Lattice discretization of 2d Integrable field theories in 4d CS theory

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[hep-th/2309.14412]

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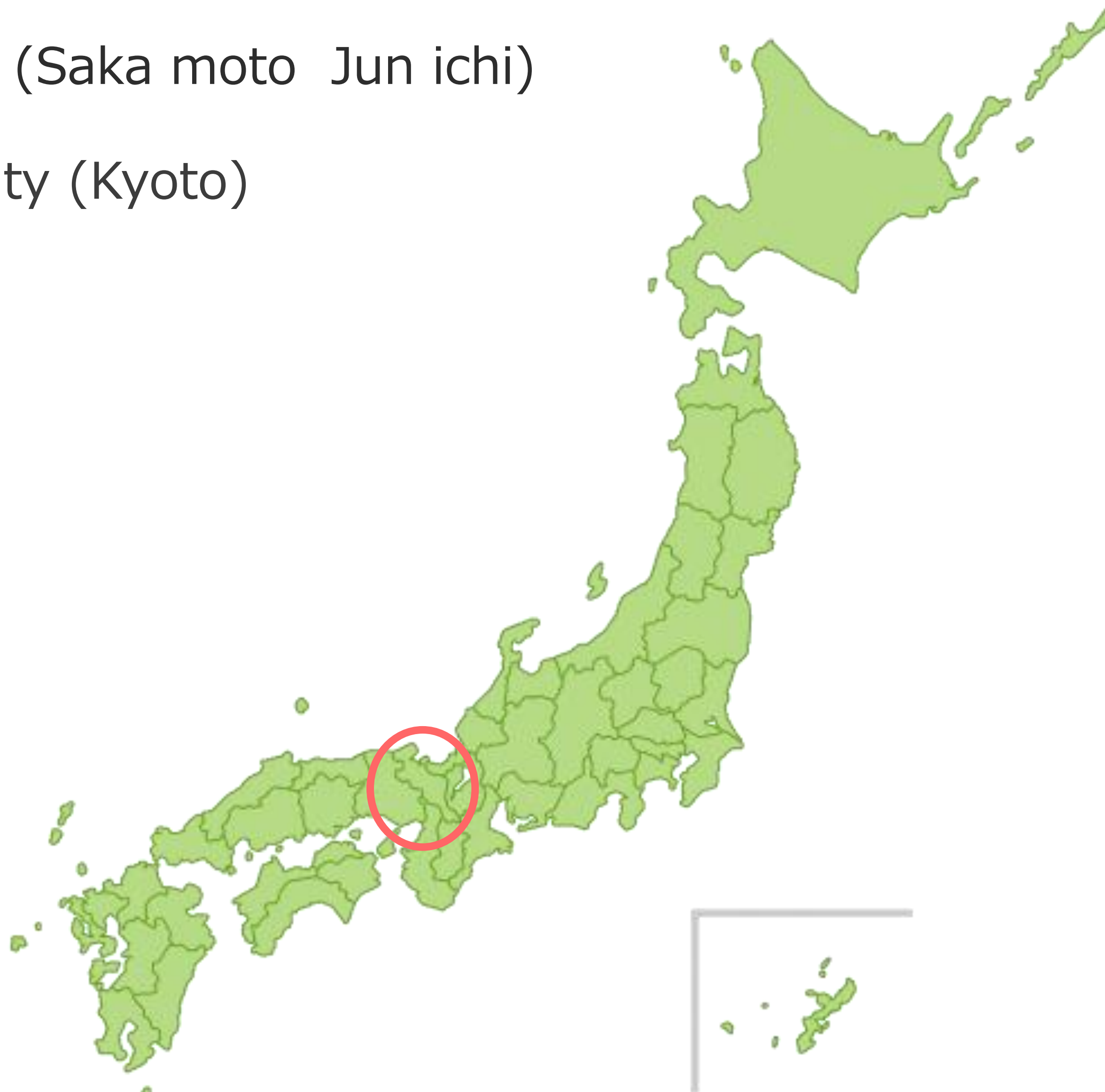
Theory retreat in Santo Stefano 2023/11/12@Santo Stefano

Myself introduction

Name　　　坂本　純一 (Saka moto Jun ichi)

Birthplace　Kameoka City (Kyoto)



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Research career

2019	Ph.D. in Department of Physics, Kyoto University
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2022-present	INFN- Sezione di Torino

Research subjects

- Integrable deformations of 2d integrable sigma models
- Relation between string dualities and integrable deformations
- Higher dimensional conformal field theory
  (Geodesic Witten diagrams, flat space limit etc.)
- TTbar deformation and 2d dilaton gravity
- 4d Chern-Simons theory and 2d integrable systems

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Introduction

Introduction

What is integrability?

#d.o.f. of a given system = #Conserved charges

→ For the system with finite d.o.f., we can solve an initial value problem (Liouville integrability)

Ex. 1d Harmonic oscillator

$$\frac{d^2 q}{dt^2} + \omega^2 q = 0$$

d.o.f. = 1 Conserved charge = E

Introduction

2d Integrable field theory (IFT)

A field theory possesses an infinite number of conserved charges

constructed from Lax pair $\mathcal{L}(z) = \mathcal{L}_\tau d\tau + \mathcal{L}_\sigma d\sigma$ (classical integrability)

(I) Meromorphic function of spectral parameter $z \in \mathbb{CP}^1$

(II) on-shell flat current $\partial_\tau \mathcal{L}_\sigma - \partial_\sigma \mathcal{L}_\tau + [\mathcal{L}_\tau, \mathcal{L}_\sigma] = 0$

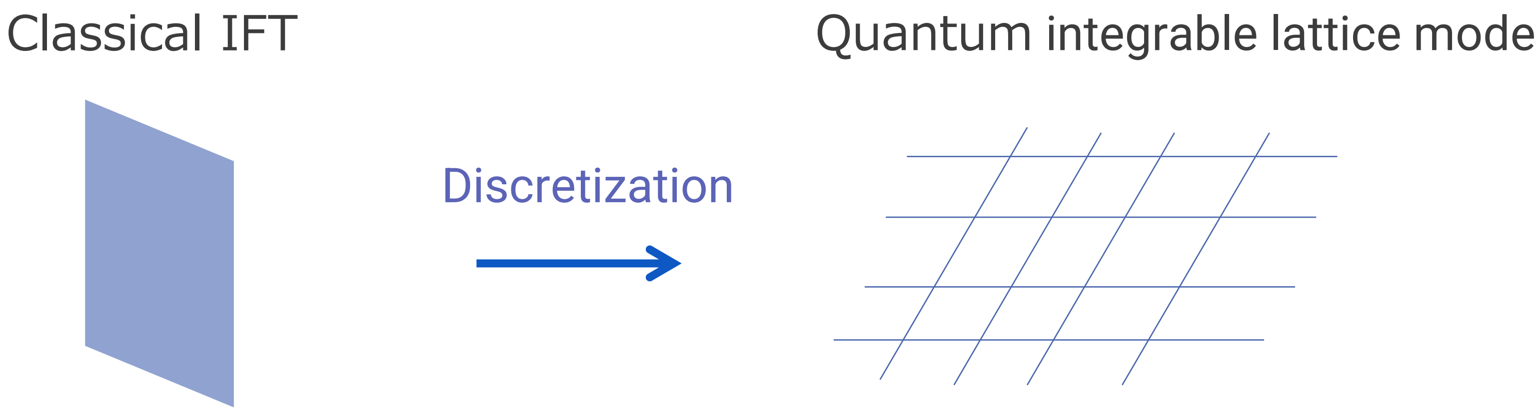
→ generating function of conserved charges

$$T(z) = \exp\left(\oint d\sigma \mathcal{L}_\sigma\right) = \sum_n Q^{(n)} z^n \qquad \frac{d}{d\tau} T(z) = 0$$

Ex. Thirring model, Principal chiral model etc.

Introduction

➢ Quantization of classical integrable field theories



✓ UV cut-off = lattice spacing

✓ Quantum integrability

△ In the continuous limit, the original IFTs must be reproduced

Our purpose

To establish a systematic quantization method for 2d classical IFTs

4d Chern-Simons theory

4d Chern-Simons theory

[Costello, 1303.2632]
[Costello-Witten-Yamazaki,1709.09993,1802.01579]
[Costello-Yamazaki,1908.02289]

2d classical IFTs and 2d quantum lattice models
can be described within a single theoretical system

$$S_{\mathrm{CS}}[A] = \frac{i}{4\pi} \int_{\mathcal{M}\times\mathbb{C}P^1} dz \wedge \mathrm{Tr}\left[A \wedge \left(dA + \frac{2}{3} A \wedge A\right)\right]$$

- $\mathcal{M}$: 2d surface $(\tau\,,\sigma)$ The 2d integrable systems are defined on $\mathcal{M}$
- $\mathbb{C}P^1 = \mathbb{C} \cup \{\infty\}$ $(z\,,\bar{z})$ Space of a spectral parameter
- gauge field $A \in \mathfrak{g}$ $\mathfrak{g}$: Lie algebra

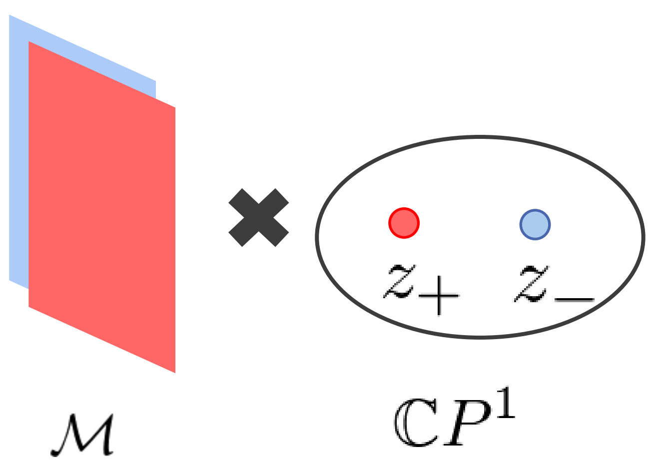
Overview of the derivation of 2d IFTs [Costello-Yamazaki,1908.02289]

4d CS + 2d defects ⟶ 2d IFT

SET UP $S_{\text{4d-2d}}[A] = S_{\text{CS}}[A] + S_+[A_-,\phi] + S_-[A_+,\chi]$

Chiral defect $S_+[A_-,\phi] = S_{\text{2d-defect}}[\phi] + \int_{\mathcal{M}\times\{z_+\}} d^2\sigma\, J_a(\phi)A^a_-$

Anti-chiral defect $S_-[A_+,\chi] = S_{\text{2d-defect}}[\chi] + \int_{\mathcal{M}\times\{z_-\}} d^2\sigma\, J_a(\chi)A^a_+$



Overview of the derivation of 2d IFTs [Costello-Yamazaki,1908.02289]

4d CS + 2d defects ⟶ 2d IFT

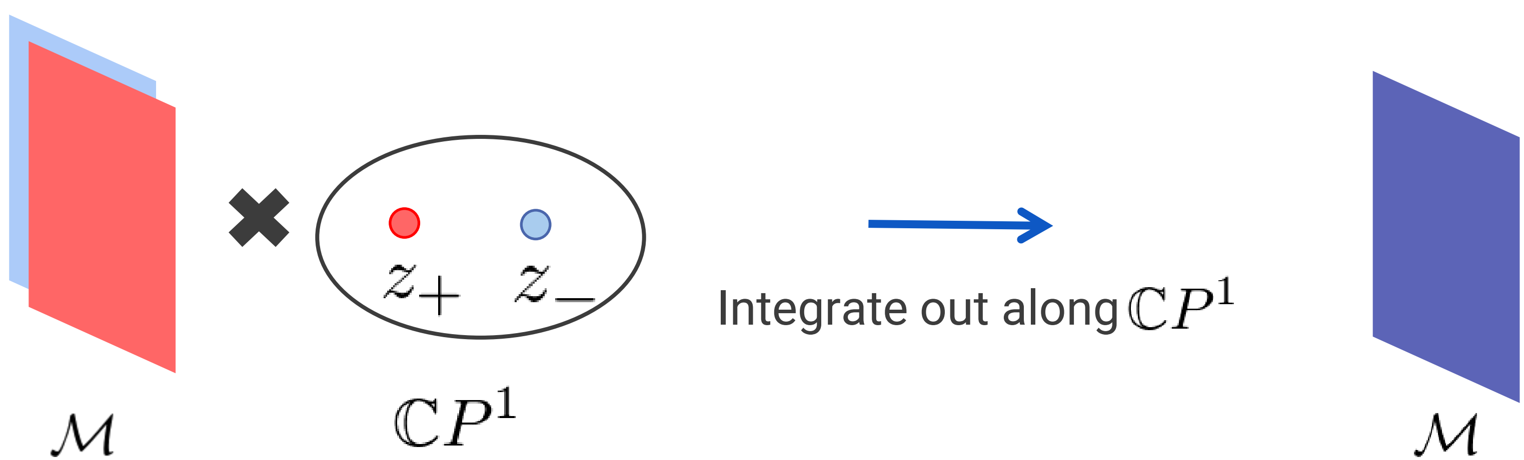
STEP1 Construct a solution to the EOM for A along $\mathbb{C}P^1$

$$A = \mathcal{L} \quad = \text{Lax pair}$$

$\because \; F_{+-} = 0$
$(F(A) = dA + A \wedge A)$

STEP2 Integrating $S_{\text{4d-2d}}[\mathcal{L}]$ over $\mathbb{C}P^1$

$$S_{\text{4d-2d}}[\mathcal{L}] = S_{\text{2dIFT}}[\phi, \chi]$$



Overview of the derivation of 2d IFTs [Costello-Yamazaki,1908.02289]

Example

$$S_{\text{4d-2d}} = S_{\text{CS}}[A] + S^f_+[A_-, \psi_L] + S^f_-[A_+, \psi_R]$$

Chiral defect $$S^f_+ = \int_{\mathcal{M}\times\{z_+\}} d^2\sigma\, \psi^\dagger_L i(\partial_- + A_-)\psi_L$$

Anti-chiral defect $$S^f_- = \int_{\mathcal{M}\times\{z_-\}} d^2\sigma\, \psi^\dagger_R i(\partial_+ + A_+)\psi_R$$

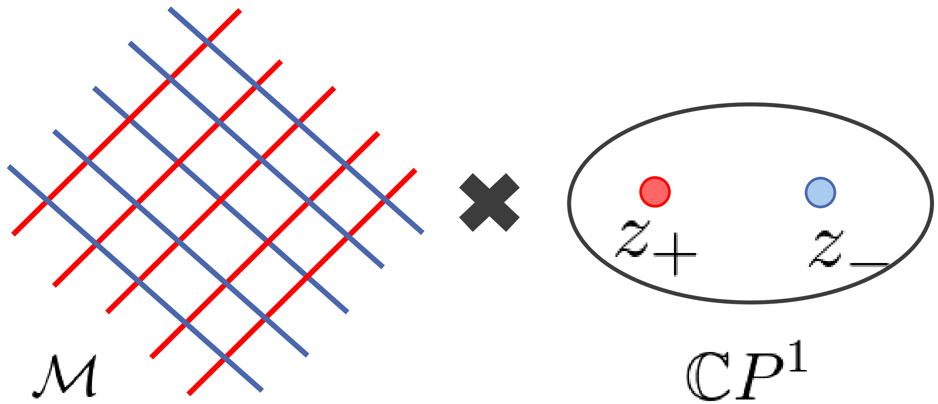
Massless Thirring model

$$S_{\text{Th}}[\psi] = \int_{\mathcal{M}} d^2\sigma \left(\psi^\dagger_L i\partial_- \psi_L + \psi^\dagger_R i\partial_- \psi_R - \frac{g^2}{2}\psi^\dagger_L T_a \psi_L \psi^\dagger_R T^a \psi_R\right)$$

Integrable lattice models in 4d CS theory

4d CS theory + Wilson lines ⟶ (1+1)d lattice models

[Costello, 1303.2632]
[Costello-Witten-Yamazaki]



Expectation value of Wilson lines = Partition function of lattice model

$$Z^{\mathrm{lat}} = \int DA \prod_n W_{L,n}[A_-|_{z_+}] \prod_m W_{R,m}[A_+|_{z_-}] \exp\left(\frac{i}{\hbar} S_{CS}[A]\right)$$

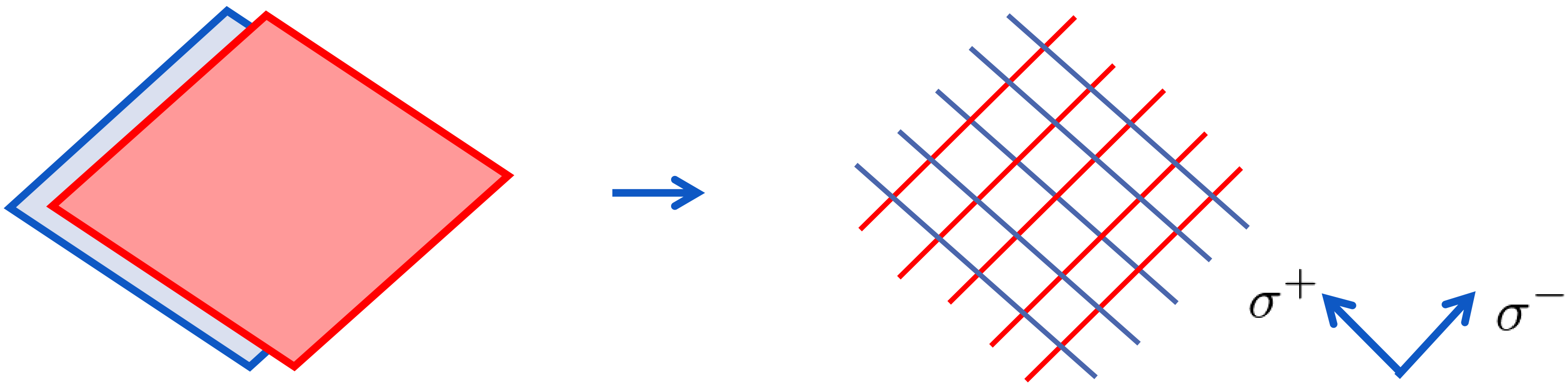
$$W[A] = \exp\left(\int d\sigma^\mu A_\mu\right)$$

Lattice discretization in 4d CS theory

Lattice discretization in 4d CS theory

Basic Ideas

The lattice discretization of the 2d classical IFTs should result in a discretization of the 2d defects along the l.c. directions



$$\int_{\mathbb{R}^2\times\{z_+\}} \psi_L^* i(\partial_- + A_-)\psi_L d\sigma^+ d\sigma^- \qquad \sum_{j=1}^{N}\int_{\{\sigma_j^+\}\times\mathbb{R}\times\{z_+\}} \psi_{L,j}^* i(\partial_- + A_-)\psi_{L,j} d\sigma^-$$

$$\int_{\mathbb{R}^2\times\{z_-\}} \psi_R^* i(\partial_+ + A_+)\psi_R d\sigma^+ d\sigma^- \qquad \sum_{j=1}^{N}\int_{\mathbb{R}\times\{\sigma_j^-\}\times\{z_-\}} \psi_{R,j}^* i(\partial_+ + A_+)\psi_{R,j} d\sigma^+$$

Lattice discretization in 4d CS theory

[Dikonov-Petrov 1989 etc.]

Partition function of 1d QM = Wilson line

$$Z^{\mathrm{QM}}[A] = \int D\psi D\psi^* \exp\left(i \int dt \psi^* i(\partial_t + A_t)\psi \right)$$

$$= \mathrm{Tr}_{\mathcal{H}_R}\left(\exp\left(\int dt\, \hat{A}_t \right)\right) = W_R[A]$$

$\mathcal{H}_R$: Hilbert space with rep. R

Then, the original partition function

$$Z^{\text{2d-4d}}[A, \psi_{L,R}] = \int DAD\psi_L D\psi_R \exp\left(\frac{i}{\hbar} S_{\text{2d-4d}}[A, \psi_{L,R}] \right)$$

can be discretized to

$$Z^{\mathrm{lat}} = \int DA \prod_n W_{L,n}[A_-|_{z_+}] \prod_m W_{R,m}[A_+|_{z_-}] \exp\left(\frac{i}{\hbar} S_{CS}[A] \right)$$

Summary

- We have discussed a systematic derivation of 2d IFTs from 4d CS theory coupled with the 2d systems.

- Discussed lattice discretization of a certain kind of 2d IFTs in 4d CS theory

  → Can be understood as a discretization of 2d defects coupled to 4d CS theory

- Dualities of 2d integrable field theory are translated to dualities of 2d defects

  In particular, we found a hint that a similar approach might be useful for the discretization problem of a certain class of non-ultralocal IFTs.

Thank you

Lax pair of harmonic oscillator

Well, consider a Lax pair for the Harmonic Oscillator;

$$L = \begin{pmatrix} p & \omega q \\ \omega q & -p \end{pmatrix}, \quad M = \frac{\omega}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Since the Hamiltonian is

$$H(q,p) = \frac{1}{2}(p^2 + \omega^2 q^2)$$

It is eay to check that the Lax equation $\dot{L} = [M, L]$ is equivalent to the Hamiltonian system

$$\dot{q} = \frac{\partial H}{\partial p} = p, \quad \dot{p} = -\frac{\partial H}{\partial q} = -\omega^2 q$$

It is a trivial exercise to show that

$$\ddot{q} + \omega^2 q = 0$$

https://physics.stackexchange.com/questions/182589/lax-representation-of-the-harmonic-oscillator

Lax pair of Principle chiral model

Ex. Principle chiral model

$$S = \int d^2\sigma \, \mathrm{Tr}(g^{-1}\partial_\mu g g^{-1}\partial^\mu g) \qquad g \in G$$

Lax pair $\mathcal{L} = -\frac{j_+}{z-1}d\sigma^+ + \frac{j_-}{z+1}d\sigma^- \qquad j_\pm = g^{-1}\partial_\pm g$

$$\because \quad \partial_+\mathcal{L}_- - \partial_-\mathcal{L}_+ + [\mathcal{L}_+, \mathcal{L}_-]$$

$$= \frac{1}{1-z^2}\left[\underline{(\partial_+ j_- - \partial_- j_+ + [j_+, j_-])} - z(\partial_+ j_- + \partial_- j_+)\right]$$

= 0 EOM