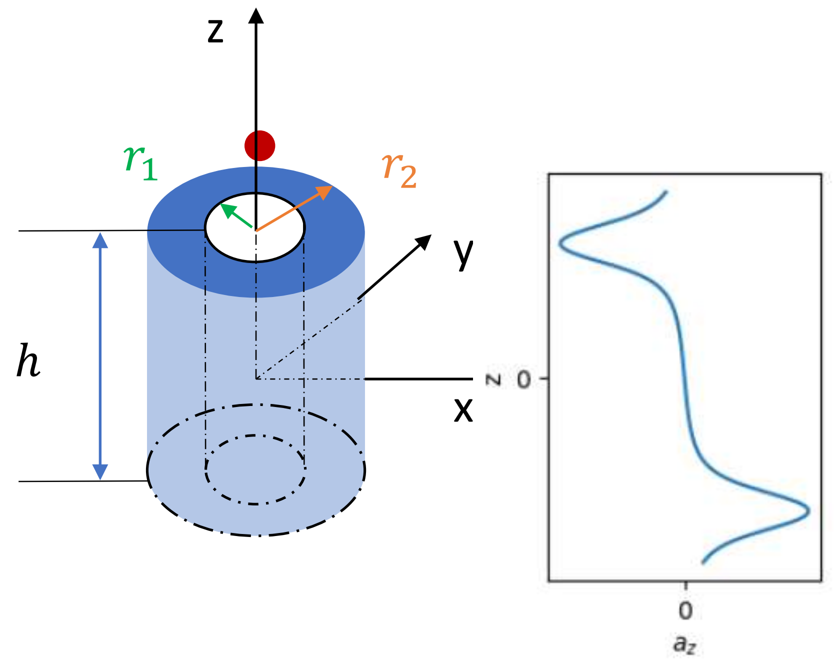
$$\Phi(r, \phi, z) = \mathrm{b_o} + \mathrm{b_1 r} + \mathrm{b_2 r^2} + \mathrm{b_3 r^3} + .. = \sum b_i \, r^i$$

Applying the Laplacian yields

$$b_1 r^{-1} + \sum r^i \left(\frac{\partial^2 b_i}{\partial z^2} + (i+2)^2 b_{i+2}\right) = 0$$

0 for symmetry

The hollow cylinder



$$\sum r^i \left(\frac{\partial^2 b_i}{\partial z^2} + (i+2)^2 b_{i+2} \right) = 0$$

This equation has to be true independent of r

$$b_{i+2} = -\frac{1}{(i+2)^2} \frac{\partial^2 b_i}{\partial z^2}$$

Let's use the following abbreviation:

$$V = \phi(0,0,z)$$

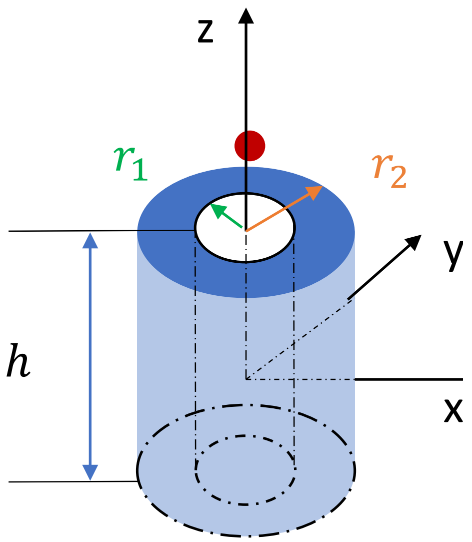
$$b_0 = V$$

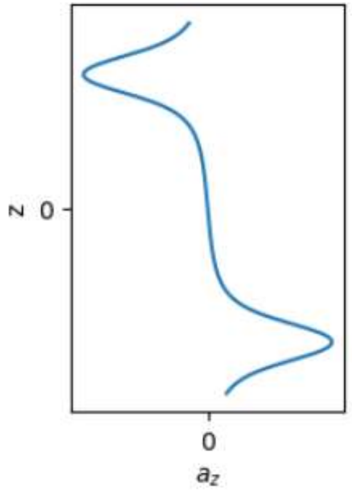
$$b_2 = -\frac{1}{4} \frac{\partial^2 V}{\partial z^2}$$

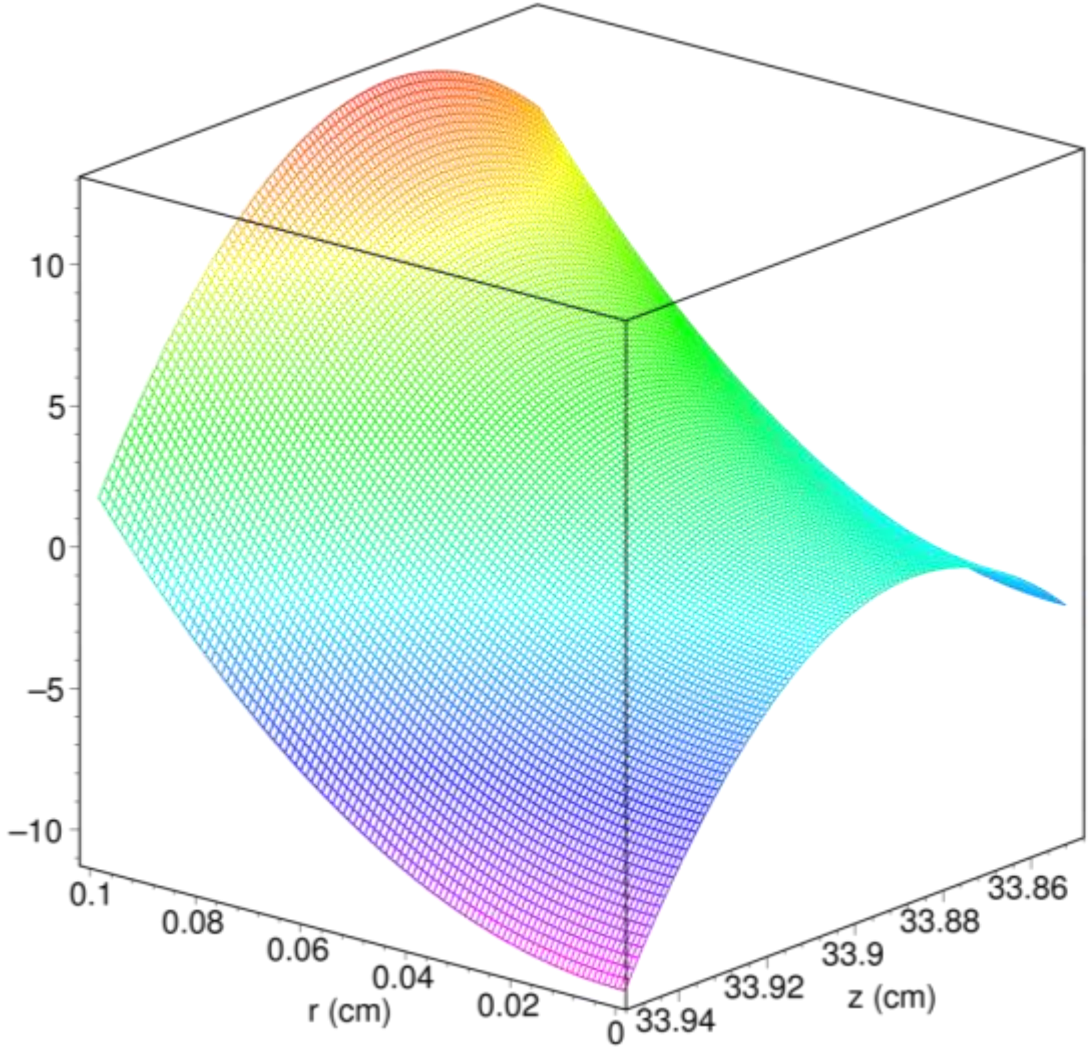
$$b_4 = \frac{1}{64} \frac{\partial^2 V}{\partial z^4}$$

....

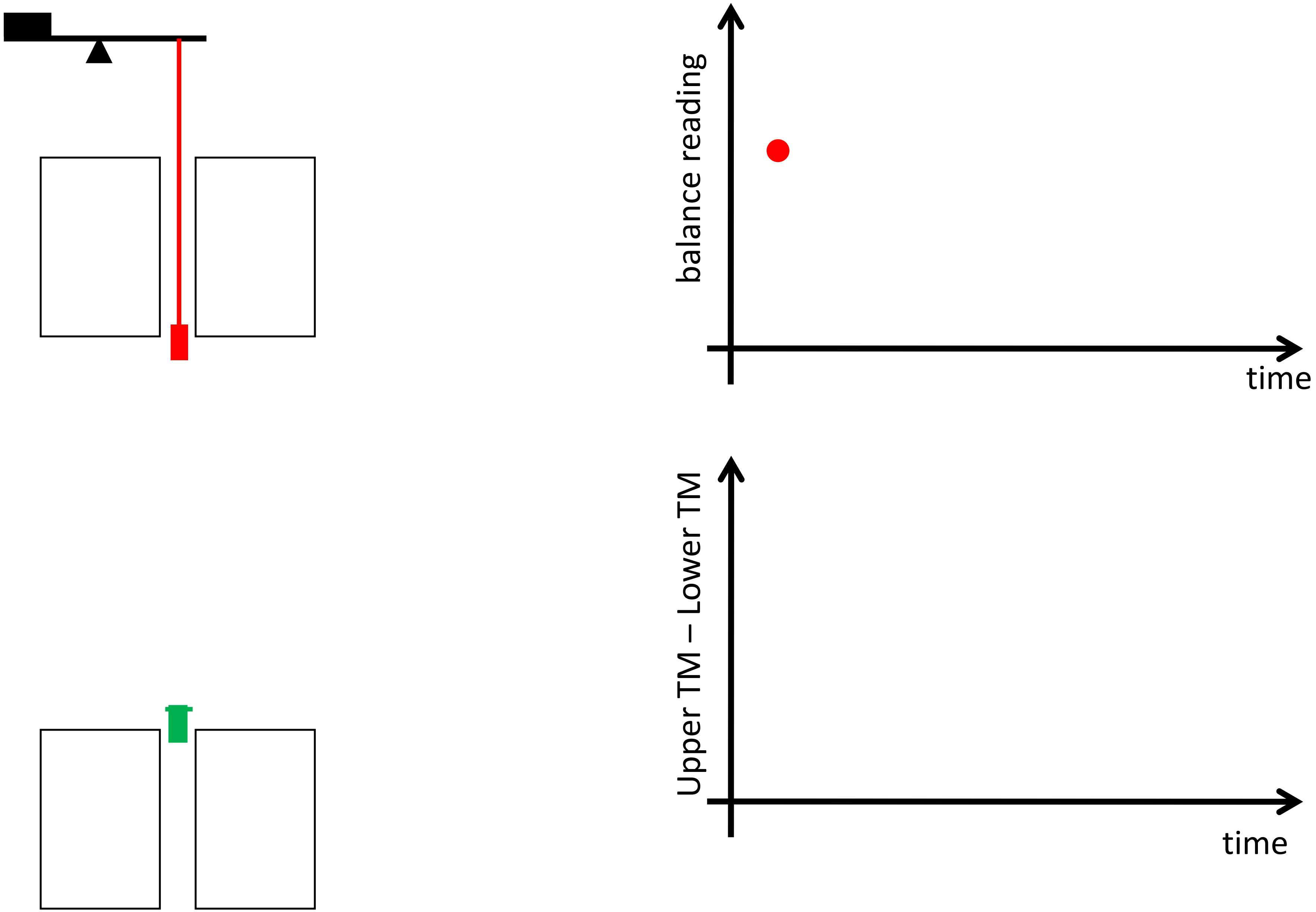
The hollow cylinder







The principle of the experiment



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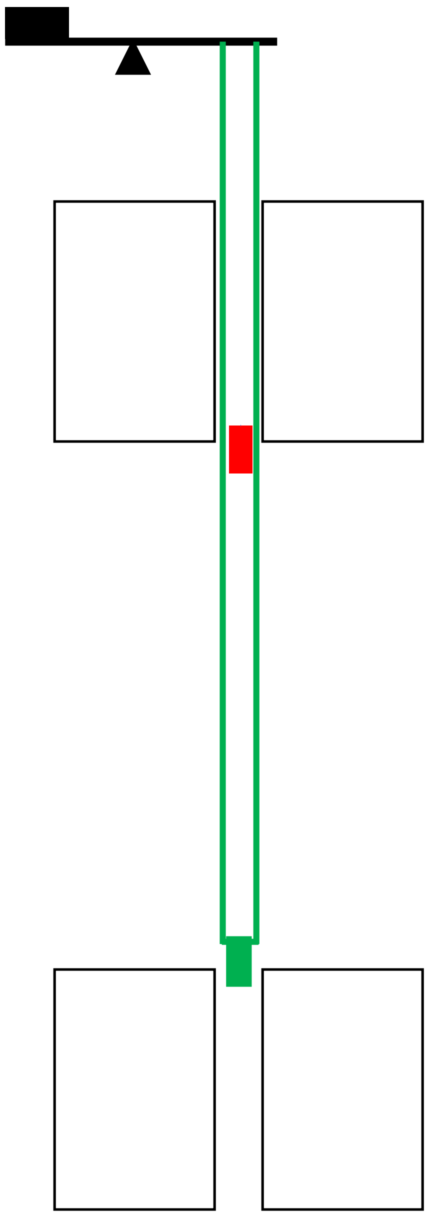

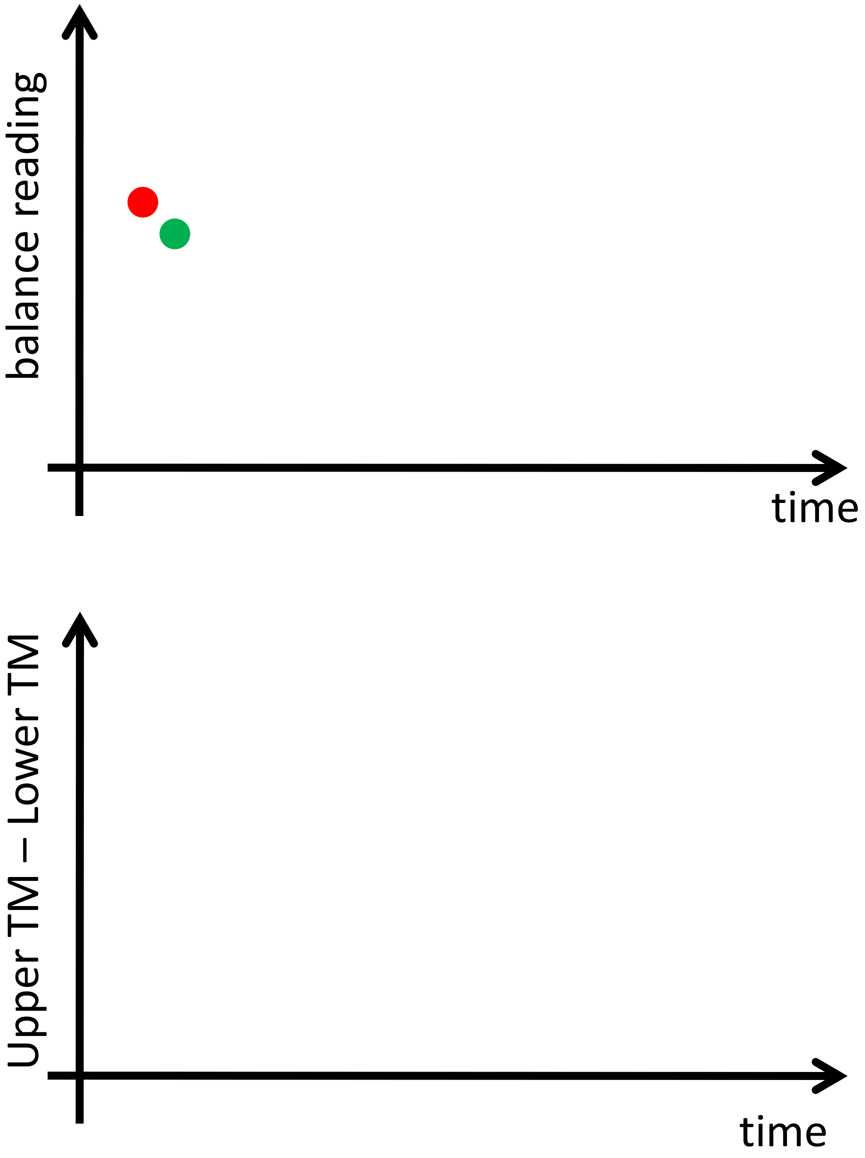

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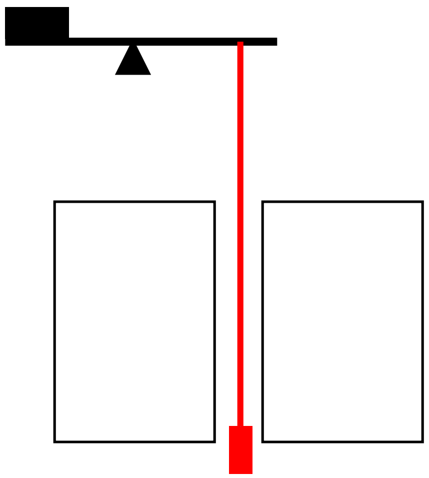
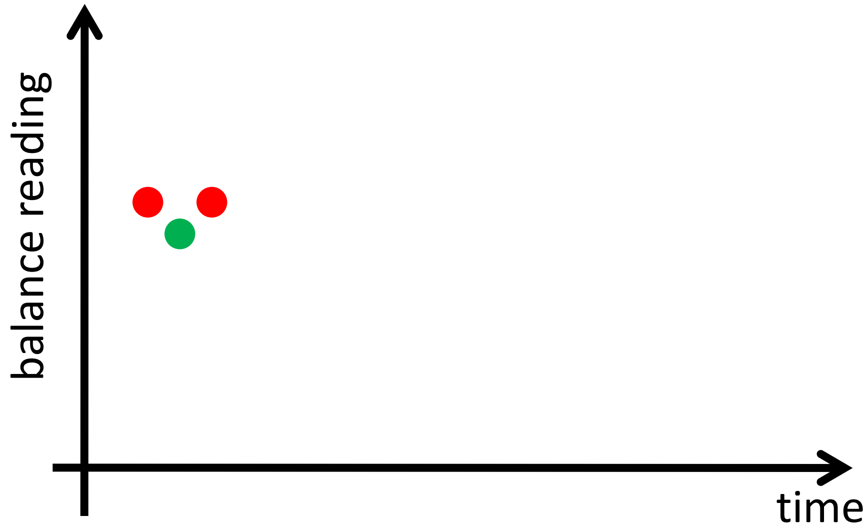
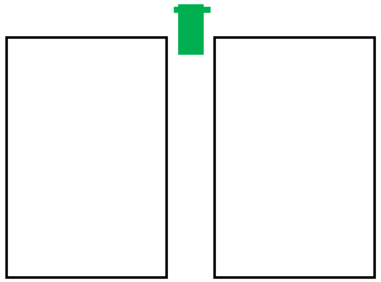
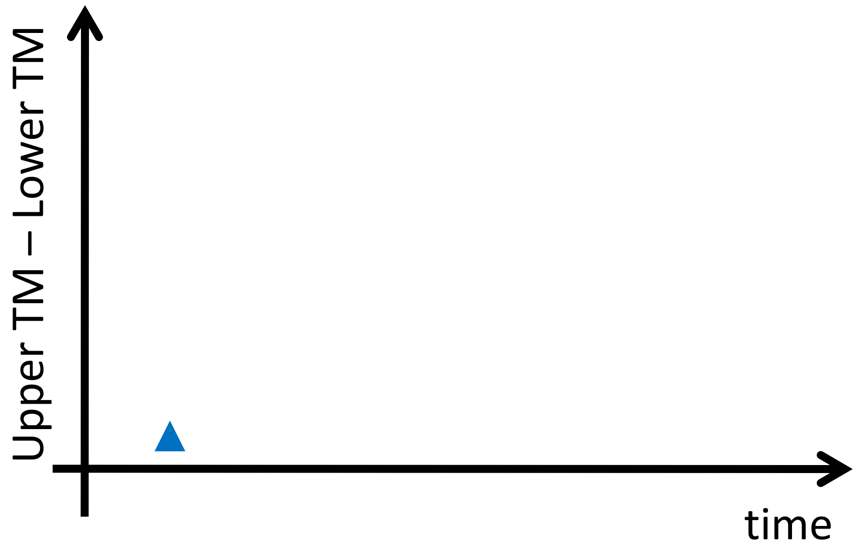
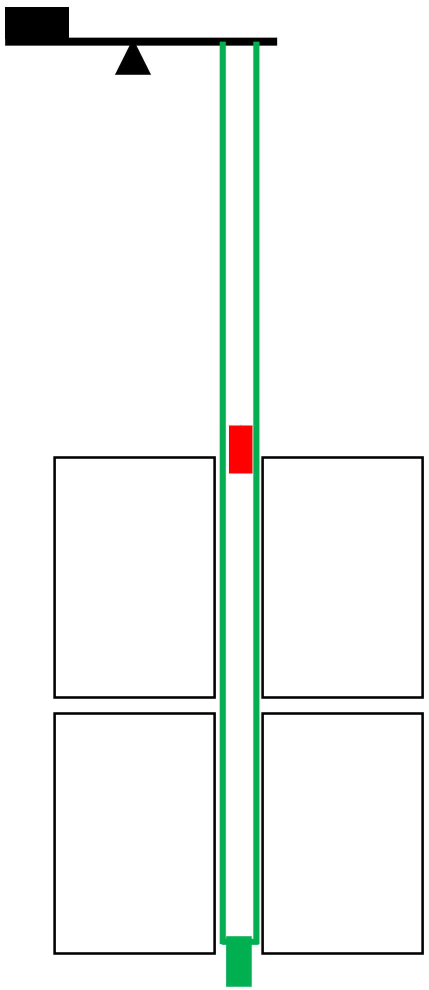


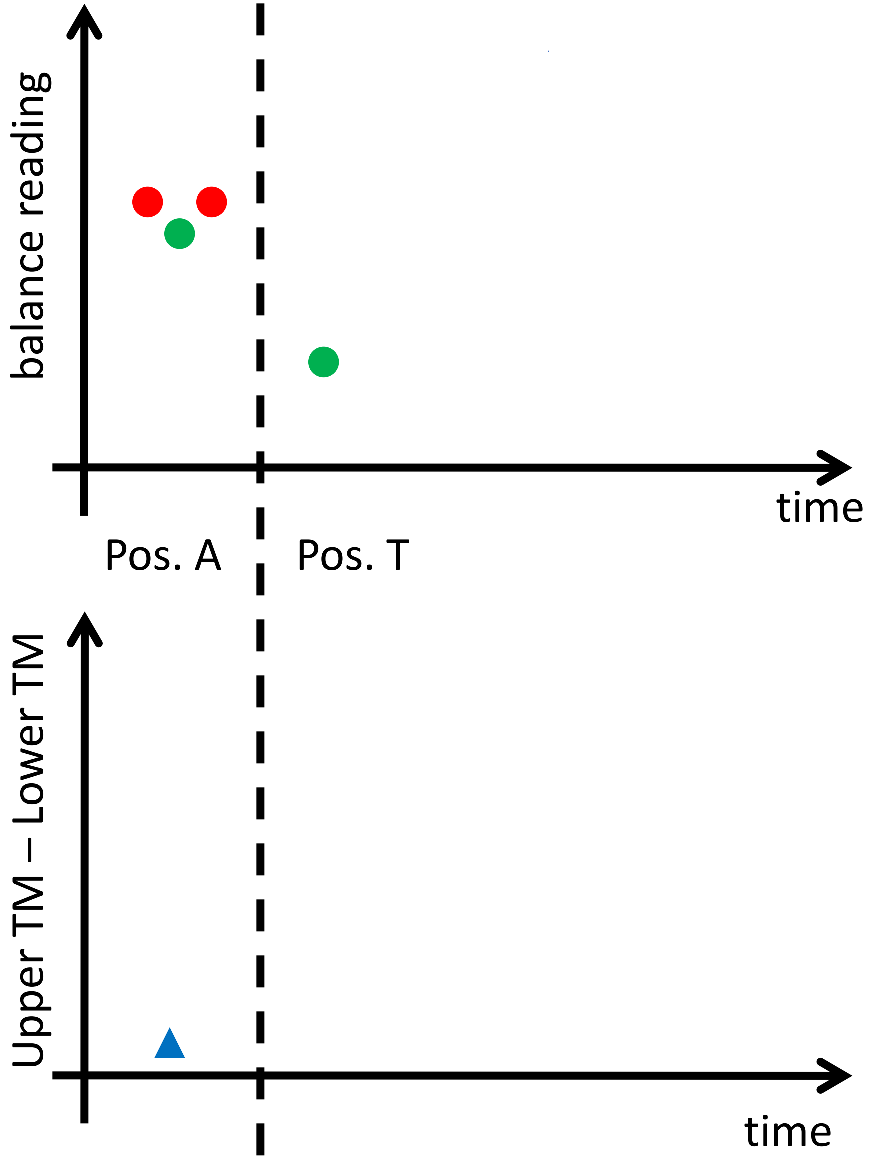




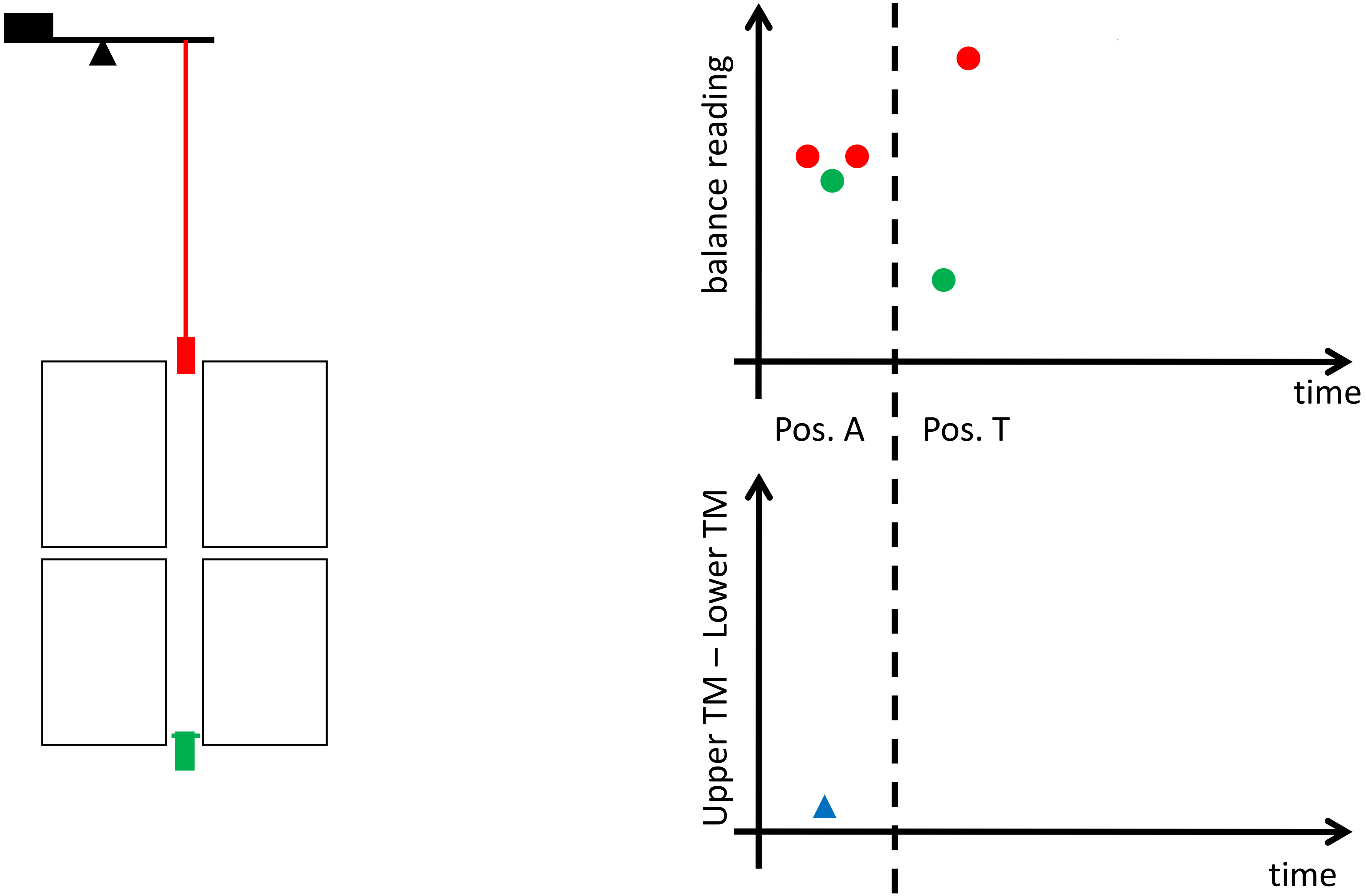


The principle of the experiment

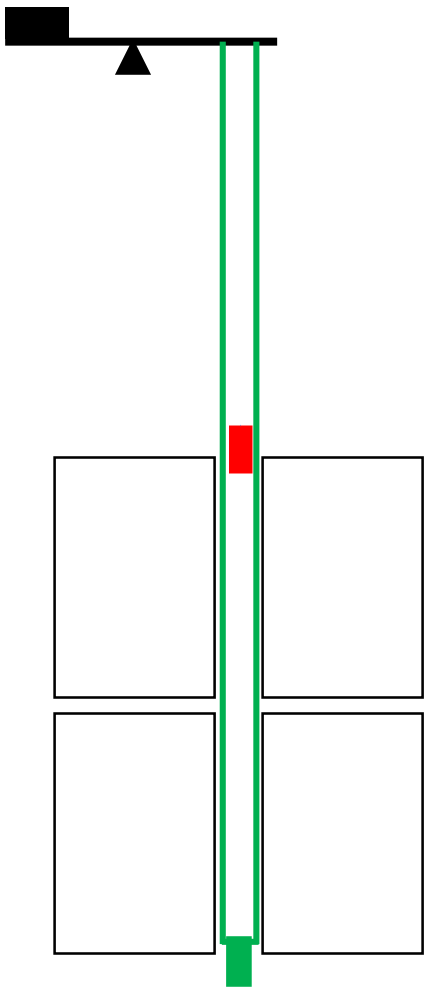


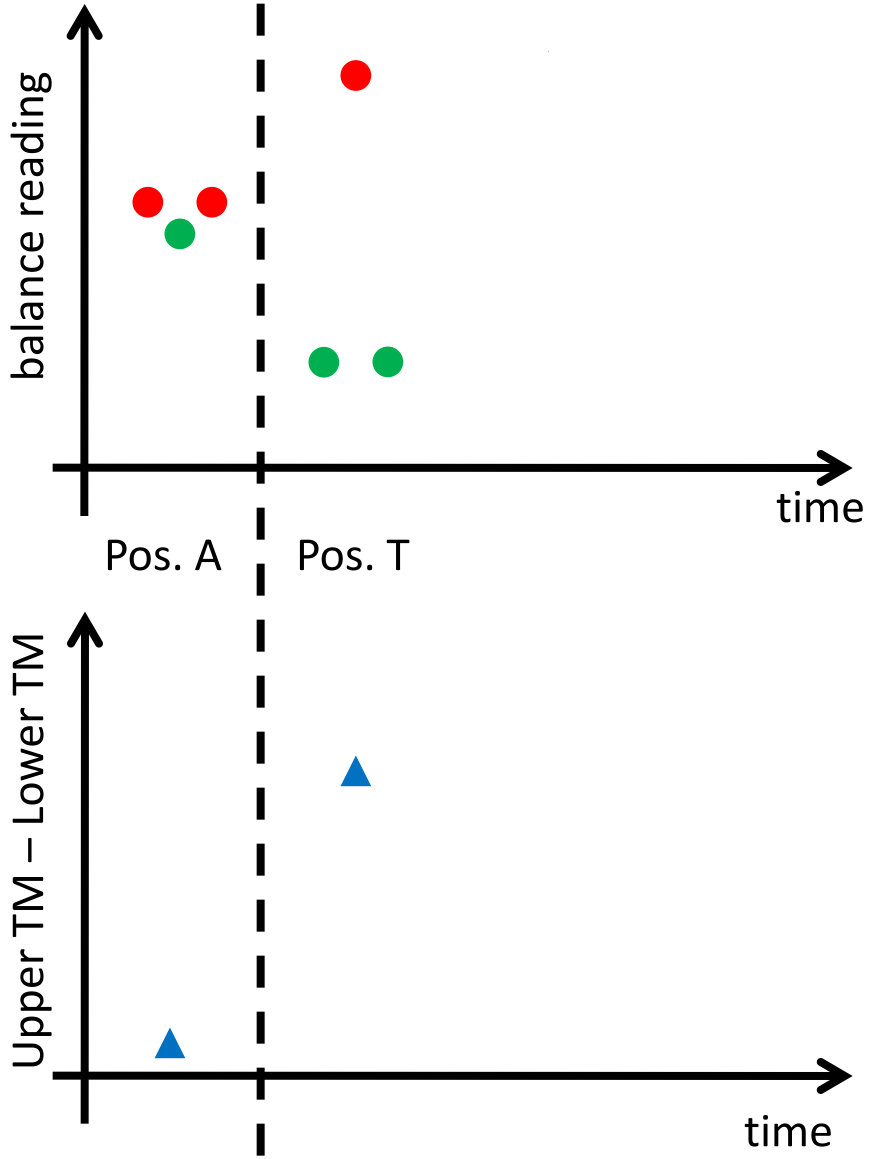


The principle of the experiment

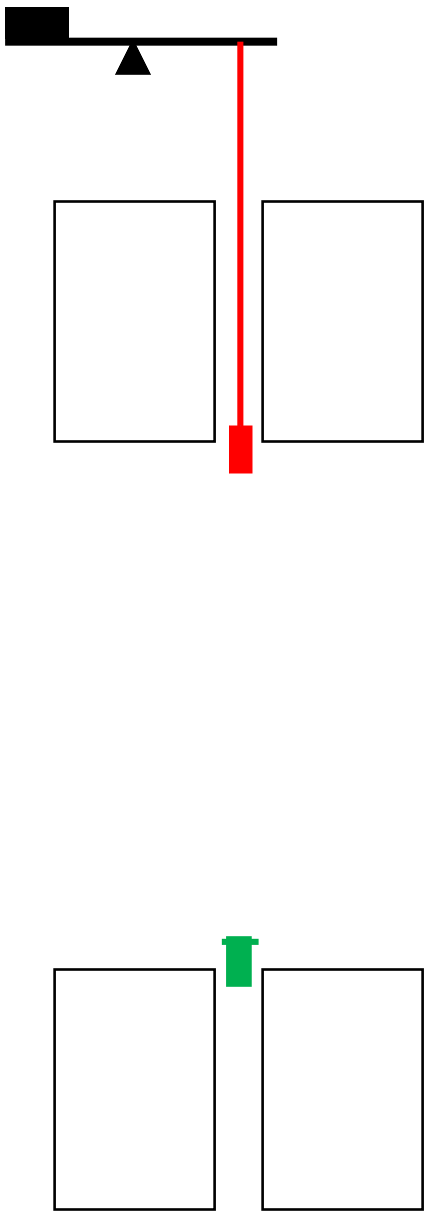


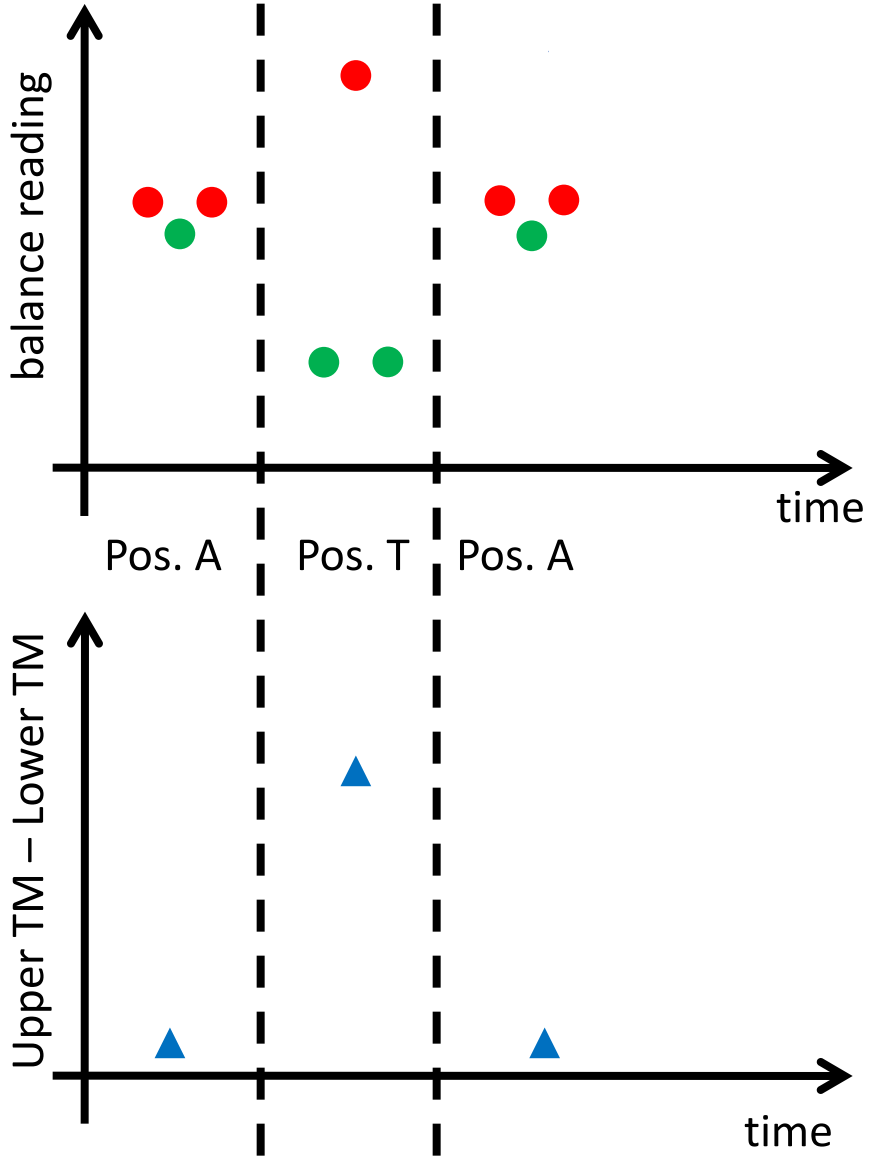
The principle of the experiment



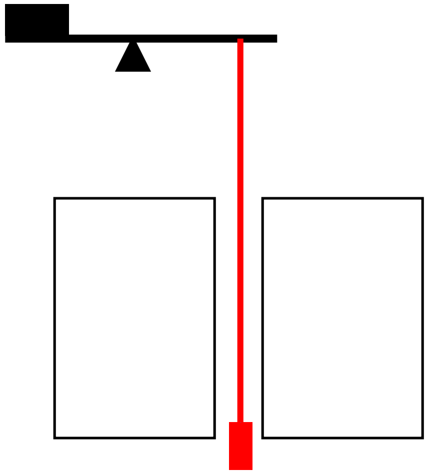


The principle of the experiment

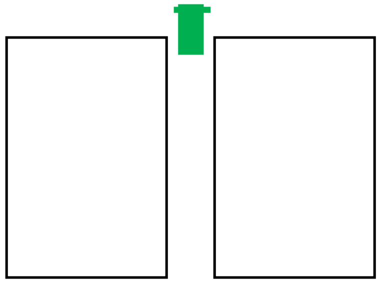


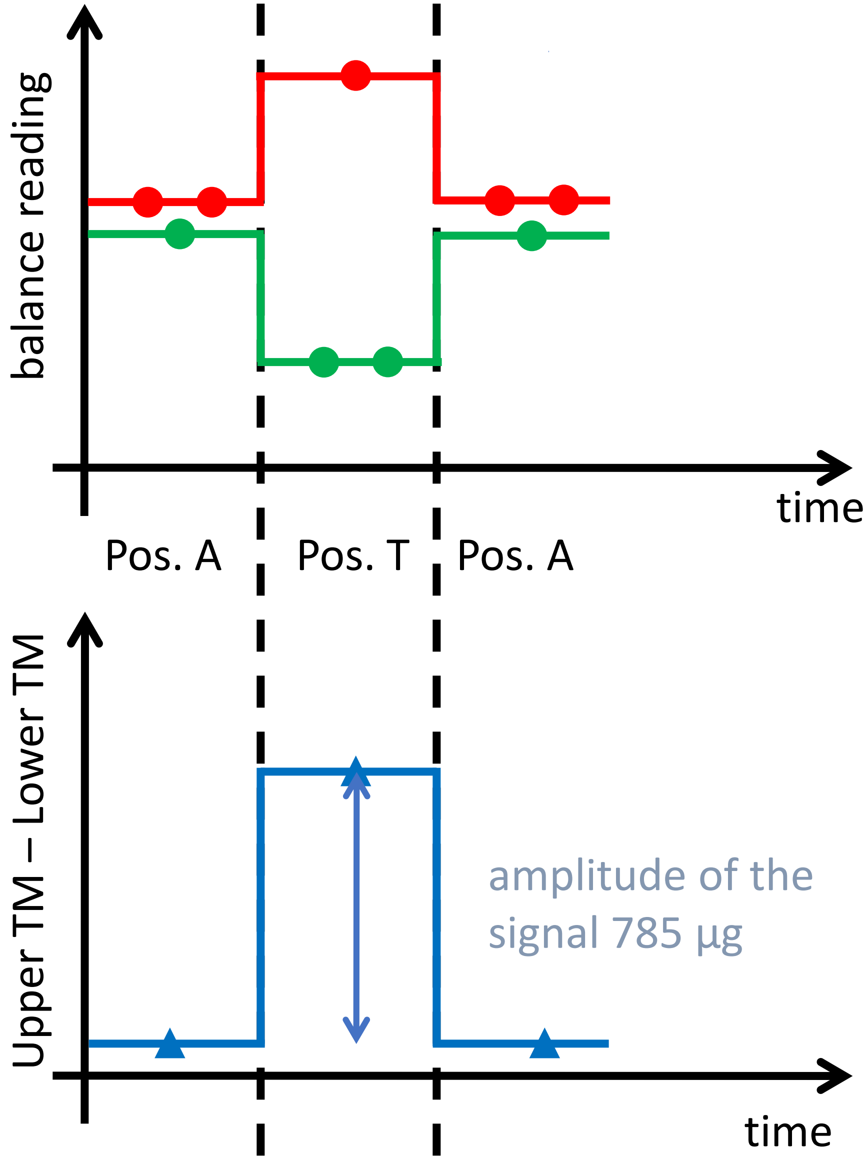


The principle of the experiment

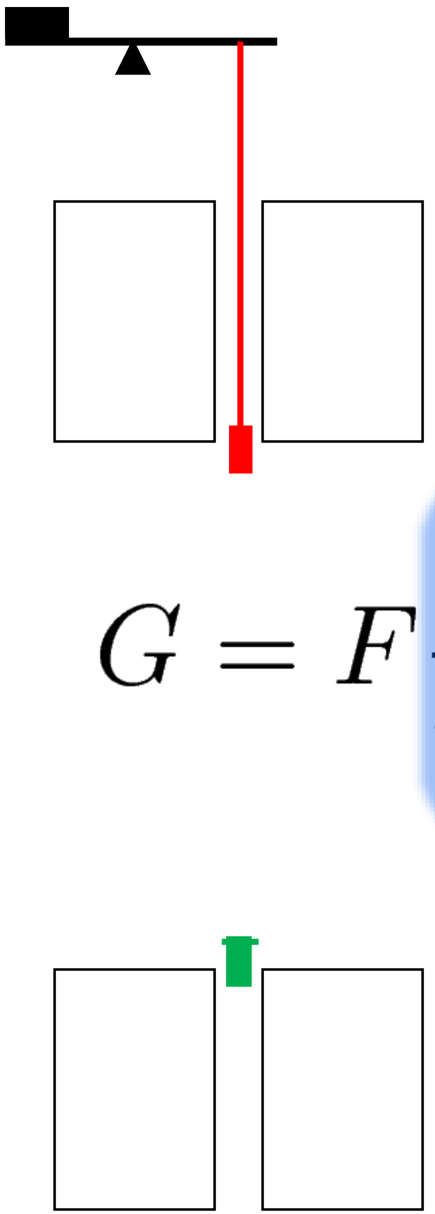


$$G = F\frac{r^2}{m_1 m_2}$$



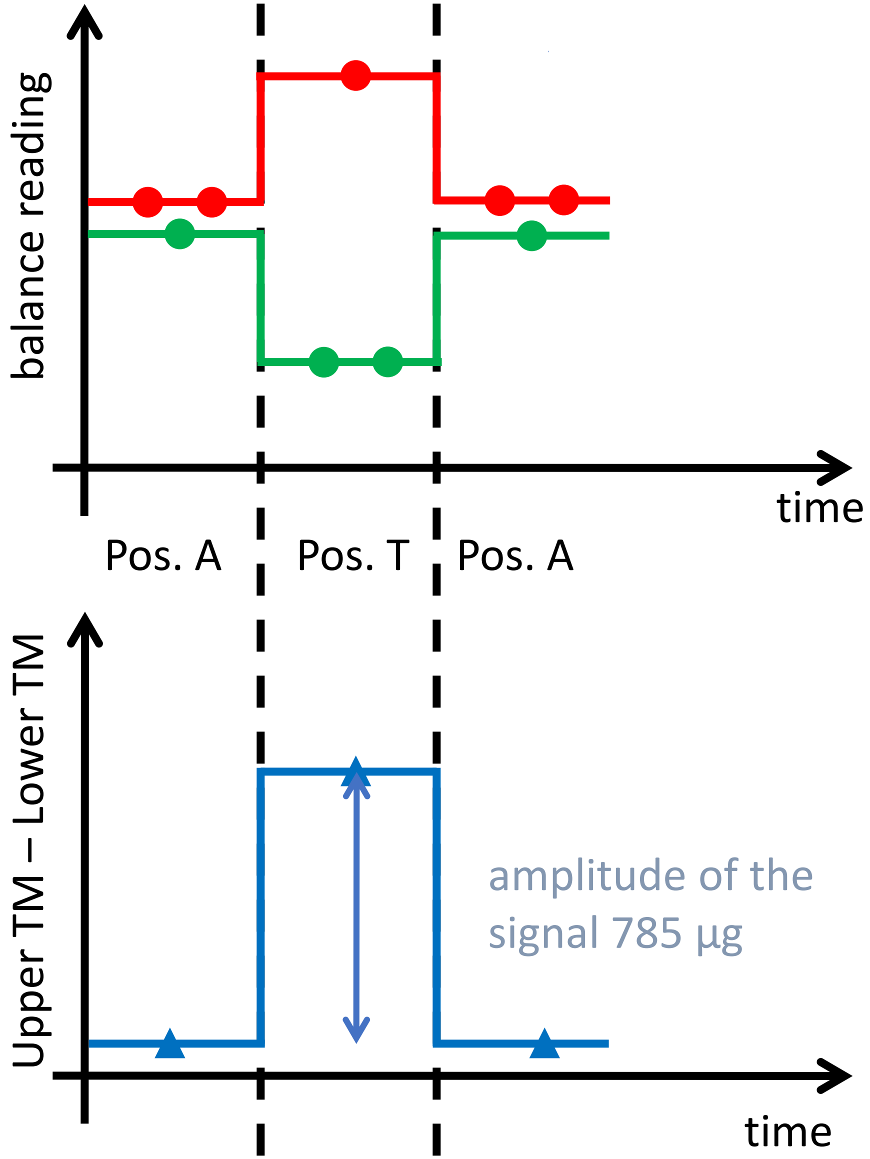


The principle of the experiment

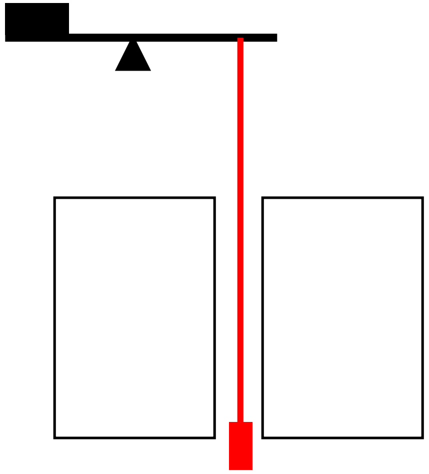


Mass integration

$$G = F\frac{r^2}{m_1 m_2}$$

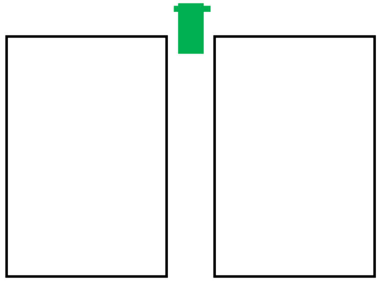


The principle of the experiment

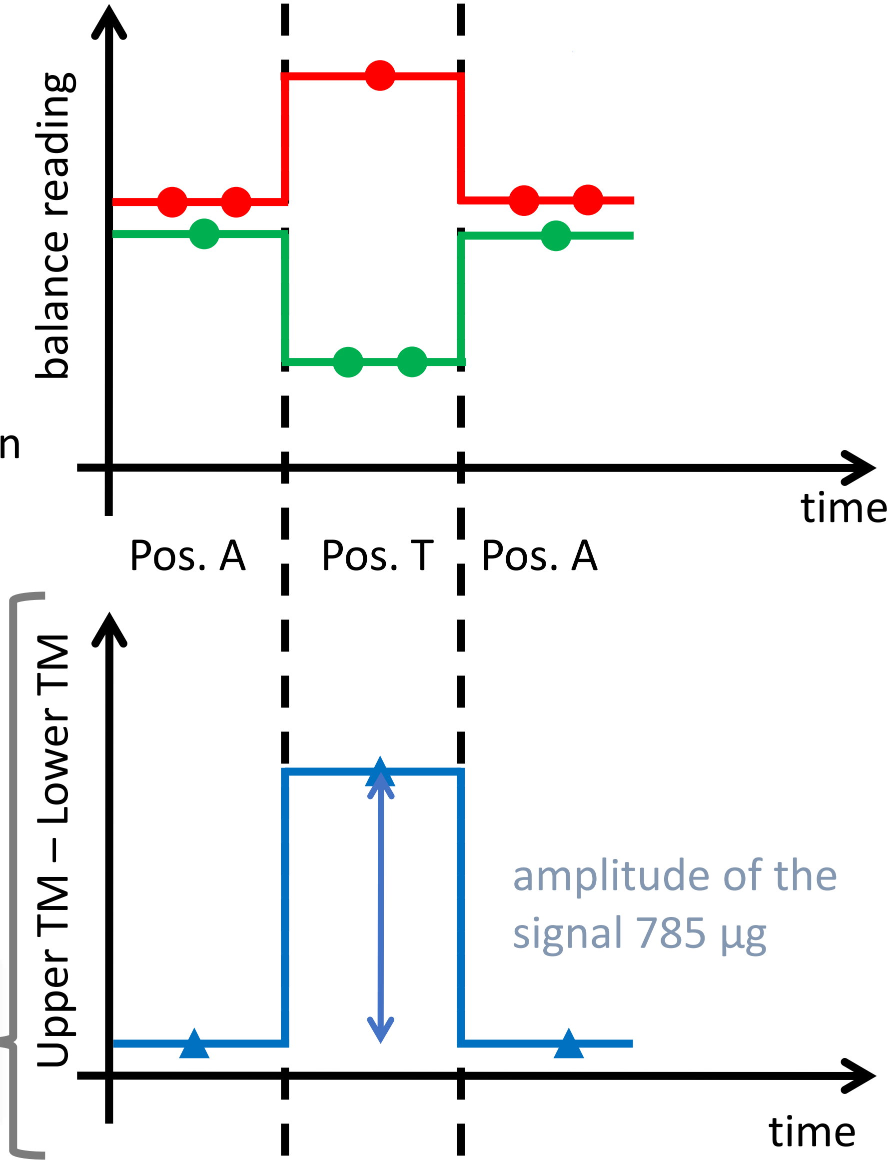


Mass integration

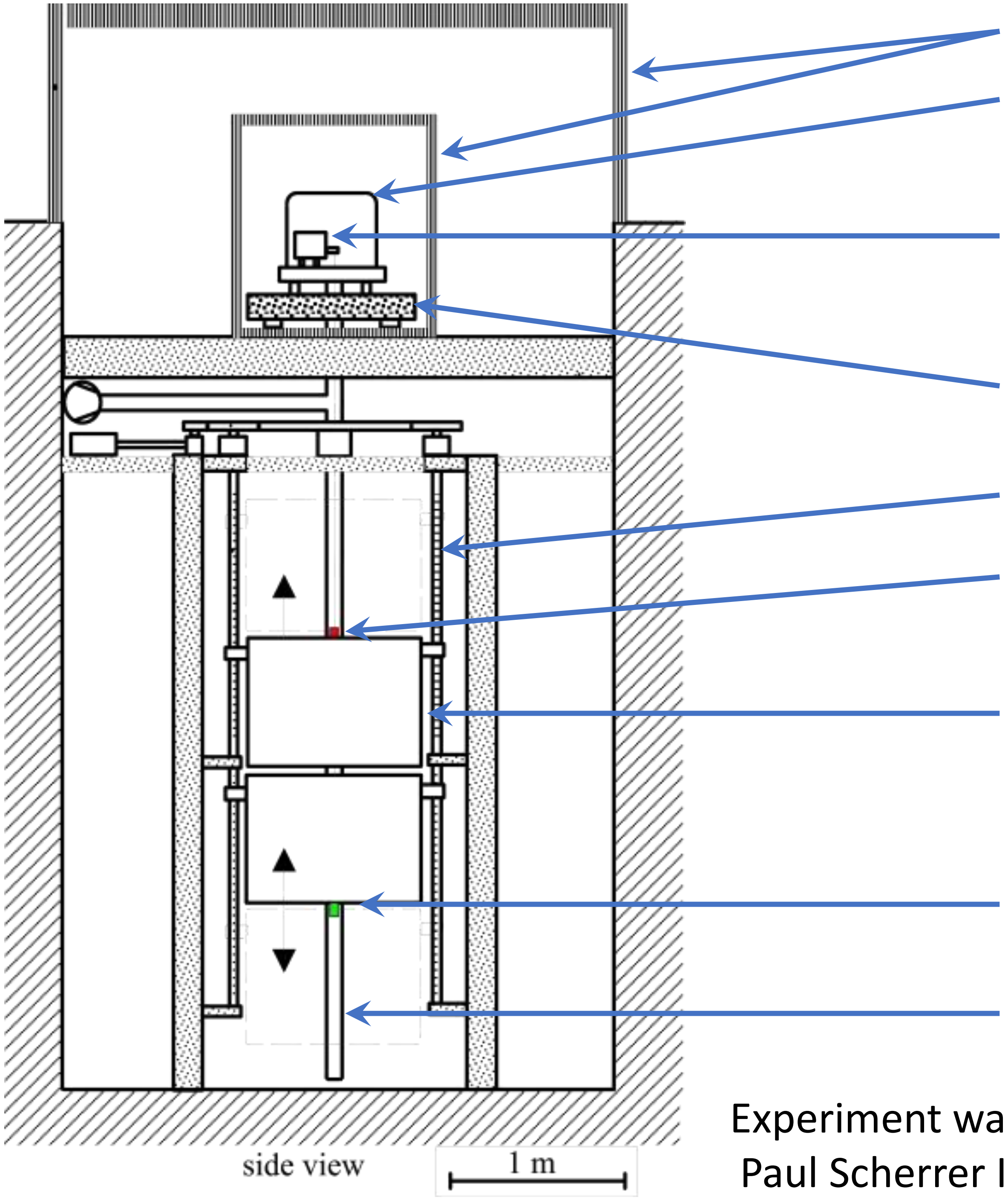
$$G = F\frac{r^2}{m_1 m_2}$$



$$F = g\,\Delta m$$



Measured with a rel. uncertainty of 6.1×10^{-8} (E. Klingelé, ETHZ)



Layered thermal shielding

Vacuum bell jar

10^{-6} mbar = 10^{-4} Pa

Mass Comparator

Mettler Toledo AT1006

Granite base plate

Spindle to move field mass

Upper test mass

1.1 kg copper cylinder

Field mass

800 kg stainless vessel

6760 kg mercury

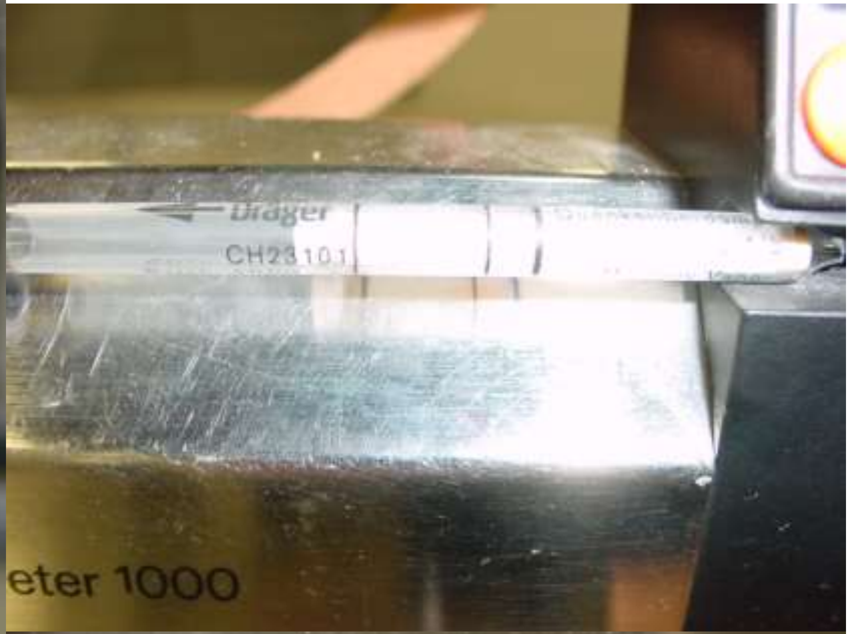
Lower test mass

1.1 kg copper cylinder

Vacuum tube

Experiment was located at the
Paul Scherrer Institute (PSI)


one vessel, after mostly
emptying the mercury
ground surface
orings
central tube
bottom plate
one of three brackets each holding a
recirculating ball nut.



The mercury vapor concentration was monitored with gas detection tubes.

One of 395 flasks of mercury used to fill the tank.

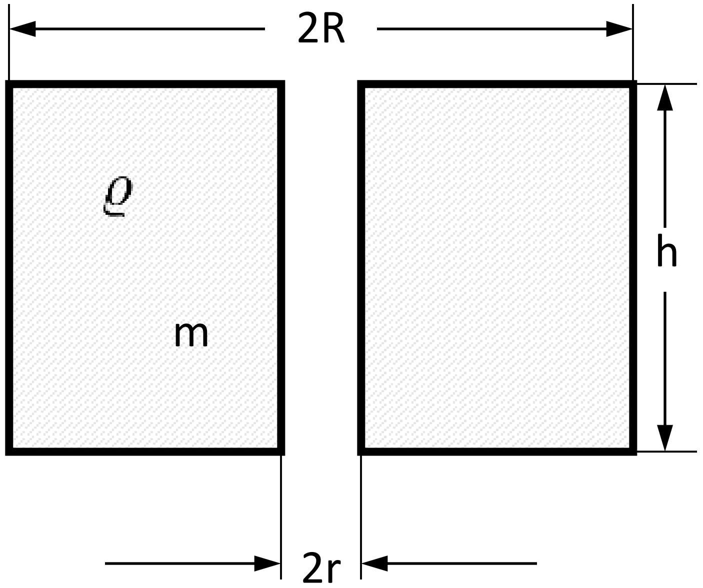
The mercury was leased. After the experiment, it was shipped back to the mine

Weighing the mercury

- F. Nolting and J. Schurr have done an excellent job weighing the mercury.
- The mercury was delivered in 395 flasks.
- Each flasks was weighed before and after the mercury was transferred to the vessel.
- A dedicated calibration weight was made for this task.

Type of uncertainty	**Upper Vessel (g)**		**Lower Vessel (g)**	
	(g)	**(10^{-6})**	**(g)**	**(10^{-6})**
Loss of mercury and residue from flasks	8.0	1.2	11.0	1.6
Approximate equation	7.0	1.0	8.0	1.2
Mass variation of flasks	4.0	0.6	4.0	0.6
Uncertainty of calibration weights	3.7	0.5	3.7	0.5
Buoyancy correction	1.2	0.2	1.2	0.2
Weighing statistics	0.4	0.1	0.4	0.1
Total uncertainty	**12.0**	**1.8**	**15.0**	**2.2**

Over constraint system



Measured to $3\text{x}10^{-6}$ (ϱ)

Measured to $2.2\text{x}10^{-6}$ (m)

$$\varrho = \frac{m}{\pi(R^2 - r^2)h}$$

Measured to $18.1\text{x}10^{-6}$ (R)

Measured to $18.3\text{x}10^{-6}$ (r)

Measured to $36.9\text{x}10^{-6}$ (h)

Least square adjustment (LSA) with constraint:

$$\chi^2 = \left(\frac{r - r_0}{\sigma_r}\right)^2 + \left(\frac{R - R_0}{\sigma_R}\right)^2 + \left(\frac{h - h_0}{\sigma_h}\right)^2 + \left(\frac{m - m_0}{\sigma_m}\right)^2 + \left(\frac{\rho - \rho_0}{\sigma_\rho}\right)^2$$

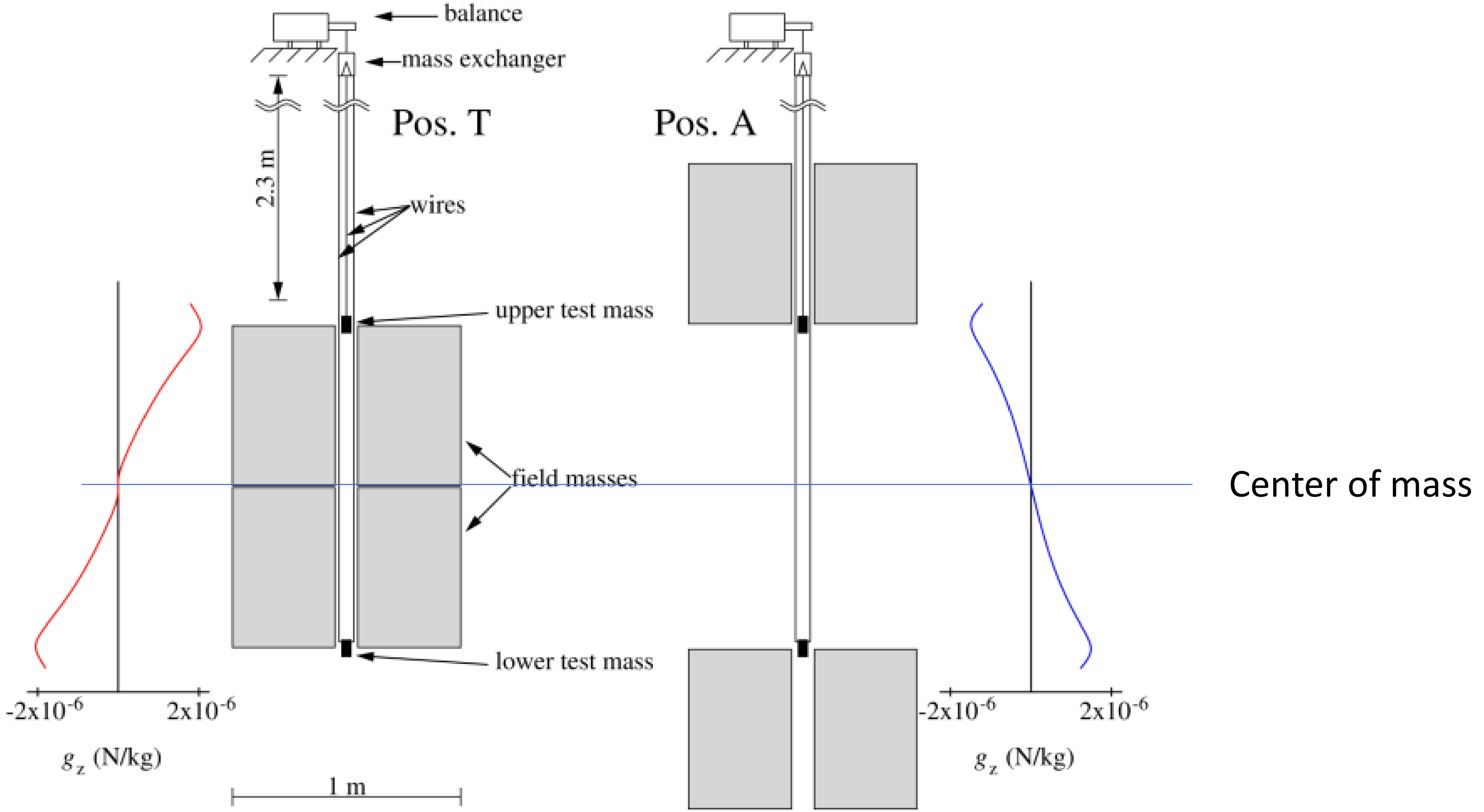
LSA with mercury density reduced the uncertainty of the mass integration by a factor 7.

Mass integration

- Both vessels were broken up in 1200 objects.
- Most object were simple, rotational symmetric shapes with rectangular, triangular, or circular cross section.
- Positive density, where there was material, negative density where material was removed (threaded holes, O-ring grooves,…).
- 90 % of the signal was due to mercury and 10 % of the signal due to the vessel.
- The total mass integration was done at least twice.

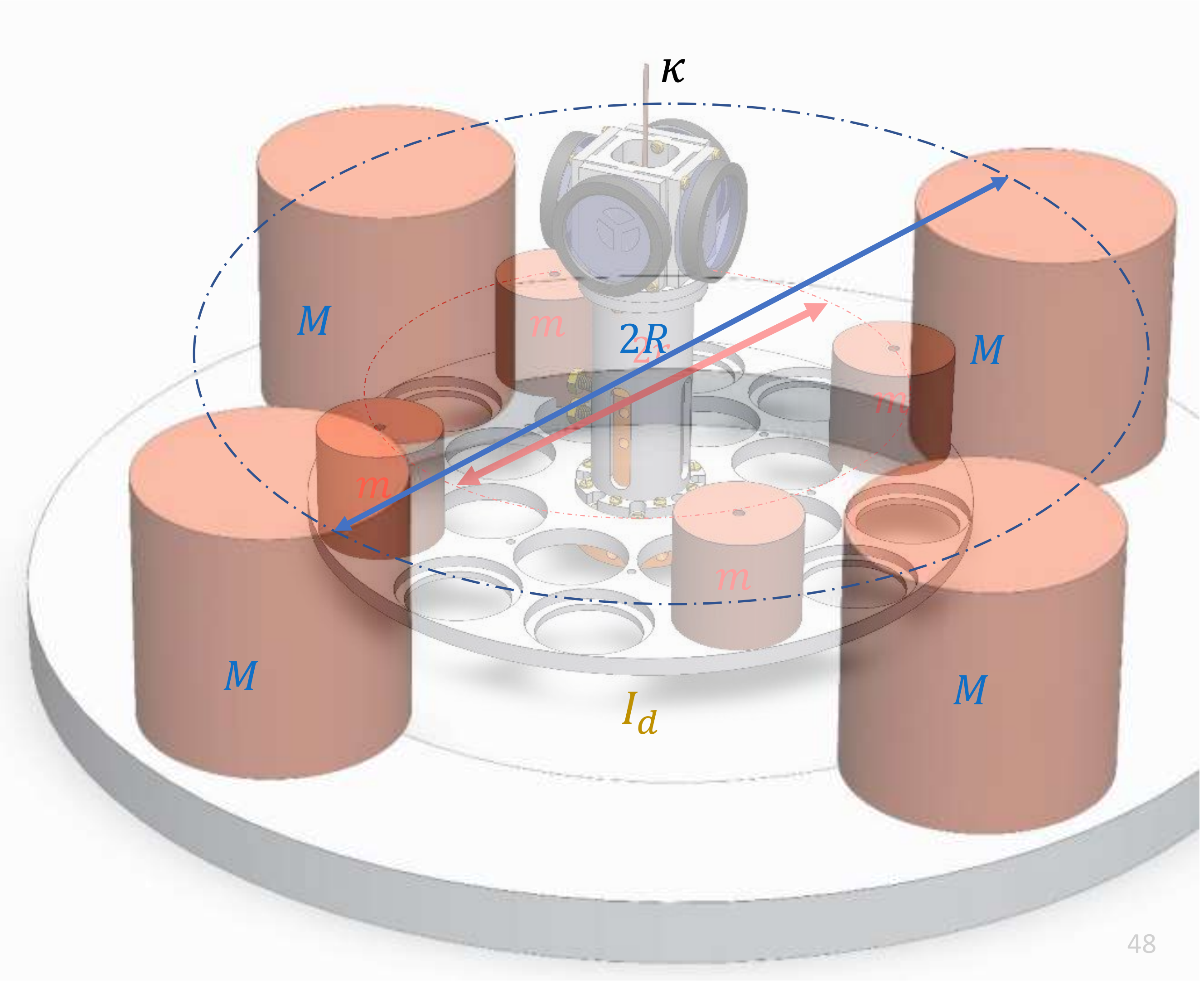
.

Symmetry & Geometry



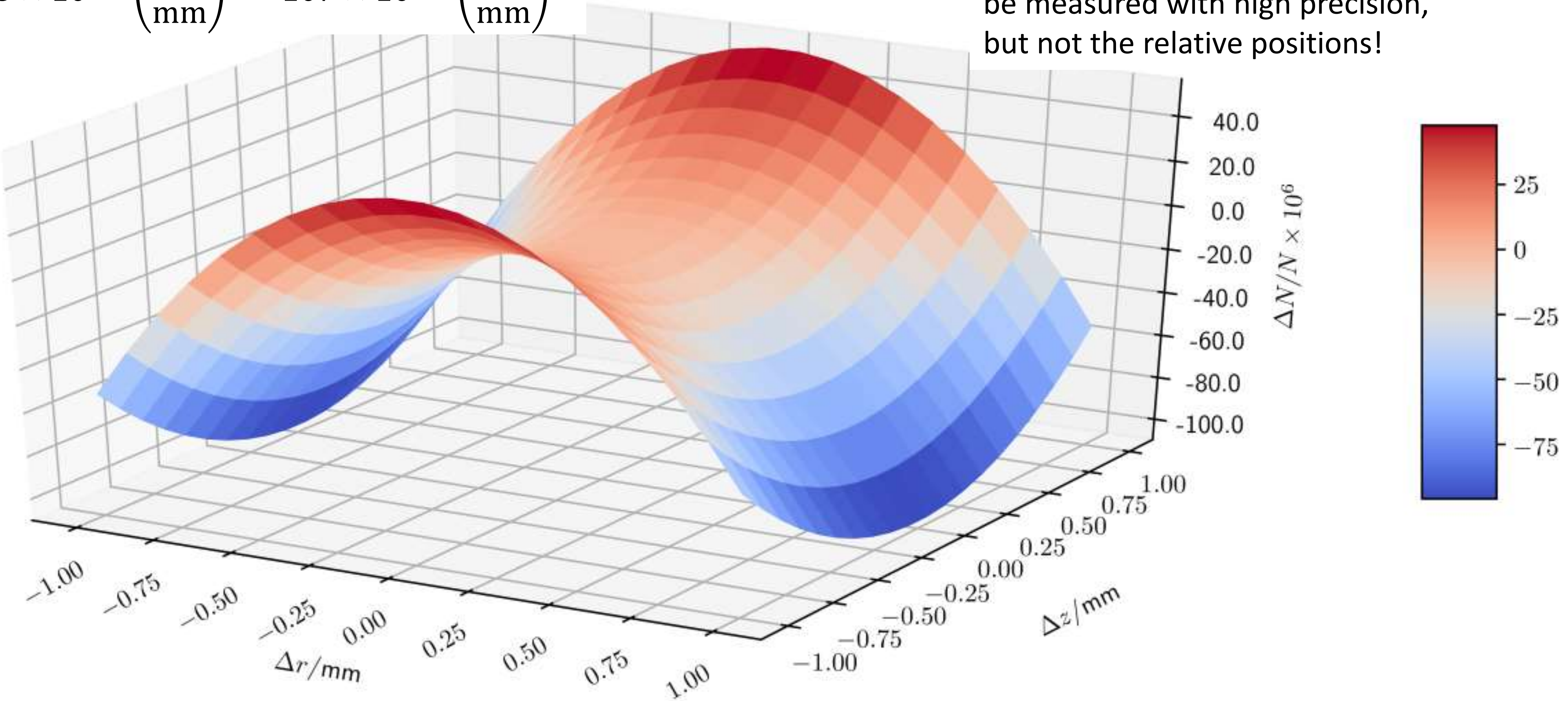
Center of mass of the FMs remains at the same position:

- No effect due to tilt or other coupling to the floor.
- Only a small motor needed to move FMs. Not much heat generated.

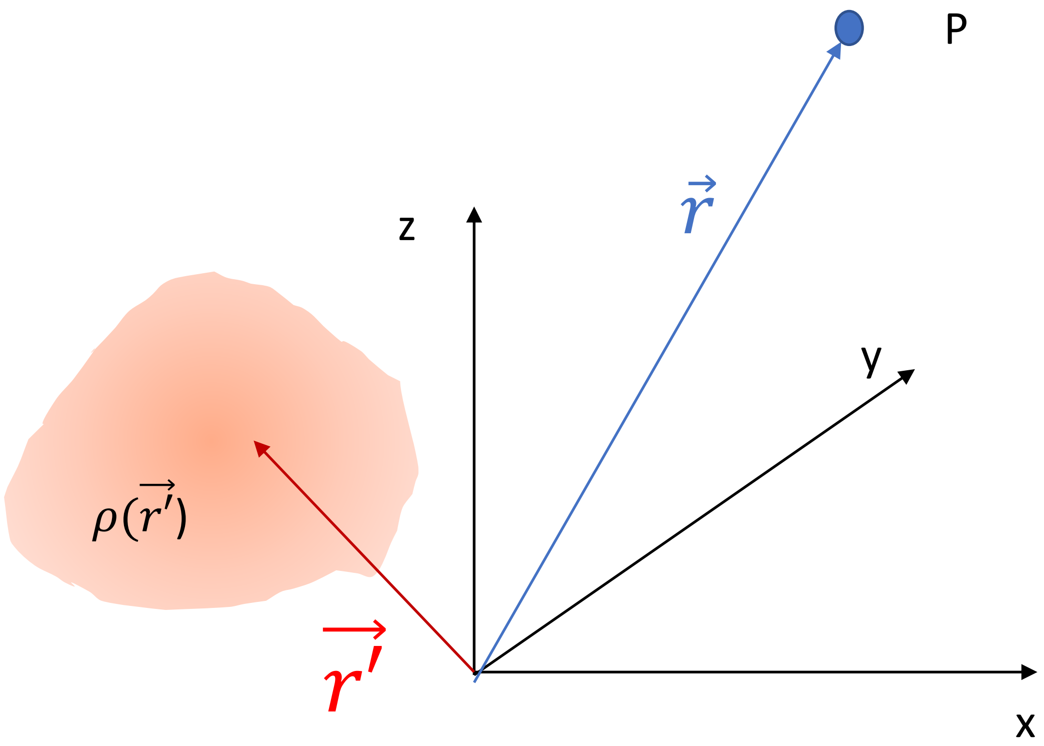

κ
M
M
m
2R
m
m
M
m
M
M
I_d

$$\frac{\Delta N}{N} = 53.5 \times 10^{-6} \left(\frac{\Delta z}{\mathrm{mm}}\right)^2 - 107 \times 10^{-6} \left(\frac{\Delta r}{\mathrm{mm}}\right)^2$$

The radius of each mass assembly has to be measured with high precision, but not the relative positions!



Multipole Formalism 1



$$V(\vec{r}) = G \iiint \frac{\rho(\overrightarrow{r'})}{|\vec{r} - \vec{r}'|} \, dV'$$

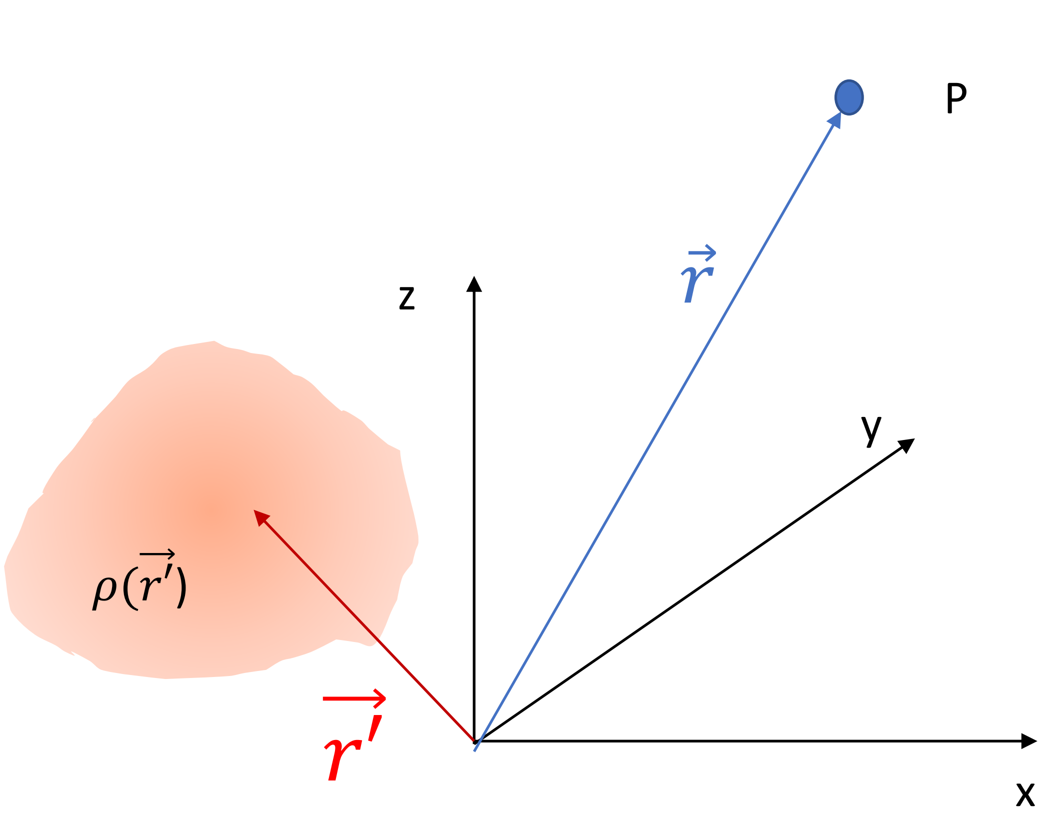
$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2\vec{r}\vec{r}' + r'^2}$$

$$|\vec{r} - \vec{r}'| = r\sqrt{1 - 2\,\frac{\hat{r}}{r}\vec{r}' + \left(\frac{r'}{r}\right)^2}$$

Binomial expansion

$$\frac{1}{\sqrt{1 - 2\,\frac{\hat{r}}{r}\vec{r}' + \left(\frac{r'}{r}\right)^2}} \approx 1 - \frac{\hat{r}}{r}\vec{r}' + \frac{1}{2r^2}\left(r'^2 - 3\left(\hat{r}\overrightarrow{r'}\right)^2 + O\left(\frac{r'}{r}\right)^3\right.$$

Multipole Formalism 2



$$V(\vec{r}) = G \iiint \frac{\rho(\vec{r'})}{|\vec{r} - \vec{r}'|}\, dV'$$

$$V(\vec{r}) \approx \frac{G}{r} \iiint \rho(\vec{r'}) \left(1 - \frac{\hat{r}}{r}\vec{r}' + \frac{1}{2r^2}\left(r'^2 - 3\left(\hat{r}\vec{r'}\right)^2 + O\left(\left(\frac{r'}{r}\right)^3\right)\right)\right) dV'$$

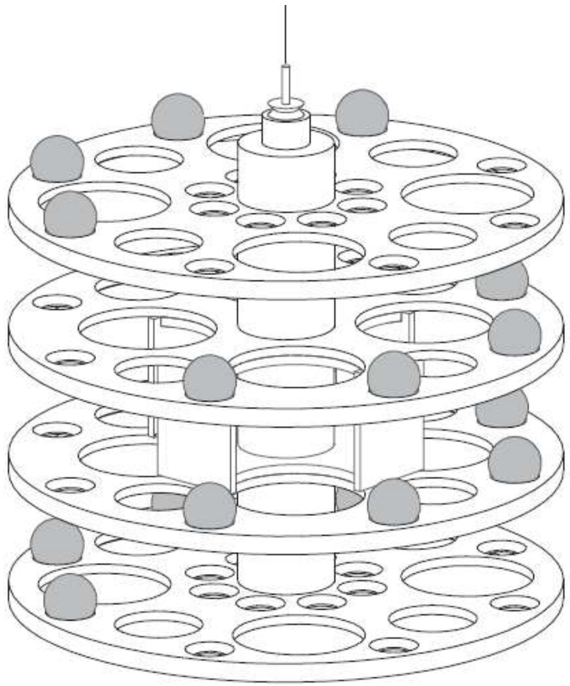
$$V(\vec{r}) \approx V_{mon}(\vec{r}) + V_{dip}(\vec{r}) + V_{quad}(\vec{r}) + \cdots$$

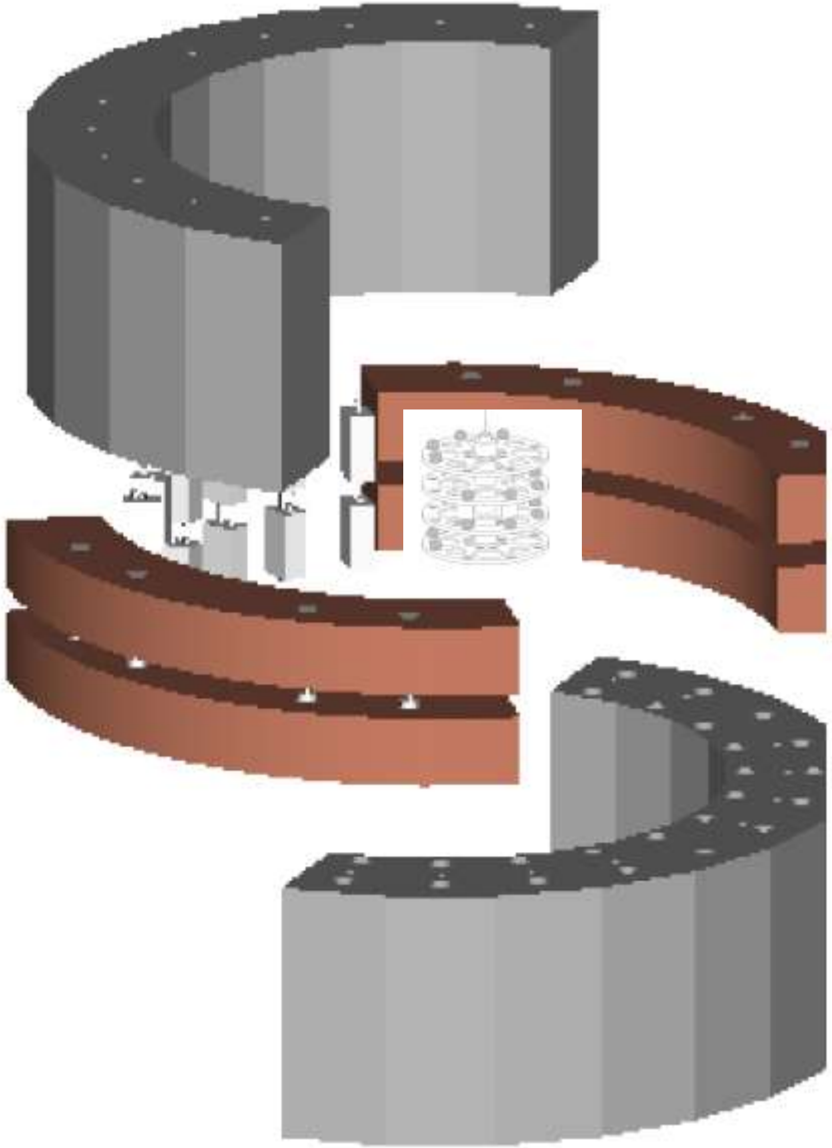
$$V_{mon}(\vec{r}) = \frac{G}{r} \iiint \rho(\vec{r'})\, dV'$$

$$V_{dip}(\vec{r}) = -\frac{G}{r^2} \iiint \rho(\vec{r'})(\hat{r}\,\vec{r}\,)dV'$$

$$V_{quad}(\vec{r}) = \frac{G}{2\,r^3} \iiint \rho(\vec{r'})\,(r'^2 - 3\left(\hat{r}\vec{r'}\right)^2)dV'$$

For a torsion pendulum 1





For a torsion pendulum 2

$$V(|\vec{r}-\vec{r}\,'|) = -G\int d^3r\,\rho_p(\vec{r})\int d^3r'\rho_s(\vec{r}\,')\frac{1}{|\vec{r}-\vec{r}\,'|},$$

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}\frac{1}{2l+1}\frac{r^l}{r'^{l+1}}Y^*_{lm}(\hat{r})Y_{lm}(\hat{r}\,')$$

$$q_{lm} = \int d^3r\,\rho_p(\vec{r})\,r^l\,Y^*_{lm}(\hat{r}),$$

gravity gradient moment

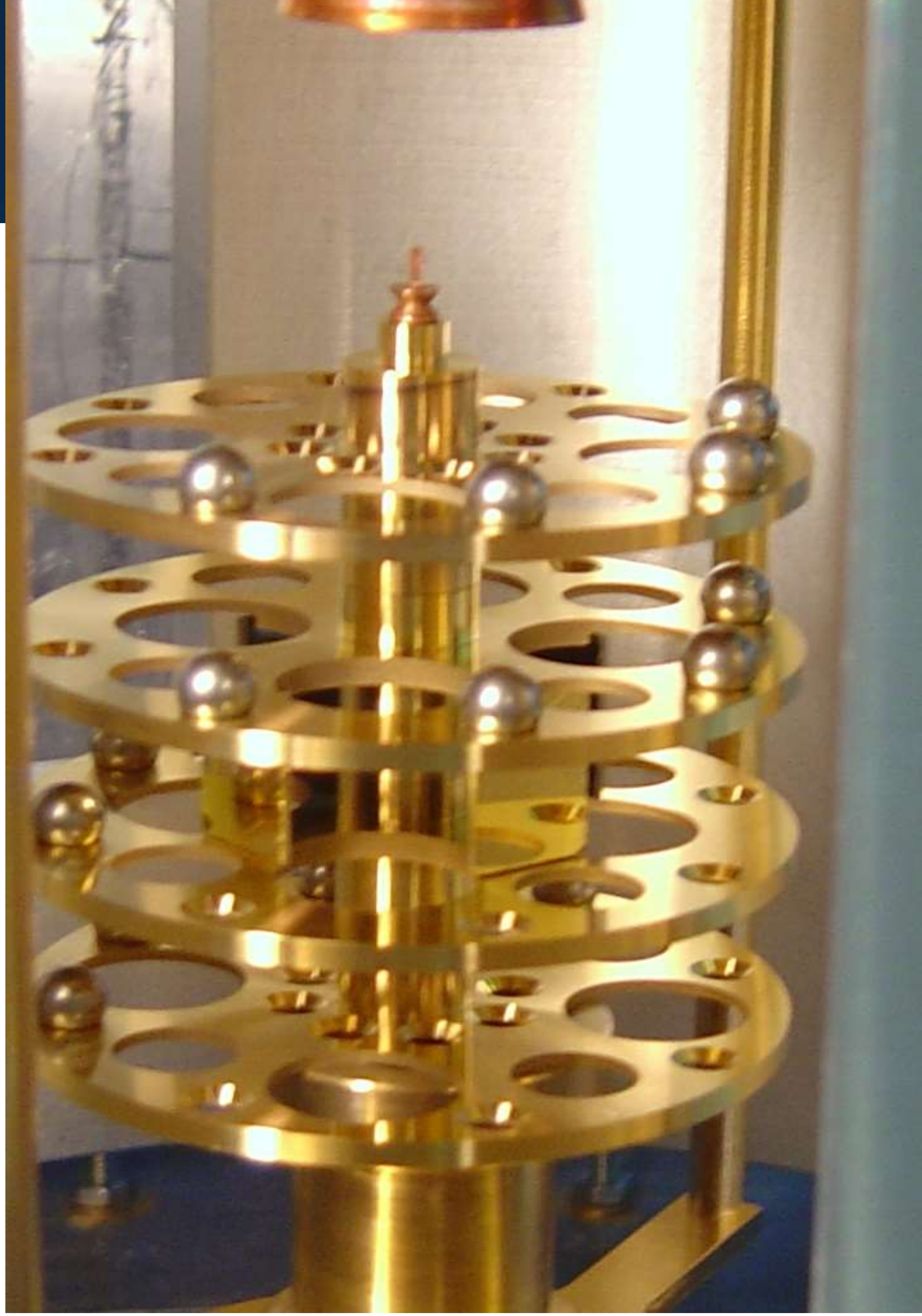
$$q_{lm} = \bar{q}_{lm}e^{-im\phi},$$

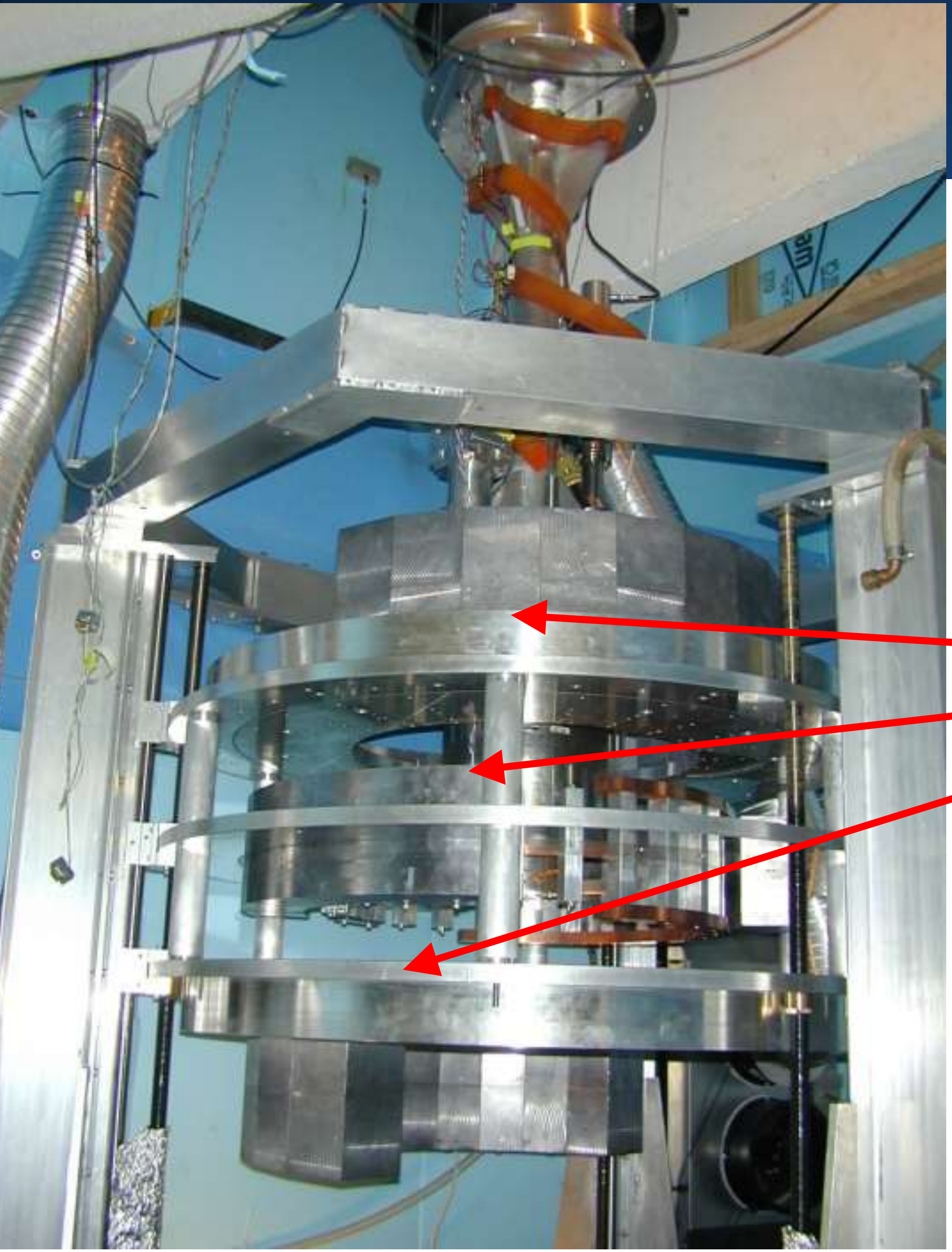
$$Q_{lm} = \int d^3r'\,\frac{\rho_s(\vec{r}\,')}{r'^{l+1}}Y_{lm}(\hat{r}\,').$$

gravity gradient fields

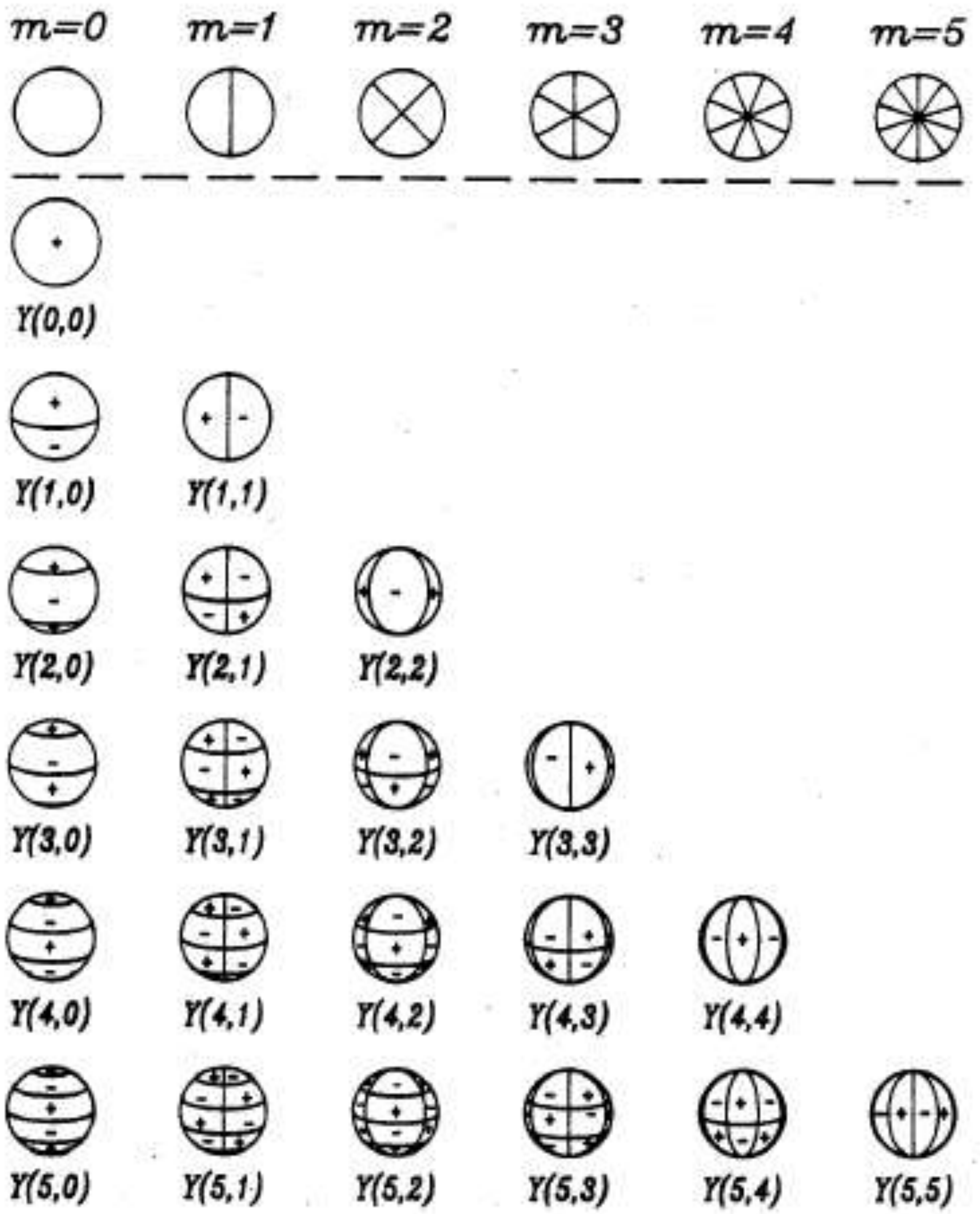
q_{41}

q_{21}




gravity gradient
compensators

Q_{lm} and q_{lm}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m=1
m=2
m=3
m=4
m=5
Y(0,0)
Y(1,0)
Y(1,1)
Y(2,0)
Y(2,1)
Y(2,2)
Y(3,0)
Y(3,1)
Y(3,2)
Y(3,3)
Y(4,0)
Y(4,1)
Y(4,2)
Y(4,3)
Y(4,4)
Y(5,0)
Y(5,1)
Y(5,2)
Y(5,3)
Y(5,4)
Y(5,5)

Thanks

Thanks for listening to the source mass lecture!

Questions?