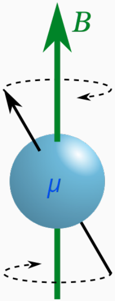
The muon $(g-2)$ theoretical calculation

Giuseppe Gagliardi, INFN Sezione di Roma Tre

XXV Roma Tre topical workshop: The muon anomalous magnetic moment.
December 11th

Introduction: the magnetic moment of a lepton



The magnetic moment $\boldsymbol{\mu}$ of a charged object parameterizes the torque that a static magnetic field exerts on it.

For a charged spin-$1/2$ particle:

$$\boldsymbol{\mu} = g \frac{e}{2m} \boldsymbol{S}$$

g is the well-known gyromagnetic factor.

In QFT the response of a charged lepton (say a muon μ) to a static and uniform e.m. field is encoded in ($k = p_1 - p_2$)

$$\langle \mu(p_2) | J^{\nu}_{\mathrm{em}}(0) | \mu(p_1) \rangle = -ie\bar{u}(p_1)\Gamma^{\nu}(p_1, p_2)u(p_2)$$

Lorentz invariance and e.m. current conservation constrain Γ^{ν}-structure:

$$\Gamma^{\nu}(p_1, p_2) = F_1(k^2)\gamma^{\nu} + \frac{i}{2m_{\mu}} F_2(k^2)\sigma^{\nu\rho}k_{\rho} + \text{P-violating terms}$$

The muon anomalous magnetic moment

Gyromagnetic factor g_μ related to form-factors $F_1(k^2)$ and $F_2(k^2)$ through

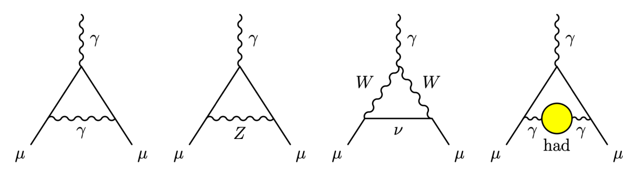
$$g_\mu = 2\left[F_1(0) + F_2(0)\right]$$

- Electric charge conservation $\implies F_1(0) = 1$.
- At tree level in the SM: $F_2(0) = 0 \implies g_\mu = g_\mu^{\text{Dirac}} \equiv 2$.

The muon anomalous magnetic moment:

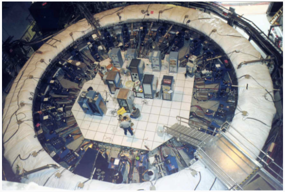
$$a_\mu = \frac{g_\mu - 2}{2} = F_2(0)$$

non-zero only at loop level. Contributions from all SM (and BSM) fields. E.g.



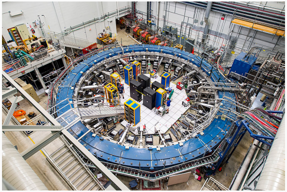
If very precisely measured can be a crucial probe of the completeness of the SM. Is it?

Latest update (August ’23) from FNAL experiment

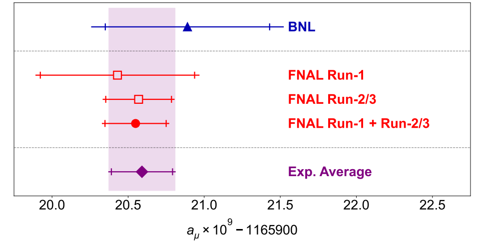


$g_\mu - 2$ @BNL (up to 2006) $\Longrightarrow$

transfer to Fermilab $\Longrightarrow$



$g_\mu - 2$ @Fermilab



$$a_\mu^{\text{exp}} = 116\,592\,059(22) \times 10^{-11} \quad [0.19\text{ppm}]$$

Congratulations!!

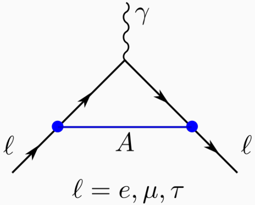
Results from Run-4/5/6 expected in 2025

Why did we pick the muon (and not e, τ) ?

Electron anomalous magnetic moment is measured with even higher precision (×1000):

$$a_e^{\text{exp}} = 1\,159\,652\,180.73(28) \times 10^{-12} \quad [0.0002\,\text{ppm}]$$

However, NP contributions expected to be



$$\Longrightarrow \quad a_\ell^A \propto m_\ell^2 / m_A^2$$

$$m_\mu^2 / m_e^2 \simeq 43\,000$$

a_τ would have a much higher enhancement due to NP but decays too fast...

$$-0.052 < a_\tau^{\text{exp}} < 0.013$$

Can we match, on the theory side, the experimental accuracy on a_μ?

The Muon $g-2$ Theory Initiative

The muon $g-2$ TI has been established in 2017 with the aim of matching the precision of the SM-theory prediction for a_μ with the experimental one.

https://muon-gm2-theory.illinois.edu

- Composed by experts in lattice QCD, dispersive approach, perturbative calculations, ...
- First white paper out in '20 [Physics Reports 887 (2020)]. Second in preparation.

The anomalous magnetic moment of the muon in the Standard Model

T. Aoyama[1,2,3], N. Asmussen[4], M. Benayoun[5], J. Bijnens[6], T. Blum[7,8], M. Bruno[9], I. Caprini[10],
C. M. Carloni Calame[11], M. Cè[9,12,13], G. Colangelo[†14], F. Curciarello[15,16], H. Czyż[17], I. Danilkin[12], M. Davier[†18],
C. T. H. Davies[19], M. Della Morte[20], S. I. Eidelman[†21,22], A. X. El-Khadra[†23,24], A. Gérardin[25], D. Giusti[26,27],
M. Golterman[28], Steven Gottlieb[29], V. Gülpers[30], F. Hagelstein[14], M. Hayakawa[31,2], G. Herdoíza[32], D. W. Hertzog[33],
A. Hoecker[34], M. Hoferichter[†14,35], B.-L. Hoid[36], R. J. Hudspith[12,13], F. Ignatov[21], T. Izubuchi[37,8], F. Jegerlehner[38],
L. Jin[7,8], A. Keshavarzi[39], T. Kinoshita[40,41], B. Kubis[36], A. Kupich[21], A. Kupść[42,43], L. Laub[14], C. Lehner[†26,37],
L. Lellouch[25], I. Logashenko[21], B. Malaescu[5], K. Maltman[44,45], M. K. Marinković[46,47], P. Masjuan[48,49],
A. S. Meyer[37], H. B. Meyer[12,13], T. Mibe[†1], K. Miura[12,13,3], S. E. Müller[50], M. Nio[2,51], D. Nomura[52,53],
A. Nyffeler[†12], V. Pascalutsa[12], M. Passera[54], E. Perez del Rio[55], S. Peris[48,49], A. Portelli[30], M. Procura[56],
C. F. Redmer[12], B. L. Roberts[†57], P. Sánchez-Puertas[49], S. Serednyakov[21], B. Shwartz[21], S. Simula[27],
D. Stöckinger[58], H. Stöckinger-Kim[58], P. Stoffer[59], T. Teubner[†60], R. Van de Water[24], M. Vanderhaeghen[12,13],
G. Venanzoni[61], G. von Hippel[12], H. Wittig[12,13], Z. Zhang[18],

M. N. Achasov[21], A. Bashir[62], N. Cardoso[47], B. Chakraborty[63], E.-H. Chao[12], J. Charles[25], A. Crivellin[64,65],
O. Deineka[12], A. Denig[12,13], C. DeTar[66], C. A. Dominguez[67], A. E. Dorokhov[68], V. P. Druzhinin[21], G. Eichmann[69,47],
M. Fael[70], C. S. Fischer[71], E. Gámiz[72], Z. Gelzer[23], J. R. Green[9], S. Guellati-Khelifa[73], D. Hatton[19],
N. Hermansson-Truedsson[14], S. Holz[36], B. Hörz[74], M. Knecht[25], J. Koponen[1], A. S. Kronfeld[24], J. Laiho[75],
S. Leupold[42], P. B. Mackenzie[24], W. J. Marciano[37], C. McNeile[76], D. Mohler[12,13], J. Monnard[14], E. T. Neil[77],
A. V. Nesterenko[68], K. Ottnad[12], V. Pauk[12], A. E. Radzhabov[78], E. de Rafael[25], K. Raya[79], A. Risch[12],
A. Rodríguez-Sánchez[6], P. Roig[80], T. San José[12,13], E. P. Solodov[21], R. Sugar[81], K. Yu. Todyshev[21], A. Vainshtein[82],
A. Vaquero Avilés-Casco[66], E. Weil[71], J. Wilhelm[12], R. Williams[71], A. S. Zhevlakov[78]

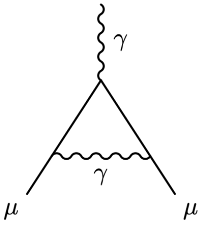
- Last TI meeting in Bern, next one at KEK (Tsukuba, Japan).

The muon magnetic moment in the SM

a_μ can be decomposed into QED, weak and hadronic contributions

$$a_\mu = \underbrace{a_\mu^{\text{QED}}}_{>99.99\%} + a_\mu^{\text{weak}} + \underbrace{a_\mu^{\text{had}}}_{\text{non-perturbative}}$$

- The QED contribution to a_μ is completely dominant. LO (1-loop) contribution evaluated by J. Schwinger in 1948

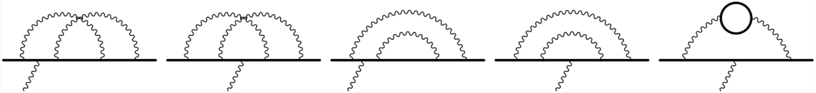


$$\Longrightarrow a_\mu^{\text{QED},1-\text{loop}} = \frac{\alpha}{2\pi}$$

- Since Schwinger's calculation many more QED-loops included...

The QED contribution a_μ^{QED}

Two-loops QED contributions to a_μ



To match experimental accuracy $\Delta a_\mu^{\exp} \simeq \mathcal{O}(10^{-10})$ several orders in the perturbative α expansion need to be considered

$$a_\mu^{\mathrm{QED}} = \frac{\alpha}{2\pi} + \sum_{n=2}^{\infty} C_\mu^n \left(\frac{\alpha}{\pi}\right)^n$$

- Number of Feynman diagrams quickly rises with n: $1, 7, 72, 891, 12672, \ldots$
- Heroic effort to compute them up to five-loops [T. Aoyama et al. PRLs, 2012]

$C_\mu^6 \left(\frac{\alpha}{\pi}\right)^6 \simeq C_\mu^6 \times 10^{-16}$ requires unnaturally large $C_\mu^6 \simeq \mathcal{O}(10^6)$ to be relevant!!

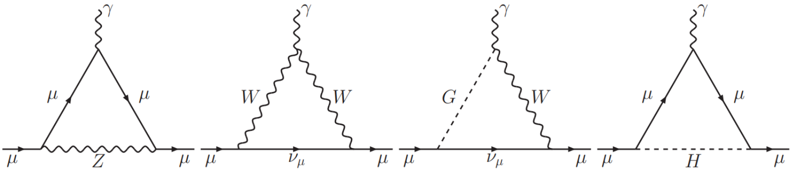
$$a_\mu^{\mathrm{QED}} = 116\,584\,718.931(104) \times 10^{-11} \checkmark$$

a_μ^{weak} defined as the sum of all loop diagrams containing at least a W, H, Z.

- Smallest of the three contributions due to Fermi-scale suppression:

$$a_\mu^{\text{weak}} \propto \alpha_W^2 \frac{m_\mu^2}{M_W^2} \simeq \mathcal{O}(10^{-9})$$

Sample of one-loop weak diagrams:



- At target precision of ~ 0.1 ppm two-loops calculation is sufficient [Czarnecki et al PRD (2006), Gnendiger et al PRD (2013)].

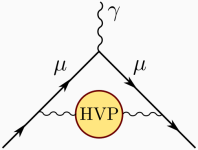
$$a_\mu^{\text{weak}} = 153.6(1.0) \times 10^{-11} \checkmark$$

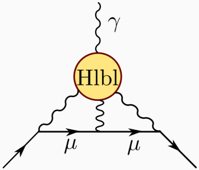
The hadronic contribution a_μ^{had}

Contributions to a_μ^{had} at target accuracy of $\mathcal{O}(10^{-10})$:

$$a_\mu^{\mathrm{had}} = \underbrace{a_\mu^{\mathrm{HVP-LO}}}_{\mathcal{O}(7\times 10^{-8})} + \underbrace{a_\mu^{\mathrm{Hlbl}}}_{\mathcal{O}(10^{-9})} + \underbrace{a_\mu^{\mathrm{HVP-NLO}}}_{\mathcal{O}(10{-}9)} + \underbrace{a_\mu^{\mathrm{HVP-NNLO}}}_{\mathcal{O}(10{-}10)}$$

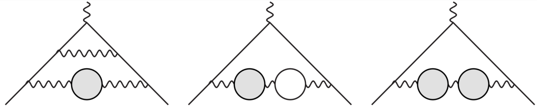
HVP-LO $\Longrightarrow$





$\Longleftarrow$ Hlbl

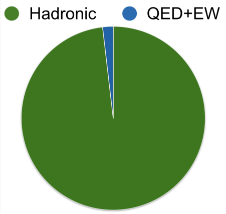
- NLO and NNLO HVP contributions relevant at target accuracy. At NLO:



- However, they can obtained from same non-perturbative input of $a_\mu^{\mathrm{HVP-LO}}$. Hence we shall discuss only the latter.

How important are hadronic contributions?

The uncertainty in the theory prediction for a_μ dominated by the hadronic contribution, despite its smallness



Dominant source of uncertainty is $a_\mu^{\mathrm{HVP-LO}}$

- Hadronic contributions are fully non-perturbative.
- Two main approaches to evaluate them:

Dispersive approach:

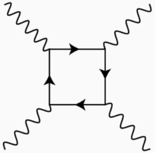
- Relates full $a_\mu^{\mathrm{HVP-LO}}$ to $e^+e^- \to$ hadrons cross-section via optical theorem.
- For Hlbl (only) low-lying intermediate-states contributions can expressed in terms of transition form-factors TFFs.

Lattice QCD:

- Only known first-principles SM method to evaluate both a_μ^{HVP} and a_μ^{Hlbl}.
- In the past the accuracy of the predictions were not good enough. The situation has recently changed.

The hadronic light-by-light contribution

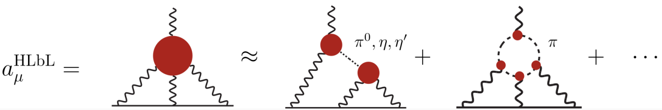
$a_\mu^{\rm Hlbl}$ occurs at $\mathcal{O}(\alpha^3)$. Related to $2 \to 2$ (generally virtual) photons scattering



It involves the fourth-rank VP tensor:

$$T\langle 0|J^\mu J^\nu J^\rho J^\sigma|0\rangle = \Pi^{\mu\nu\rho\sigma}(k_1,\dots,k_4)$$

- In the dispersive framework [Colangelo et al. JHEP09 (2015)] one isolates the dominant intermediate-states contributions:



- parameterized by transition form-factors **TFFs**. For dominant π^0-pole contr.

$$i\int d^4x e^{iqx} T\langle 0|J^\mu(x)J^\nu(0)|\pi^0(p)\rangle = \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta F_{\pi^0\gamma^*\gamma^*}(q^2,(q-p)^2)$$

TFFs from dispersion relations (using available exp. input) or recently from LQCD.

The hadronic light-by-light on the lattice

The cleanest, assumptions-independent, way of computing a_μ^{Hlbl} is given by Lattice QCD. The lattice QCD input is the 4-point correlation function of e.m. currents

$$\Pi^{\mu\nu\rho\sigma}(x,y,z,w) = T\langle 0|J^\mu(x)J^\nu(y)J^\rho(z)J^\nu(w)|0\rangle$$

- Long distance contribution very noisy. Noise rapidly increases reaching m_π^{phys}.
- Clever tricks employed to reduce computational cost. Lattice input can be compressed into

$$i\hat{\Pi}^{\rho,\mu\nu\lambda\sigma}(x,y) = \int dz\, z^\rho \langle 0|J^\mu(x)J^\nu(y)J^\sigma(z)J^\lambda(0)|0\rangle$$

$$a_\mu^{\mathrm{Hlbl}} = \frac{m_\mu e^6}{3}\int d^4y \int d^4x \underbrace{\mathcal{L}_{[\rho,\sigma],\mu\nu\lambda}(x,y)}_{\text{QED kernel}}\, i\underbrace{\hat{\Pi}^{\rho,\mu\nu\lambda\sigma}(x,y)}_{\text{QCD input}}$$

- So far two lattice Collaborations have fully computed a_μ^{Hlbl}:

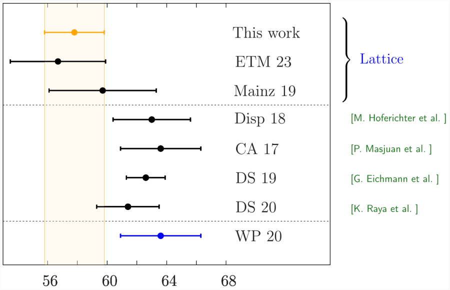
RBC/UKQCD ('21, '23) and MAINZ ('22).

Summary of current status for a_μ^{Hlbl}

[Taken from A. Gerardin talk at TI Meeting '23, Bern]

Dispersive framework ('21)	$a_\mu \times 10^{11}$
π^0, η, η'	93.8 ± 4
pion/kaon loops	-16.4 ± 0.2
S-wave $\pi\pi$	-8 ± 1
axial vector	6 ± 6
scalar + tensor	-1 ± 3
q-loops / short. dist. cstr	15 ± 10
charm + heavy q	3 ± 1
sum	92 ± 19
Mainz '22	109.6 ± 15.9
RBC/UKQCD '23	124.7 ± 15.2

$a_\mu^{\text{Hlbl};\pi^0} \times 10^{11}$ [This work=BMWc]

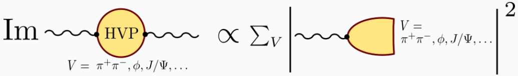


- Lattice calculations of a_μ^{Hlbl} in line, though with somewhat larger central values, with the dispersive result from WP '20.
- Lattice calculations of $a_\mu^{\text{Hlbl};\pi^0}$ slightly smaller (1.7σ) than dispersive one.
- 10% accuracy goal for a_μ^{Hlbl} seems achievable. Many lattice Coll. working on both full a_μ^{Hlbl} and pseudoscalar TFFs (π^0, η, η').

The LO hadronic-vacuum-polarization (HVP) contribution

$a_\mu^{\mathrm{HVP-LO}}$ is the largest of the hadronic contributions.

- Until '20 LQCD calculations above percent level accuracy.
- However, $a_\mu^{\mathrm{HVP-LO}}$ is related to $\sigma(\gamma^* \to \text{hadrons})$ through optical theorem...

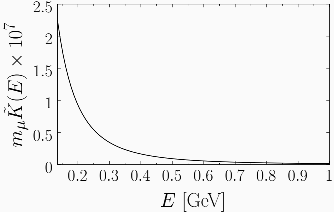


- In terms of the $e^+e^- \to$ hadron cross-section or actually the R-ratio:

$$R(E) = \frac{\sigma(e^+e^-(E) \to \text{hadrons})}{\sigma(e^+e^-(E) \to \mu^+\mu^-)}$$

- one has a very simple formula for $a_\mu^{\mathrm{HVP-LO}}$

$$a_\mu^{\mathrm{HVP-LO}} = \int_{m_\pi}^{\infty} dE\, R(E)\, \underbrace{\tilde{K}(E)}_{\text{analytic function}}$$



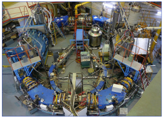
$a_\mu^{\mathrm{HVP-LO}}$ from the dispersive approach (I)

The central idea is to replace $R(E) \to R^{\exp}(E)$ and use previous formula.

$e^+e^- \to$ hadrons measured since '60 in various experiments

KLOE @ DAΦNE
FRASCATI

BABAR @ SLAC
STANFORD

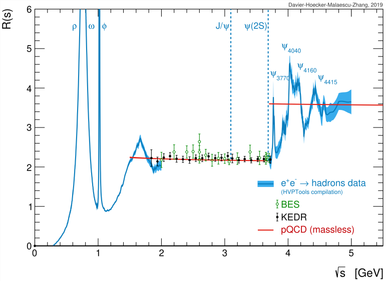


CMD3 @ VEPP-2000
NOVOSIBIRSK

Inclusive measurement of $R^{\exp}(E)$ obtained summing more than fourty exclusive channel measurements (comb. of various exp. , dominated by π's).

WP '20, pre-CMD3

	DHMZ19	KNT19
$\pi^+\pi^-$	507.85(0.83)(3.23)(0.55)	504.23(1.90)
$\pi^+\pi^-\pi^0$	46.21(0.40)(1.10)(0.86)	46.63(94)
$\pi^+\pi^-\pi^+\pi^-$	13.68(0.03)(0.27)(0.14)	13.99(19)
$\pi^+\pi^-\pi^0\pi^0$	18.03(0.06)(0.48)(0.26)	18.15(74)
K^+K^-	23.08(0.20)(0.33)(0.21)	23.00(22)
$K_S K_L$	12.82(0.06)(0.18)(0.15)	13.04(19)
$\pi^0\gamma$	4.41(0.06)(0.04)(0.07)	4.58(10)
Sum of the above	626.08(0.95)(3.48)(1.47)	623.62(2.27)
[1.8, 3.7] GeV (without $c\bar{c}$)	33.45(71)	34.45(56)
J/ψ, $\psi(2S)$	7.76(12)	7.84(19)
$[3.7, \infty)$ GeV	17.15(31)	16.95(19)
Total $a_\mu^{\mathrm{HVP,\,LO}}$	$694.0(1.0)(3.5)(1.6)(0.1)_\psi(0.7)_{\mathrm{DV+QCD}}$	692.8(2.4)



Two main groups involved in the analysis: DHMZ, KNT.

DHMZ = Davier-Hoecker-Malaescu-Zhang, KNT = Keshavarzi-Nomura-Teubner

$a_\mu^{\mathrm{HVP-LO}}$ from the dispersive approach (II)

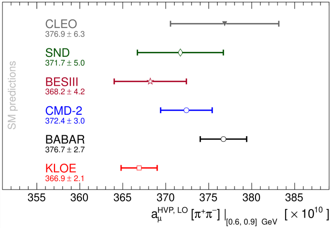
Combination of DHMZ and KNT results gives:

$$a_\mu^{\mathrm{HVP-LO}}[\mathrm{disp.}] = 6931(40) \times 10^{-11} \qquad [\mathrm{WP\ '20}]$$

A word of caution here, after all we are trading what should be a SM prediction with the results from (many) experiment. Replacement OK if:

- All relevant decay channels identified.
- No underestimated uncertainty in any of the relevant channels (ISR & hadron/lepton VP insertion subtracted properly?).
- No NP contamination in the measurement (e.g. $e^+e^- \to A^*_{NP} \to$ hadrons).

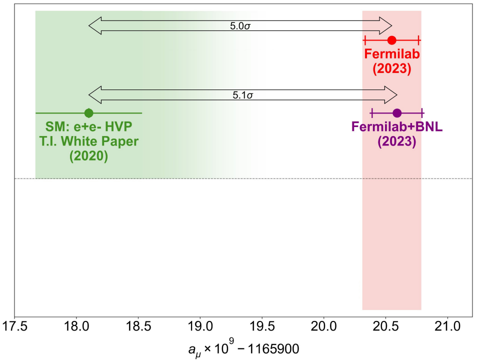
Pre-CMD3



BABAR-KLOE discrepancy in $\pi\pi$-channel considered "acceptable" in WP '20

Under these assumptions...

The $g-2$ puzzle



- Using a_μ^{HVP} from dispersive analysis as in WP '20 a $> 5\sigma$ discrepancy present.
- Did we find NP?
- In the meantime the $g-2$ puzzle has evolved because Lattice QCD entered the game. . .

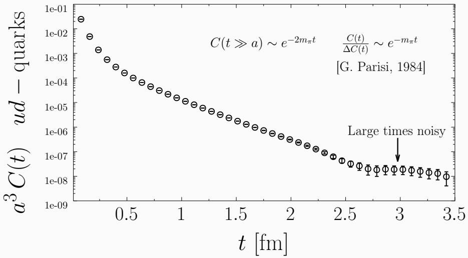
$a_\mu^{\mathrm{HVP-LO}}$ from lattice QCD

On the lattice, evaluating $a_\mu^{\mathrm{HVP-LO}}$ is much easier than a_μ^{Hlbl}.

The QCD input is the 2-point Euclidean correlation function of e.m. currents:

$$C(t) = \frac{1}{3}\int d^3x\, \langle 0|J^i_{\mathrm{em}}(t,\boldsymbol{x})J^i_{\mathrm{em}}(0)|0\rangle \qquad J^i_{\mathrm{em}} = \frac{2}{3}\bar{u}\gamma^i u - \frac{1}{3}\bar{d}\gamma^i d - \frac{1}{3}\bar{s}\gamma^i s + \frac{2}{3}\bar{c}\gamma^i c$$

$$a_\mu^{\mathrm{HVP-LO}} = \int_0^\infty dt\, \underbrace{K(t)}_{\text{analytic kernel}}\, C(t) \qquad K(t) \overset{t\gg m_\mu^{-1}}{\rightarrow} t^2 \qquad \text{[Enhancement of } C(t) \text{ tail]}$$



Main difficulties for subpercent accuracy:

- S/N problem at large times.
- Large lattice volumes $V = L^3$ required to fit the light $\pi\pi$ states.
- Isospin-breaking effects $\alpha^3, \alpha^2(m_d - m_u)$ needs to be computed at target accuracy.

BMWc crosses the Rubicon [Nature 593 (2021)]

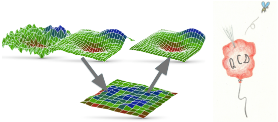
Leading hadronic contribution to the muon magnetic moment from lattice QCD

Sz. Borsanyi, Z. Fodor, J. N. Guenther, C. Hoelbling, S. D. Katz, L. Lellouch, T. Lippert, K. Miura, L. Parato, K. K. Szabo, F. Stokes, B. C. Toth, Cs. Torok & L. Varnhorst

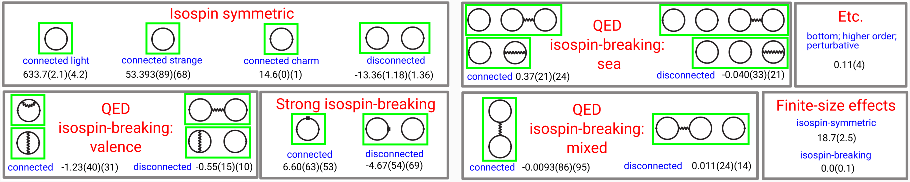
Nature **593**, 51–55 (2021) | Cite this article

- Order of magnitude improvement in stat. accuracy
- Large lattice volumes up to $L \simeq 11$ fm
- Seven lattice spacings to control UV cut-off effects.

Modern algorithms and new methods

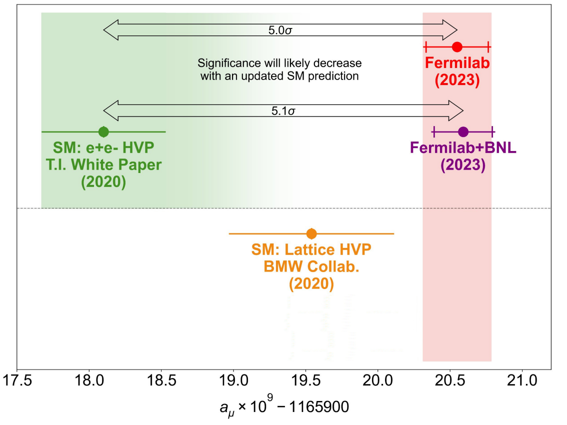


Adaptative solvers & eigendeflation



$$10^{10} \times a_\mu^{\text{LO-HVP}} = 707.5(2.3)_{\text{stat}}(5.0)_{\text{sys}}[5.5]_{\text{tot}}$$

The a_μ discrepancy after BMWc's result



- BMWc's result is 2.1σ larger then $a_\mu[\text{disp.}]$.
- ...and only 1.7σ smaller than FNAL+BNL results

A pretty confusing situation. . .

To clear up the situation we need. . .**double-checks!!**

At this point it is also very important to divide the puzzle into two different branches

$a_\mu^{\mathrm{HVP-LO}}$:

- BMWc is, as of today, the only Coll. that computed $a_\mu^{\mathrm{HVP-LO}}$ at subpercent accuracy.
- Given the complexity of the calculation, independent lattice calculations are fundamental to establish the faith of the a_μ anomaly.
- Four Coll., RBC/UKQCD, FNAL, ETMC, MAINZ, expected to give an update in '24.

$e^+e^- \to$ hadrons:

- We need stringent tests of the experimental $R(E)$ against SM (i.e. lattice) predictions.
- Can we cook-up $R(E)$-based observables capable in principle of magnifying the previous discrepancy?
- E.g. by easing the lattice calculation. . .
- Tests of e^+e^- totally independent of a_μ^{exp}.

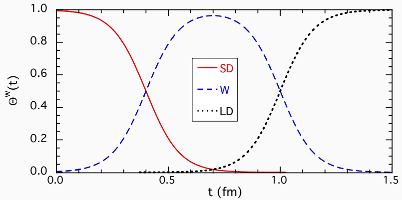
The Euclidean windows to test $e^+e^- \to$ hadrons

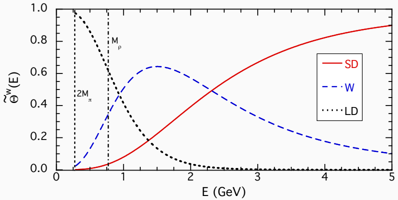
To perform stringent tests of $R(E)$ we are not bound to $a_\mu^{\mathrm{HVP-LO}}$

$$\underbrace{\int_0^\infty dt\; K(t)\; C(t)}_{\text{lattice, SM}} = a_\mu^{\mathrm{HVP-LO}} = \underbrace{\int_{M_\pi}^\infty dE\; \tilde{K}(E)\; R^{\mathrm{exp}}(E)}_{\text{dispersive, experimental}}$$

$$\Downarrow \qquad \Downarrow$$

$$\underbrace{\int_0^\infty dt\; K(t)\; C(t)\, \underline{\Theta^w(t)}}_{\text{lattice, SM}} = a_\mu^w = \underbrace{\int_{M_\pi}^\infty dE\; \tilde{K}(E)\; R^{\mathrm{exp}}(E)\, \underline{\tilde{\Theta}^w(E)}}_{\text{dispersive, experimental}}$$



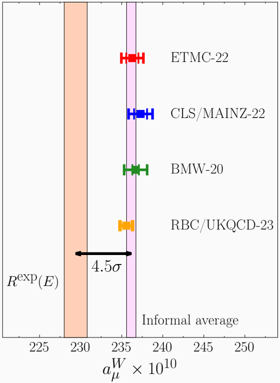


- $\Theta^{\mathrm{SD}} + \Theta^{\mathrm{W}} + \Theta^{\mathrm{LD}} = 1$. $w = \{\mathrm{SD}, \mathrm{W}, \mathrm{LD}\}$ probe $R(E)$ at different energies.
- $a^{\mathrm{SD/W}}$ very precise on the lattice $\Longrightarrow$ may enhance differences with $R^{\mathrm{exp}}(E)$.

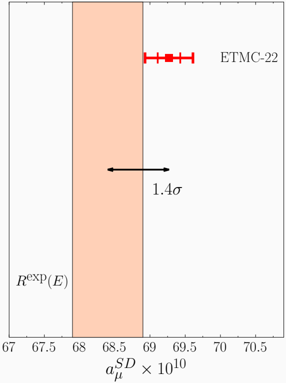
The short- and intermediate-distance windows

In '22 many lattice Collaborations computed $a_\mu^{\rm W}$ and we (ETM) also $a_\mu^{\rm SD}$.

intermediate-distance $\Longrightarrow$ $E \lesssim 1$ GeV $(\pi\pi, \pi\pi\pi)$



short-distance $\Longrightarrow$ Large $E \gtrsim 1$GeV



- Many more lattice results for ud-quark contribution. All in line ✓.
- A big achievement for the lattice community.
- Striking $\sim 4.5\sigma$ tension with $R^{\rm exp}(E)$-based results for $a_\mu^{\rm W}$.

Implications of windows results

- High-energy part of $R^{\text{exp}}(E)$ in line with SM prediction.
- Hadronic running of α at the Z-scale ($\Delta\alpha(M_Z^2)$) in line with $R^{\text{exp}}(E)$ results [Cè et al, JHEP 08 (2022)].
- EW precision tests not affected by the observed low-energy tension.
- a_μ^{W} results suggest strong deviation of $R^{\text{exp}}(E)$ from SM for $E \sim m_\rho$.
- Lattice results extremely solid. The various groups use very different simulation setups.
- How to reconcile theory and experiments ?

In [PRD 107 (2023)] we noticed that the observed differences in a_μ^w can be explained by a few percent increase in the 2π channel contribution to $R^{\text{exp}}(E)$ below $1\ \text{GeV}$.

LATTICE CALCULATION OF THE SHORT AND INTERMEDIATE … PHYS. REV. D **107,** 074506 (2023)

TABLE IV. Values of a_μ^w obtained in this work for the short and intermediate time-distance windows, $w = \{\text{SD}, \text{W}\}$, and from Ref. [14] for the full HVP term, $w = \text{HVP}$, compared with the corresponding dispersive determinations of Ref. [24], based on experimental $e^+e^- \to$ hadrons data (third column). The difference between the second and third columns, Δa_μ^w, is given in the fourth column, while the contributions of the 2π channels $a_\mu^w(2\pi)$ (below a center-of-mass energy of 1 GeV), obtained in Ref. [24], are shown in the fifth column. All quantities are in units of 10^{-10} except for the last column, where we list the values of the ratio between Δa_μ^w and the 2π contribution $a_\mu^w(2\pi)$.

Window (w)	a_μ^w(LQCD)	$a_\mu^w(e^+e^-)$ [24]	Δa_μ^w	$a_\mu^w(2\pi)$ [24]	$\Delta a_\mu^w / a_\mu^w(2\pi)$
SD	69.3(0.3)[a]	68.4(0.5)	0.9(0.6)	13.7(0.1)	0.066(43)
W	236.3(13)[a]	229.4(1.4)	6.9(1.9)	138.3(1.2)	0.050(14)
HVP	707.5(5.5) [14]	693.0(3.9)	14.5(6.7)	494.3(3.6)	0.029(14)

[a]This work.

What about computing $R(E)$ directly on the lattice?

Can we compute $R(E)$ directly on the lattice?

$$C(t) = \frac{1}{12\pi^2} \int_0^\infty dE\, e^{-Et}\, R(E)\, E^2$$

- Inverting the previous relation to obtain $R(E)$ from $C(t)$ (our lattice input) is an ill-posed problem if...

- ...$C(t)$ affected by statistical uncertainties and known only at a discrete and finite number of times (typical situation encountered in lattice calculation).

- But... this is not the end of the story.

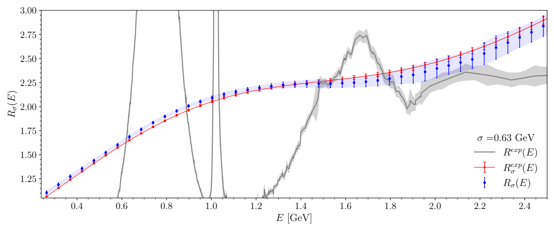
- We have a new numerical technique, the Hansen-Lupo-Tantalo (HLT) method, which allows us to obtain on the lattice an energy-smeared version of $R(E)$.

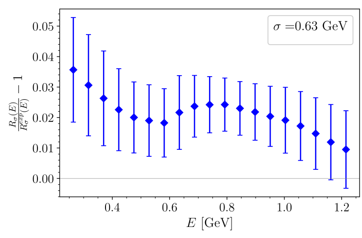
The energy-smeared $R-$ratio

In PRL 130 (2023) we (ETM) exploited the HLT method to evaluate on the lattice:

$$R_\sigma(E) = \int_0^\infty d\omega\, R(\omega)\, \underbrace{N(E-\omega,\sigma)}_{\text{Gaussian}}$$

$R_\sigma(E)$ is a "sort of" energy-binned version of $R(E)$ (with bin-size $\sim \sigma$).

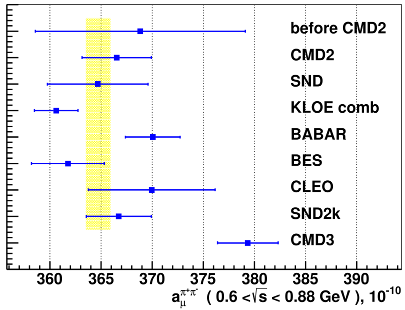


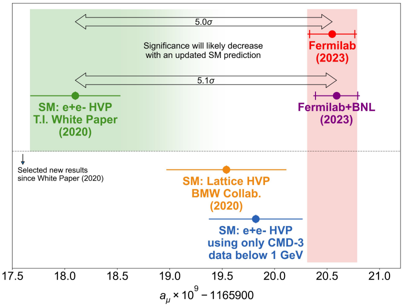


- In the low-energy region, for $\sigma \simeq 0.6$ GeV, we observe a $\approx 3\sigma$ (or $2.5-3\%$) deviation w.r.t. e^+e^- experimental results.
- Similar conclusions as from $a_\mu^{\rm W} \implies$ **higher SM value** w.r.t. $R^{\rm exp}(E)$ results around the ρ resonance.

The CMD-3 result [talk by F. Ignatov this afternoon]

A new measurement of $e^+e^- \to \pi^+\pi^-$ with CMD detector at VEPP-2000 [F. Ignatov et al, arXiv:2302.08834] can shed a new light on the puzzle.





- CMD-3 result incompatible with previous measurements (some of them were already in tension, e.g. BABAR-KLOE).
- If confirmed will drammatically reduce the strength of the a_μ anomaly (and also a_μ^{W} tension).
- At the moment the situation of exp. $e^+e^- \to$ hadrons needs to be clarified.

Summary

Where do we stand?

$a_\mu^{\mathrm{HVP-LO}}$:

- Many Collaborations expected to give an update next year, to confirm or not the BMWc result.
- It is conceivable that the SM value of $a_\mu^{\mathrm{HVP-LO}}$ in the next WP update will be entirely based on lattice results.
- $R^{\mathrm{exp}}(E)$-based results for $a_\mu^{\mathrm{HVP-LO}}$ need clarifications.

$e^+e^- \to$ hadrons:

- Lattice QCD has signalled an inconsistency between previous $e^+e^- \to$ hadron measurements and the SM value.
- NP, unknown systematic in measurements?
- The new CMD-3 result can provide an explanation.
- Double-checks needed. BABAR expected to give an update next year and KLOE re-analysis started.
- Energy-smeared $R(E)$ on the lattice can be improved (smaller σ, higher accuracy).

Thank you for the attention and Happy Holidays!