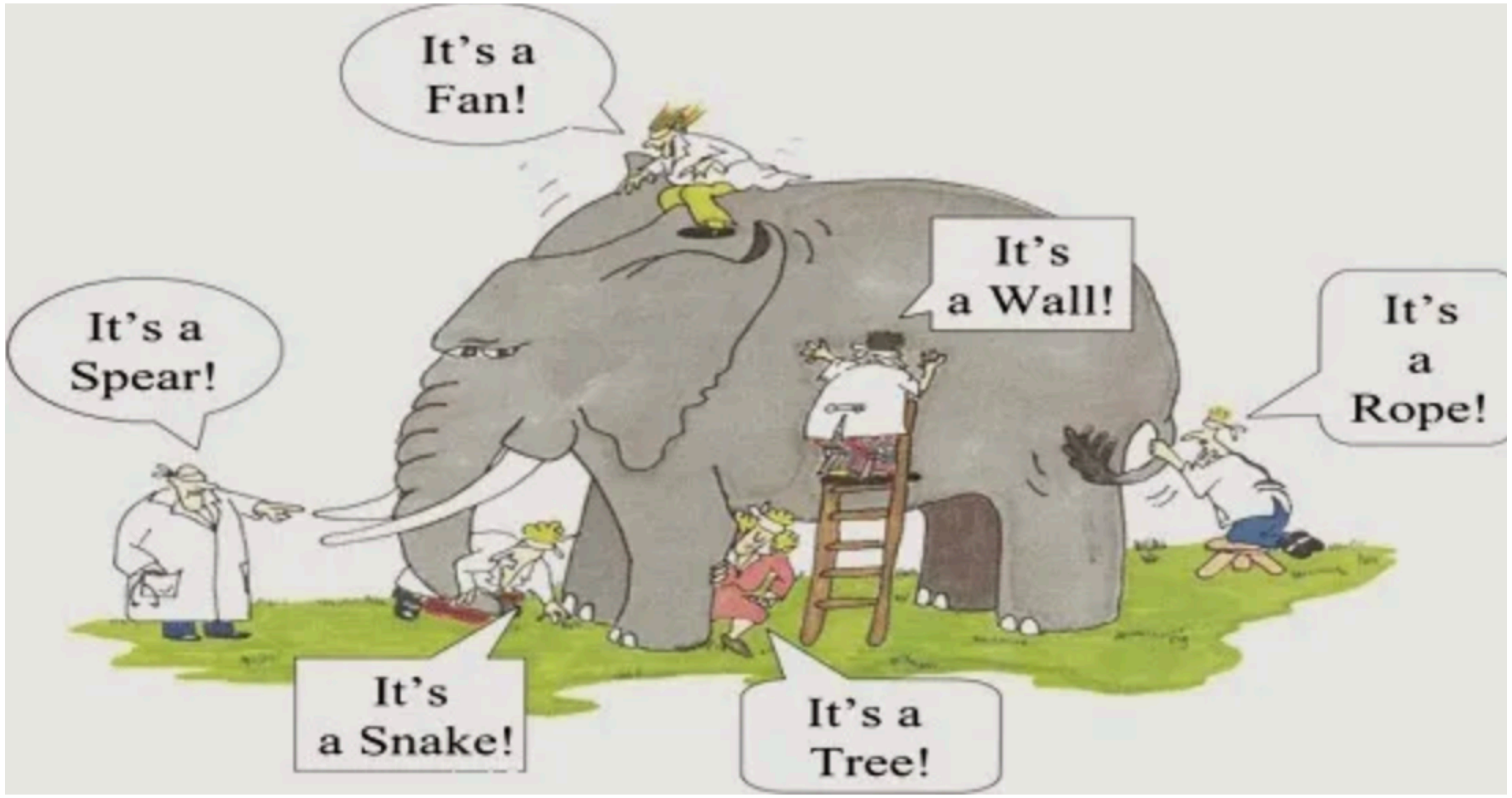
N=4 SYM
(AND OTHER ANIMALS):

TRYING TO SOLVE "SOLVABLE" GAUGE THEORIES

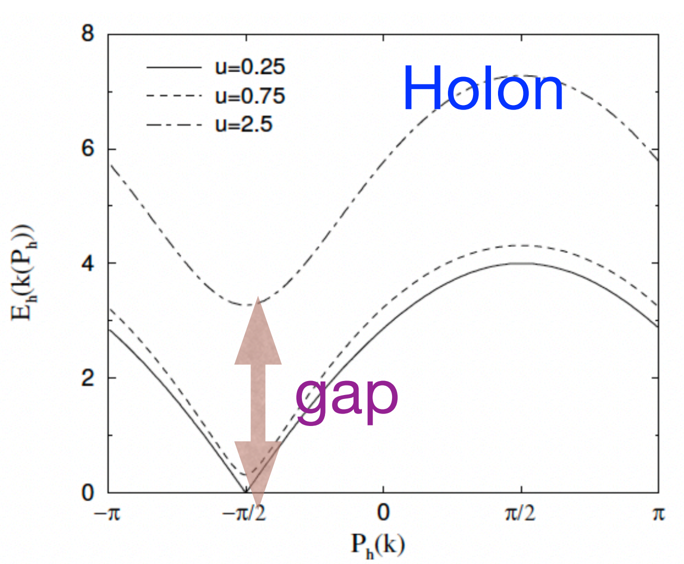


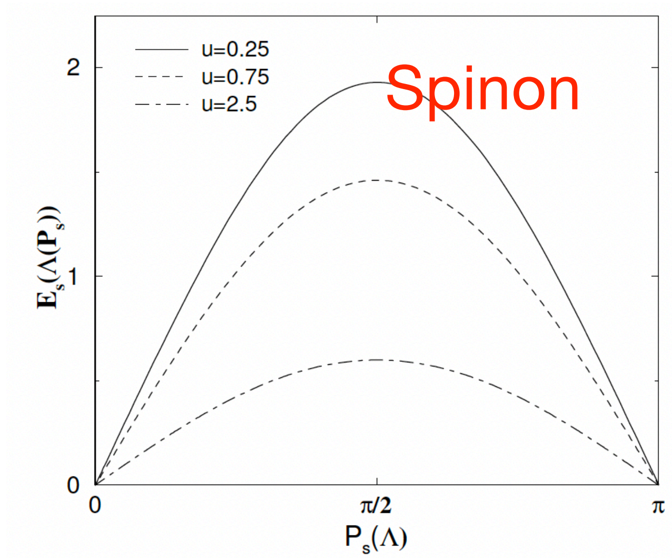
Andrea Cavaglia `, Torino Theory Retreat 2023

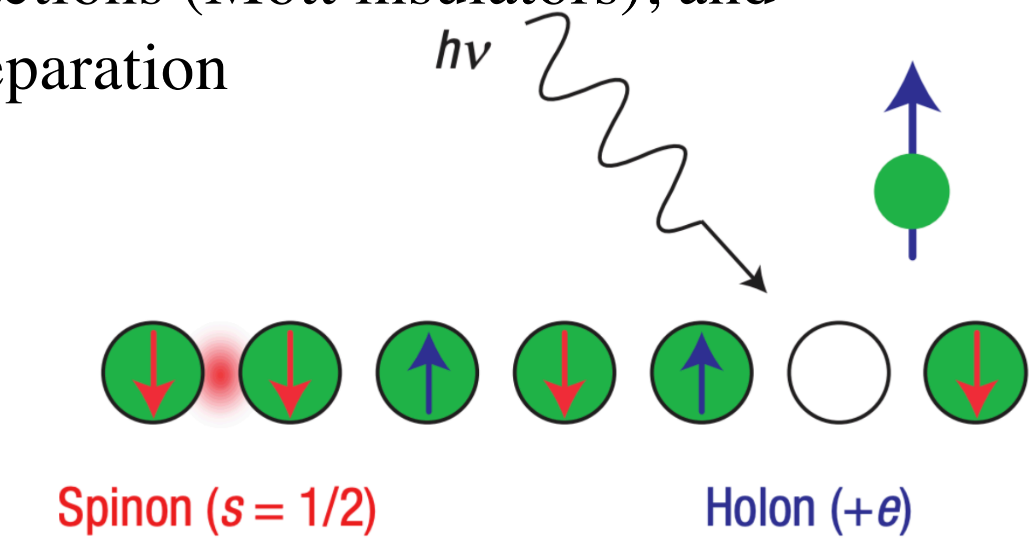
Why integrable systems: some of the reasons

Exact study of non-perturbative phenomena

Example: new phases of solids due to electronic interactions (Mott insulators), and their counterintuitive properties such as spin-charge separation



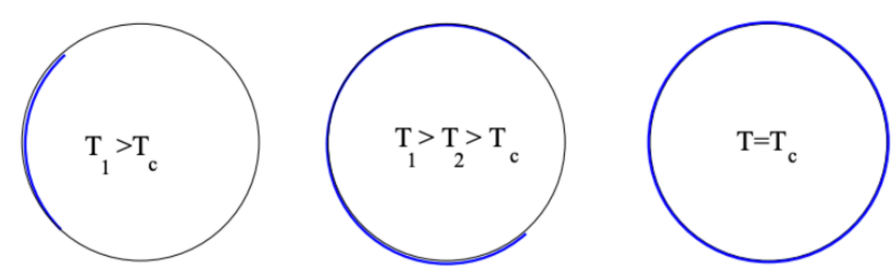




Experimental observation only in *2006!*

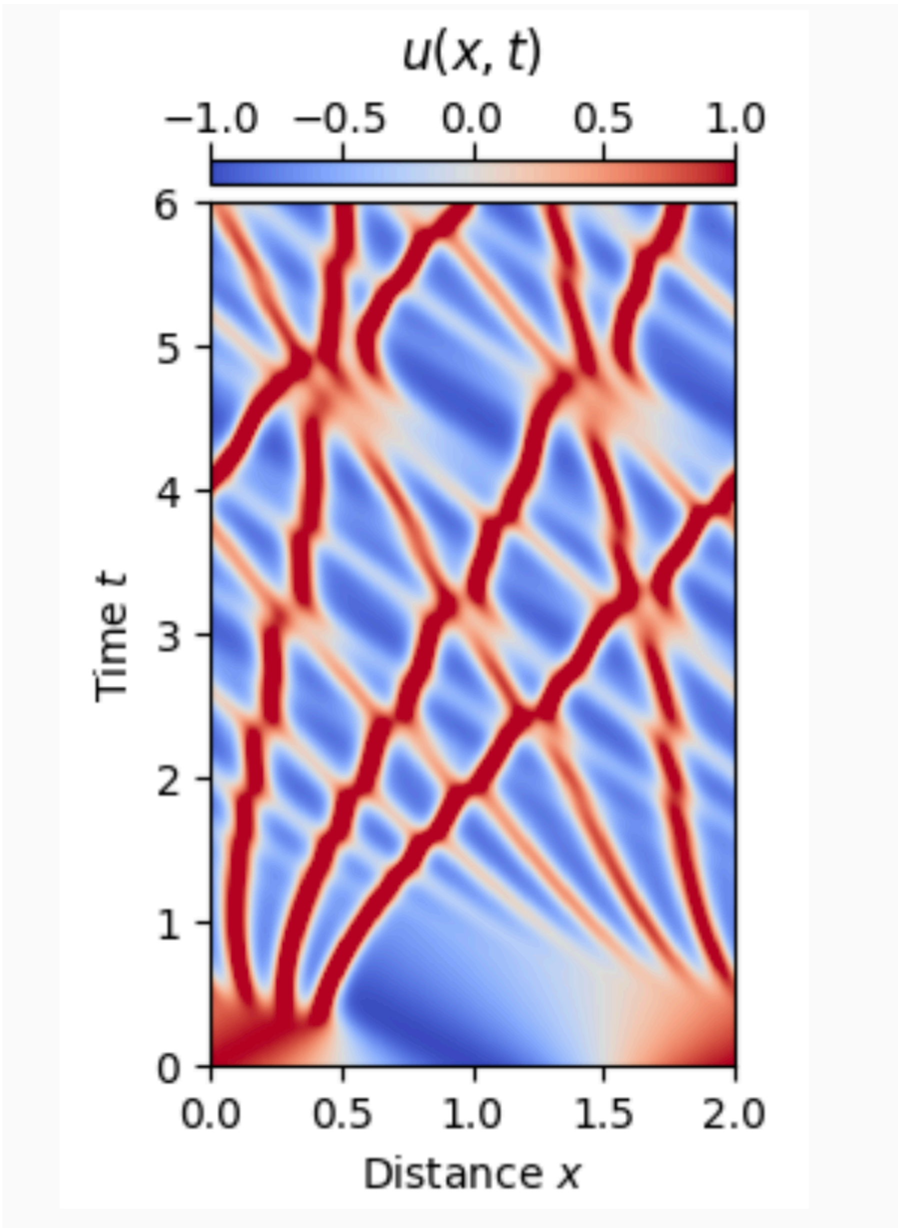
Validate physical models and understand their mathematical workings

e.g., solution of Ising 2D showed us **how statistical mechanics can "do" phase transitions**

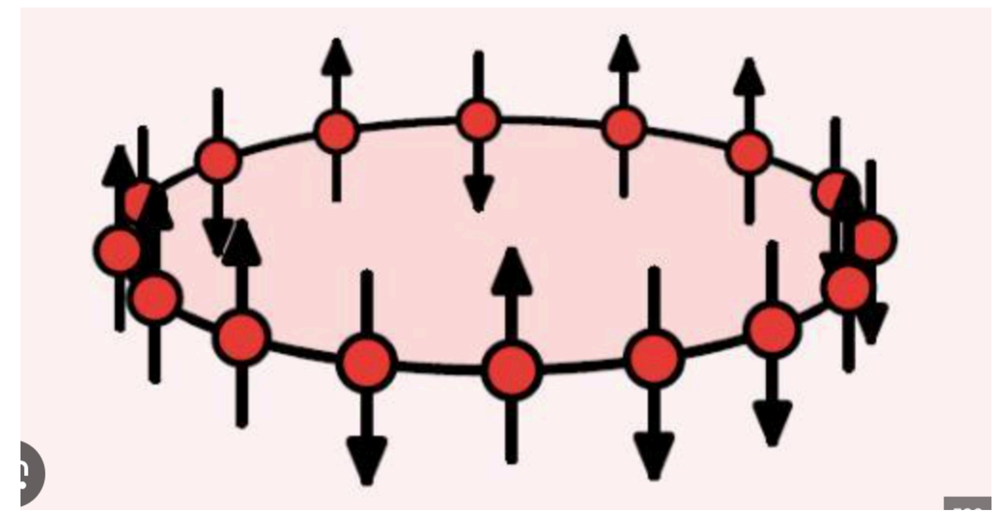


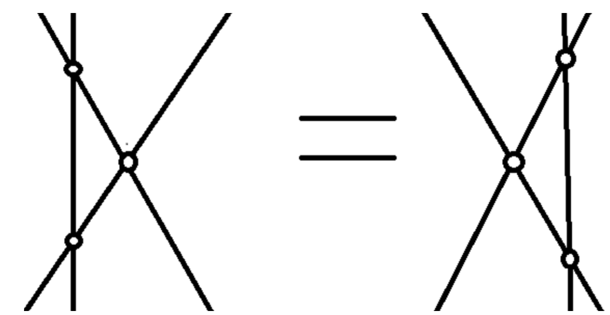
Starting point to solve more general models

e.g., perturbation theory around integrable systems, Truncated Conformal Space Approach, …



Solitons in the KdV equation

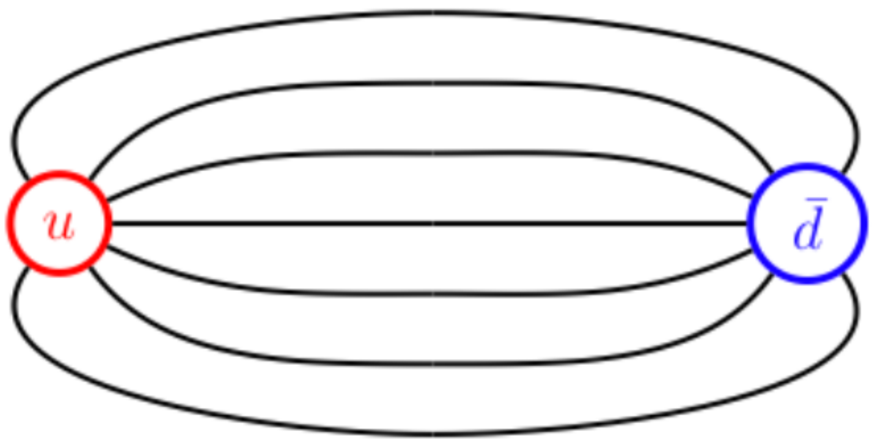




Spin waves in the Heisenberg quantum ferromagnet

the magic of integrability (i.e. some powerful mathematical structures) looks tightly related to low dimensional dynamics

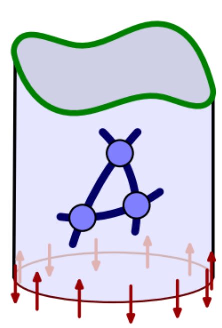
.. but there are 2D structures also in higher-D gauge theories



e.g. flux tube (effective string)…

Integrability seems to play some role, but is only approximate

Works by M. Caselle, F. Gliozzi, R. Tateo & Dubovsky et al



… or we can have an exact duality with strings for some gauge theories at $N_c \to \infty$ *[Maldacena '97]*

Integrability in D >1+1

$\mathcal{N}$=4 Super Yang-Mills theory (SYM) in D=4

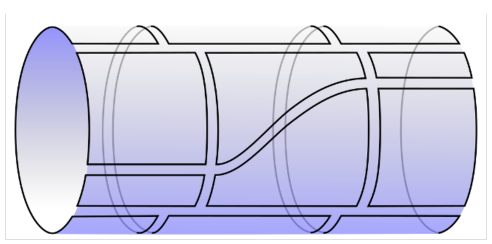
$$S_{SYM} = \frac{1}{g_{YM}^2}\int d^4x\,\mathrm{tr}\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}D_\mu\Phi_i D^\mu\Phi^i + \frac{i}{2}\bar{\Psi}\Gamma^\mu D_\mu\Psi + (\text{Yukawa int.}) + (\text{quartic int.})\right]$$

In the planar limit $N_c \to \infty$

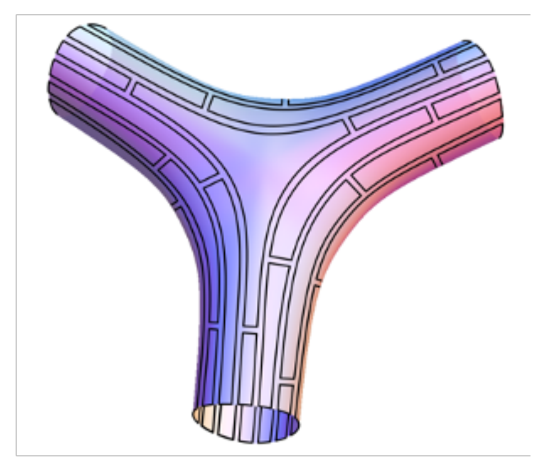
't Hooft coupling $\lambda = g_{YM}^2 N_c$

Local Observables

2 pt



3 pt



.. n pt

...

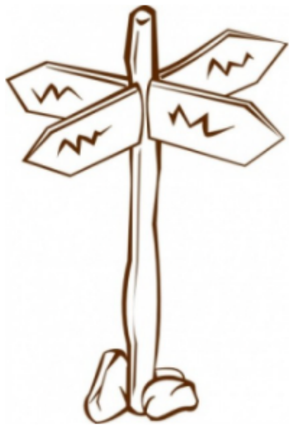
Planar Feynman diagrams / string worldsheets *[Maldacena '97]*

Integrable structures in 1+1 d! *[Minahan,Zarembo '02]*

At the crossroads

QCD

Similarities (formal and sometimes phenomenological) for gluonic observables
e.g. Regge high-energy limits, Wilson lines, Amplitudes,…



Conformal Field Theory (CFT)

It is a critical theory.
Important benchmark for conformal bootstrap
(more on this below)

N=4 SYM

String Theory on Curved Space

Rare opportunity to solve it

AdS/CFT

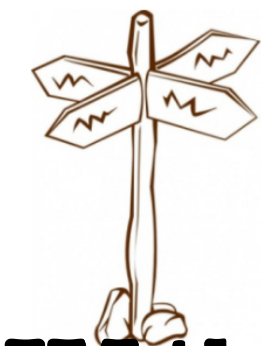
How can gauge theory do gravity?
What do we learn about one or the other?

More integrable crossroads

Chern-Simons+Matter

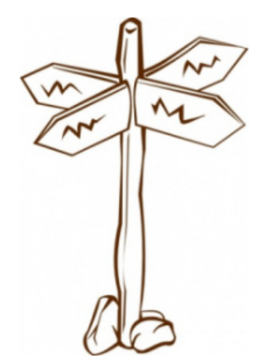
A different type of gauge theory in 3D
(arising in condensed matter)

AdS_4/CFT_3



ABJM theory (**A**harony **B**ergman **J**afferis **M**aldacena)

Strings



AdS_3/CFT_2

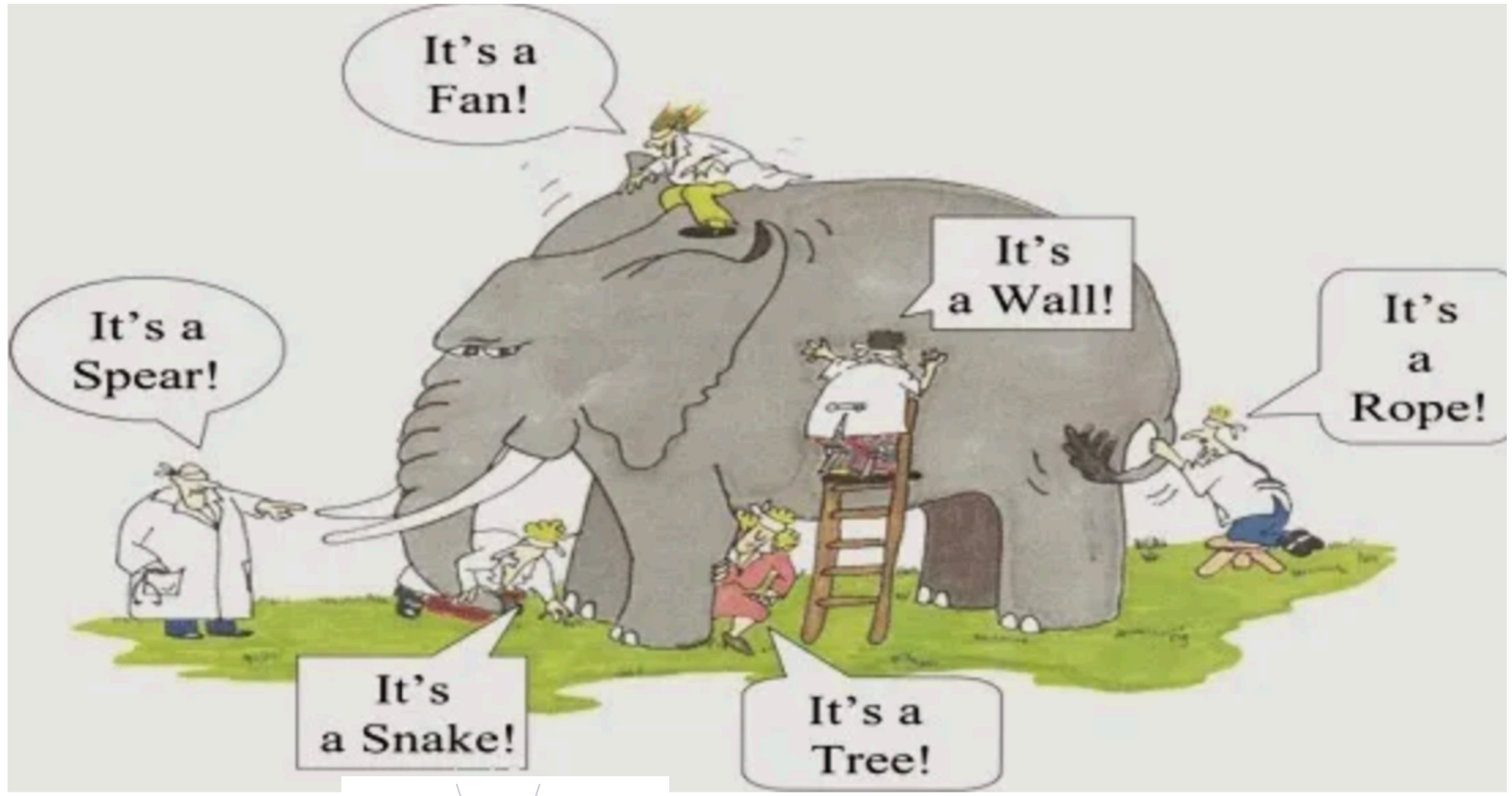
CFT_2

Mysterious non-Lagrangian theories living
on D-branes in string theory

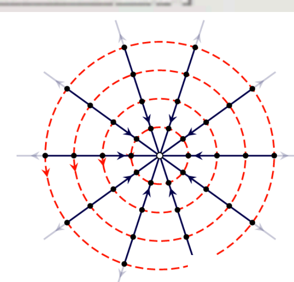
AdS_3/CFT_2

Strings different models with large parameter space

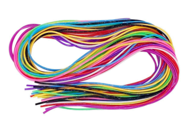
How do we solve these new types of integrable theories?



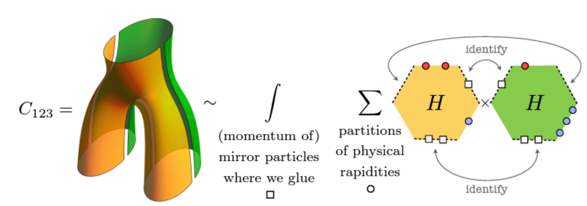
"It is a spin chain/lattice model!"



"It is a string!"



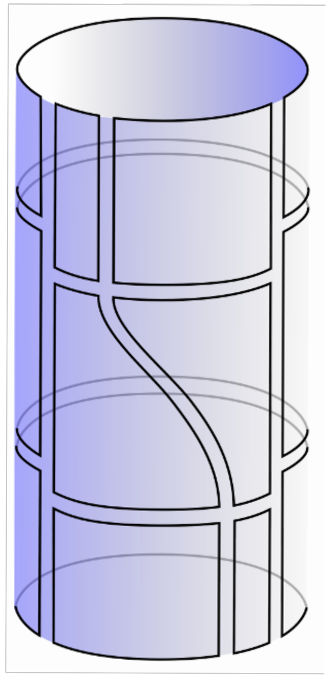
"The building blocks are hexagons!"



A lot of evidence for solvability of <u>all</u> correlators, at all orders in 1/Nc… but not there yet

Let's see what we can solve very well (non-perturbatively), how, and where we go from there

What we can really solve very well: the spectrum



$$\langle \mathcal{O}(x)\bar{\mathcal{O}}(y)\rangle \propto \frac{1}{|x-y|^{2\Delta_{\mathcal{O}}(g)}}$$

First conceptual solution:
works by R. Tateo et al. , Arutyunov et al. , Kazakov et al. '09

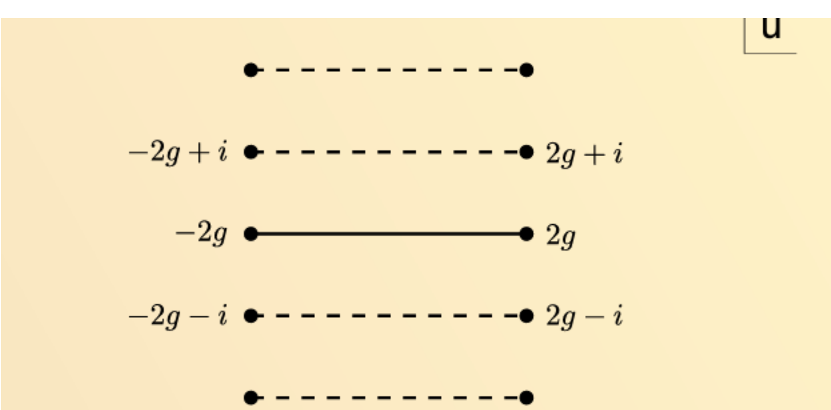
Modern very powerful method is called
*"**Quantum Spectral Curve**"*

[Gromov, Kazakov, Leurent, Volin'13]	$\mathcal{N}$=4 SYM
[AC, Gromov, Fioravanti, Tateo'14]	ABJM
[AC, Gromov, Stefanski Torrielli] +[Ekhammar,Volin] '21	AdS_3/CFT_2

What is it?

In each case, a complex analysis problem for "Q-functions" $Q_i(u)$

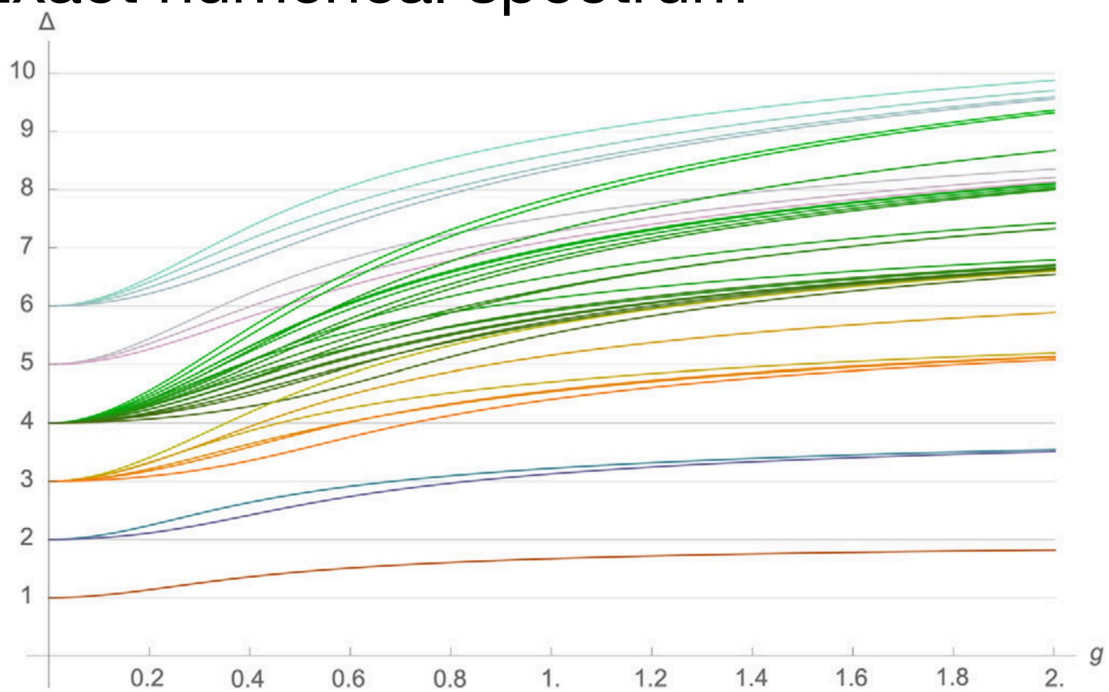
spectral parameter



solution <-> $Q_i(u)$ <-> point in the spectrum

Can solve (almost) any question on the spectrum

Exact numerical spectrum

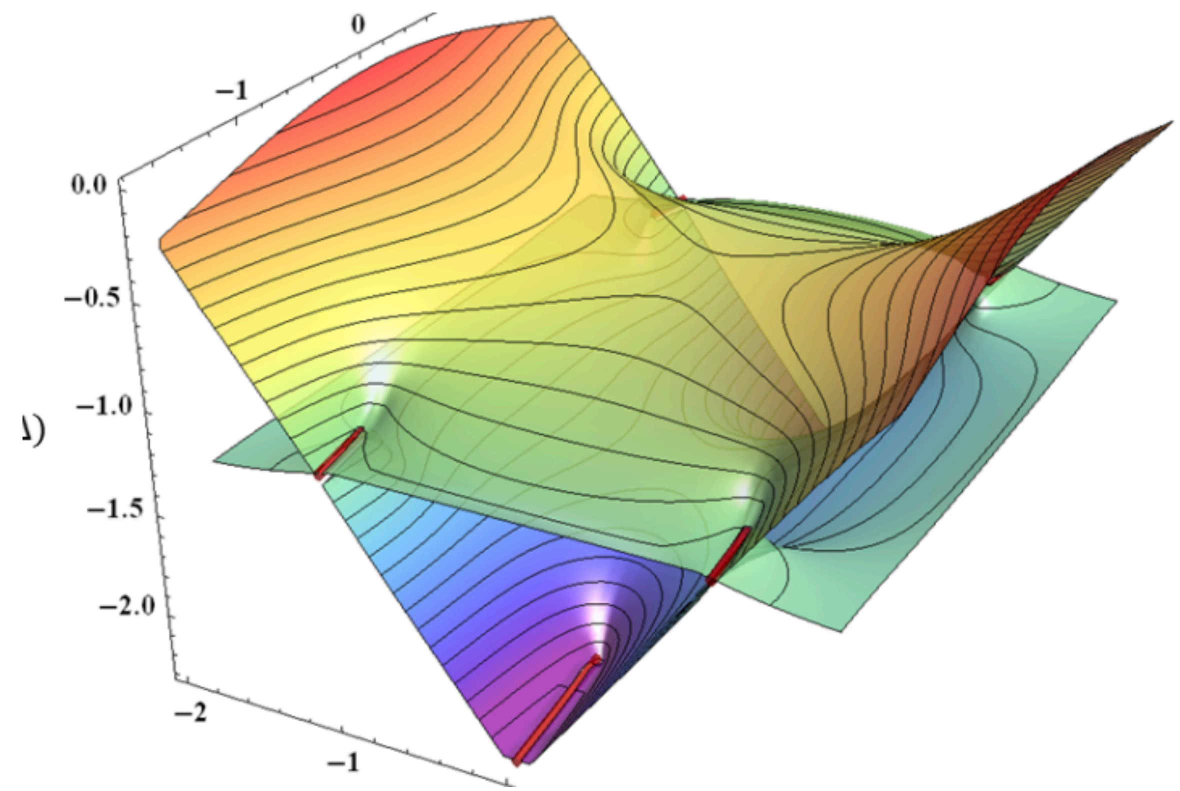


Spectrum of operators on Wilson line: [AC, Julius, Gromov, Preti '21]

Analytic continuation in Spin: Regge Trajectories

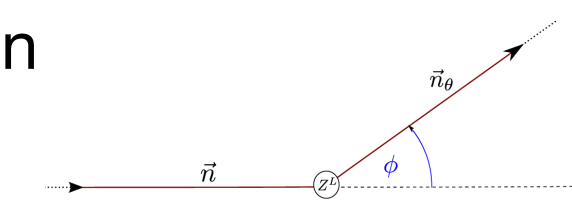
From [Gromov, Levkovich-Maslyuk, Sizov '15]

(also w.i.p. with R. Tateo and N. Brizio, V. Tripodi)



Cusp anomalous dimension

[Gromov, Levkovich-Maslyuk'15]



$$
\begin{aligned}
\Delta \;=\;& 4+12g^2-48g^4+336g^6+g^8\left(-2496+576\,\zeta_3-1440\,\zeta_5\right)\\
&+g^{10}\left(15168+6912\,\zeta_3-5184\,\zeta_3^2-8640\,\zeta_5+30240\,\zeta_7\right)\\
&+g^{12}\left(-7680-262656\,\zeta_3-20736\,\zeta_3^2+112320\,\zeta_5+155520\,\zeta_3\,\zeta_5+75600\,\zeta_7-489888\,\zeta_9\right)\\
&+g^{14}\left(-2135040+5230080\,\zeta_3-421632\,\zeta_3^2+124416\,\zeta_3^3-229248\,\zeta_5+411264\,\zeta_3\,\zeta_5\right.\\
&\quad\left.-993600\,\zeta_5^2-1254960\,\zeta_7-1935360\,\zeta_3\,\zeta_7-835488\,\zeta_9+7318080\,\zeta_{11}\right)\\
&+g^{16}\left(54408192-83496960\,\zeta_3+7934976\,\zeta_3^2+1990656\,\zeta_3^3-19678464\,\zeta_5-4354560\,\zeta_3\,\zeta_5\right.\\
&\quad-3255552\,\zeta_3^2\,\zeta_5+2384640\,\zeta_5^2+21868704\,\zeta_7-6229440\,\zeta_3\,\zeta_7+22256640\,\zeta_5\,\zeta_7\\
&\quad\left.+9327744\,\zeta_9+23224320\,\zeta_3\,\zeta_9+\frac{65929248}{5}\,\zeta_{11}-106007616\,\zeta_{13}-\frac{684288}{5}\,Z_{11}^{(2)}\right)
\end{aligned}
$$

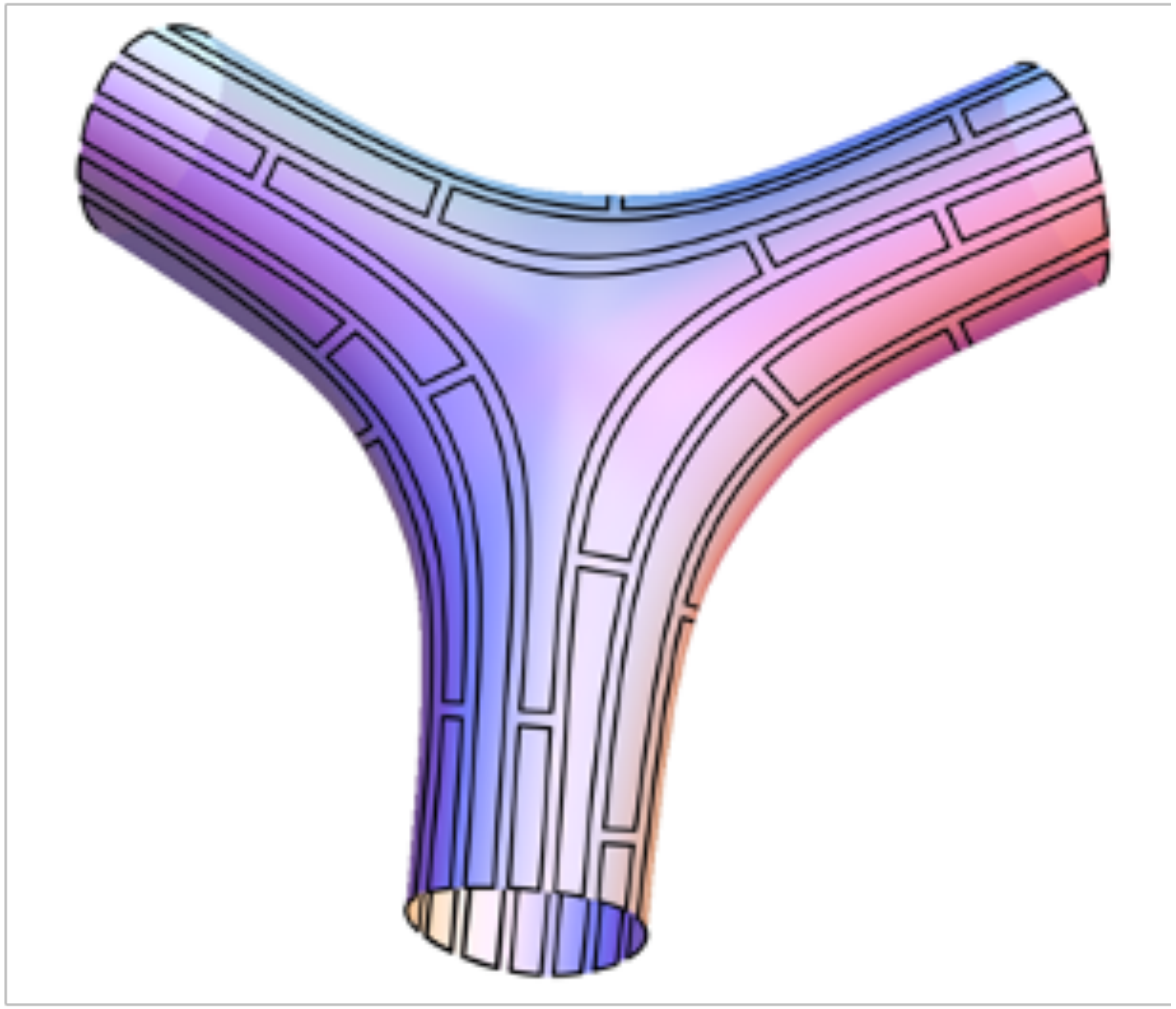
Analytic computations and numerology:

N=4 SYM: Multiple Zeta Values

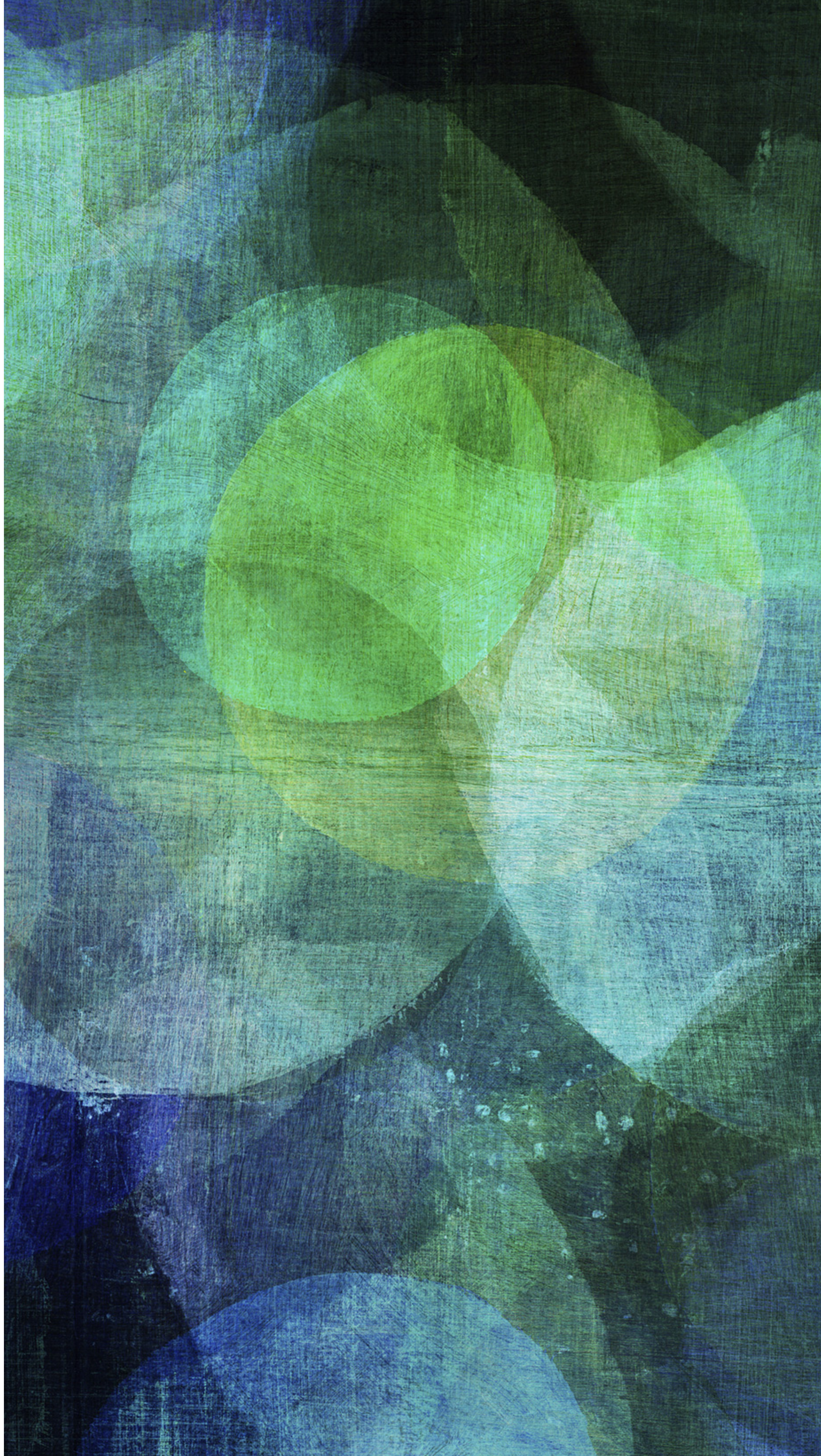
[Marboe, Volin '14]+…

ABJM: alternating Multiple Zeta Values

Anselmetti, Bombardelli, AC, Conti, Tateo]

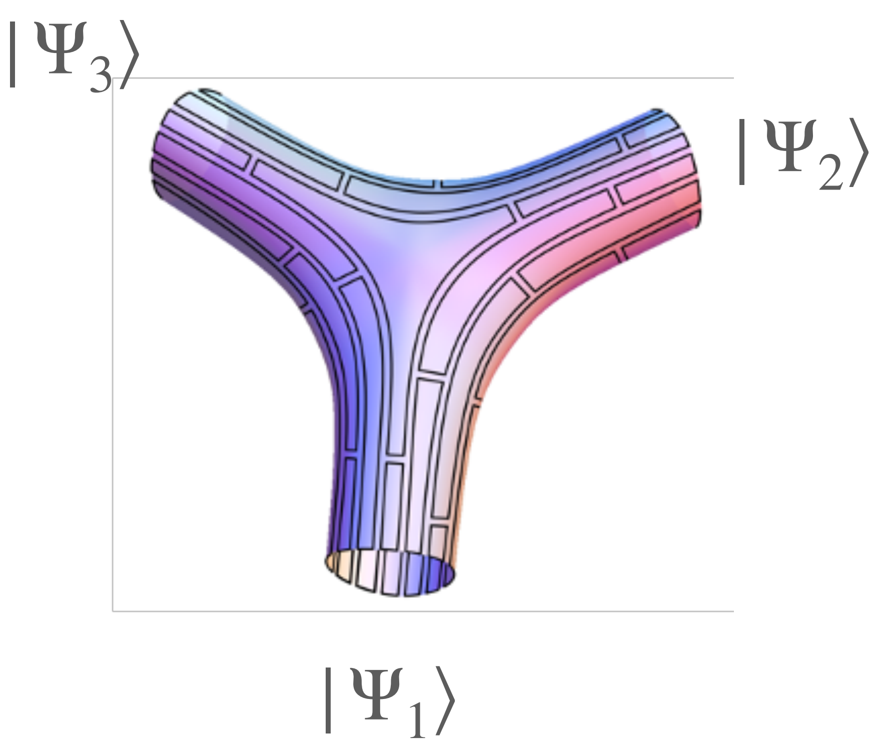


Beyond the spectrum



One of the exciting (and difficult) frontier problems of my research field is understanding how to link the Quantum Spectral Curve and correlation functions

Q-functions ↔ operator wave functions in a special basis

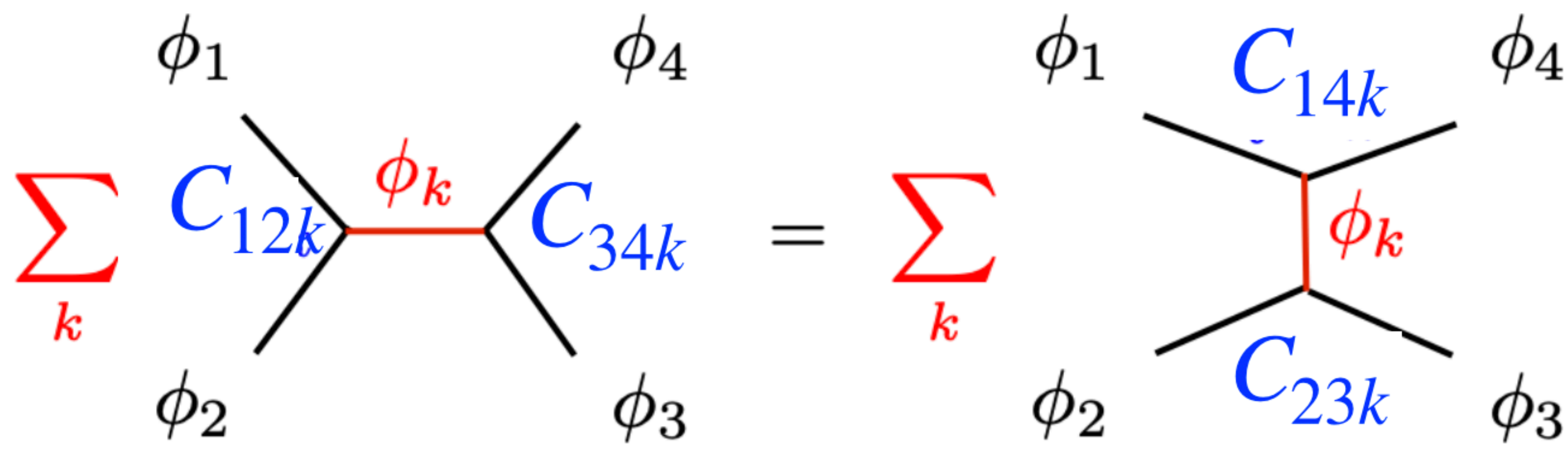


$$\propto \int \prod Q_1(\vec{x}) \prod Q_2(\vec{x}) \prod Q_3(\vec{x})\, d\mu(\vec{x}) \quad \textbf{?}$$

Much work is needed, but we are starting to see concrete examples of precisely this structure *[AC,Gromov,Levkovich-Maslyuk'18,'21] [Giombi, Komatsu '18][Basso Georgoudis '22] …*

Another way: Bootstrap

Conformal Field Theories satisfy consistency conditions



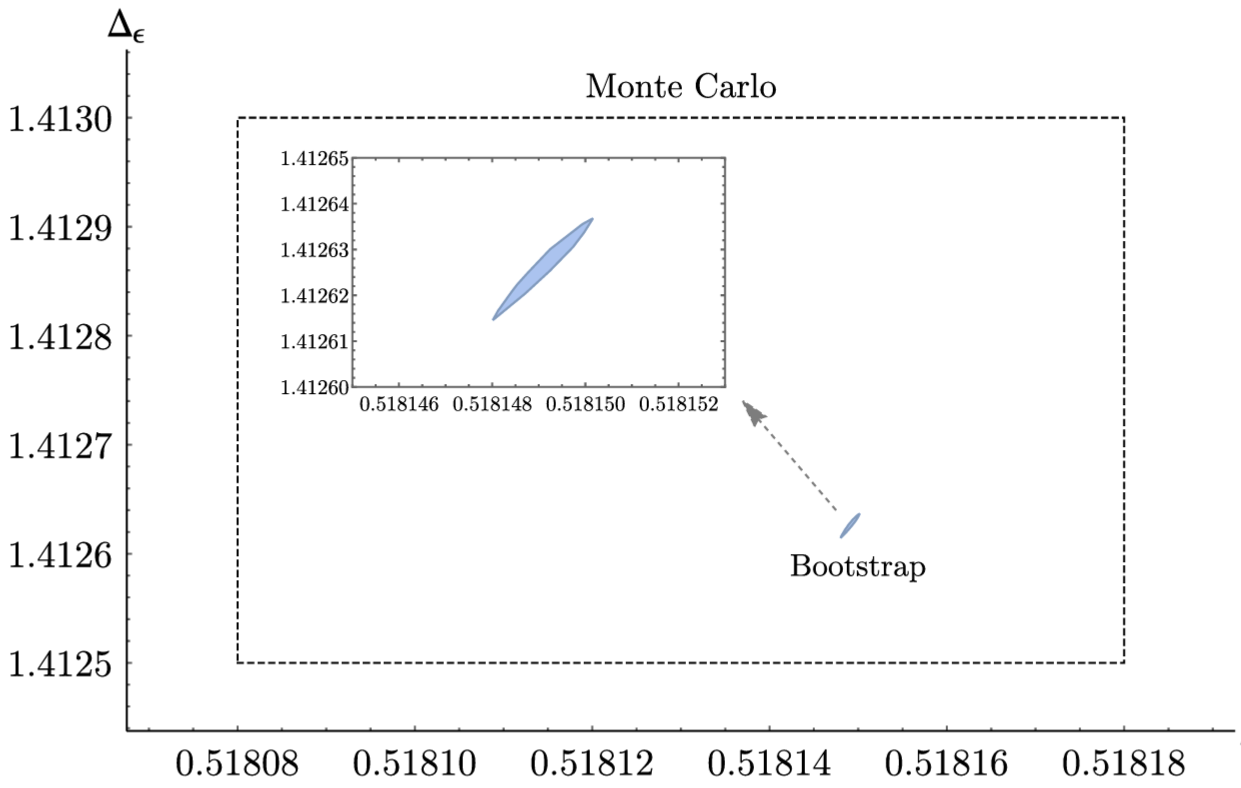
Crossing equations <-> conformal symmetry, unitarity, locality (OPE)

Not limited to large N_c .

Complicated consistency equations for the conformal data Δ_k , C_{ijk}

One of the recent revolutions in formal theory: analytical and numerical methods to exploit these constraints

e.g. critical exponents of Ising 3D [El-Showk,Paulos,Poland, Rychkov,Simmons-Duffin,Vichi '12]+…

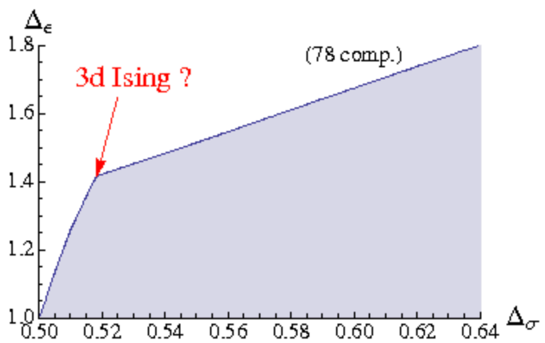


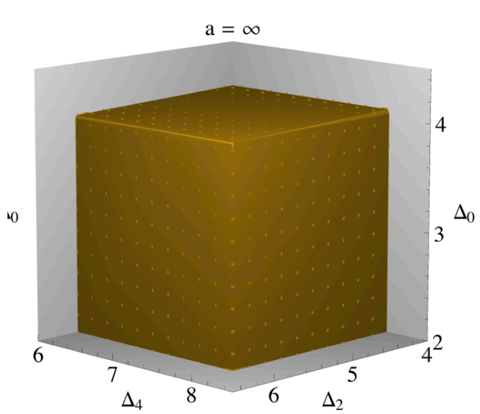
Numerical method gives **rigorous bounds.**

Do they converge to precise values? How?

What about theories with continuous parameters like N=4 SYM?

Are they just somewhere inside the bounds?

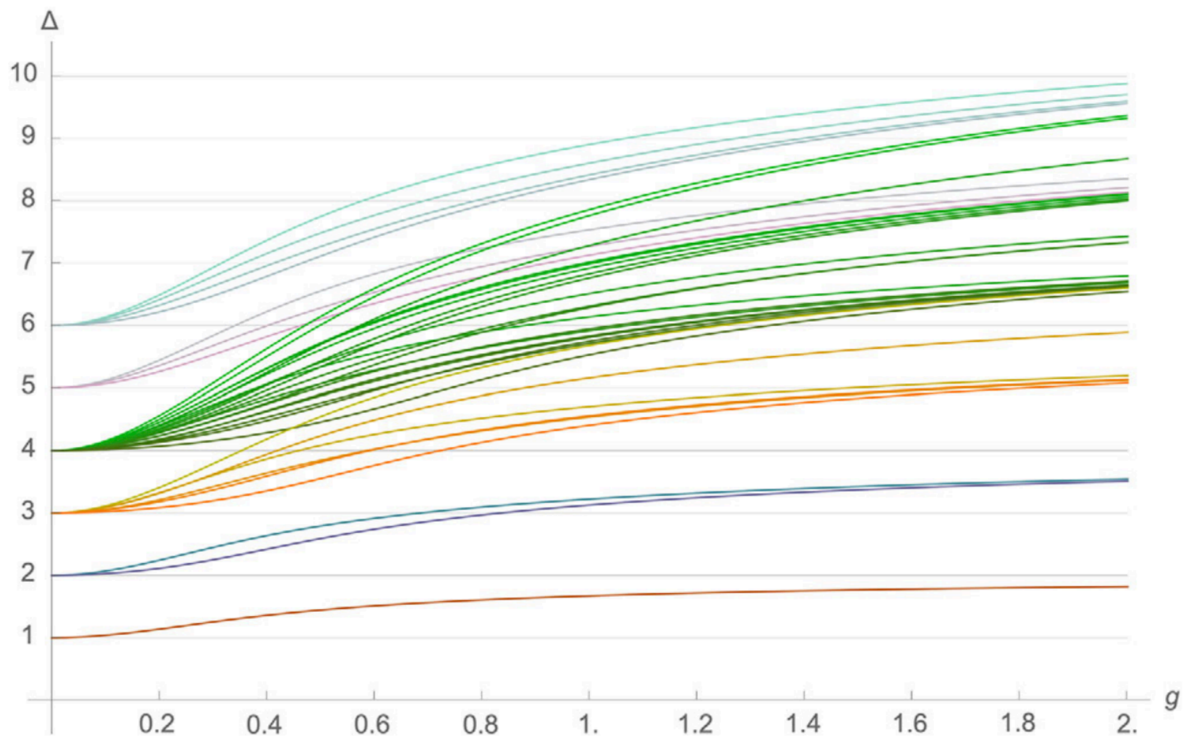


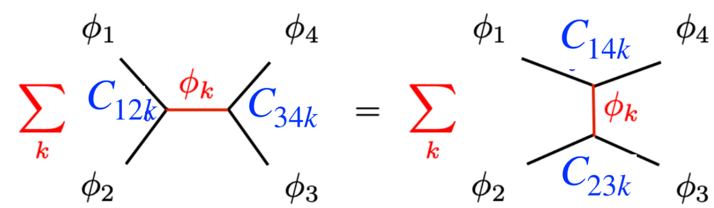


[Beem, Rastelli, van Rees '13]

Idea: combine integrability and conformal bootstrap.

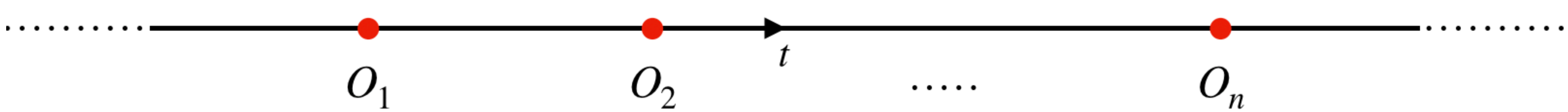
[AC, Gromov, Julius, Preti, '21-'23]





?

A nice setup (avoids mixing orders in 1/Nc) is the defect sector: operators inserted on a straight Wilson line



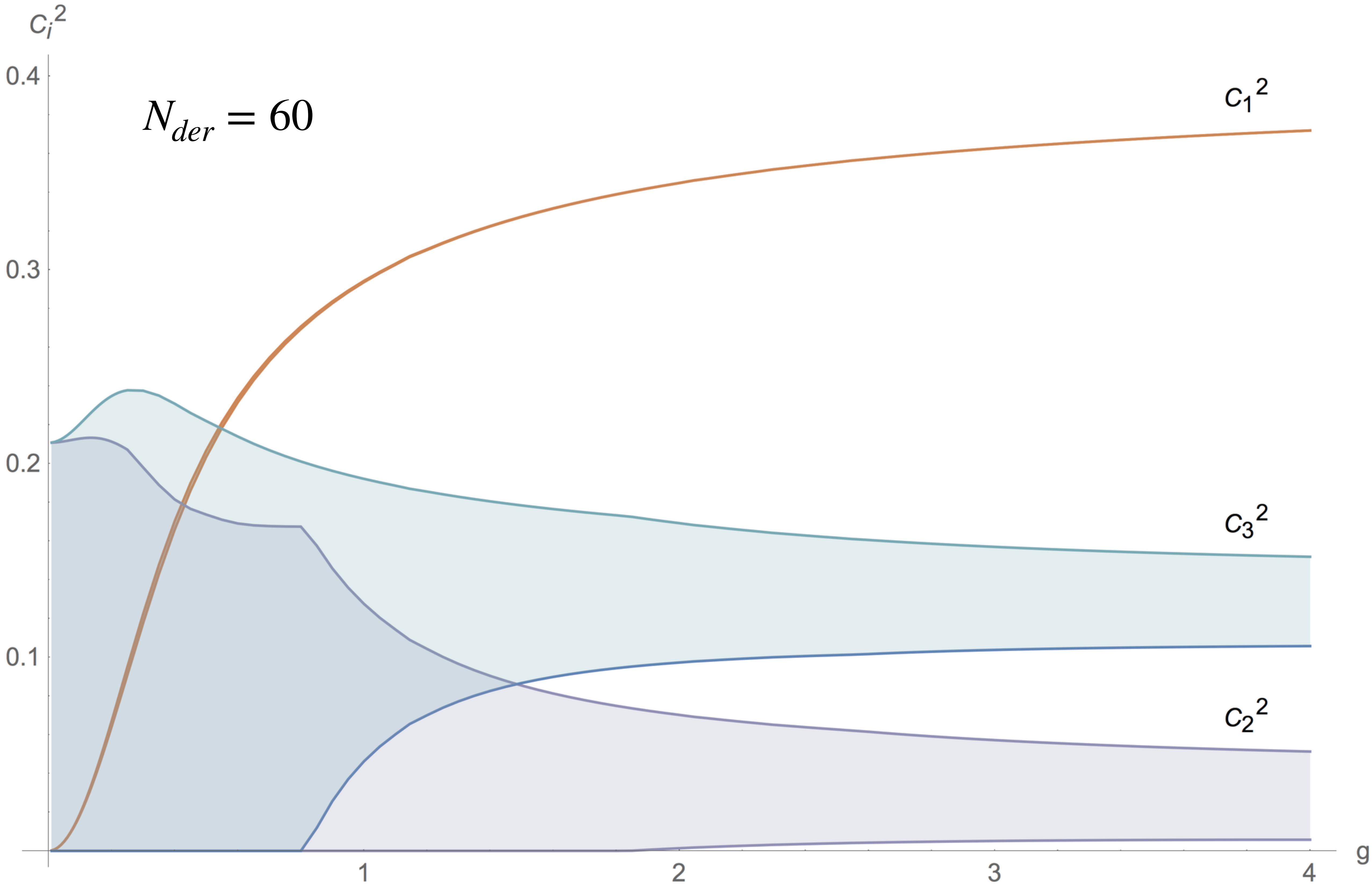
...

$$\langle\langle O_1(t_1) O_2(t_2) \dots O_n(t_n) \rangle\rangle$$

$$\equiv Tr\left[P e^{\int_{-\infty}^{t_1} dt\left(iA_t + \Phi_{||}\right)} O_1(t_1)\, e^{\int_{t_1}^{t_2} dt\left(iA_t + \Phi_{||}\right)} O_2(t_2) \dots O_n(t_n)\, e^{\int_{t_n}^{+\infty} dt\left(iA_t + \Phi_{||}\right)} \right]$$

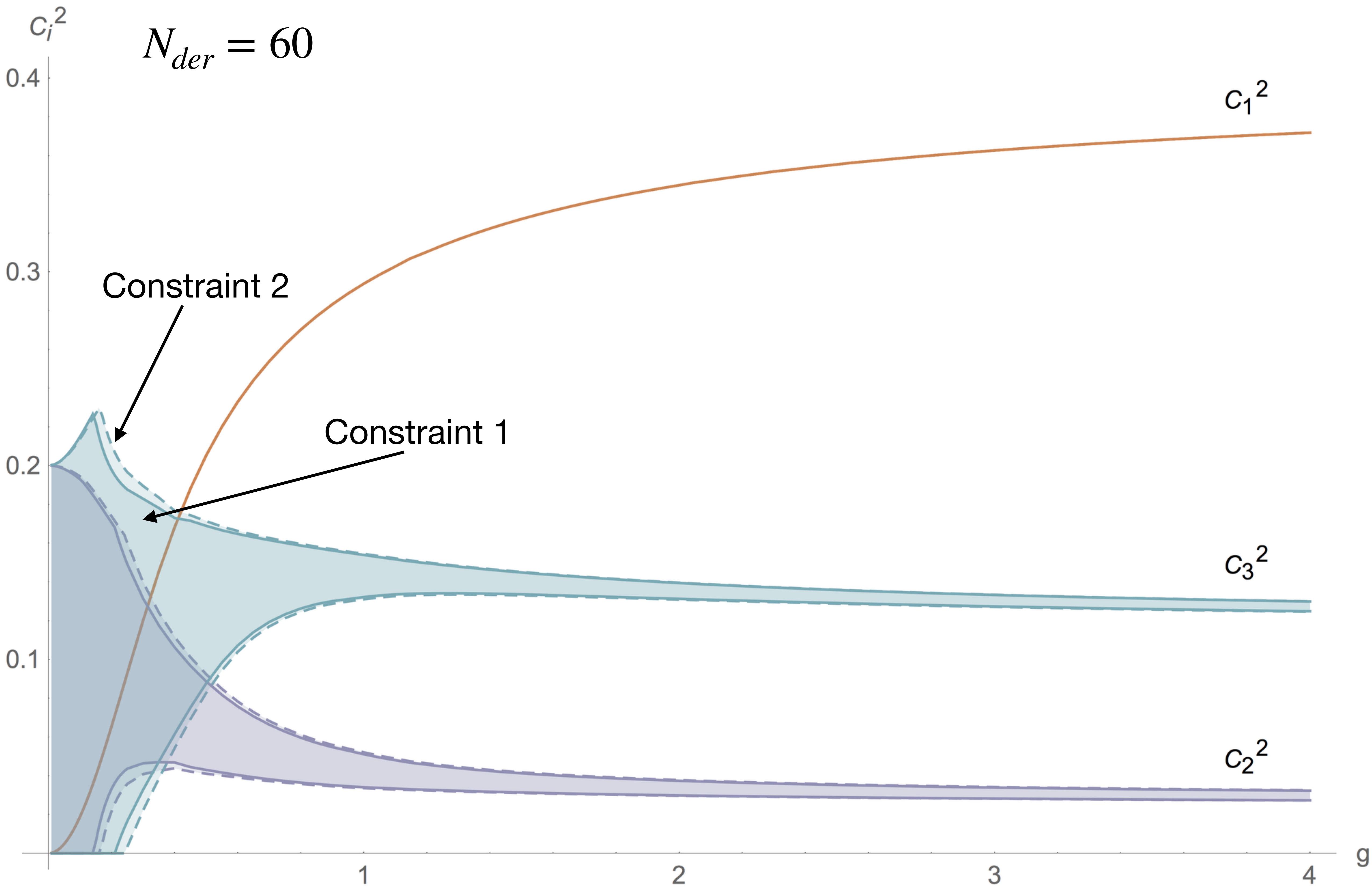
We can also get **extra constraints from integrability** considering deformations of the line into a cusp

First 3 OPE coefficients including the first 10 states

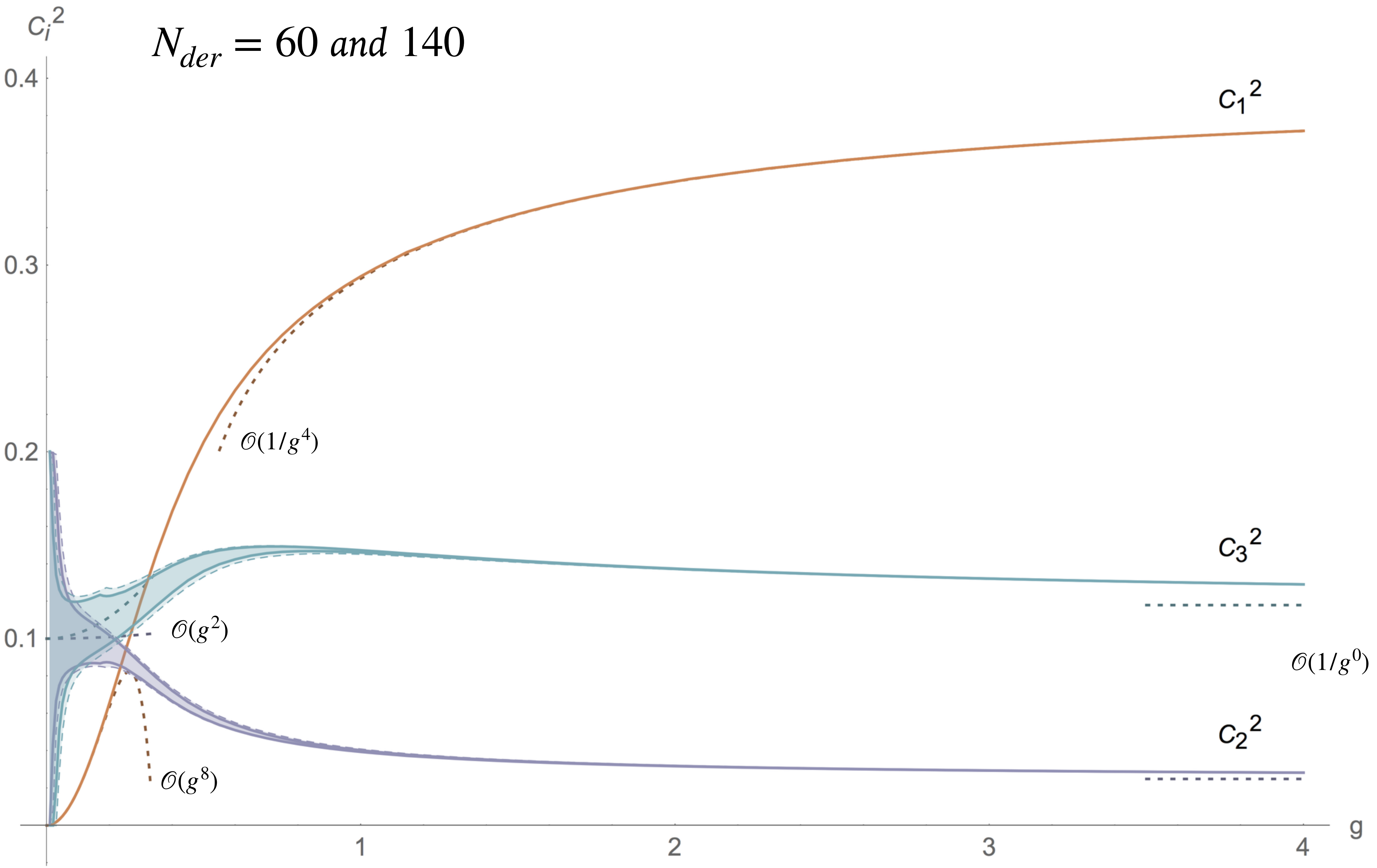


The error reduces from 16% to 75% depending on the coupling

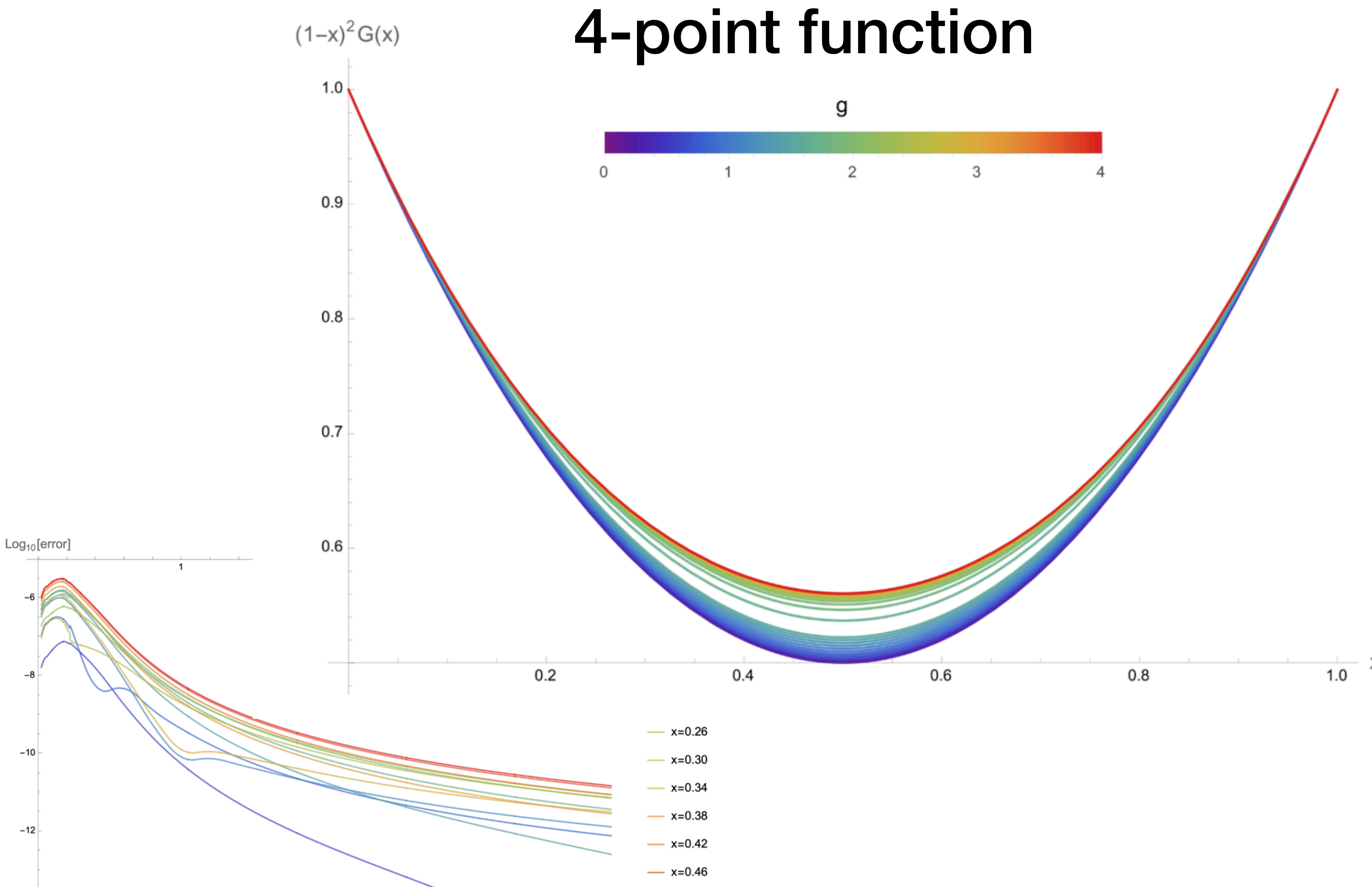
First 3 OPE coefficients including the first 10 states and 1 integrated correlator



First 3 OPE coefficients including the first 10 states and both integrated correlator



And surprisingly higher-point functions (not just constants) can be constrained tightly too



there is much more to explore

- Is there a procedure at least in principle to get arbitrary precision?
- Move off from the Wilson line to study the local theory in 4D, maybe using consistency equations with the defect?
  [Billo`, Goncalves, Lauria, Meineri ‘2016] [Bianchi ’21,’22
  See [Caron-Huot, Coronado Zahraee ’22]!
- Beyond large Nc?.
- And many more questions….

Conclusions

Integrability and conformal bootstrap seem to have a lot to say about some **gauge theories and AdS/CFT.**

We can already study **non-perturbative phenomena** and explore the **mathematical structures** of these models
e.g. perturbative series and MZV, radius of convergence, complex spin…

Until now the deepest understanding concerns the spectrum, but there are important indications that much more should be possible!

Thank you for your attention!

Thank you for your attention!

Extra

Bootstrability setup

The bootstrap equation obtained before is

$$\mathcal{G}_{1+\mathcal{B}_2}(\lambda,\chi) + \sum_{\Delta_n} C^2_{1,1,\Delta_n} \mathcal{G}_{\Delta_n}(\chi) = 0$$

Now we have access to the spectrum!

1D CFT is unitary $C^2_{1,1,\Delta} \geq 0$

Numerical bootstrap setup:

Try to find a linear functional α, satisfying desired inequalities on conf. blocks

↓

Bounds on the allowed conformal data.

[El-Showk,Paulos,Poland,Rychkov, Simmons-Duffin,Vichi '12]

This can be done efficiently with Semi Definite Programming.
We use the powerful package SDPB [Simmons-Duffin '15]

Truncation is necessary, but the **bounds are rigorous**:

$\alpha\left[f(\chi)\right] \sim \sum_{l=0}^{N_{der}} \alpha_l\, f^{(2l)}(\chi)\Big|_{\chi=\frac{1}{2}}$, bounds better and better for $N_{der} \to \infty$.

Bounds for the first OPE coefficient $C^2_{1,1,\Delta_1}$

$$\mathcal{G}_{1+\mathcal{B}_2}(\lambda,\chi) + C^2_{1,1,\Delta_1}\mathcal{G}_{\Delta_1}(\chi) + \sum_{n\geq 2} C^2_{1,1,\Delta_n}\mathcal{G}_{\Delta_n}(\chi) = 0$$

Using SDPB, find the functional such that

Upper bound

$$\alpha^{upper}[\mathcal{G}_{\Delta_1}] = 1\,,$$

$$\alpha^{upper}[\mathcal{G}_{\Delta}] \geq 0 \text{ for } \Delta \geq \Delta_* \equiv \Delta_2$$

$$\alpha^{upper}[\mathcal{G}_{1+\mathcal{B}_2}] \text{ is maximal } \equiv -\mathbf{B}_{upper}$$

Lower bound

$$\alpha^{lower}[\mathcal{G}_{\Delta_1}] = -1\,,$$

$$\alpha^{lower}[\mathcal{G}_{\Delta}] \geq 0 \text{ for } \Delta \geq \Delta_* \equiv \Delta_2$$

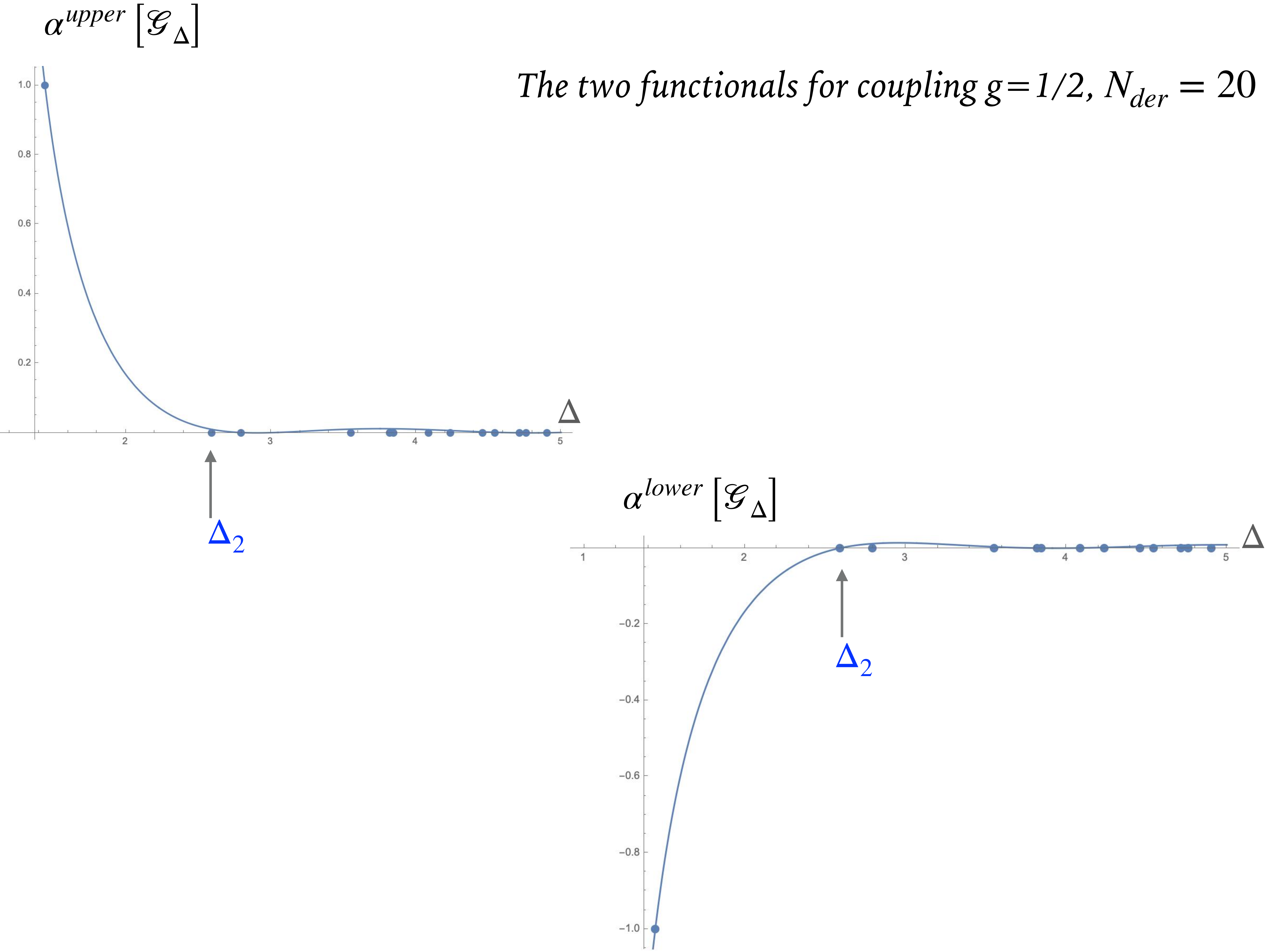
$$\alpha^{lower}[\mathcal{G}_{1+\mathcal{B}_2}] \text{ is maximal } \equiv \mathbf{B}_{lower}$$

$$-\mathbf{B}_{upper} + C^2_{1,1,\Delta_1} + (\geq 0 \quad quantity) = 0$$

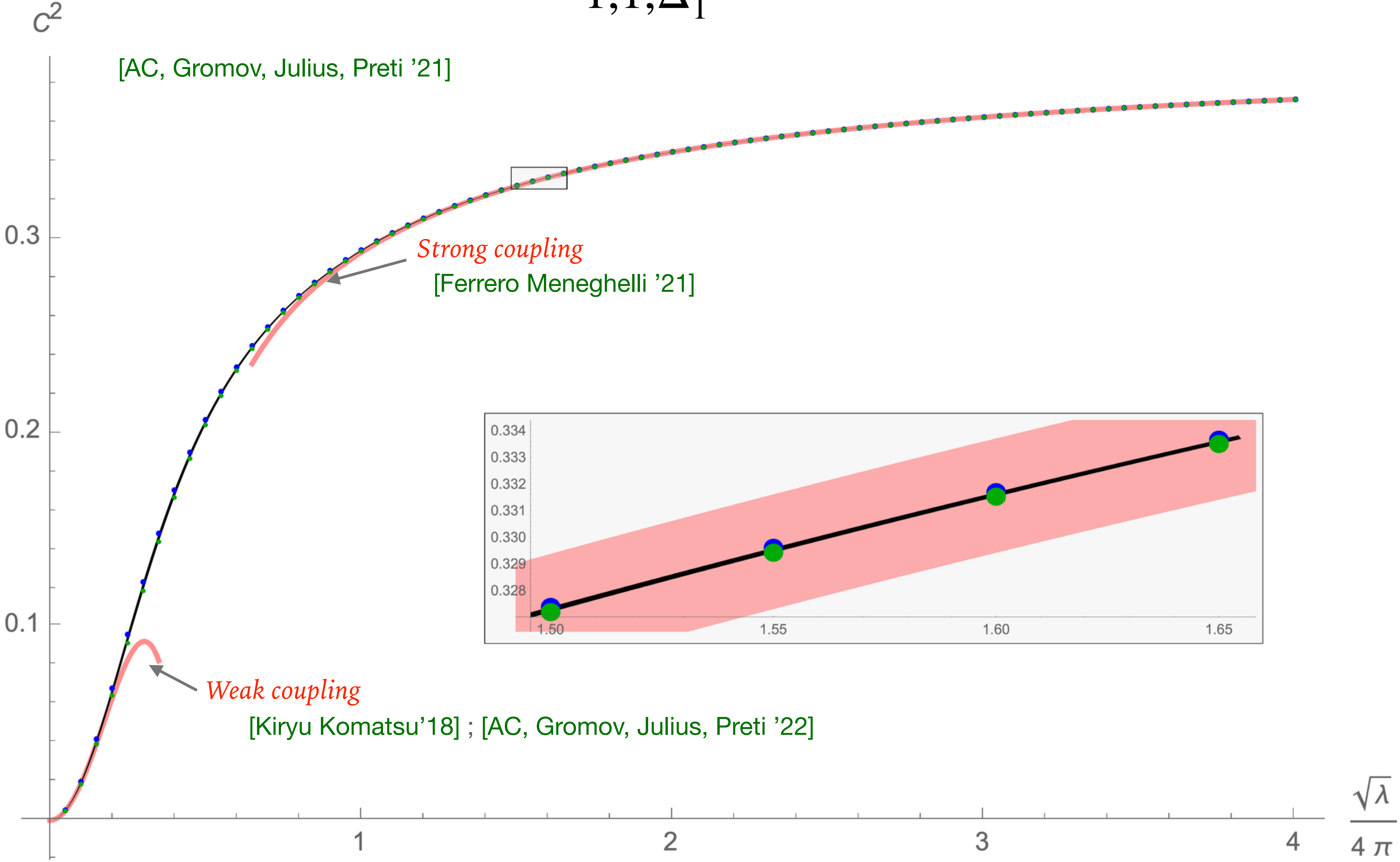
$$+\mathbf{B}_{lower} - C^2_{1,1,\Delta_1} + (\geq 0 \quad quantity) = 0$$

$$\mathbf{B}_{lower} \leq C^2_{1,1,\Delta_1} \leq \mathbf{B}_{upper}$$

The two functionals for coupling g=1/2, $N_{der} = 20$



OPE coefficient $C^2_{1,1,\Delta_1}$ including only 2 states



The error is computed measuring the thickness of the region, namely $1/2(C^2_{\text{upper}} - C^2_{\text{lower}})$

Strong coupling has higher precision (10^{-4}) than weak coupling (10^{-2})

Integrated correlators

Constraint 1

$$\int_0^1 \delta\mathscr{A}(\chi)\frac{1+\log\chi}{\chi^2}d\chi - \frac{3\mathbb{C}-\mathbb{B}}{8\ \mathbb{B}^2} = 0$$

Constraint 2

$$\int_0^1 \frac{\delta f(\chi)}{\chi}d\chi - \frac{\mathbb{C}}{4\ \mathbb{B}^2} - \mathbb{F} + 3 = 0$$