Equivariant Donaldson-Witten theory

Ekaterina Sysoeva

Università di Torino, Sezione INFN di Torino

Santo Stefano Belbo 11-12 Nov 2023

Outline

1. Review. What is equivariant Donaldson-Witten theory?
2. Equivariant localisation on $\mathbb{R}^4$. Recurrence relation for instantons
3. Localisation on a $4D$ sphere. How we can use our formula
4. Equivariant localisation on compact toric manifolds

Supersymmetric field theory

- We consider en extension of Poincaré algebra with $\mathcal{N}$ fermionic generators

$$Q_u = \begin{pmatrix} Q_{\alpha u} \\ \overline{Q}^{\dot{\alpha} u} \end{pmatrix}, \quad u = 1, \ldots, \mathcal{N}$$

- Super Poincaré algebra

$$\{Q^u_\alpha, \overline{Q}_{\dot{\beta} v}\} = 2\delta^u_v \sigma^\mu_{\alpha\dot{\beta}} P_\mu, \qquad [P_\mu, Q_{\alpha u}] = 0,$$
$$[M_{\mu\nu}, Q_{\alpha u}] = -(\sigma_{\mu\nu})^\beta_\alpha Q_{\beta u}, \qquad \{Q_{\alpha u}, Q_{\beta v}\} = 0,$$
$$[P_\mu, \overline{Q}_{\dot{\alpha} u}] = 0, \qquad [M_{\mu\nu}, \overline{Q}^{\dot{\alpha} u}] = -(\overline{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\dot{\beta}} \overline{Q}^{\dot{\beta} u}$$

- There is $U(\mathcal{N})$ symmetry called an R-symmetry

$$Q_{\alpha u} \to U^w_u Q_{\alpha w}, \quad \overline{Q}^u_{\dot{\alpha}} \to U^{*w}_u \overline{Q}^w_{\dot{\alpha}}$$

Topological field theories

There are two types of the topological field theories.

1°. Shwartz type: Action is explicitly metric independent.

2°. Cohomological type: There is an explicit metric dependence of the action, but the correlation functions do not depend on the metric.

- Q - scalar supersymmetry of the action and

$$T_{\mu\nu} = \frac{\delta S}{\delta g^{\mu\nu}} = -\mathrm{i}\, Q G \mu\nu$$

- $\mathcal{O}_i$ - Q-invariant operators

$$\frac{\delta}{\delta g^{\mu\nu}} \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} \rangle = 0 \qquad \langle \mathcal{O}_{i_1} \cdots \mathcal{O}_{i_n} (QV) \rangle = 0$$

- Topological observables

$$\mathcal{O} \in \frac{\mathrm{Ker\, Q}}{\mathrm{Im\, Q}}$$

Topological twist of the $\mathcal{N}=2$ supersymmetric theory

- Field content of $\mathcal{N}=2$ theory: gauge field A_μ, two gluinos $\lambda_{v\alpha}$, complex scalar ϕ, auxiliary field D_{uv}
- Wick rotation $\mathbb{R}^{(3,1)} \to \mathbb{R}^4$
- Total symmetry group

$$\overbrace{SU(2)_- \times SU(2)_+}^{SO(4)} \times SU(2)_R \times U(1)$$

$$\underbrace{SU(2)_+ \times SU(2)_R}_{SU(2)'_+}$$

- $SU(2)_R$ indices u, $v \to \dot\alpha$, $\dot\beta$
- Twisted supercharges

$$\mathcal{Q} = \varepsilon^{\dot\alpha\dot\beta}\overline{Q}_{\dot\alpha\dot\beta}, \quad \mathcal{Q}_\mu = \frac{1}{2}(\sigma_\mu)^{\dot\alpha\beta}Q_{\dot\alpha\beta},$$

$$\mathcal{Q}^+_{\mu\nu} = \frac{1}{2}\varepsilon^{\alpha\beta}(\sigma_\mu)^{\dot\alpha\alpha}(\sigma_\nu)^{\dot\beta\beta}(\overline{Q}_{\dot\alpha\dot\beta} + \overline{Q}_{\dot\beta\dot\alpha})$$

Twisted $\mathcal{N} = 2$ theory

- Twisted group

$$SU(2)_- \times SU(2)_+ \times SU(2)_R \times U(1)_R \to SU(2)_- \times SU(2)'_+ \times U(1)_R$$

- Twisted spin content of the fields

$$\begin{array}{rcl}
A_\mu \, (1/2, 1/2, 0)^0 & \to & A_\mu \, (1/2, 1/2)^0 \\
\lambda_{v\alpha} \, (1/2, 0, 1/2)^{-1} & \to & \psi_\mu \, (1/2, 1/2)^1 \\
\bar{\lambda}_{v\dot{\alpha}} \, (1/2, 0, 1/2)^1 & \to & \eta \, (0,0)^{-1}, \; \chi^+_{\mu\nu} \, (1,0)^{-1} \\
\phi \, (0,0,0)^{-2} & \to & \phi \, (0,0)^{-2} \\
\phi^\dagger \, (0,0,0)^2 & \to & \phi^\dagger \, (0,0)^2 \\
D_{uv} \, (0,0,1)^0 & \to & D^+_{\mu\nu} \, (1,0)^1
\end{array}$$

- No spinors, no more!

Donaldson-Witten theory on a manifold

- Twisted action is $\mathcal{Q}$-exact up to a topological term

$$S = \mathcal{Q}V - \frac{1}{2}\int F \wedge F$$

- Semi-classical approximation is exact!

$$\langle \mu_1 \cdots \mu_n \rangle = \int [\mathcal{D}] \mu_1 \cdots \mu_n \, e^{-\frac{1}{g^2}\mathcal{Q}V}$$

$$\frac{\partial}{\partial g}\langle \mu_1 \cdots \mu_n \rangle = \frac{2}{g^3}\langle \mu_1 \cdots \mu_n (\mathcal{Q}V) \rangle = 0$$

With $g \to 0$ the integral reduces to the integral over the classical solutions.

Equivariant twisted $\mathcal{N} = 2$ theory on $\mathbb{R}^4$

- We can get a theory on $\mathbb{R}^4$ from a theory in 6D space $\mathbb{C} \times \mathbb{R}^4$ with flat metric

$$\mathrm{d}s^2 = A\,\mathrm{d}z\mathrm{d}\bar{z} + \mathrm{d}x^2$$

by a standard torus identification

$$(z,\, \bar{z},\, x) \sim (z + n + m\tau,\, \bar{z} + n + m\bar{\tau},\, x)$$

- Nekrasov suggested to consider a theory on the Ω-background. Two ways to understand what it is:
  1°. Keep the flat metric, but change the identification

$$(z,\, \bar{z},\, x) \sim (z + n + m\tau,\, \bar{z} + n + m\bar{\tau},\, g_1^n g_2^m x)$$

  g_1, g_2 are two commuting rotations on $\mathbb{R}^4$

  2°. Keep the standard identification, but change the metric

$$\mathrm{d}s^2 = A\mathrm{d}z\mathrm{d}\bar{z} + g_{\mu\nu}(\mathrm{d}x^\mu + \xi^\mu_\Omega \mathrm{d}z + \bar{\xi}^\mu_\Omega \mathrm{d}\bar{z})(\mathrm{d}x^\nu + \xi^\nu_\Omega \mathrm{d}z + \bar{\xi}^\nu_\Omega \mathrm{d}\bar{z})$$

$\xi^\mu_\Omega = \Omega^\mu_\nu x^\nu$, $\bar{\xi}^\mu_\Omega = \overline{\Omega}^\mu_\nu x^\nu$ and Ω^μ_ν are the generators of the rotations g_1, g_2 on $\mathbb{R}^4$.

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \quad \overline{\Omega}^{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix}.$$

Nekrasov: in a theory on the Ω-background the supersymmetric charge preserving the action is not $\mathcal{Q}$ but

$$\tilde{\mathcal{Q}} = \mathcal{Q} + \Omega^\mu_\nu x^\nu \mathcal{Q}_\mu.$$

- Now we take observables from the equivariant cohomology $\mathcal{O} \in \frac{\mathrm{Ker}\,\tilde{\mathrm{Q}}}{\mathrm{Im}\,\tilde{\mathrm{Q}}}$

Equivariant localisation

- The scalar supersymmetric generator turns out to be an equivariant differential on the space of the field configurations.
- We can use the Localisation theorem to compute the correlators.

Localisation theorem

Let $\mathcal{M}$ be a compact oriented D-dimensional smooth manifold equipped with the action of a group G with X being the locus of the fixed points. Then the integral of a closed equivariant form α is given by the localization formula:

$$\int_{\mathcal{M}} \alpha = (-2\pi)^{\frac{D}{2}} \int_X \frac{\alpha_0(x)}{e_G(\mathcal{N}X)},$$

where $\alpha_0(x)$ is the zero-form part of α at the fixed point x.

Equivariant localisation on $\mathbb{R}^4$. Nekrasov partition function

- $\mathbf{a} = (a_1, \ldots, a_N) \in \mathbb{C}^N$ are the vacuum expectation values of the Higgs field. $SU(N)$ theory: $\sum_{u=1}^N a_u = 0$

$$\mathcal{Z}(\mathbf{a}) = Z_{\mathrm{pert}}(\mathbf{a}) Z_{\mathrm{inst}}(\mathbf{a}) \qquad Z_{\mathrm{inst}}(\mathbf{a}) = \sum_{k=0}^{\infty} q^k Z_k(\mathbf{a})$$

- Terms are parametrized by N Young diagrams $\vec{Y} = (Y_1, \ldots, Y_N)$ with the total number of boxes $|\vec{Y}| = k$.

$$Z_k(\mathbf{a}) = \sum_{\substack{\vec{Y} \\ |\vec{Y}|=k}} \Bigg(\prod_{(i,j)\in Y_u} (a_{vu} + \epsilon_1(i - \tilde{l}_{Y_v,j}) - \epsilon_2(j - 1 - l_{Y_u,i})) \cdot \prod_{(i,j)\in Y_v} (a_{vu} - \epsilon_1(i - 1 - \tilde{l}_{Y_u,j}) + \epsilon_2(j - l_{Y_v,i})) \Bigg)^{-1}$$

$l_{Y,i}$ is the length of the i-th row of diagram Y,
$\tilde{l}_{Y,i}$ is the length of the i-th column of diagram Y.

Zamolodchikov recurrence relation

- There is a recurrence relation in $SU(2)$ theory

$$Z_{\rm inst}(a) = 1 + \sum_{m,\, n=1}^{\infty} \frac{q^{mn} Z_{\rm inst}(\epsilon_{m,-n})}{(-a+\epsilon_{m,n})(a+\epsilon_{m,n})} \frac{2\epsilon_{m,n}}{\prod_{\substack{i=-m+1 \\ (i,j)\neq(0,0)}}^{m} \prod_{j=-n+1}^{n} \epsilon_{i,j}}$$

- $\epsilon_{m,n} = m\epsilon_1 + n\epsilon_2$, $a = a_1 - a_2 \triangleq a_{12}$
- Poles are simple and located at $a = \epsilon_{m,n}$ with $m \cdot n > 0$.

$$\mathrm{Res}_{a=\epsilon_{m,n}} Z_{\rm inst}(a) = -q^{mn} \frac{Z_{\rm inst}(\epsilon_{m,-n})}{\prod_{\substack{i=-m+1 \\ (i,j)\neq(0,0)}}^{m} \prod_{j=-n+1}^{n} \epsilon_{i,j}}$$

- Equivalent form

$$\lim_{\alpha\to 0} \frac{\mathcal{Z}(\alpha+\epsilon_{m,n})}{\mathcal{Z}(\alpha+\epsilon_{m,-n})} = -\mathrm{Sign}(\epsilon_1).$$

Partial Weyl permutation

- $\mathbf{m}, \mathbf{n} \in \mathbb{Z}^N$ are points of reference for $\mathbf{a}$

$$
\begin{aligned}
a_1 &= \alpha_1 + m_1\epsilon_1 + n_1\epsilon_2 & \hat{a}_1^{(uv)} &= \alpha_1 + m_1\epsilon_1 + n_1\epsilon_2 \\
&\dots & &\dots \\
a_u &= \alpha_u + m_u\epsilon_1 + \mathbf{n_u}\epsilon_2 & \hat{a}_u^{(uv)} &= \alpha_u + m_u\epsilon_1 + \mathbf{n_v}\epsilon_2 \\
&\dots \quad \rightarrow & &\dots \\
a_v &= \alpha_v + m_v\epsilon_1 + \mathbf{n_v}\epsilon_2 & \hat{a}_v^{(uv)} &= \alpha_v + m_v\epsilon_1 + \mathbf{n_u}\epsilon_2 \\
&\dots & &\dots \\
a_N &= \alpha_N + m_N\epsilon_1 + n_N\epsilon_2 & \hat{a}_N^{(uv)} &= \alpha_N + m_N\epsilon_1 + n_N\epsilon_2
\end{aligned}
$$

- α_u is arbitrary (not necessary small)

Residue formula

- Poles are simple and located at $a_{uv} = \epsilon_{mn}$ with $m \cdot n > 0$
- Residue of $Z_{\rm inst}(\mathbf{a})$ w.r.t. a_{uv} is proportional to the value $Z_{\rm inst}(\hat{\mathbf{a}}^{(\mathbf{uv})})$.

$$\lim_{\alpha_{uv}\to 0} \frac{\mathcal{Z}(\mathbf{a})}{\mathcal{Z}(\hat{\mathbf{a}}^{(uv)})} = -\mathrm{Sign}(\epsilon_1)$$

It is **exact** with respect to all variables except α_{uv}.

- In terms of only instanton part if has the form

$$\mathrm{Res}_{a_{uv}=\epsilon_{m,n}} Z_{\rm inst}(\mathbf{a}) = q^{mn} \frac{1}{\mathcal{P}_N^{(uv)}(m,n|\mathbf{a})} Z_{\rm inst}(\hat{\mathbf{a}}^{(uv)}),$$

where

$$\mathcal{P}_N^{(uv)}(m,n|\mathbf{a}) = \prod_{i=-m}^{m-1} {\prod_{j=-n}^{n-1}}' \epsilon_{i,j} \cdot \prod_{\substack{w=1\\ w\neq u,v}}^{N} \prod_{i=1}^{m} \prod_{j=1}^{n} [(a_{vw}+\epsilon_{i,j})(-a_{uw}+\epsilon_{i,j})]$$

Recurrence relation

- We chose $N-1$ independent variables to be a_{uN}, $u = 1 \dots N-1$ and assume that $a_{\hat{u}N}$, $\hat{u} = 2 \dots N-1$ are away from the poles as well as their differences $a_{\hat{u}N} - a_{\hat{v}N}$.
- $Z_{\text{inst}}(\mathbf{a})$ has poles only with respect to a_{1N} at the points $a_{1N} = \epsilon_{m,n}$ and $a_{1N} = \epsilon_{m,n} + a_{\hat{u}N}$

$$Z_{\text{inst}}(\mathbf{a}) = 1 + \sum_{w=2}^{N} \sum_{m,\,n=1}^{\infty} \frac{q^{mn} Z_{\text{inst}}(\hat{\mathbf{a}}^{(1w)})}{(a_{1N} - a_{wN} + \epsilon_{m,n})(a_{1N} - a_{wN} - \epsilon_{m,n})} \cdot \frac{2\epsilon_{m,n}}{\mathcal{P}_N^{(1w)}(m,n|\mathbf{a})}$$

[E. Sysoeva & A. Bykov. (2023). Recurrence relation for instanton partition function in SU(N) gauge theory. JHEP]

Localisation on S^4. Pestun's formula

- Partition function of **nonequivariant** $\mathcal{N} = 2$ theory on S^4

$$\mathcal{Z}_{S^4} = \frac{1}{\mathrm{vol}(G)} \int_{\mathfrak{g}} [\mathrm{d}a] e^{-2\pi \mathrm{i} \tau r^2 a^2} Z^{S^4}_{\mathrm{1loop}}(a) |Z^{\mathbb{R}^4}_{inst}(a)|^2$$
$$= \frac{1}{\mathrm{vol}(G)} \int_{\mathbb{R}^N} \mathrm{d}^N a \, |Z^{\mathbb{R}^4}_{\mathrm{pert}}(a) Z^{\mathbb{R}^4}_{\mathrm{inst}}(a)|^2$$

  is expressed via the **equivariant** Nekrasov partition function with the equivariant parameters $\epsilon_1 = \epsilon_2 = r^{-1}$.
- It is a strong instrument to compute different correlators on a sphere.
- Z_{inst} complicates calculations, especially in the case of $SU(N)$ gauge group, nut something has been done.

$$\int \prod_{i=1}^{4} \mathrm{d}x_i \mu(\{x_i\}) \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4) \rangle = \frac{1}{4} \Delta_\tau \partial_m^2 \log \mathcal{Z}_{S^4} \Big|_{m=0}$$

$4D$ compact toric manifold

Compact toric manifold - algebraic torus $(\mathbb{C}^*)^2$ appropriately compactified by gluing with $\mathbb{CP}^1$ surfaces (divisors).

- Toric manifold comes with $U(1)^2$ action.
- There are χ fixed points of $U(1)^2$ and a standard cover by χ coordinate patches with a fixed point at the origin.
- Each patch is a copy of $\mathbb{C}^2$ with $U(1)^2$ acting with the local weights ϵ_1^ℓ, ϵ_2^ℓ.
- It is a complex manifold which admit a Kähler form ω.
- Examples of compact toric manifolds: $\mathbb{CP}^2$, $S^2 \times S^2$, ...

Partition function of the theory

We want to compute

$$\mathcal{Z}_{\mathcal{M}} = \int [\mathrm{d}A][\mathrm{d}\Psi][\mathrm{d}\Phi][\mathrm{d}\bar{\Phi}][\mathrm{d}\eta][\mathrm{d}\chi^+][\mathrm{d}B^+] e^{-\int_{\mathcal{M}} \mathcal{L}}$$

where the Lagrangian is

$$\mathcal{L} = (\text{topological term}) + \mathcal{Q}\mathcal{V},$$

and

$$\mathcal{V} = -\mathsf{Tr}\Big[\mathrm{i}\chi^{(0,2)} \wedge F^{(2,0)} + \mathrm{i}\chi(\omega \wedge F - \lambda\, \omega \wedge \omega\, \mathbb{1}) + \Psi \wedge \star(\mathcal{Q}\Psi)^\dagger + \eta \wedge \star(\mathcal{Q}\eta)^\dagger\Big].$$

- Nekrasov conjectured

$$\mathcal{Z}_{\mathcal{M}} = \sum_{\mathbf{k} \in \mathcal{R}} \mathrm{Res}_{\alpha=0} \prod_{\ell=1}^{\chi} \mathcal{Z}^{\mathbb{R}^4}(\alpha + k^\ell \epsilon_1^\ell + k^{\ell+1} \epsilon_2^\ell),$$

$\mathcal{R}$ is restricted by the stability conditions.

DI on compact toric manifolds in $SU(N)$

After a very long computation the answer turns out to be

$$\mathcal{Z}_{\mathcal{M}} = \lim_{\delta\to 0} \sum_{\mathbf{k}\in\mathbb{Z}^{\chi(N-1)}} \mathrm{JK}\left[e^{i\delta\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}}\mathcal{Z}(\boldsymbol{\alpha},\mathbf{k})\right]$$

where

$$\mathcal{Z}(\boldsymbol{\alpha},\mathbf{k}) = \prod_{\ell=1}^{\chi} \mathcal{Z}^{\mathbb{R}^4}(\alpha + k^\ell\epsilon_1^\ell + k^{\ell+1}\epsilon_2^\ell, q e^{\Omega_\ell})$$

- Stability conditions are hidden in $e^{i\delta\boldsymbol{\kappa}\cdot\boldsymbol{\alpha}}$

$$\boldsymbol{\kappa}\cdot\boldsymbol{\alpha} = \sum_{u=1}^{N}\kappa_u\alpha_u \qquad \kappa_u = \sum_{\ell=1}^{\chi} k_u^\ell \int_{\mathcal{D}_\ell}\omega$$

Some more explicit answers

- With the $SU(2)$ gauge group we get

$$\mathcal{Z}_{\mathcal{M}} = \sum_{k^\ell \in \mathbb{Z}^\chi} \mathrm{sign}(\kappa) \mathrm{Res}_{\alpha=0} \mathcal{Z}(\alpha, \mathbf{k})$$

[Bonelli et al. (2021). Gauge theories on compact toric manifolds.]

- With the $SU(3)$ gauge group we get

$$\mathcal{Z}_{\mathcal{M}} = \sum_{\mathbf{k} \in \mathbb{Z}^{2\chi}} \mathrm{sign}(\kappa_2 + \kappa_3 - 2\kappa_1)\, \mathrm{sign}(\kappa_1 + \kappa_3 - 2\kappa_2) \mathrm{Res}_{\alpha_{12}=0} \mathrm{Res}_{\alpha_{23}=0} \mathcal{Z}(\boldsymbol{\alpha}, \mathbf{k})$$

- Number of terms is actually finite.
- Numerical computations are very demanding in terms of the computational power even with the $SU(3)$ gauge group.

Possible directions of future research

- Direct application of the recurrence relation to some computational problems
- Understand what the recurrence relation means in terms of the dual CFT theory
- Consider non-commutative instantons on compact toric manifolds
- Build the 5D and 6D theories on compact toric manifolds

Thank you for the attention!