Follow-up of the electromagnetic counterparts of GW sources

Scuola di Analisi Dati
(onde gravitazionali e astrofisica multimessaggera)

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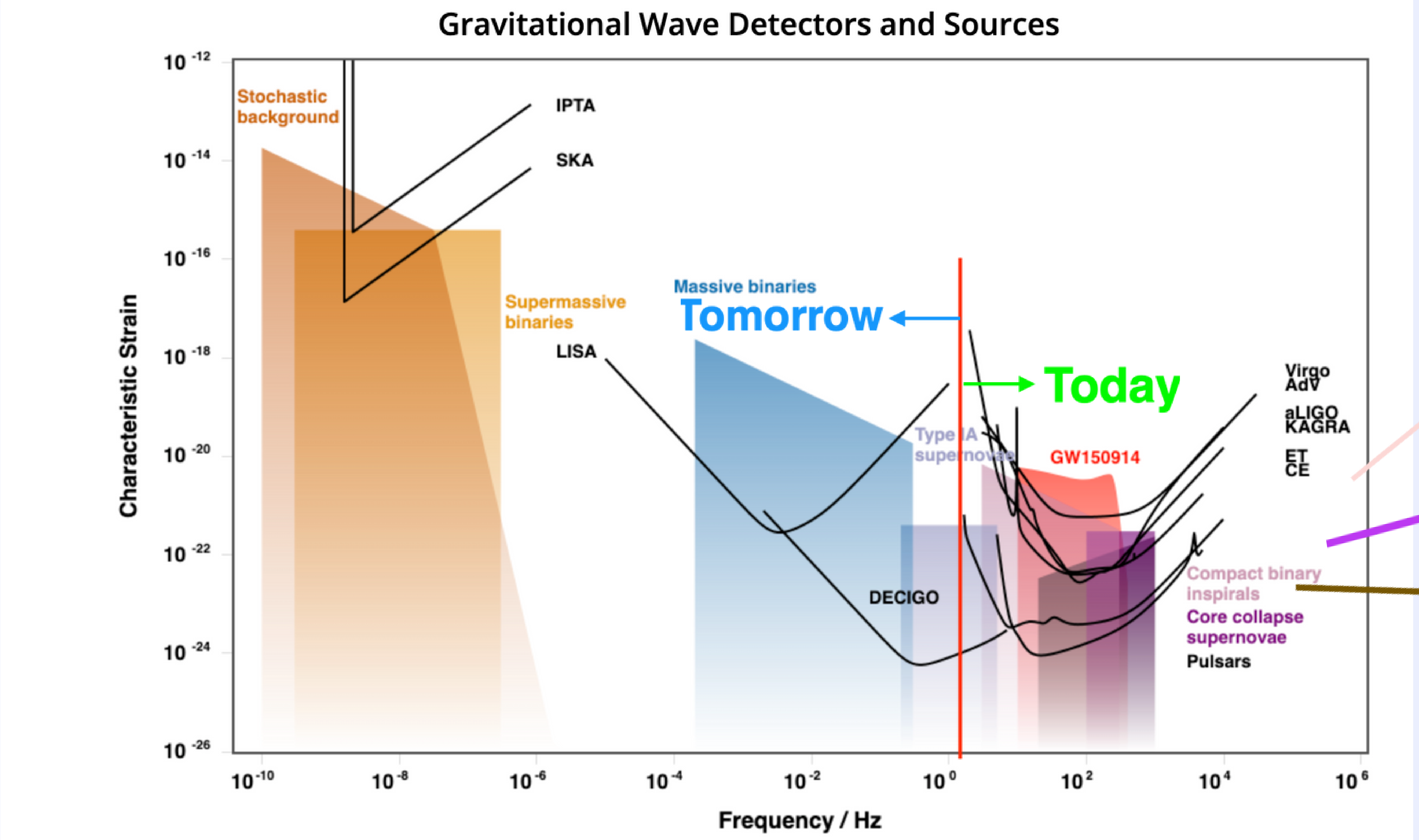
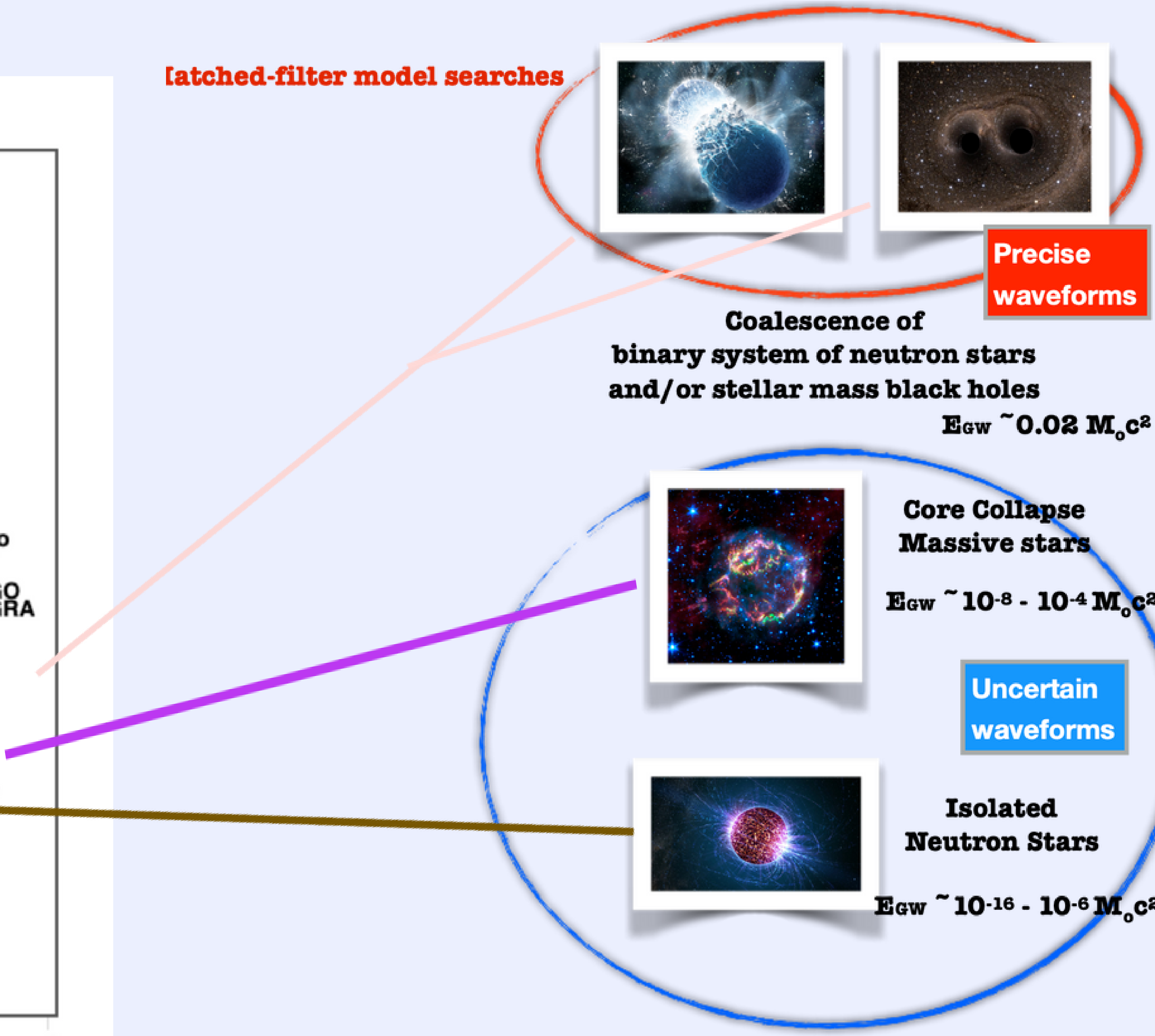
INAF - Osservatorio Astronomico di Roma

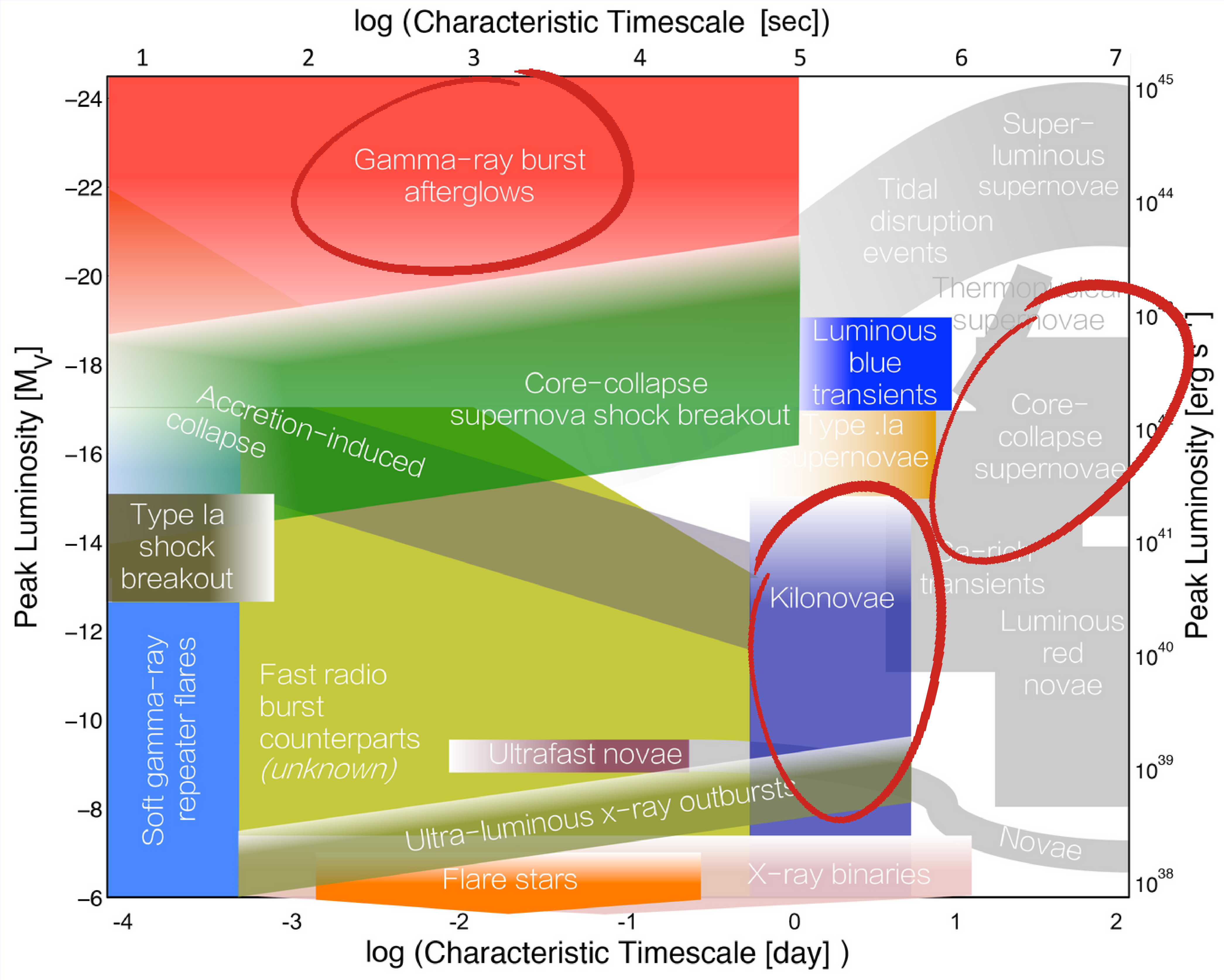
MULTIMESSENGER ASTRONOMY

Ligo & Virgo
(signals within 10-1000 Hz)





THE TRANSIENT SKY



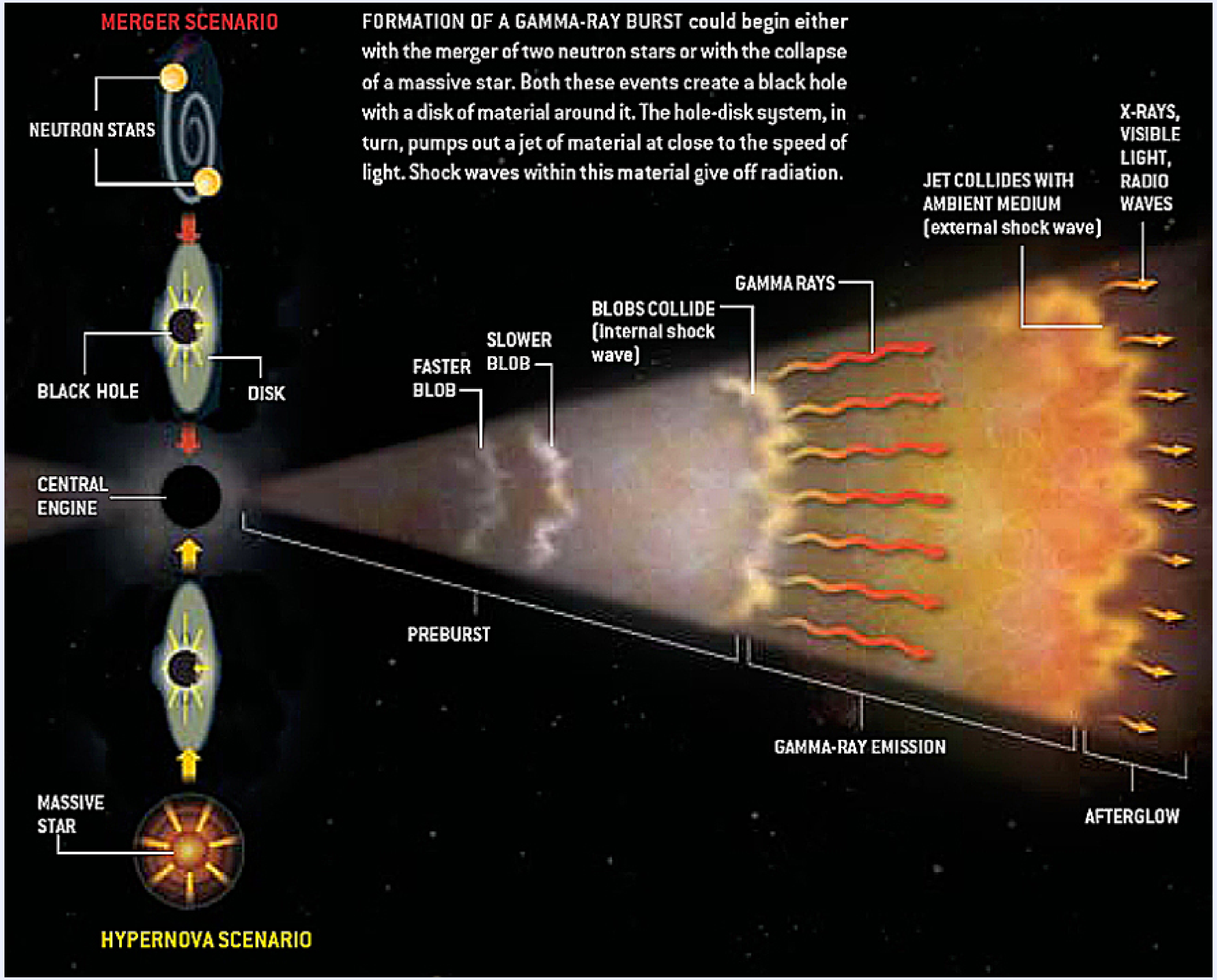
Gamma-ray bursts (GRBs)

LONG GRB

Hypernovae/collapsar: evolve much faster, going off in their formation site
(SN bump needed, no Fe X-ray lines)
=> "mass-rich environment"

or

SupraNova : the GRB is preceded by a SN explosion leaving a dense shell of matter of many solar masses in the nearby
(Fe X-ray lines, no SN bump)
=> "mass-rich environment"



SHORT GRB

NS-NS (BH-NS & BH-WD): travel far from their formation sites before producing GRBs => "clean environment"

Both the hypernova and the merger scenarios lead to the a central engine (rapidly accreting BH surrounded by a massive disc) able to launch both long and short GRBs

GRBs Jet Model

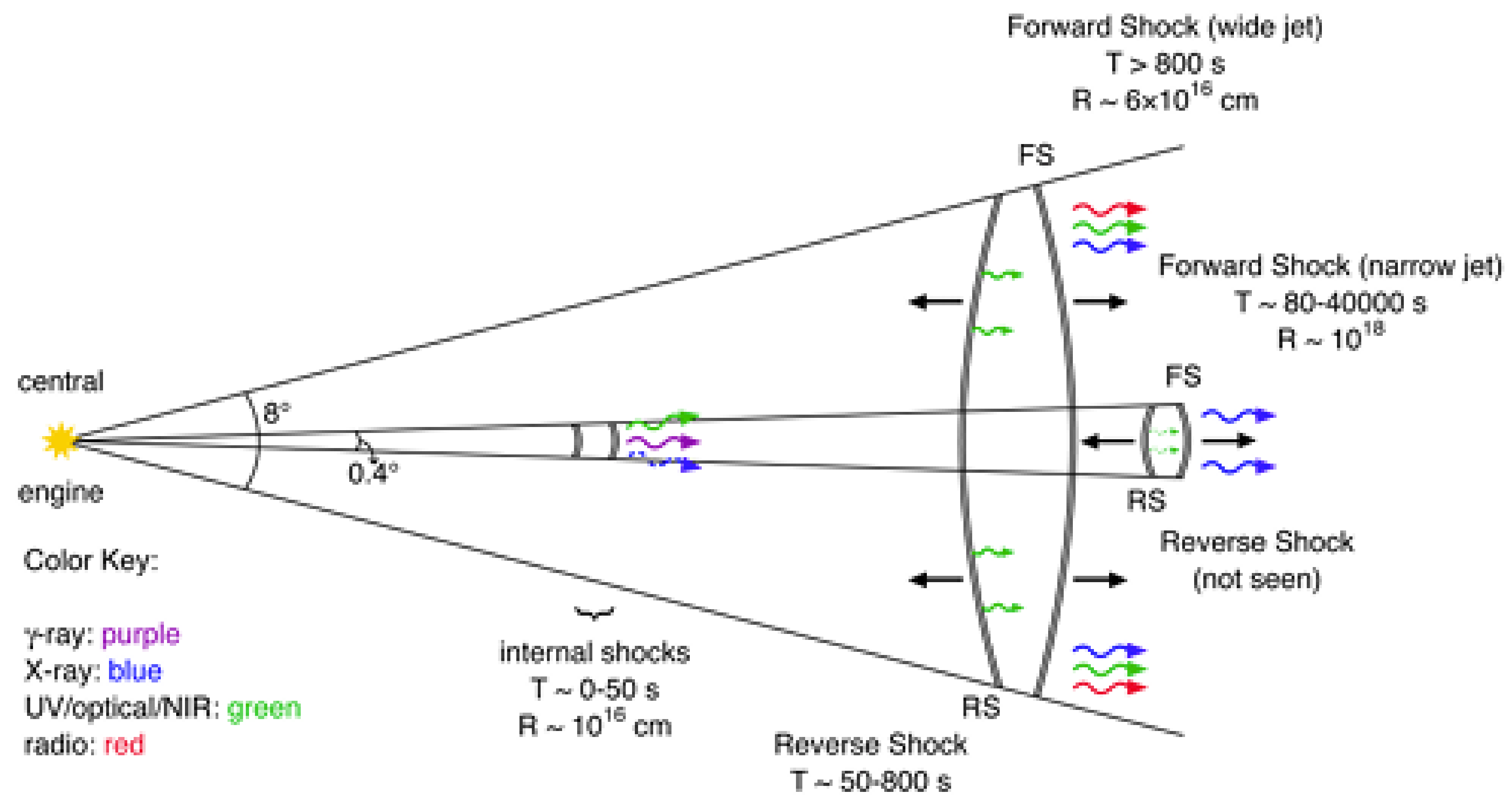


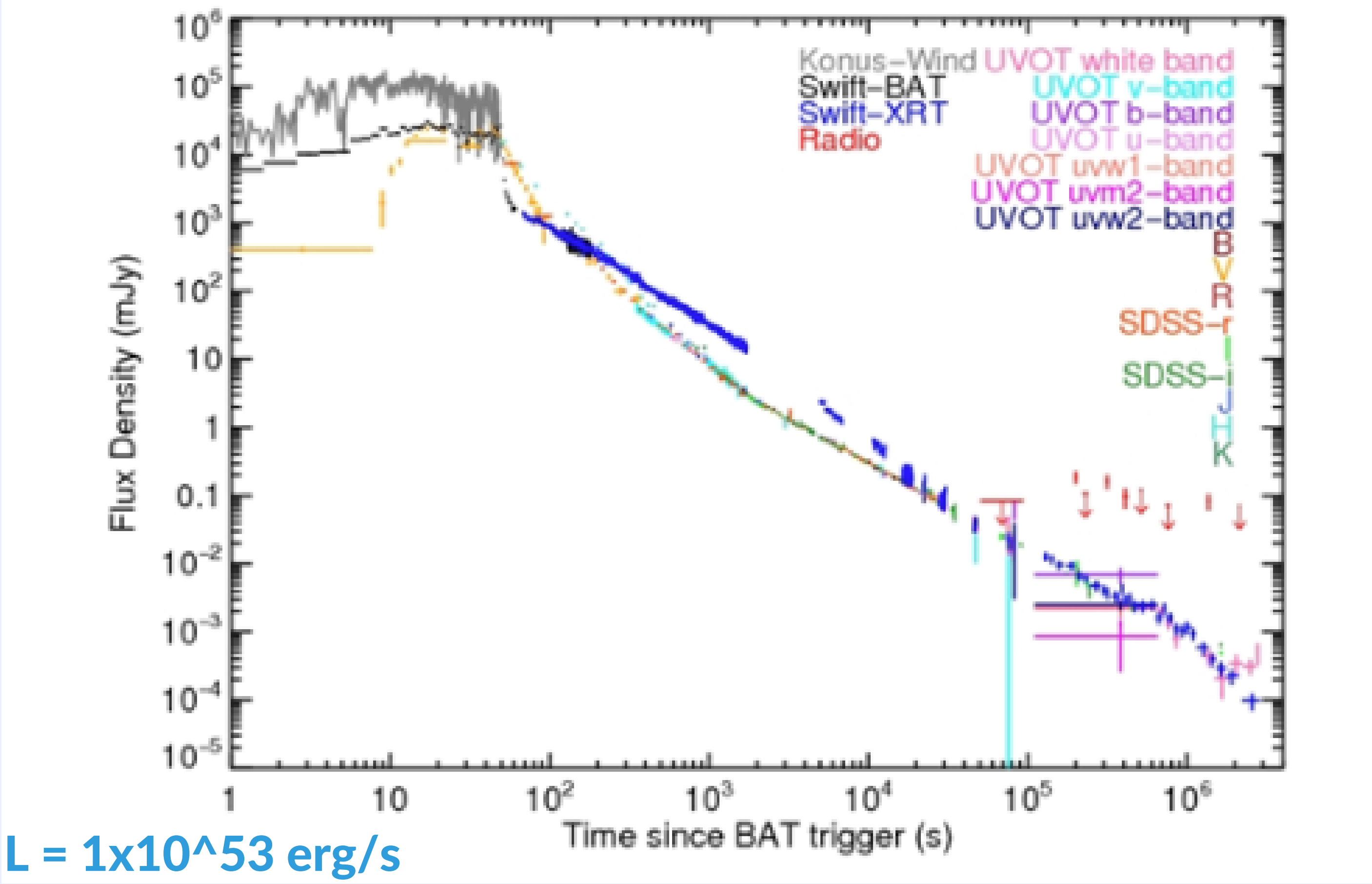
Figure 4 | Schematic of Two-Component Jet Model. Summary diagram showing spectral and temporal elements of our two-component jet model. The prompt γ-ray emission is due to the internal shocks in the narrow jet, and the afterglow is a result of the forward and reverse shocks from both the narrow and wide jets. The reverse shock from the narrow jet is too faint to detect compared to the bright wide jet reverse shock and the prompt emission. If X-ray observations had begun earlier, we would have detected X-ray emission during the prompt burst. These expected (but unobserved) emission sources are indicated by the dashed photon lines. Diagram is courtesy of J.D. Myers (NASA).

Racusin et al 2008

Long-GRBs Lightcurves

GRB 080319B: the Naked-Eye GRB

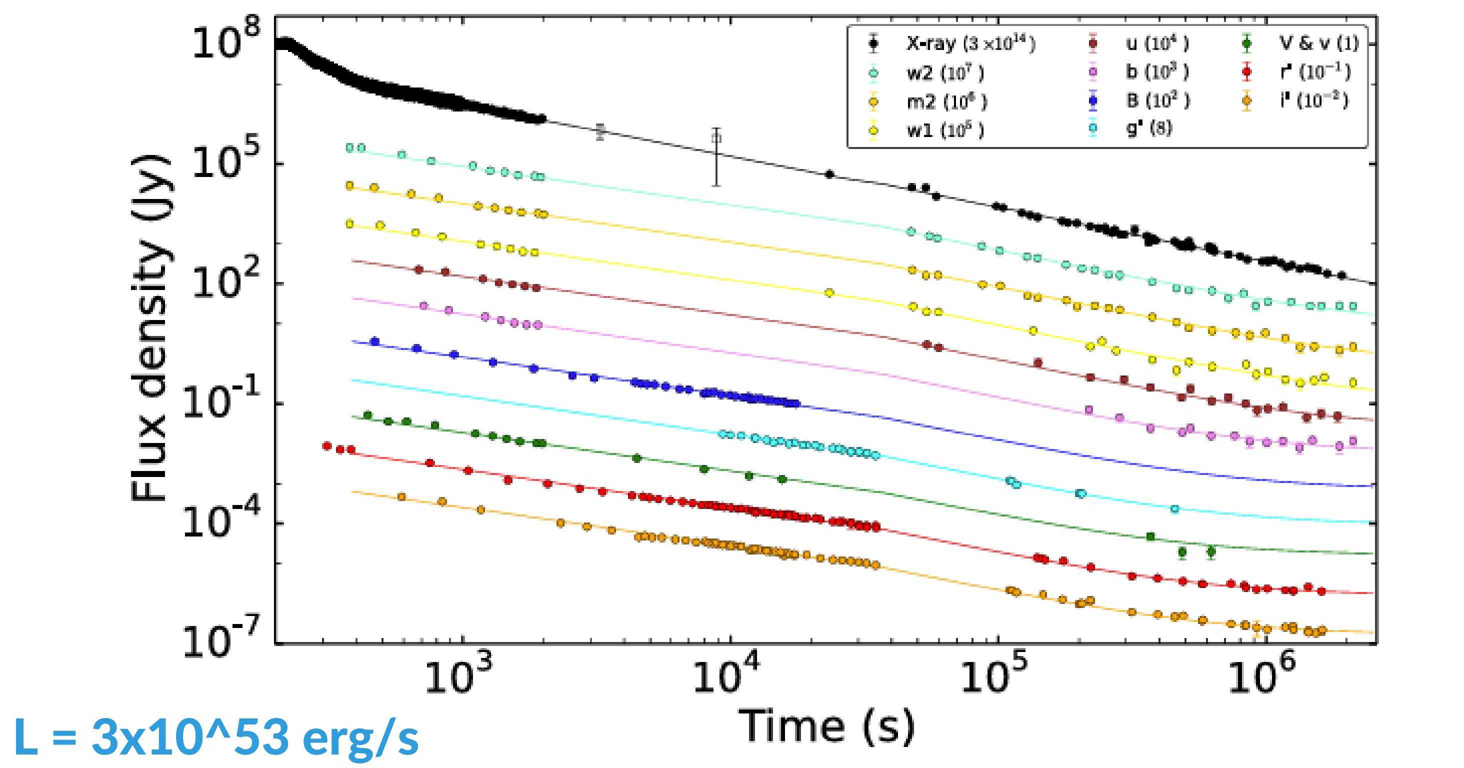
Racusin et al 2008 (z=0.937)



L = 1x10^53 erg/s

GRB130427A: The Monster GRB

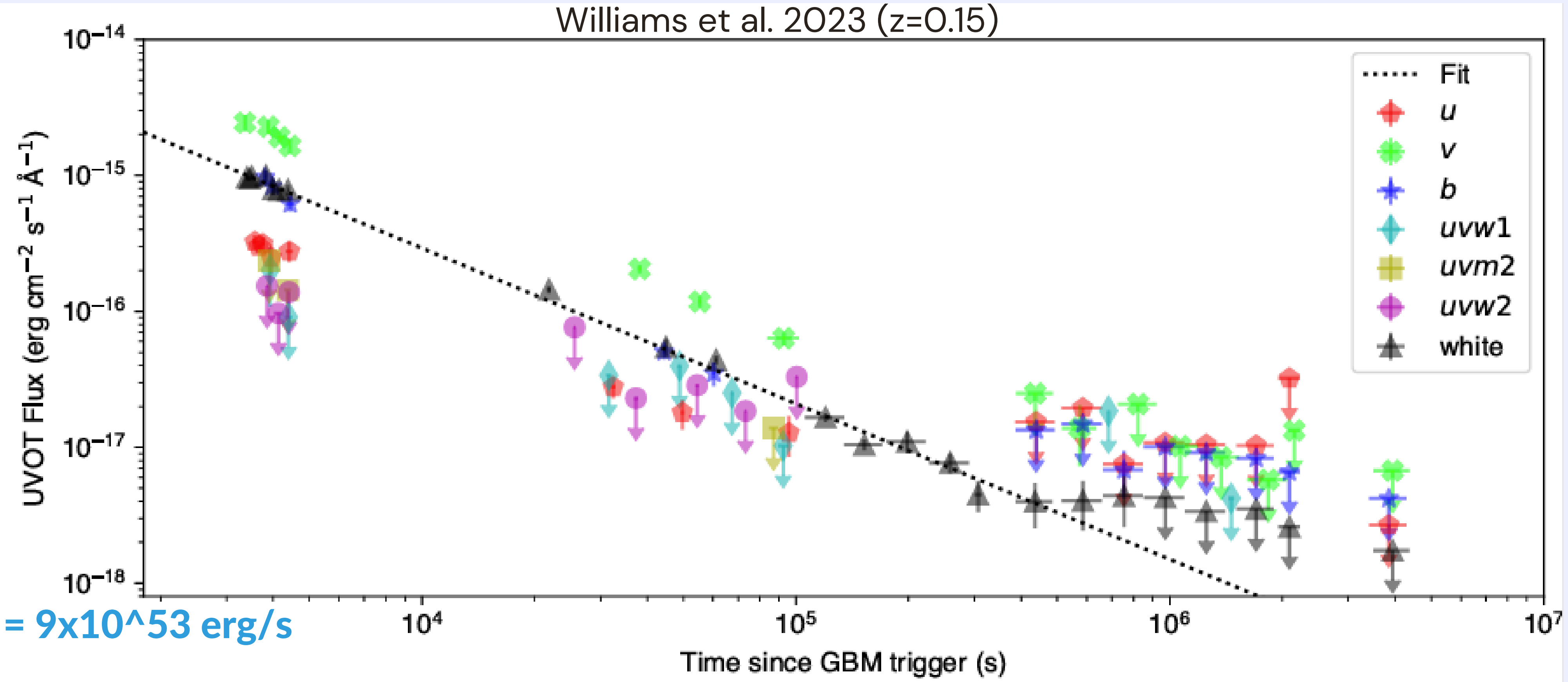
Maselli et al. 2014 (z=0.34)



L = 3x10^53 erg/s

GRB221009A: the BOAT brightest of all time

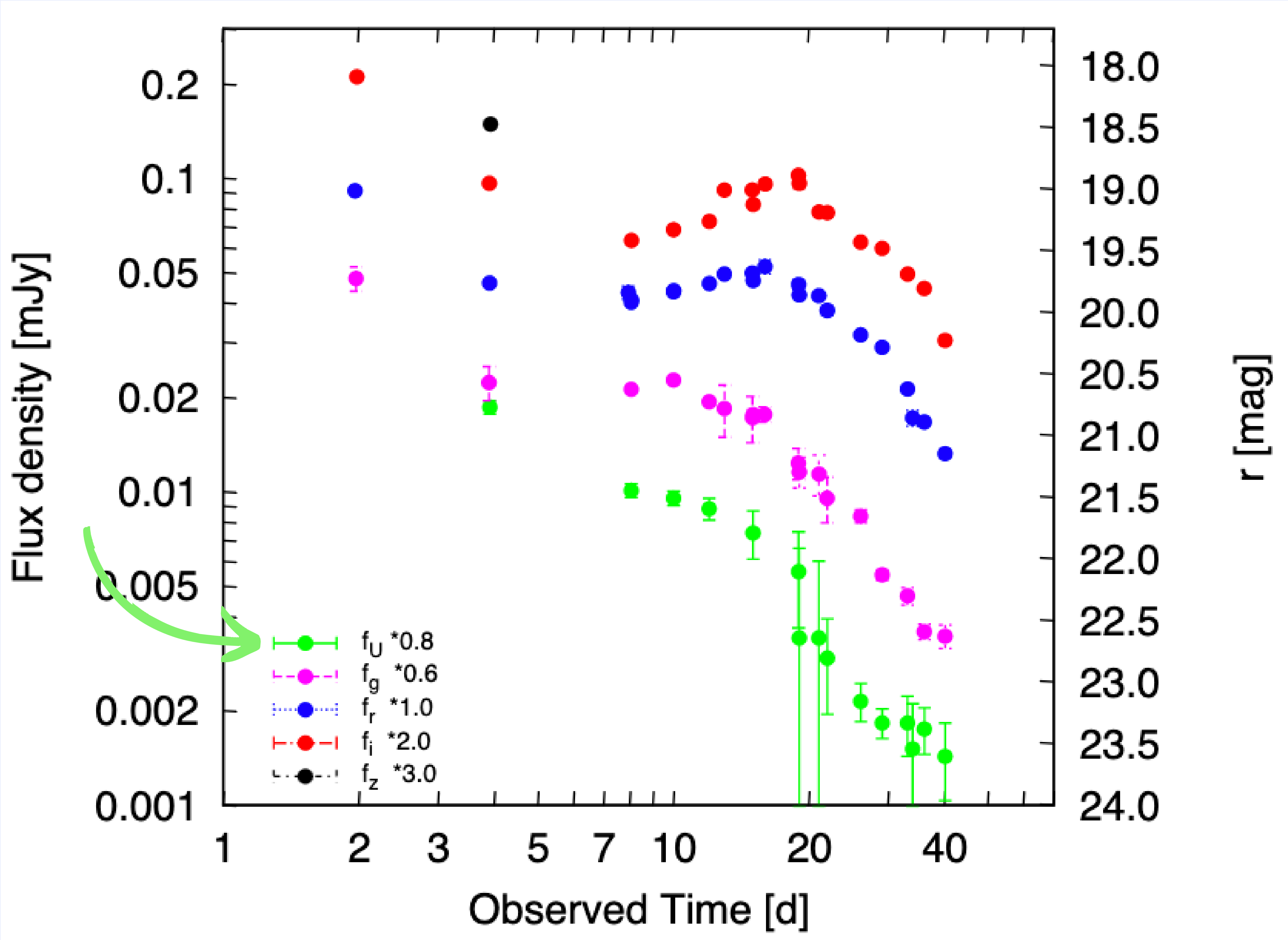
Williams et al. 2023 (z=0.15)

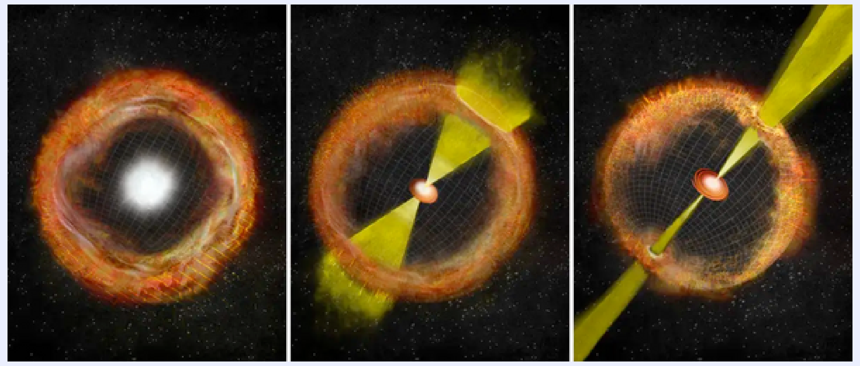


L = 9x10^53 erg/s

LGRBs: supernovae connection

GRB130702A/SN2013dx





- Bump in the light curve after ~ 15*(1+z) days
- Spectroscopic obs
- Unmistakable SN features
- Typical Bolometric light curve

Some GRBs (~15%) are related to Type Ibc Core Collapse SNe

Compact Binary Model -> SGRBs

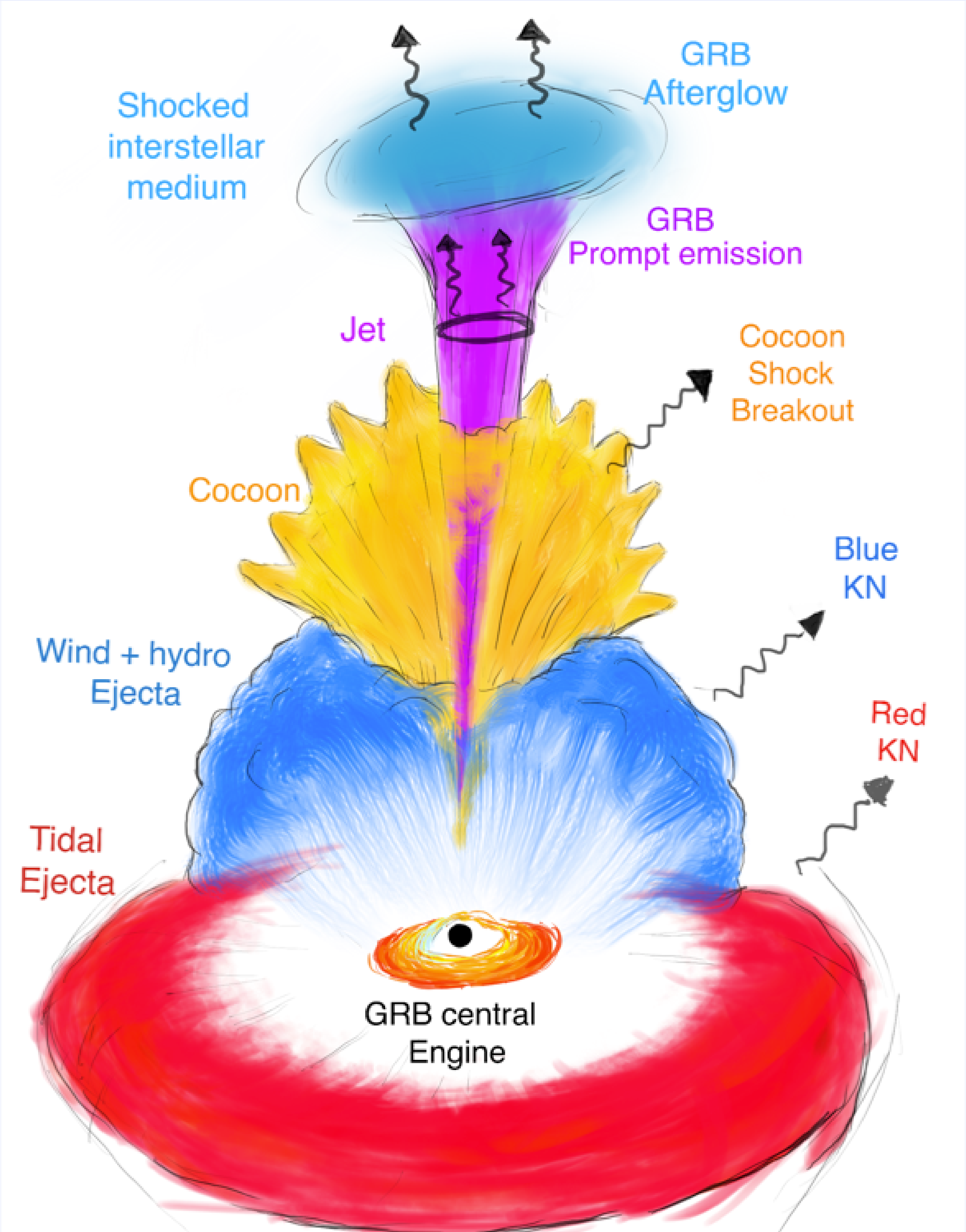
NS-NS / NS-BH mergers: what do we expect to see ?

Collimated EM emission from short GRBs

WITH a detectable X-ray/UV/optical/IR/radio counterpart (afterglow)

Fast and Furious

Fast: from few ms to hundreds of s
Energetic: 10^(−7) - 10^(−3) erg cm^(−2)
Bright: 10^(−8) - 10^(−4) erg cm^(−2) s^(−1)



Courtesy of S. Ascenzi

Optical/NIR isotropic emissions

WITH a significant mass (0.01-0.1Mo) dinamically ejected during mergers at subrelativistic velocity (0.1-0.3c)

↓

R-Process

Neutron capture rate much faster than decay, special conditions: T> 10 K, high neutron density 10^22 cm^(-3)

Nucleosynthesis of heavy nuclei

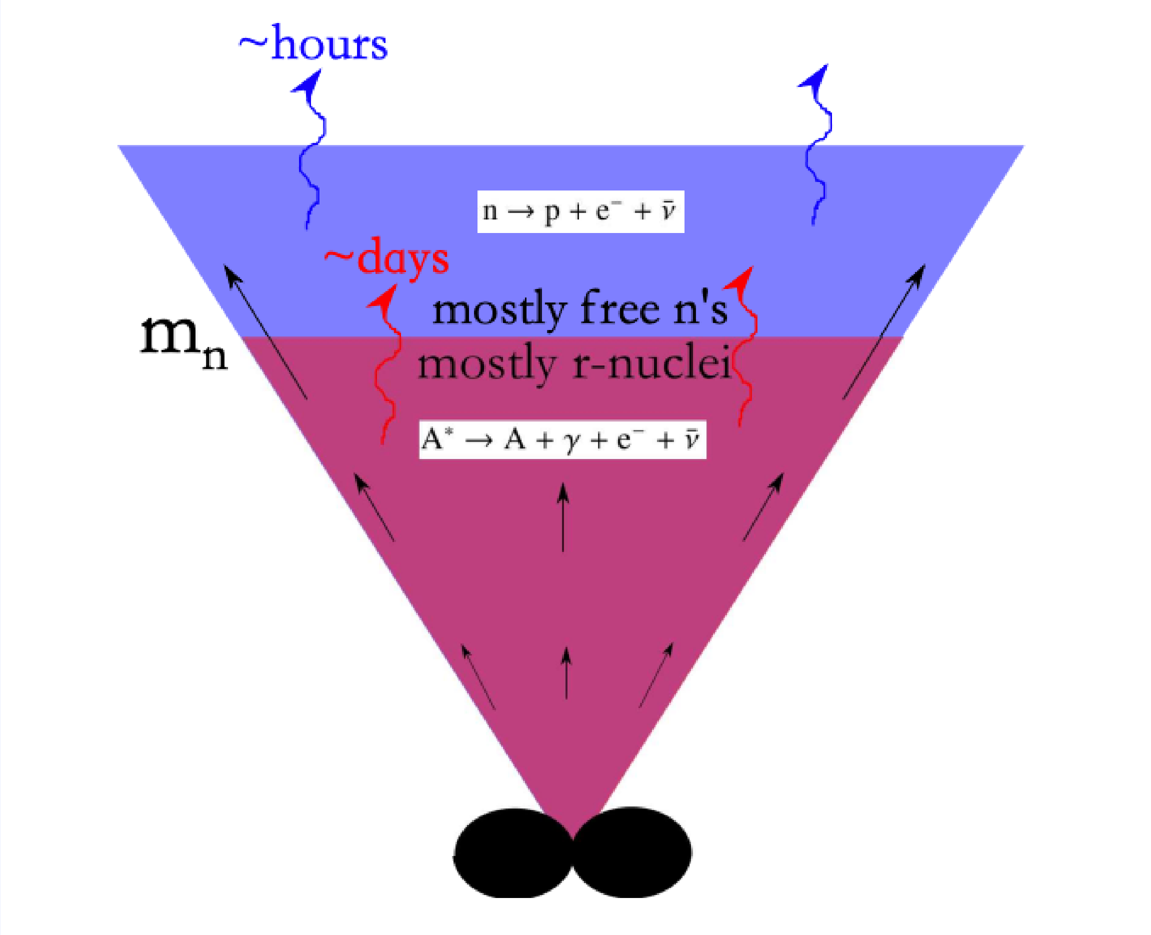
↓

Radioactive decay of heavy elements

↓

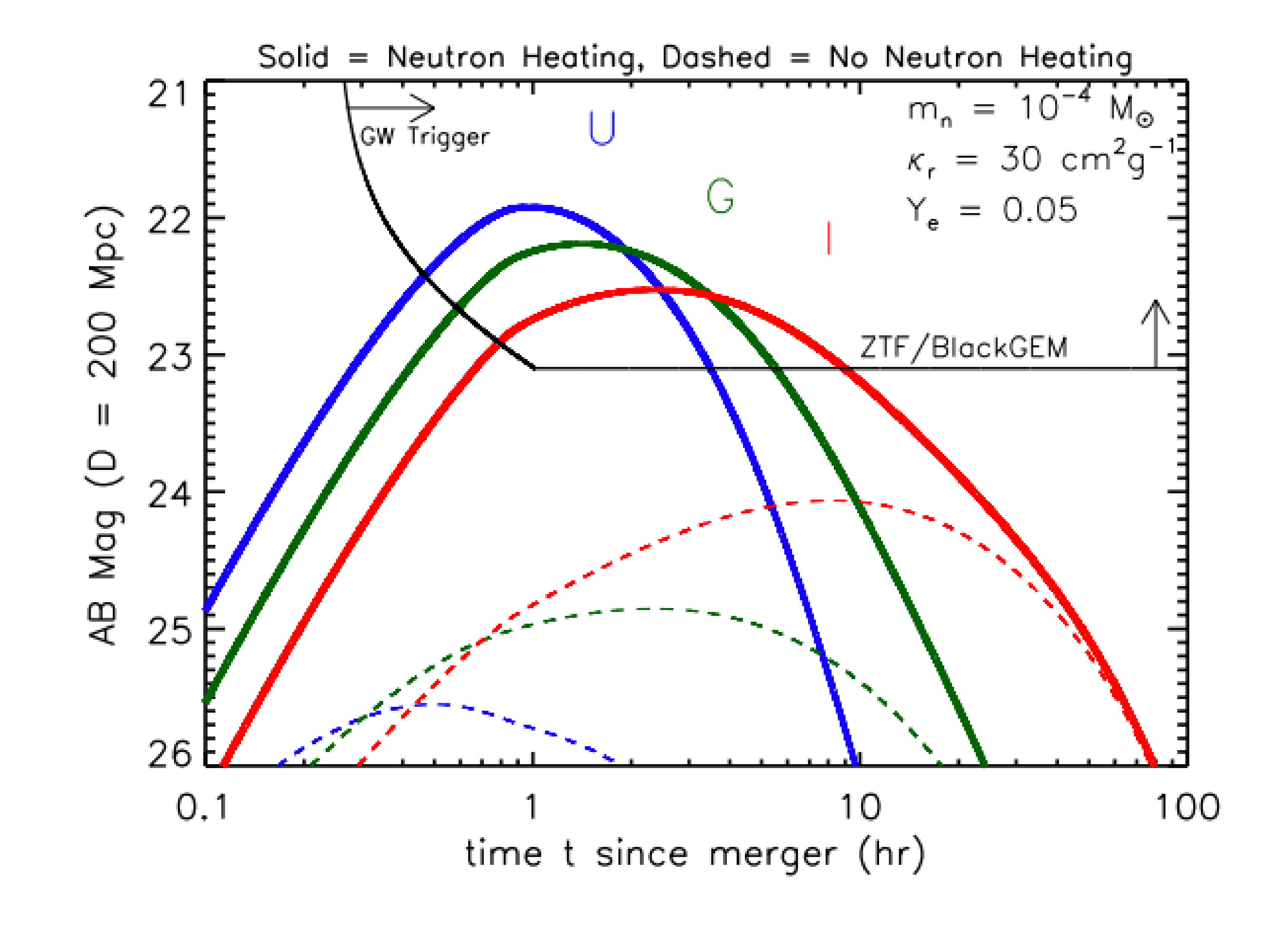
KILONOVA

KN neutron precursor



Matter originates from the **shock-heated interface** between the merging NSs.

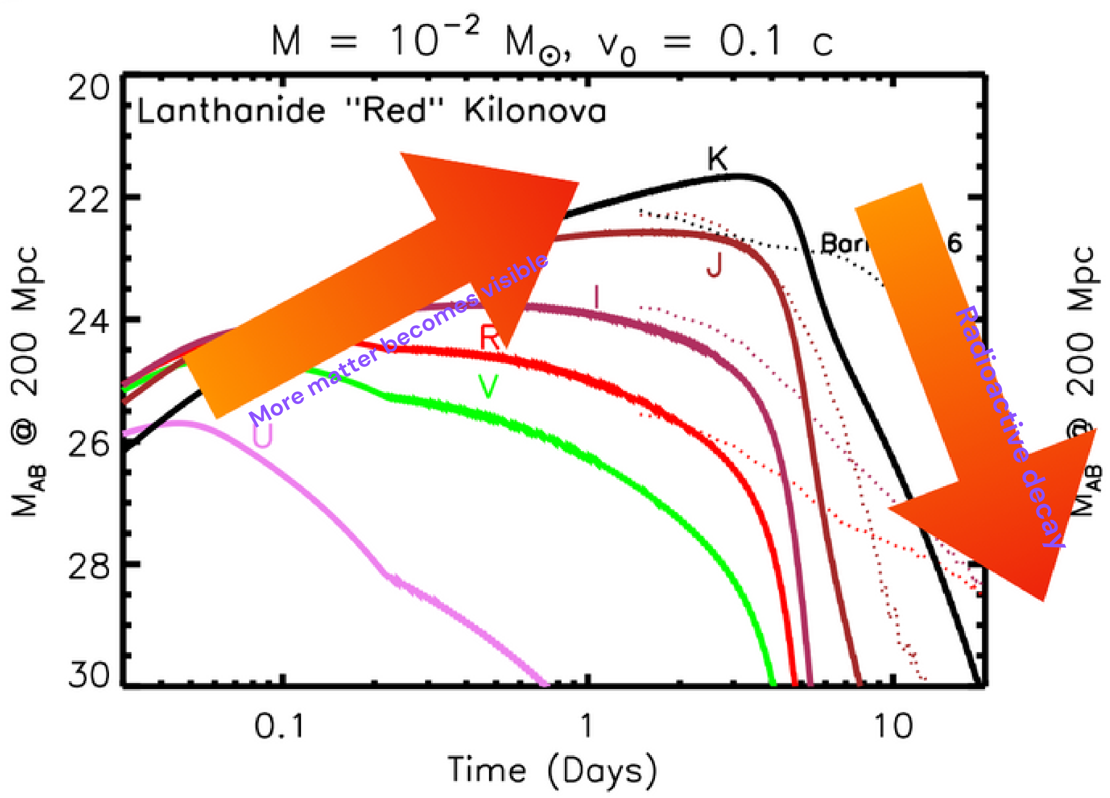
β-decay of these **free neutrons** in the outermost ejecta powers a '**precursor**' to the main kilonova emission



Metzger et al. 2014

Lightcurves

Metzger 2017, Living Reviews in Relativity

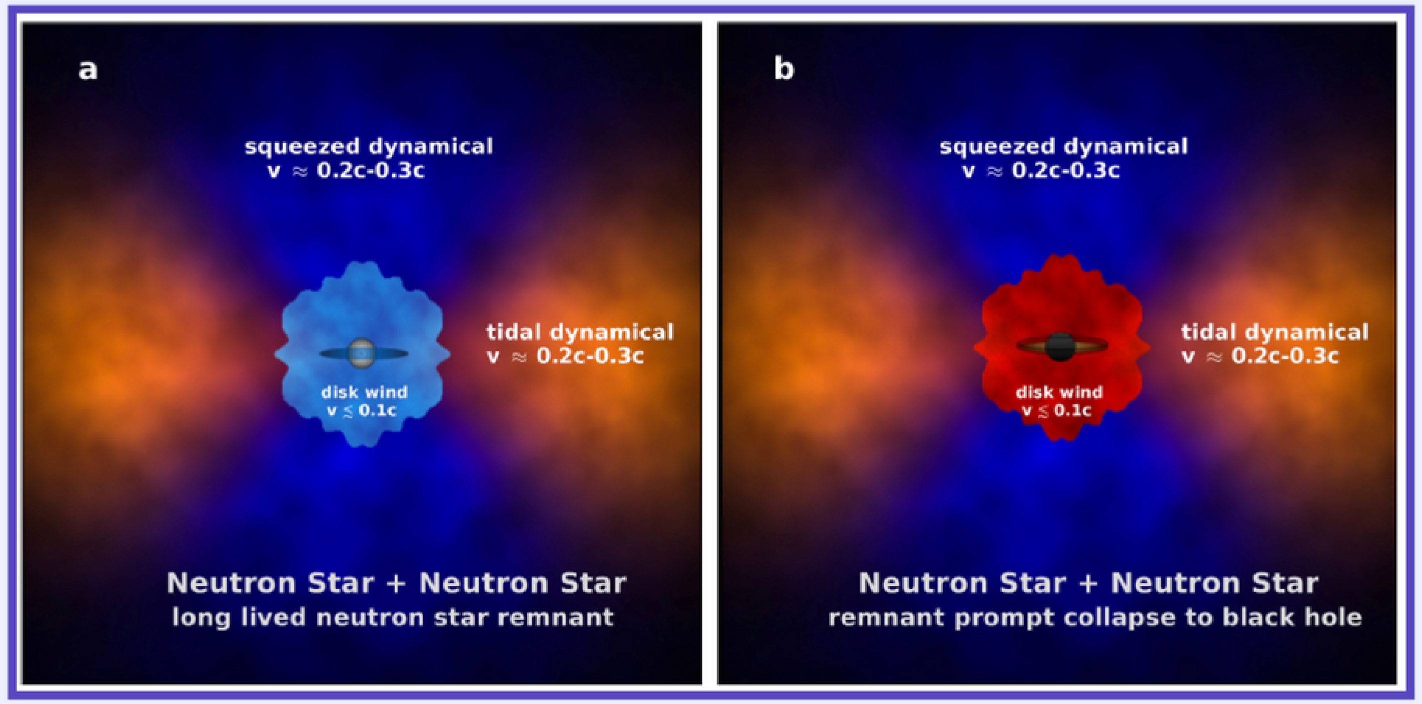


Lpeak = 10^41 erg/s (Metzger 2017) Lpeak > 10^41 erg/s

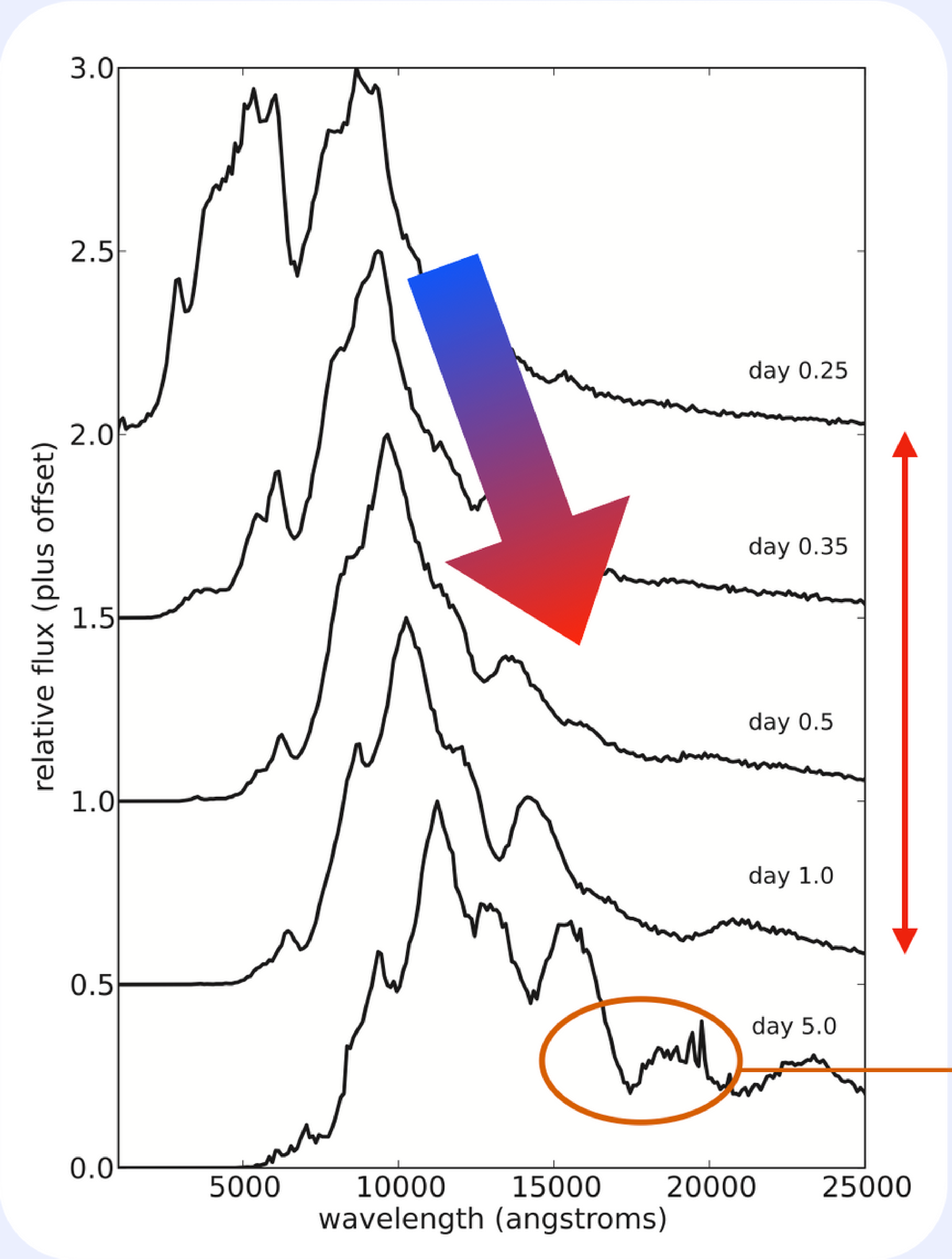
- Neutron-rich ejecta **with** Lanthanide elements
- Neutron-rich ejecta **without** Lanthanide elements

Depending on the viewing angle of the observer **both components may be present in a single merger** (if they originate from different locations of the ejecta)

Kasen et al. 2017, Nature 551, 80



KN SPECTRA



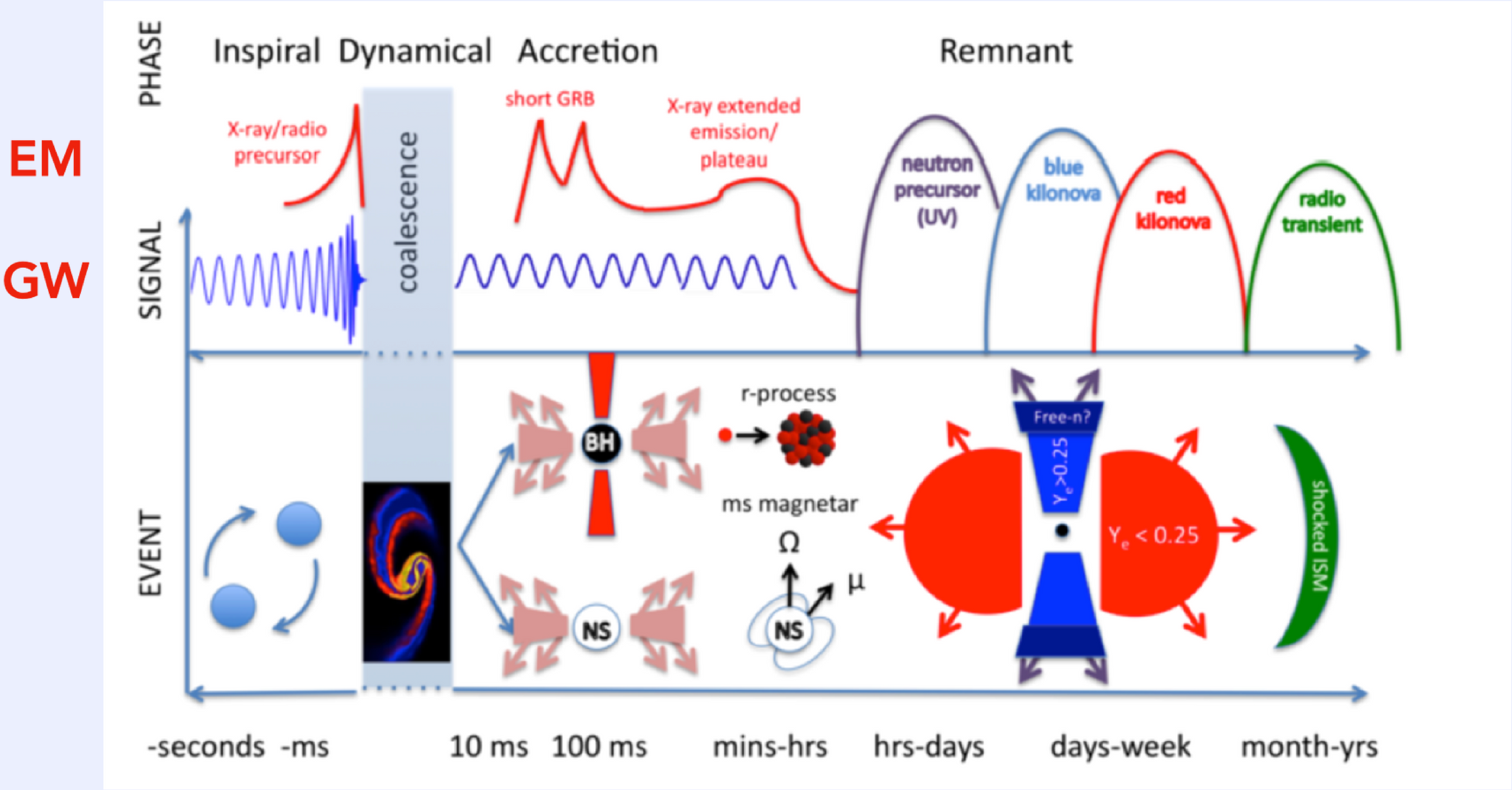
Rapid evolution toward red colors

1 day !

Broad spectral features arising from blending of many lines

(Kasen et al. 2013)

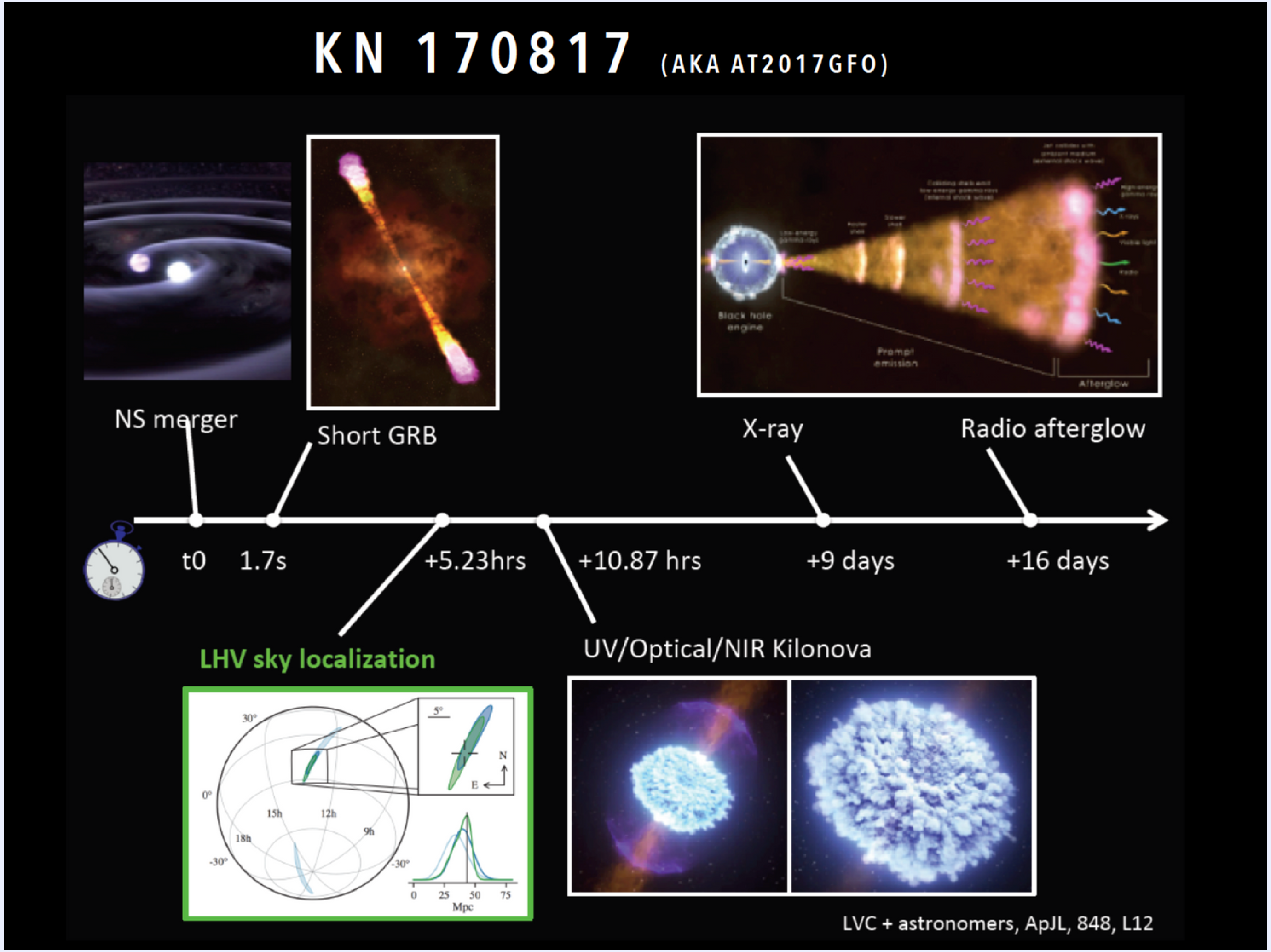
NS-NS NS-BH Merger: a global picture



Fernandez & Metzger 2016

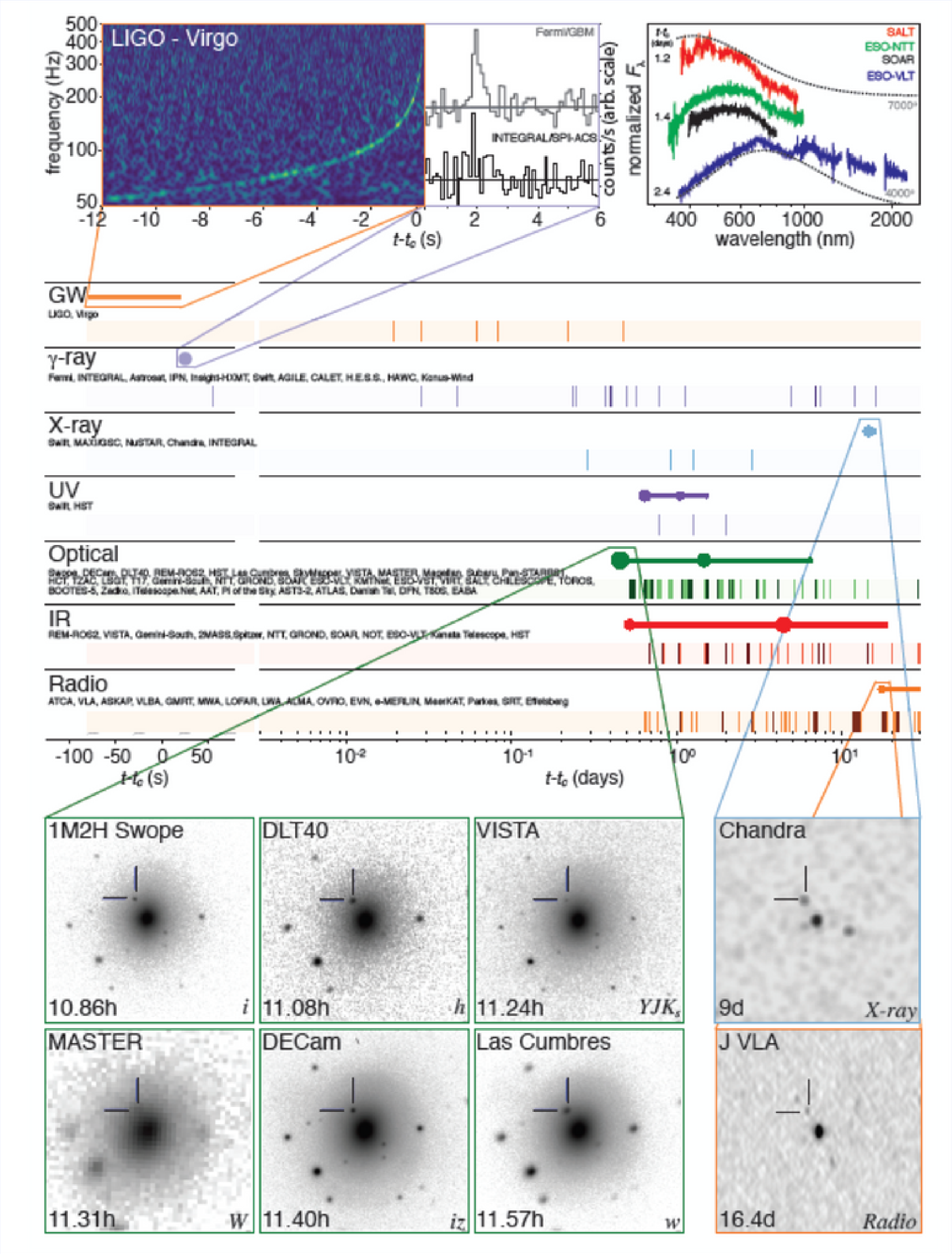
Request for network of multi-wavelenght observatories which cover huge region of the sky and repeat observations over different timescales

GW170817-GRB170817 !!!



GW170817-GRB170817

World-wide observational campaign



Multi-messenger Observations paper, 2017, ApJ, 848, L12

Publications on 16 October 2017:

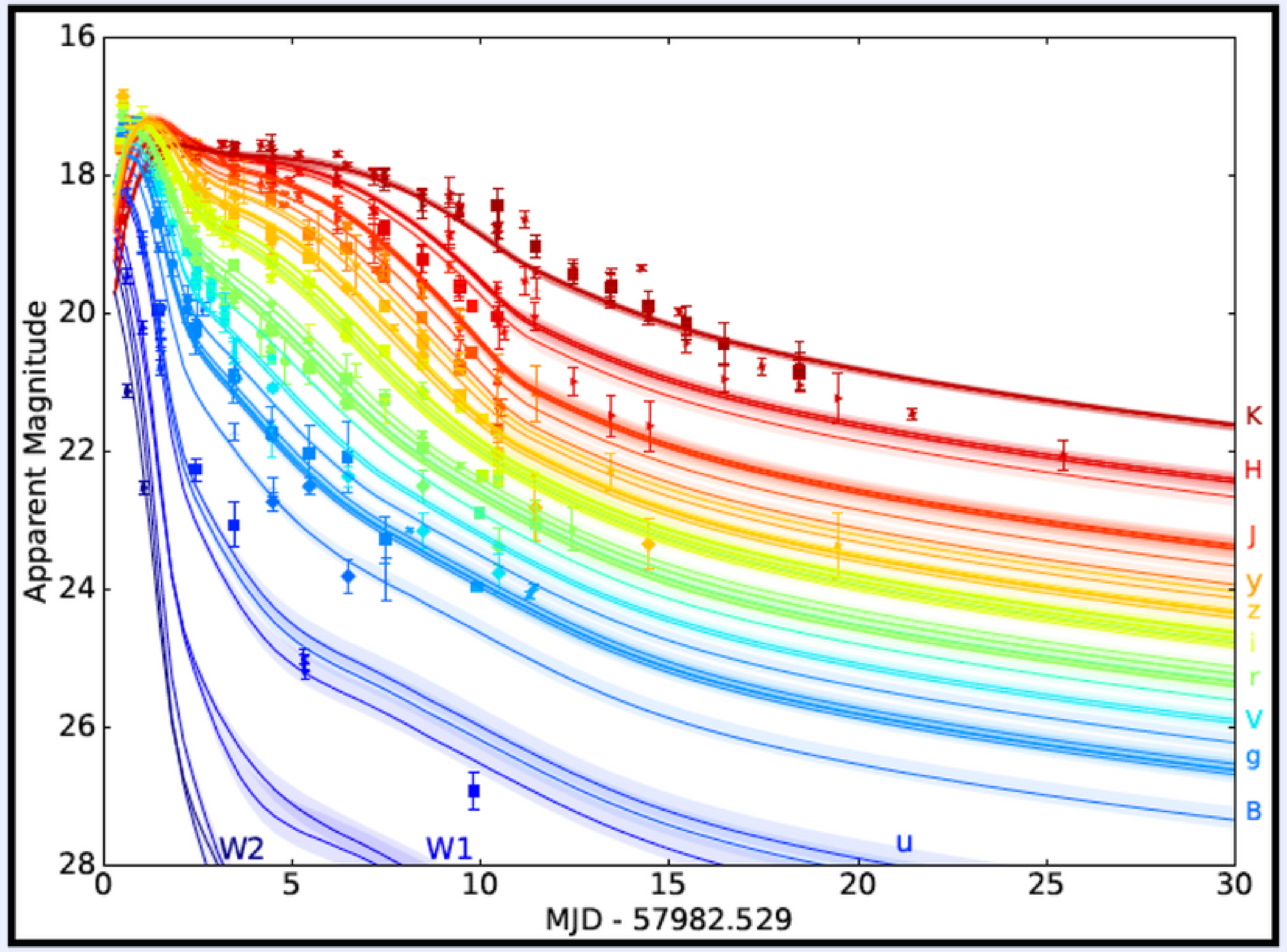
Abbott et al. 2017, PhReL, 119, 1101; **Abbott** et al. 2017, ApJL, 848, L13;
Alexander et al. 2017, ApJ, 848, L21; **Arcavi** et al. 2017, ApJ, 848, L33;
Blanchard et al. 2017, ApJ, 848, L22; **Chornock** et al. 2017, ApJ, 848, L19;
Coulter et al. 2017, Science, 10.1126, aap9811; **Covino** et al. 2017, Nat. Astr., 1, 791;
Cowperthwaite et al. 2017, ApJ, 848, L17; **Diaz** et al. 2017, ApJ, 848, L29;
Drout et al. 2017, Science, 10.1126, aaq0049; **Goldstein** et al. 2017, ApJ, 848, L14;
Evans et al. 2017, Science, 10.1126, aap9580; **Fong** et al. 2017, ApJ, 848, L23
Haggard et al. 2017, ApJ, 848, L25; **Hallinan** et al. 2017, Science, 10.1126, aap9855;
Hjorth et al. 2017, ApJ, 848, L31; **Kasen** et al. 2017, Nature 551, 80;
Kasliwal et al. 2017, Science, 10.1126, aap9455; **Levan** et al. 2017, ApJ, 848, L28;
Kilpatrick et al. 2017, Science, 10.1126, aaq0073; **Margutti** et al. 2017, ApJ, 848, L20;
Marguia-Berthier et al. 2017, ApJ, 848, L34; **McCully** et al. 2017, ApJ, 848, L32;
Nicholl et al. 2017, ApJ, 848, L18; **Pian, D'Avanzo** et al. 2017, Nature, 551, 67;
Pan et al. 2017, ApJ, 848, L30; **Shappe** et al. 2017, Science, 10.1126, aaq0186;
Savchenko et al. 2017, ApJ, 848, L15; **Soares-Santos** et al. 2017, ApJ, 848, L16;
Siebert et al. 2017, ApJ, 848, L26; **Smartt** et al. 2017, Nature, 551, 75;
Tanvir et al. 2017, ApJ, 848, L27; **Troja** et al. 2017, Nature, 551, 71;
Valenti et al. 2017, ApJ, 848, L24;

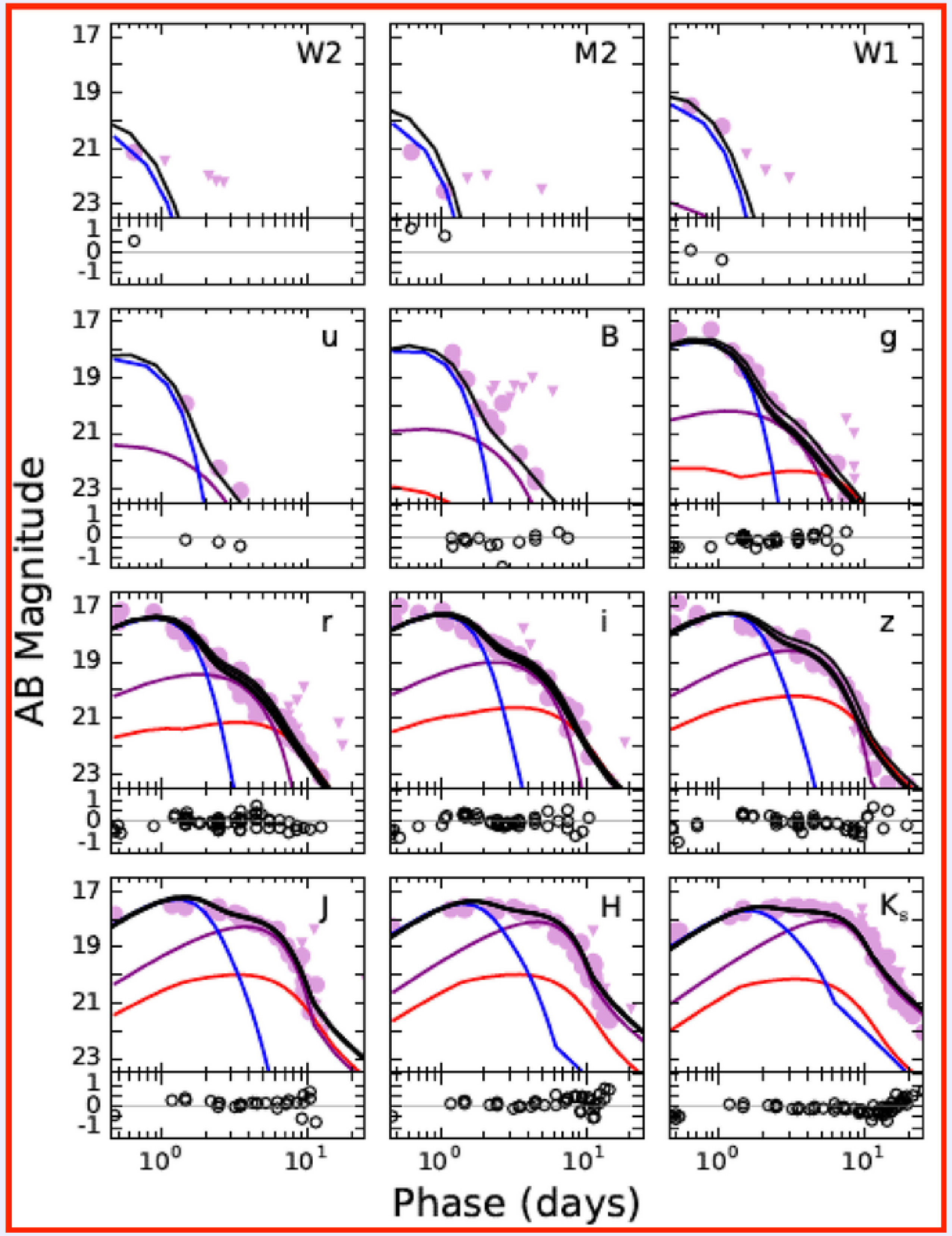
+ 70 papers on astro-ph between 16-18 October 2017

~110 published paper in the first week after the end of the embargo

Blue KN + Purple KN + Red KN

Villar et al. 2017, arXiv 1710.11576





Do we need a third component?

Not so significantly different, uncertainties are not small!

KN170817 SPECTRA

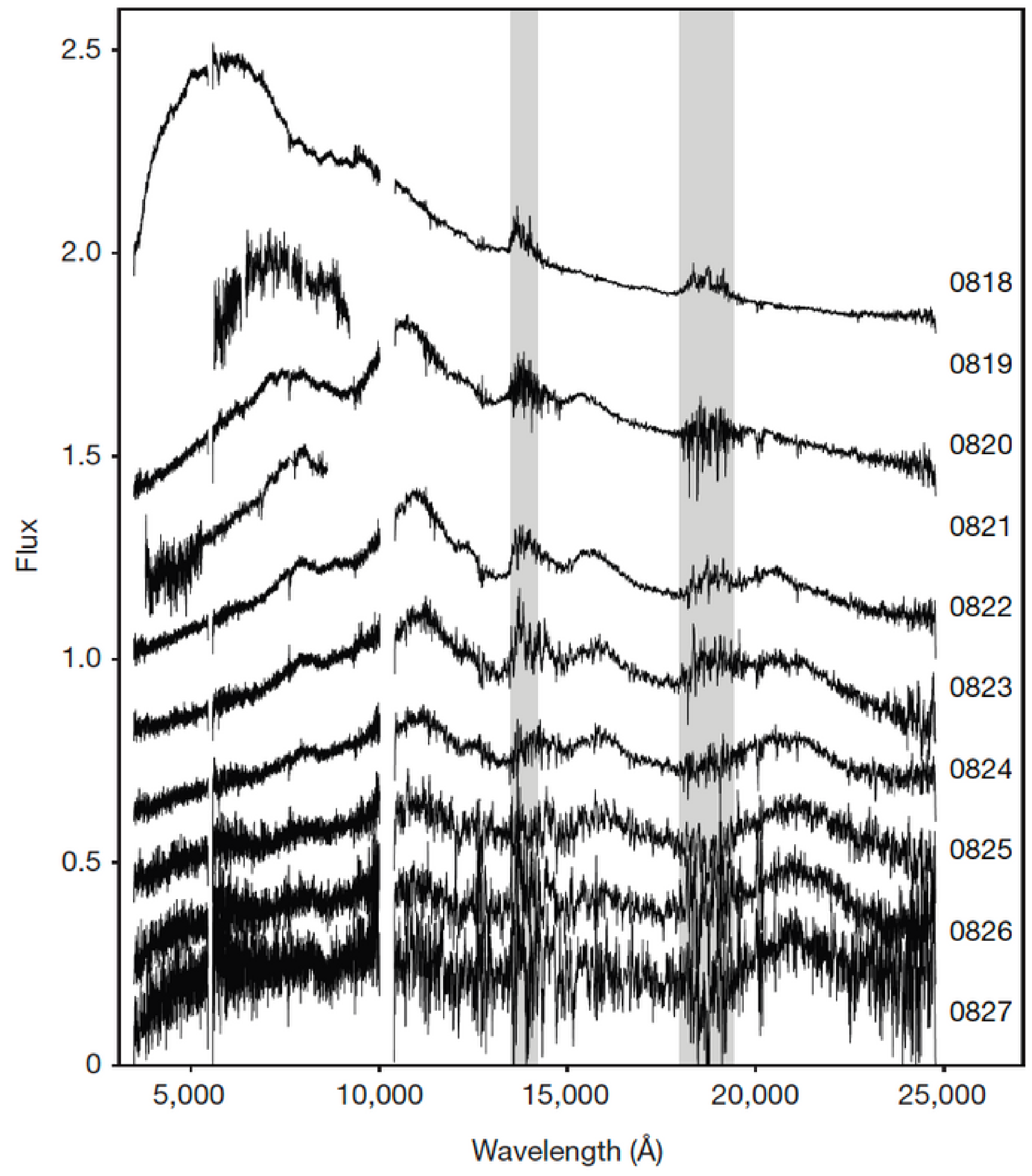
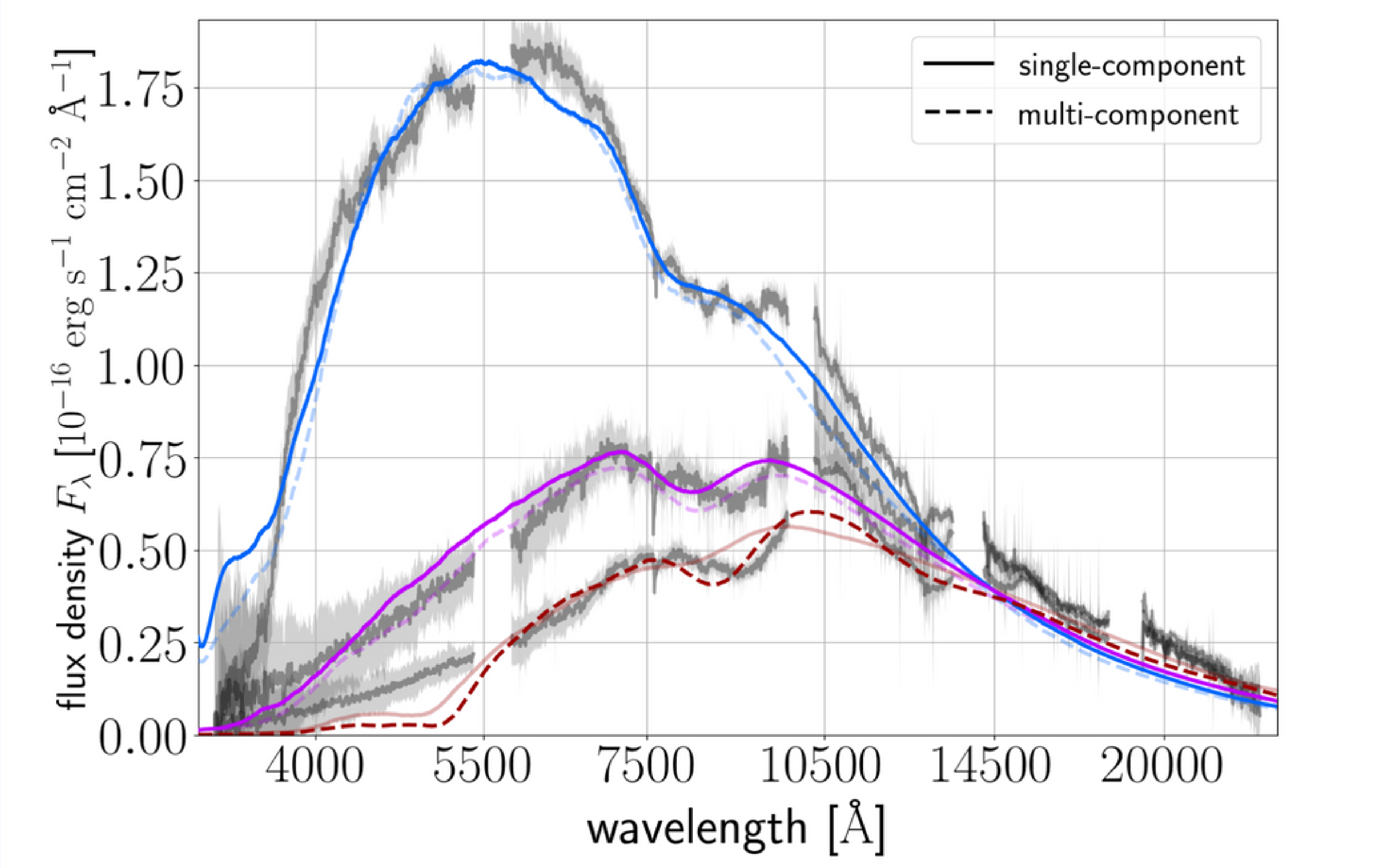


Figure 2 | Time evolution of the AT 2017gfo spectra. VLT/X-shooter,

Pian et al. 2017



Vieira et al. 2023

WHO WE ARE

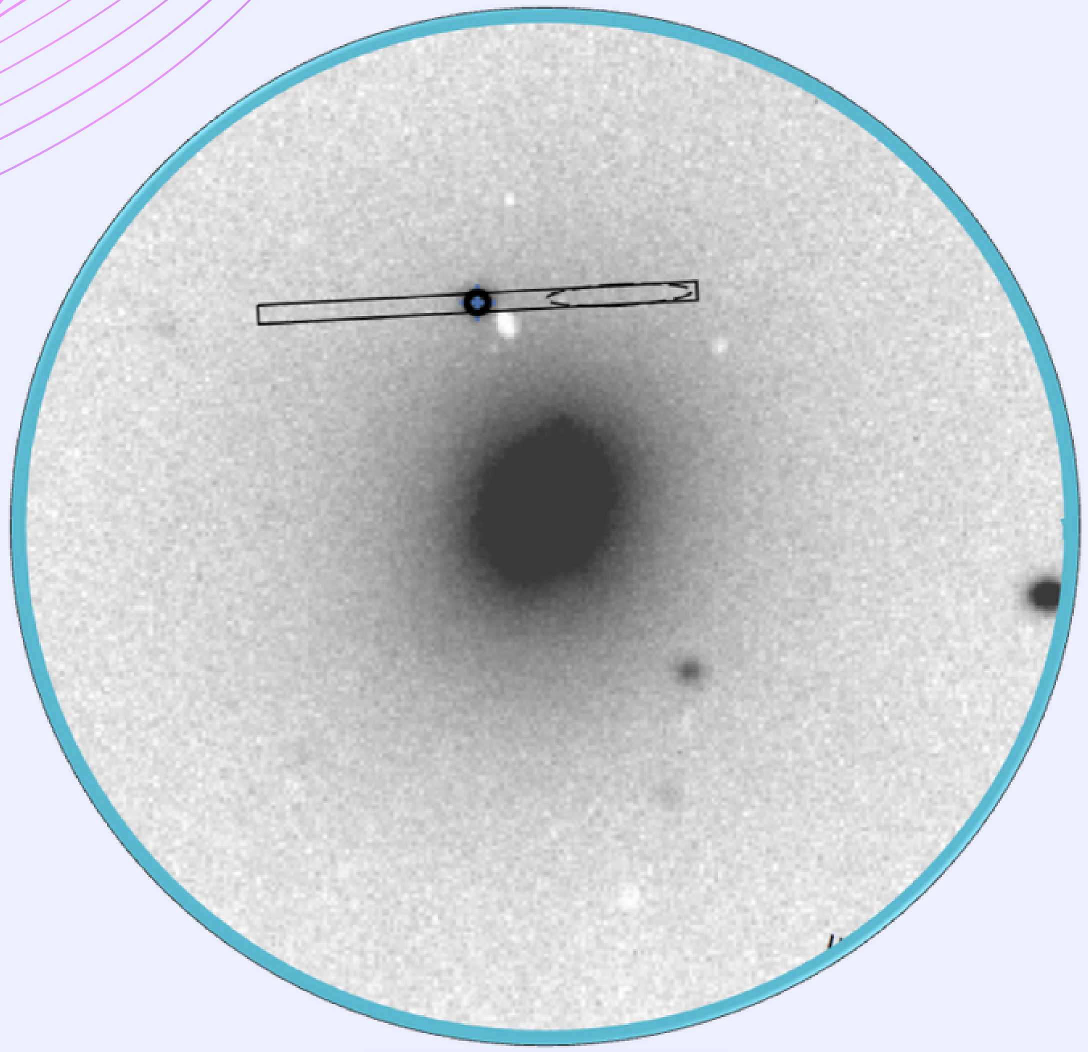
GRAvitational Waves Inaf TeAm

members: 100+ scientists from 20 research institutions

know how: Transients, GRBs, FRBs and SNe MW follow-up observations and theoretical models and data interpretation

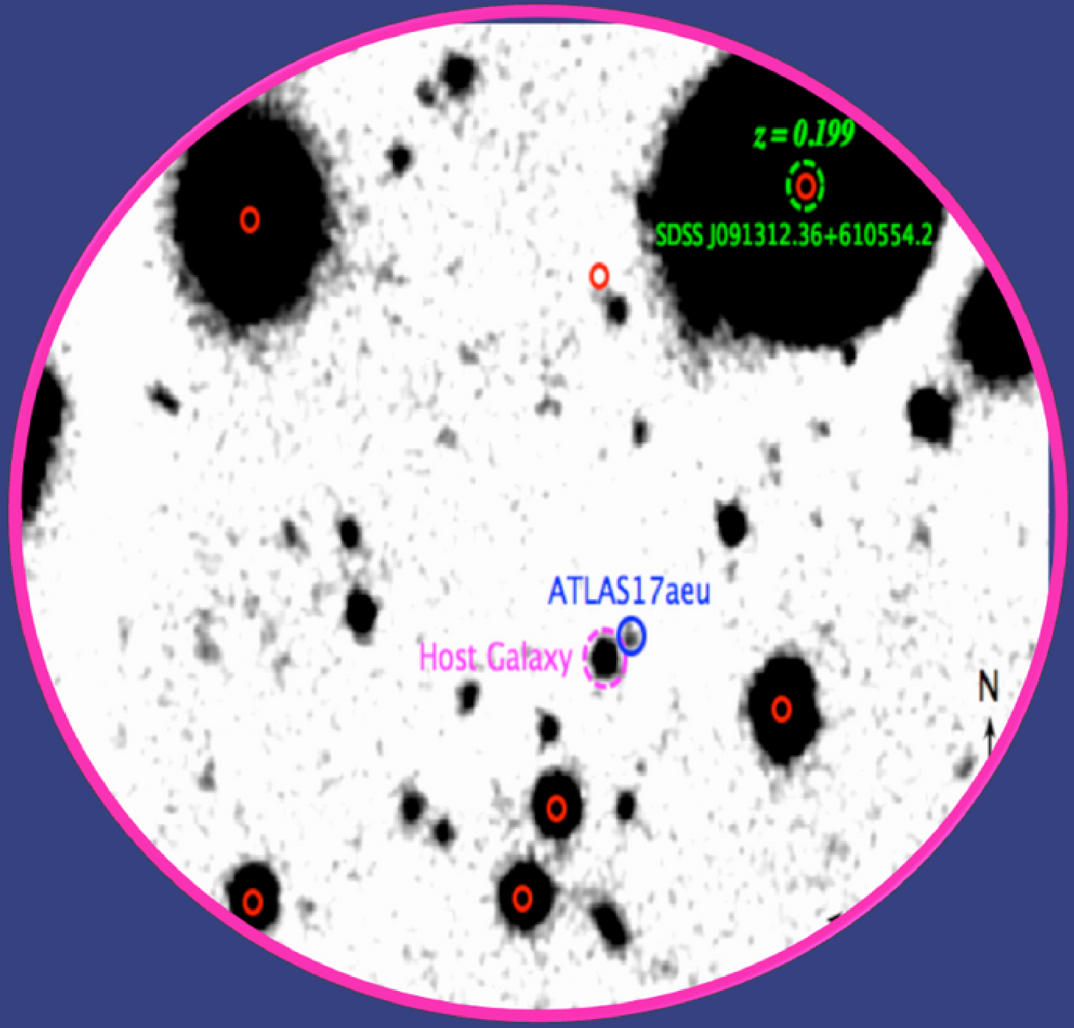
goals: Committed to taking part in the search and the study of electromagnetic counterparts of the GW events by using different observational facilities

collaborations:



Targeted Search

- most probable galaxies
- ranked by Luminosity and Distance
- imaging and (eventually) spectroscopy



Single Objects

- potential EM counterparts
- candidates found by WG2
- candidates by other teams
- imaging and spectroscopy

REM

CHARACTERISTICS

- 0.60 m
- f.o.v : 10x10 arcmin
- optical : griz simultaneous
- infrared : JHK
- **imaging**

WHAT CAN WE DO: <u>EVERYTHING</u> (OVERRIDE PRIVILEGES)

ASIAGO SCHMIDT

CHARACTERISTICS

- 0.67 - 0.92 m
- f.o.v : 59x59 arcmin
- optical : BVugri
- **imaging**

WHAT CAN WE DO: EVERYTHING (OVERRIDE PRIVILEGES)

ASIAGO COPERNICO

CHARACTERISTICS

- 1.82 m
- f.o.v : 8.6x8.6 arcmin
- optical : uBVgriz
- imaging and spectroscopy

WHAT CAN WE DO: EVERYTHING (OVERRIDE PRIVILEGES)

LOIANO

CHARACTERISTICS

- 1.52 m
- f.o.v : 13x13 arcmin
- optical : UBVRI
- **imaging** and **spectroscopy**

WHAT CAN WE DO: EVERYTHING (SOMEONE NEEDED ON SITE!!)

TNG

CHARACTERISTICS

- 3.6 m
- f.o.v : 8.6x8.6 / 4.2x4.2 arcmin
- optical : typically ugriz
- infrared : JHK
- imaging and spectroscopy

WHAT CAN WE DO: EVERYTHING (OVERRIDE PRIVILEGES)

LBT

CHARACTERISTICS

- 8.4 m
- f.o.v : 23x25 / 2x2 arcmin
- optical : UBVgrizY
- infrared : zJHK
- **imaging** and **spectroscopy**

WHAT CAN WE DO: TOO —> LBC (OPTICAL IMAGING); MODS (OPTICAL IMAGING + SPECTROSCOPY); LUCIFER (INFRARED IMAGING + SPECTROSCOPY)

NO PROPOSAL NEEDED

CAMPO IMPERATORE

- **SCHMIDT** (0.61-0.91 m) / **AZT-24** (1.1 m)
- f.o.v : 70x70 / 4x4 arcmin
- optical (ugriz) and infrared (JHK)
- **imaging**

PARCO ASTRONOMICO LILIO

- 0.5 m
- f.o.v : 21.1x21.1 arcmin
- optical (ugri)
- **imaging**

NOT

CHARACTERISTICS

- 2.5 m
- f.o.v : 6.4x6.4 arcmin
- optical : UBVRi
- infrared : JHK
- imaging and spectroscopy

WHAT CAN WE DO: ~EVERYTHING

NTT (EPESSTO+)

CHARACTERISTICS

- 3.6 m
- f.o.v : 4.1x4.1 arcmin
- optical : UBVRgriz
- imaging and spectroscopy

WHAT CAN WE DO: TOO COORDINATION WITH EPESSTO+ COLLABORATION

VLT (ENGRAVE)

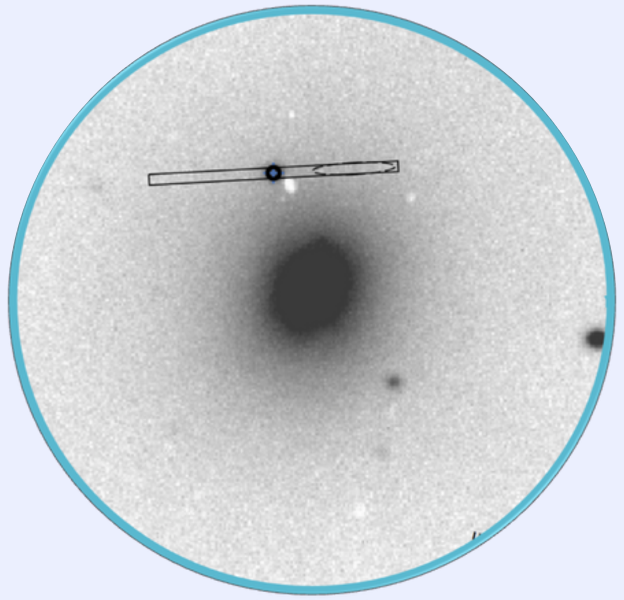
+ HST + JWST

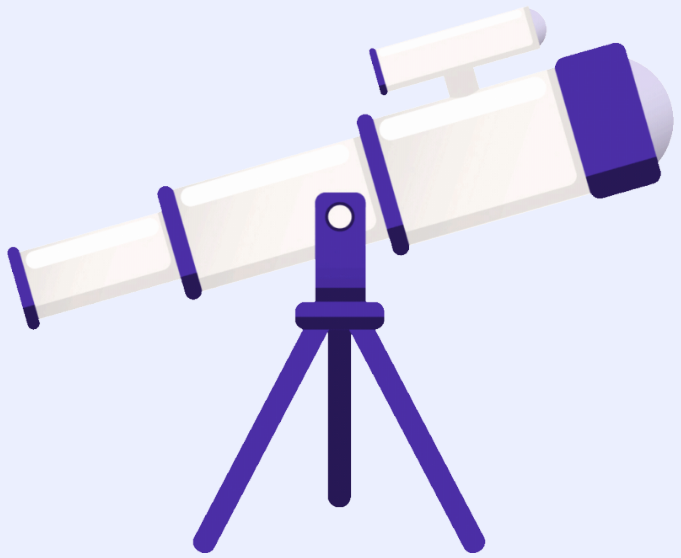
CHARACTERISTICS

- 8.2 m
- f.o.v : 6.8x6.8 arcmin
- optical : ubvgRIz
- **imaging** and **spectroscopy**

WHAT CAN WE DO: TOO IN COORDINATION WITH ENGRAVE COLLABORATION

STRATEGY





EM YES

- observing visibility
- available telescope time
- best multi-band coverage
- coordination with others

EM NO

- rank based on probability
- host brightness
- source distance
- observing visibility
- available telescope time

EM MAYBE

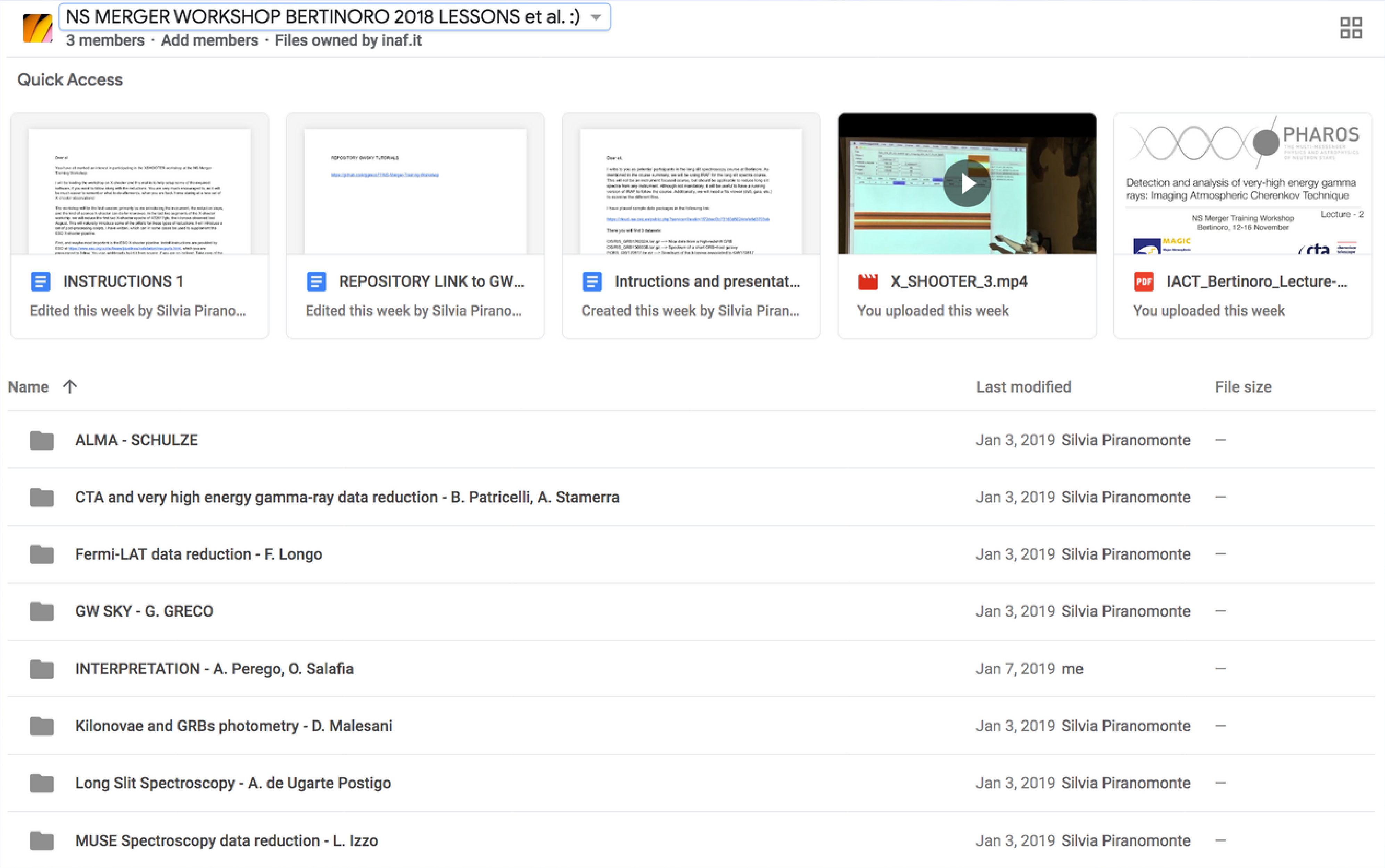
- characterization of possible candidates
- coordination with others

DATA REDUCTION

& transient classification

- Our work will take place from “day 2” onwards
- Subgroups for imaging and spectroscopy reduction
- For each observation: duties for data reduction are assigned (depending on members expertise)
- Transients classification based on light curve and spectroscopic templates (KNe, GRB afterglows, SNe, novae, etc…)
- Interesting/peculiar sources can be followed for longer (byproduct science)

DATA REDUCTION TUTORIALS



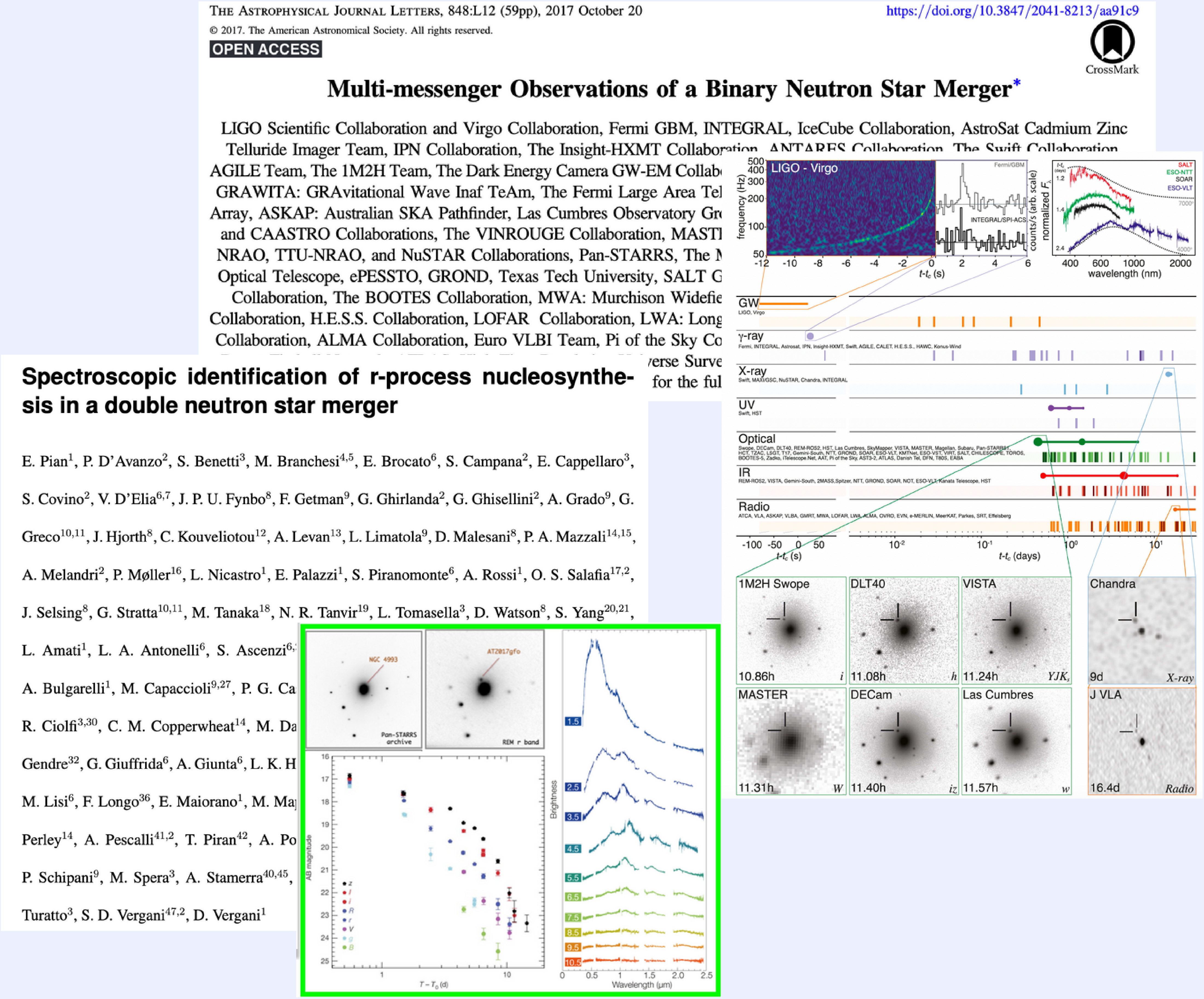
https://drive.google.com/drive/folders/0AKB_He3lnc6XUk9PVA

https://drive.google.com/drive/folders/0AKB_He3lnc6XUk9PVA

FOLLOW-UP RESULTS

DURING O1, O2 AND O3

- 53 GCNS
- "REACTION" TO 11 GW TRIGGERS

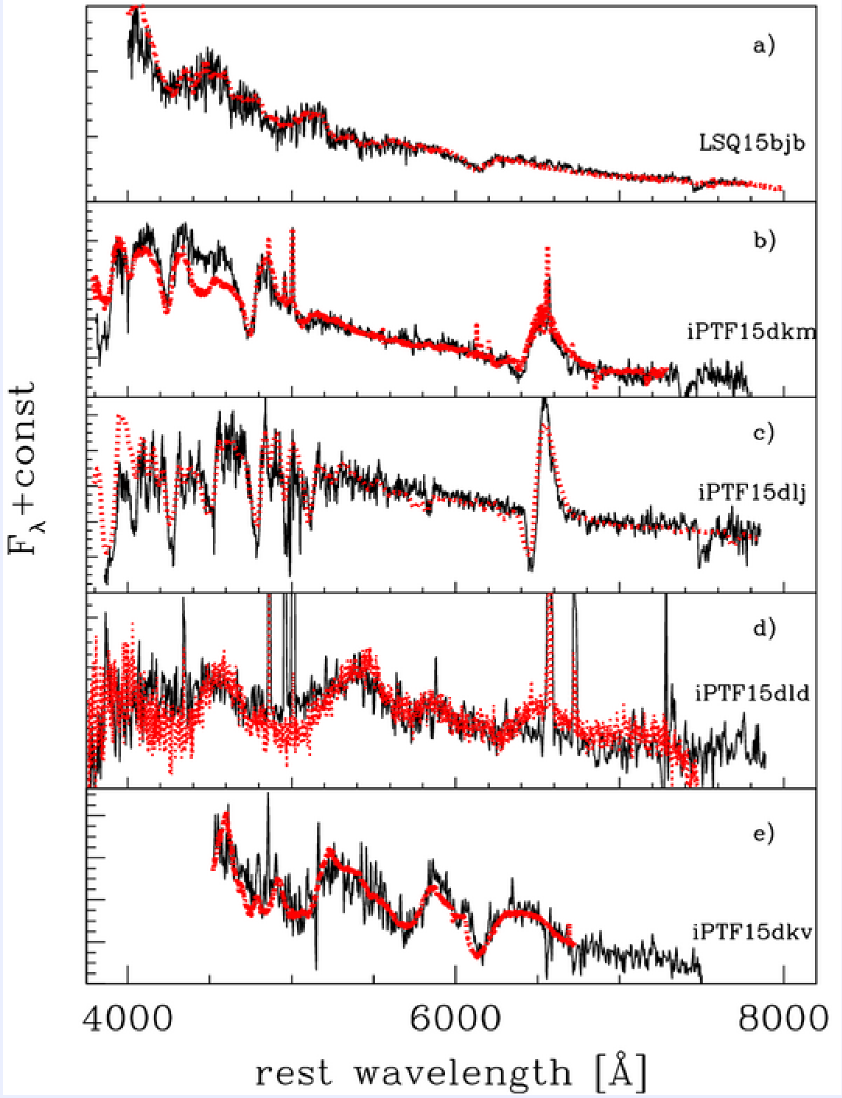


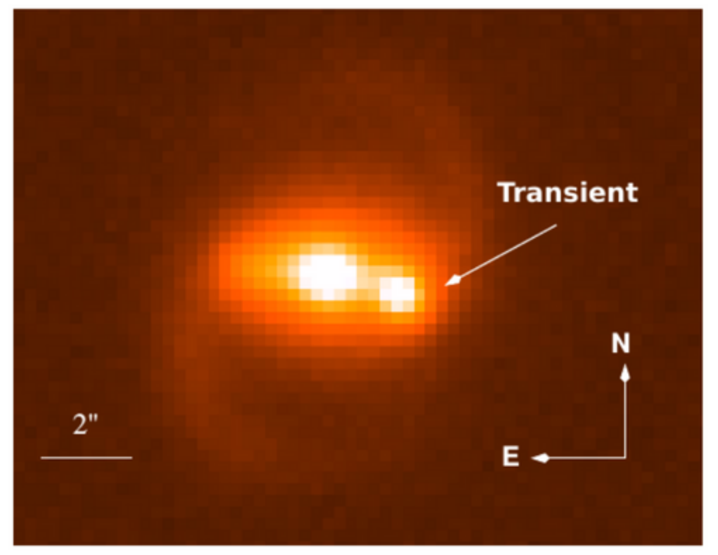
FOLLOW-UP RESULTS

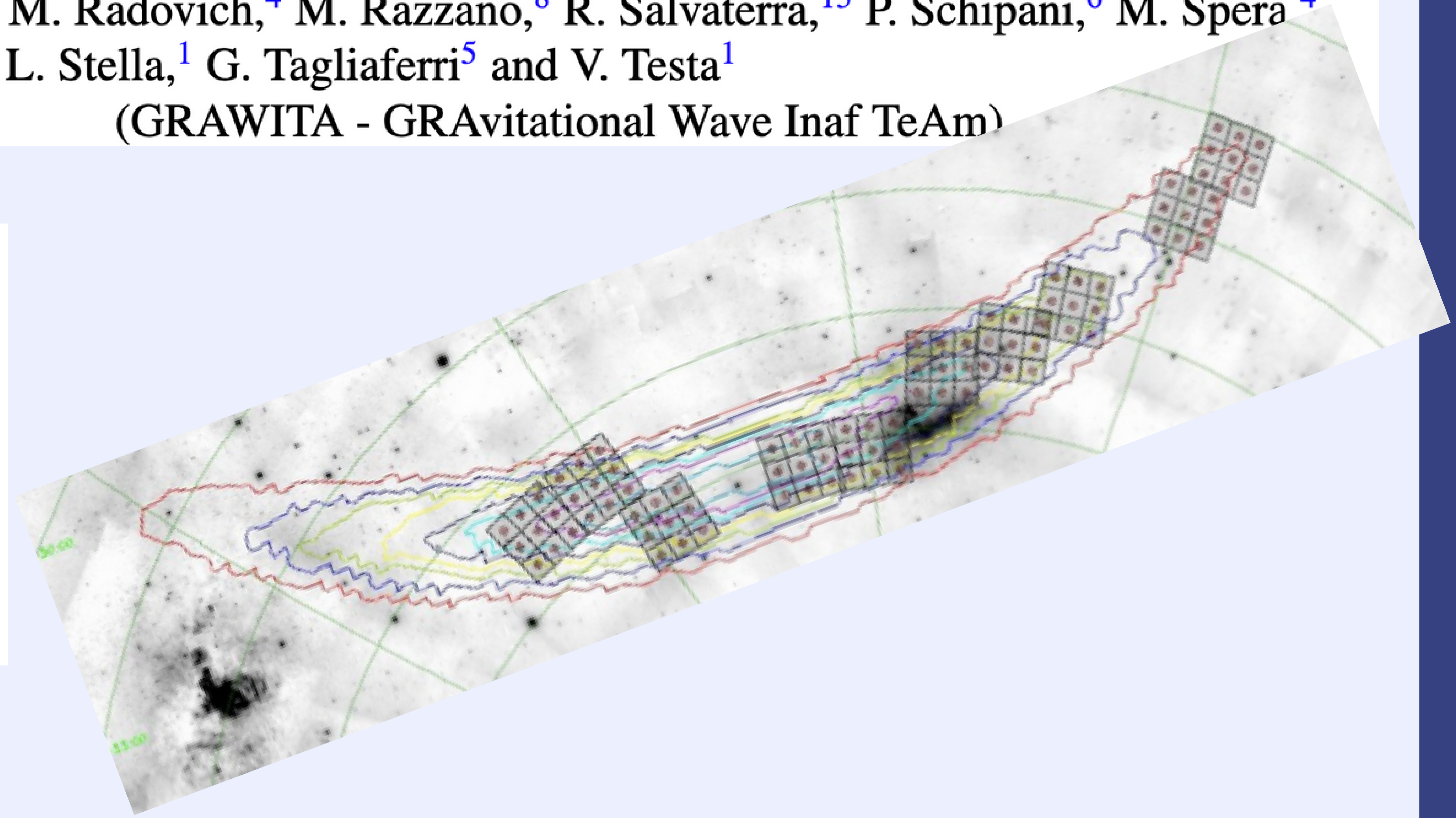
DURING O1, O2 AND O3

GRAWITA: VLT Survey Telescope observations of the gravitational wave sources GW150914 and GW151226

E. Brocato,[1★] M. Branchesi,[2,3] E. Cappellaro,[4] S. Covino,[5★] A. Grado,[6] G. Greco,[2,3] L. Limatola,[6] G. Stratta,[2,3] S. Yang,[4] S. Campana,[5] P. D'Avanzo,[5] F. Getman,[6] A. Melandri,[5] L. Nicastro,[7] E. Palazzi,[7] E. Pian,[7,8] S. Piranomonte,[1] L. Pulone,[1] A. Rossi,[7] L. Tomasella,[4] L. Amati,[7] L. A. Antonelli,[1] S. Ascenzi,[1] S. Benetti,[4] A. Bulgarelli,[7] M. Capaccioli,[9] G. Cella,[8] M. Dadina,[7] G. De Cesare,[7] V. D'Elia,[1] G. Ghirlanda,[5] G. Ghisellini,[5] G. Giuffrida,[1] G. Iannicola,[1] G. Israel,[1] M. Lisi,[1] F. Longo,[10] M. Mapelli,[4] S. Marinoni,[1] P. Marrese,[1] N. Masetti,[7,11] B. Patricelli,[8] A. Possenti,[12] M. Radovich,[4] M. Razzano,[8] R. Salvaterra,[13] P. Schipani,[6] M. Spera,[4] A. Stamerra,[8] L. Stella,[1] G. Tagliaferri[5] and V. Testa[1]
(GRAWITA - GRAvitational Wave Inaf TeAm)







FOLLOW-UP RESULTS

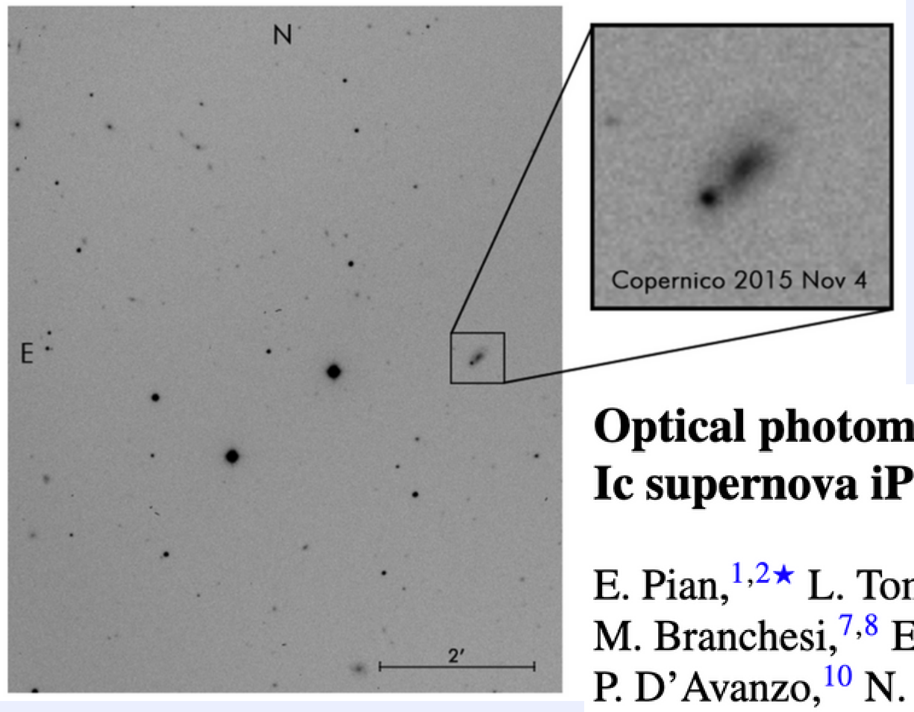
DURING O1, O2 AND O3

A comparison between short GRB afterglows and kilonova AT2017gfo: shedding light on kilonovae properties

A. Rossi,[1,2★] G. Stratta,[1,3] E. Maiorano,[1] D. Spighi,[1] N. Masetti,[1,4] E. Palazzi,[1] A. Gardini,[5] A. Melandri,[6] L. Nicastro,[1] E. Pian,[1] M. Branchesi,[7] M. Dadina,[1] V. Testa,[2] E. Brocato,[2,8] S. Benetti,[9] R. Ciolfi,[9,10] S. Covino,[6] V. D'Elia,[2,11] A. Grado,[12] L. Izzo,[5] A. Perego,[13] S. Piranomonte,[2] R. Salvaterra,[14] J. Selsing,[15] L. Tomasella,[9] S. Yang,[9] D. Vergani,[1] L. Amati[1] and J. B. Stephen[1]
on behalf of the Gravitational Wave Inaf TeAm (GRAWITA)

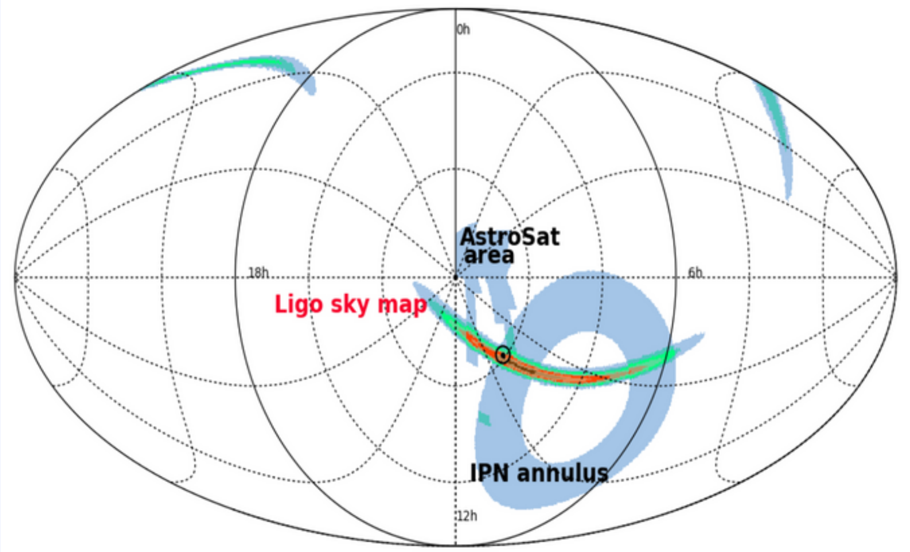
Unveiling the enigma of ATLAS17aeu★,★★

A. Melandri[1], A. Rossi[2], S. Benetti[3], V. D'Elia[4,5], S. Piranomonte[5], E. Palazzi[2], A. J. Levan[6], M. Branchesi[7,8], A. J. Castro-Tirado[9], P. D'Avanzo[1], Y.-D. Hu[9], G. Raimondo[10], N. R. Tanvir[11], L. Tomasella[3], L. Amati[2], S. Campana[1], R. Carini[3], S. Covino[1], F. Cusano[2], M. Dadina[2], M. Della Valle[12,9], X. Fan[13], P. Garnavich[14], A. Grado[12], G. Greco[15], J. Hjorth[16], J. D. Lyman[6], N. Masetti[2,17], P. O'Brien[11], E. Pian[2], A. Perego[18,19], R. Salvaterra[20], L. Stella[5], G. Stratta[15], S. Yang[3], A. di Paola[5], M. D. Caballero-García[21], A. S. Fruchter[22], A. Giunta[5], F. Longo[23], M. Pinamonti[24], V. V. Sokolov[25], V. Testa[5], A. F. Valeev[25], and E. Brocato[5]
on behalf of the Gravitational Wave InAf TeAm (GRAWITA)



Optical photometry and spectroscopy of the low-luminosity, broad-lined Ic supernova iPTF15dld

E. Pian,[1,2★] L. Tomasella,[3] E. Cappellaro,[3] S. Benetti,[3] P. A. Mazzali,[4,5] C. Baltay,[6] M. Branchesi,[7,8] E. Brocato,[9] S. Campana,[10] C. Copperwheat,[4] S. Covino,[10] P. D'Avanzo,[10] N. Ellman,[6] A. Grado,[11] A. Melandri,[10] E. Palazzi,[1] A. Piascik,[4] S. Piranomonte,[9] D. Rabinowitz,[6] G. Raimondo,[12] S. J. Smartt,[13] I. A. Steele,[4] M. Stritzinger,[14] S. Yang,[3] S. Ascenzi,[9] M. Della Valle,[11,15] A. Gal-Yam,[16] F. Getman,[11] G. Greco,[7,8] C. Inserra,[13] E. Kankare,[13] L. Limatola,[11] L. Nicastro,[1] A. Pastorello,[3] L. Pulone,[9] A. Stamerra,[2,17] L. Stella,[9] G. Stratta,[7,8] L. Tartaglia[3] and M. Turatto[3,18]



RUBIN-LSST

- 100000 transients/night
- Photometric accuracy = 10 mMag
- Astrometric accuracy = 50 mas
- Information within 60s (brokers)

INAF-GURU project (GRAWITA using Rubin-LSST):

1. as a discovery machine
2. as a follow-up machine

INAF

Up to 10 million alerts, 20 TB of data … every night

VERA C. RUBIN OBSERVATORY

u	23.9	26.1
g	25.0	27.4
r	24.7	27.5
i	24.0	26.8
z	23.3	26.1
y	22.1	24.9

Science Collaborations (SCs)

- Galaxies
- Stars, Milky Way & Local Volume
- Solar System
- Dark Energy
- Active Galactic Nuclei
- Transient/Variable Stars
- Strong Lensing
- Informatics & Statistics

✓ 6-band (0.3-1.1 micron) wide-field deep astronomical survey of over 20,000 square degrees of the southern sky.

✓ Each patch of sky will be visited about 1000 times in ten years.

Up to 10 million alerts, 20 TB of data … every night

VERA C. RUBIN OBSERVATORY

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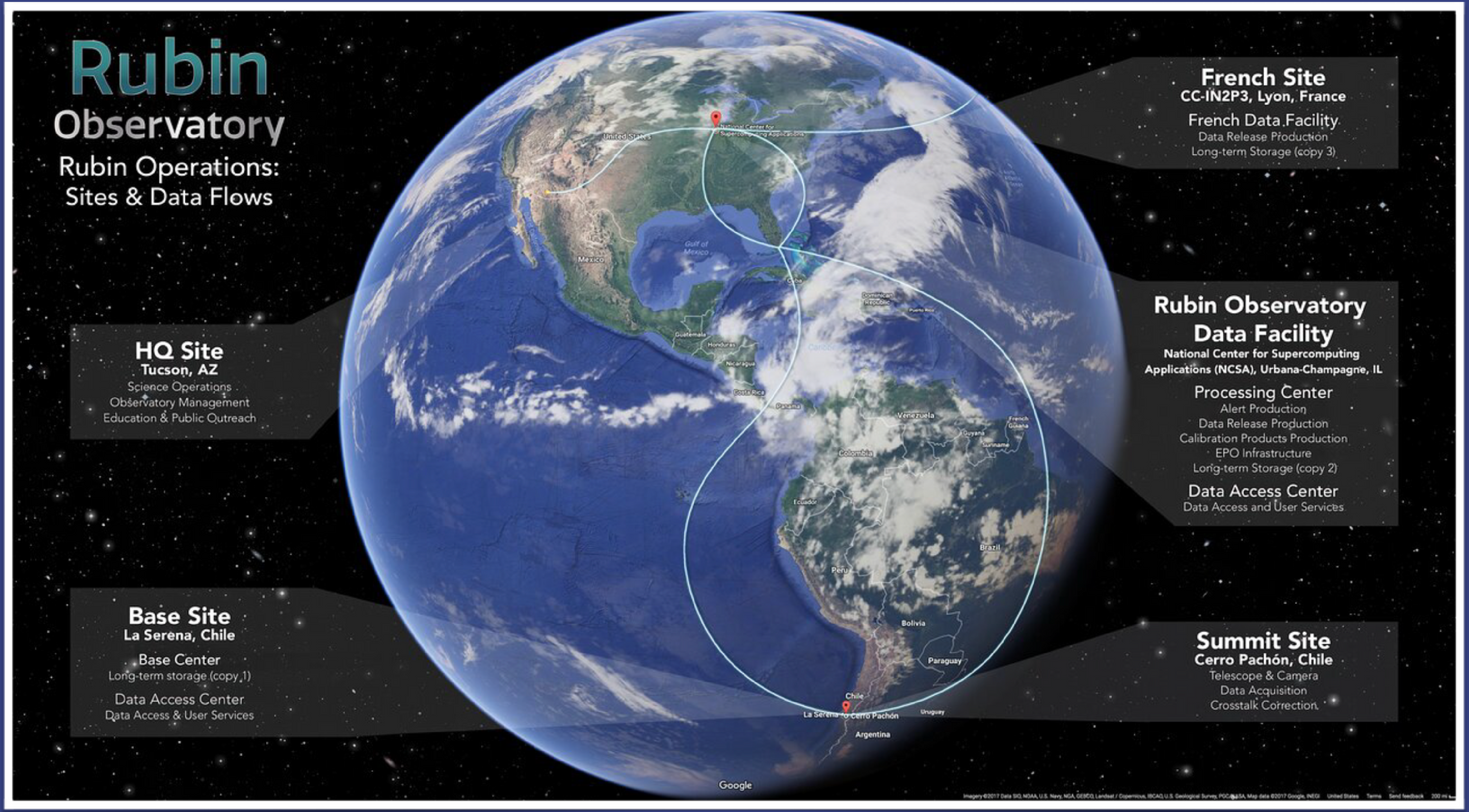
Science Collaborations (SCs)

- Galaxies
- Stars, Milky Way & Local Volume
- Solar System
- Dark Energy
- Active Galactic Nuclei
- Transient/Variable Stars
- Strong Lensing
- Informatics & Statistics

FoV 9.6 deg^2
0.2”/pixel pitch
3.2 Gpixel camera
8.4m primary mirror
10 year survey of the sky
37 billion stars and galaxies
Site El Penon, Cerro Pachon, Chile
Each image has size of 40 full moons

LSST Operations: Sites & Data FLows

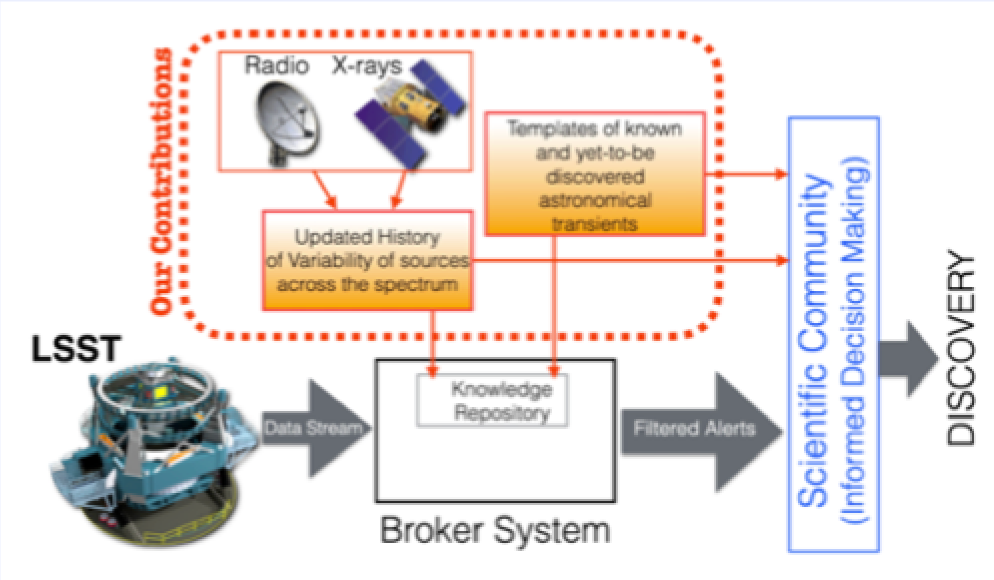


The total data volume after processing will be several hundred PB, processed using about 150 TFLOPS (trillion floating point operations per second) of computing power for the first DR, increasing to 950 TFLOPS by DR 11

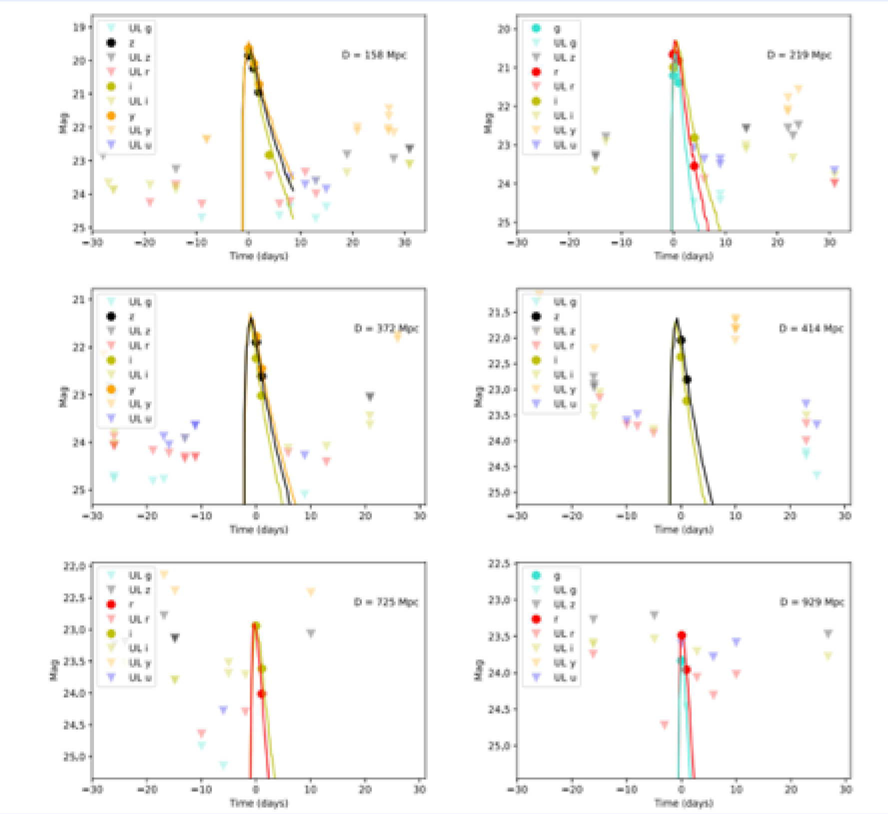
- Final volume of raw image data =60 PB
- Final image collection (DR11) =0.5 Exabytes
- Final catalog size (DR11) = 15 PB
- Final disk storage = 0.4 Exabytes
- Peak number of nodes = 1750 nodes
- Peak compute power in LSST data centers = about 2 TFLOPS

HOW CAN WE WORK?

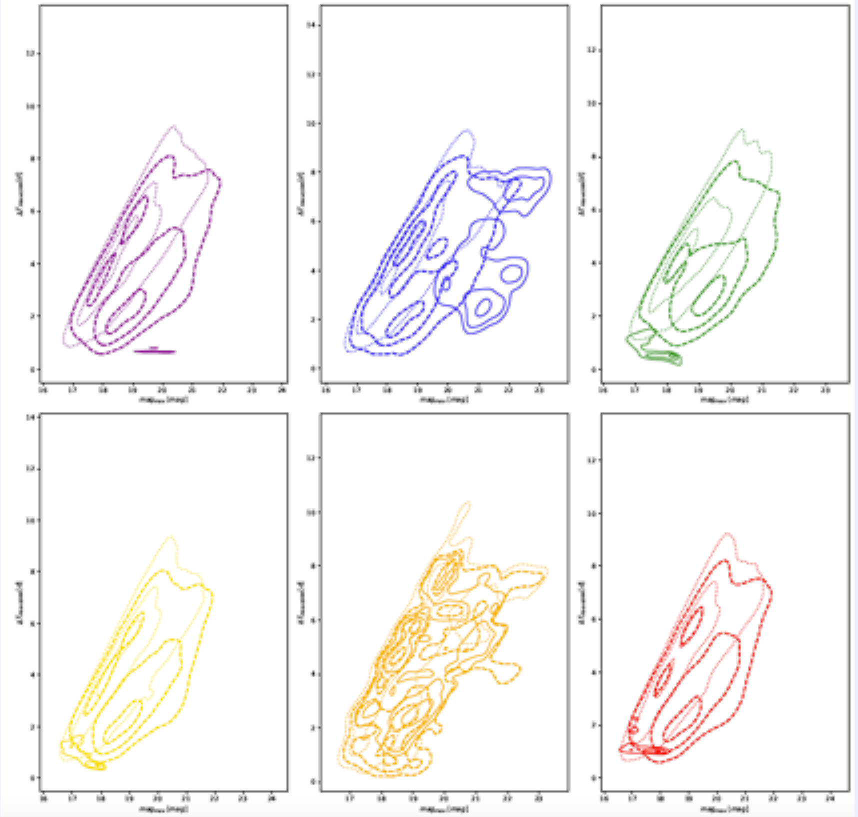
Rubin-LSST = discovery machine



- Maximize chances of first or second night counterpart candidate identification for multi-wavelength follow-up with other telescopes
- Ability to distinguish the counterpart from most "contaminants"
- Kilonova parameters estimation



Examples of synthetic GW170817-like kilonovae light-curves found serendipitously in the Rubin-LSST simulated baseline cadence.
(Fig. 1 from Andreoni et al. 2021)



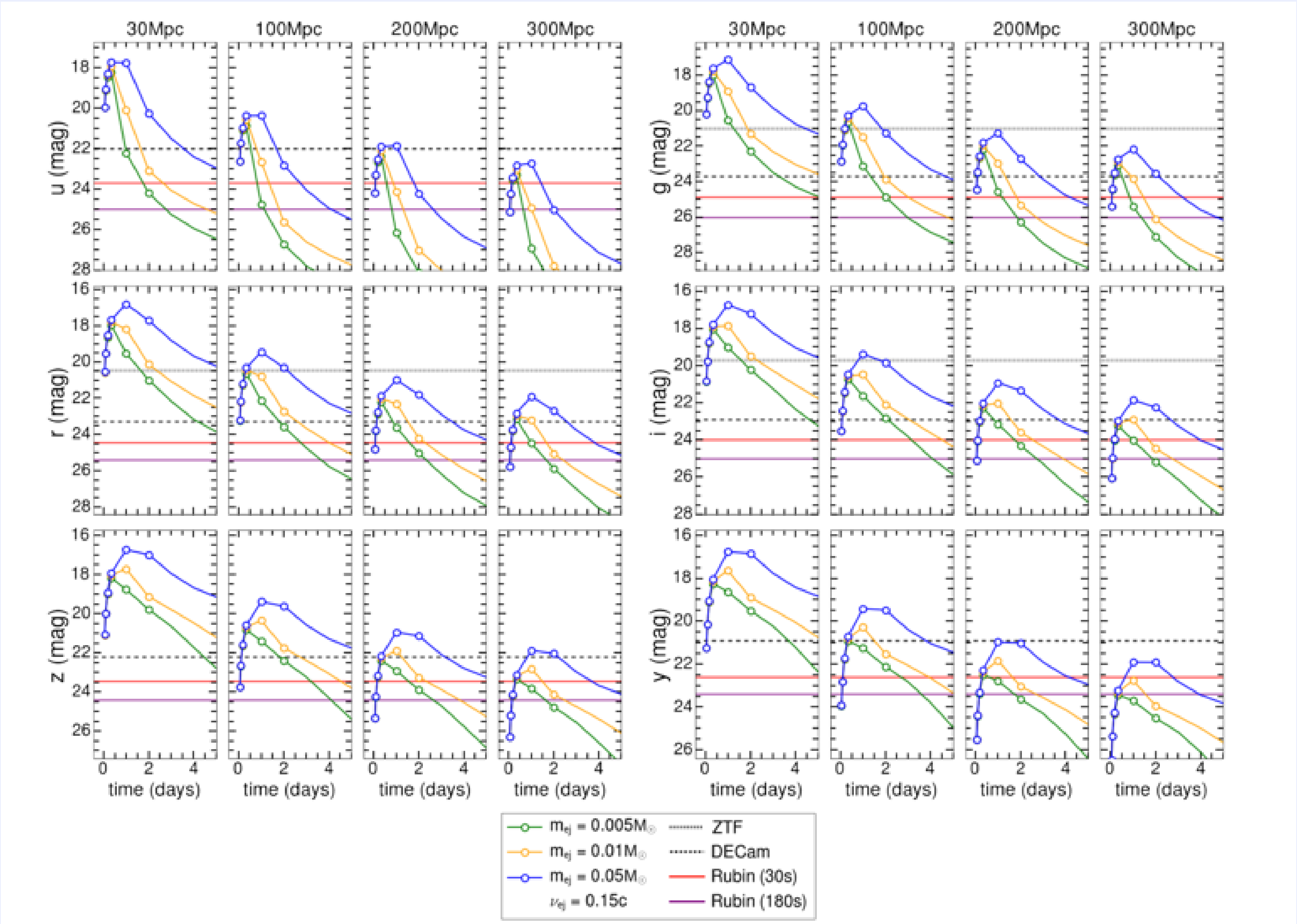
Peak magnitude (X axis) and duration (Y axis) distributions extracted from KN light curve above the limiting magnitude are shown.
(Fig. 6 from Ragosta et al. 2024)

HOW CAN WE WORK?

Rubin-LSST = follow-up machine

GW detectors find sources

- Selection of which GW triggers to follow-up (localization < 100 deg^2)
- Rapid ToO observations



Simulated KN light curves in the six Rubin filters for different properties of the ejecta (mass and velocity) at four representative distances (30, 100, 200, and 300 Mpc). (Fig 1. from Andreoni et al. 2022)

ET (2035)... WE NEED MODERN APPROACHES

We need experience to create and manipulate **large data sets** for the follow-up of the electromagnetic counterparts of GW sources.

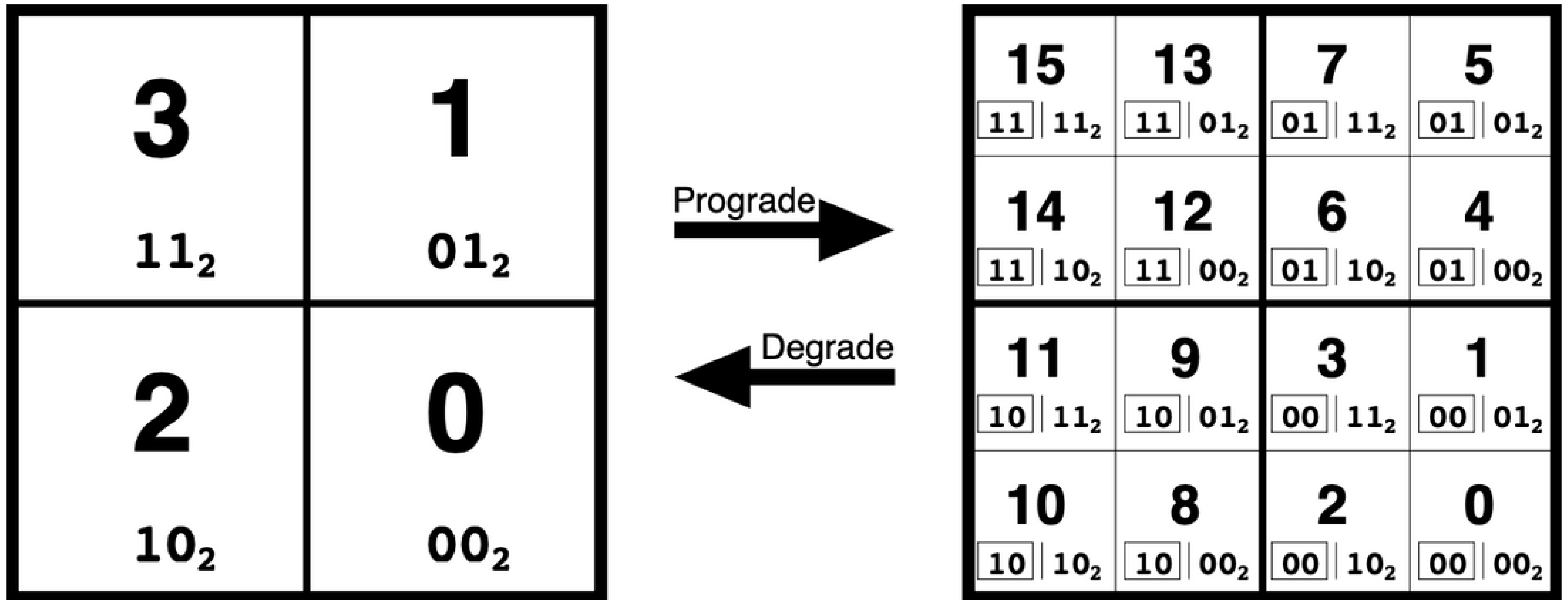
HANDS ON of TODAY

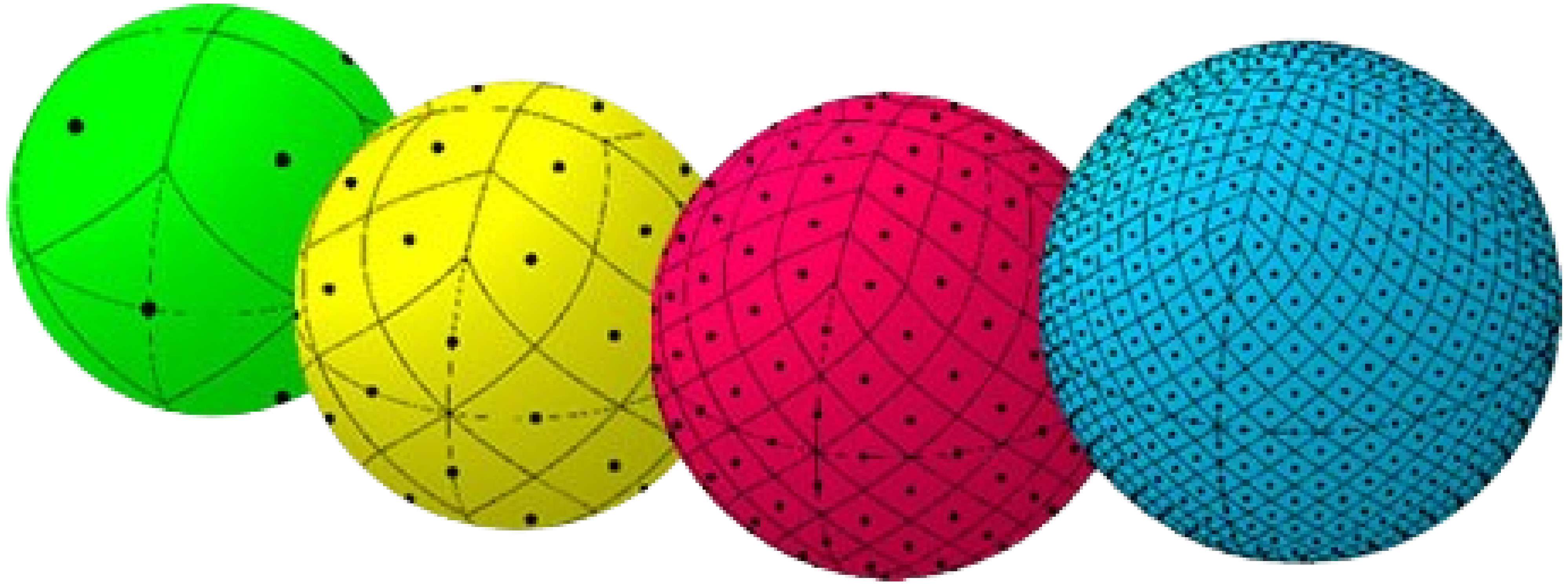
Learn how to build a light curve using Alerts stream.
Learn how classification tools work.

Fabio the floor is yours!

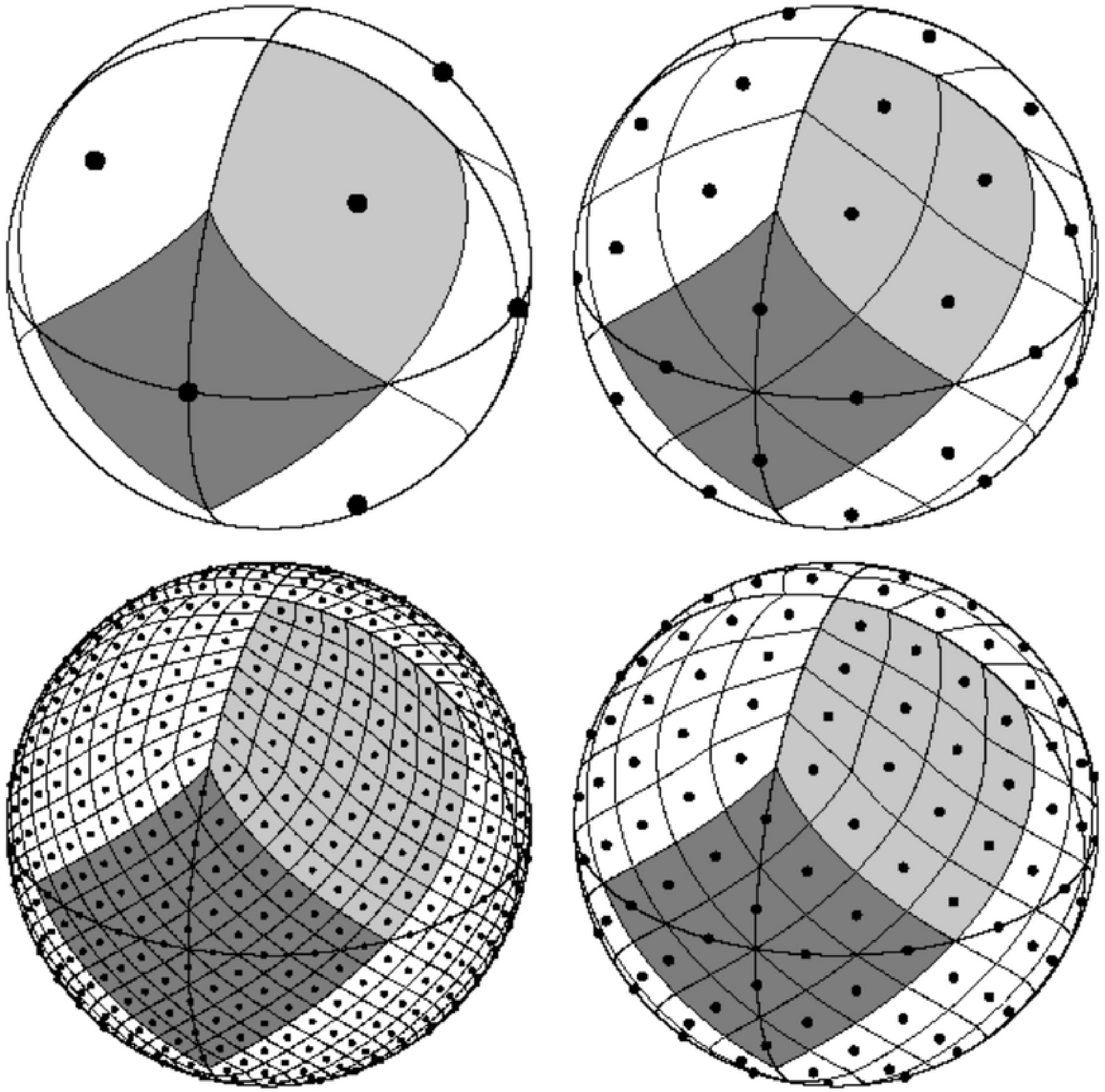
HEALPix for Galaxy ranking

- **H**ierarichical **E**qual **A**rea **Pix**elization
- Encoding All Sky image in one array so that white noise is integrated spatially and depend only on the the shape of tasselization
- **CHOICE OF INDEX SCHEME**
  - **Nasted:** Consecutive pairs of bits in the pixel indices encode the address of healpix tree
  - **Ring:** Pixel indices advande in (RA,DEC)
  - **Uniq:** Pixel resolution/ipix are converted in a single integer





HEALPix for Galaxy ranking



Moving clockwise from the upper left panel the grid is hierarchically subdivided with the grid resolution parameter equal to Nside = 1, 2, 4, 8, and the total number of pixels equal to Npix = 12 × N^2 side = 12, 48, 192, 768.

All pixel centers are located on Nring = 4 × Nside − 1 rings of constant latitude.
Within each panel the areas of all pixels are identical.

(RA, DEC) ⇄ (nside, ipix, scheme)

conversion from/to sky coordinates

`pix2ang`(nside, ipix[, nest, lonlat])	pix2ang : nside,ipix,nest=False,lonlat=False -> theta[rad],phi[rad] (default RING)
`pix2vec`(nside, ipix[, nest])	pix2vec : nside,ipix,nest=False -> x,y,z (default RING)
`ang2pix`(nside, theta, phi[, nest, lonlat])	ang2pix : nside,theta[rad],phi[rad],nest=False,lonlat=False -> ipix (default:RING)
`vec2pix`(nside, x, y, z[, nest])	vec2pix : nside,x,y,z,nest=False -> ipix (default:RING)

nside/npix/resolution ¶

`nside2npix`(nside)	Give the number of pixels for the given nside.
`npix2nside`(npix)	Give the nside parameter for the given number of pixels.

HEALPix for Galaxy ranking

HEALPix Columns

FITS Name	Symbol	Units	Description
PROB	ρ_i	pixel^{-1}	Probability that the source is contained in pixel i, centered on the direction $\boldsymbol{n}_i$
DISTMU	$\hat{\mu}_i$	Mpc	Ansatz location parameter of conditional distance distribution in direction $\boldsymbol{n}_i$, or ∞ if invalid
DISTSIGMA	$\hat{\sigma}_i$	Mpc	Ansatz scale parameter of conditional distance distribution in direction $\boldsymbol{n}_i$, or 1 if invalid
DISTNORM	$\hat{N}_i$	Mpc^{-2}	Ansatz normalization coefficient, or 0 if invalid

Conditional distance distribution

$$p(r|\boldsymbol{n}) = \frac{N(\boldsymbol{n})}{\sqrt{2\pi}\,\sigma(\boldsymbol{n})} \exp\left[-\frac{(r-\mu(\boldsymbol{n}))^2}{2\sigma(\boldsymbol{n})^2}\right] r^2 \quad \text{for } r \geqslant 0.$$

3D probability density

$$P(r, \boldsymbol{n}_i)\, dr = \rho_i \frac{\hat{N}_i}{\sqrt{2\pi}\,\hat{\sigma}_i} \exp\left[-\frac{(r-\hat{\mu}_i)^2}{2\hat{\sigma}_i^2}\right] r^2\, dr.$$

Probability density per unit Eucledian Volume

$$dV = r^2\, dr\, \Delta\Omega = \frac{4\pi}{N_{\text{pix}}} r^2\, dr.$$

$$\frac{dP}{dV} = \rho_i \frac{N_{\text{pix}}}{4\pi} \frac{\hat{N}_i}{\sqrt{2\pi}\,\hat{\sigma}_i} \exp\left[-\frac{(r-\hat{\mu}_i)^2}{2\hat{\sigma}_i^2}\right].$$