Fisher matrix for gravitational wave forecasting

Jacopo Tissino, Ulyana Dupletsa

2024-02-20

When we build Einstein Telescope, how many compact binary signals will it be able to detect?
How well will it localize them in the sky?
How well will it measure their parameters?

Matched filtering

The gravitational-wave **detection** problem: we have data

$$d(t) = \underbrace{h_\theta(t)}_{\text{GW}} + \underbrace{n_{\text{Gaussian}}(t) + n_{\text{non-Gaussian}}(t)}_{\text{noise } n(t)}$$

and we want to find where $h(t)$ is, while typically $|h| \ll |n|$. The Gaussian component has a spectral density $S_n(f)$. Let's assume we want to use a **linear filter**, so that in the time domain it reads:

$$\hat{\rho}(\tau) = \int d(t+\tau) f(t) \mathrm{d}t$$

for some function $f(t)$.

We want to maximize the "**distinguishability**" of the signal: we can quantify it with the signal-to-noise ratio

$$\frac{S}{N} = \frac{\hat{\rho}(\text{a signal is present})}{\text{root-mean-square of } \hat{\rho}}$$

Ignoring the non-Gaussian part of the noise, the **optimal** solution is $\hat{\rho} \propto (d|h)$, where

$$(a|b) = 4\Re \int_0^\infty \frac{a(f)b^*(f)}{S_n(f)} \mathrm{d}f \,,$$

Optimal signal-to-noise ratio

The **signal-to-noise** ratio statistic is

$$\rho = \frac{S}{N} = \frac{(d|h)}{\sqrt{(h|h)}}$$

With the expected noise realization ($\langle n(t) \rangle = 0$):

$$\rho_{\mathsf{opt}} = \sqrt{(h|h)} = 2\sqrt{\int_0^\infty \frac{|h(f)|^2}{S_n(f)} \mathrm{d}f} \,.$$

If we do not have the data, this is a good proxy. For a real detector, we do injection studies and compute a False Alarm Rate (FAR).

SNR thresholds

What is a "high enough" value for the SNR?

Without time shifts nor non-Gaussianities, the SNR would simply follow a χ^2 distribution with two degrees of freedom: "five σ" significance with a threshold of $\rho = 5.5$.

In real data this has to be estimated through **injections**:

$$\mathrm{FAR} = \mathrm{FAR}_8 \exp\left(-\frac{\rho - 8}{\alpha}\right) .$$

For BNS in O1: $\alpha = 0.13$ and $\mathrm{FAR}_8 = 30000 \mathrm{yr}^{-1}$.

Gravitational wave data analysis

Suppose we measure $d = h_\theta + n$, where our model for $h_\theta = h(t;\theta)$ depends on several parameters (typically, between 10 and 15).

We can estimate the parameters θ by exploring the **posterior distribution**

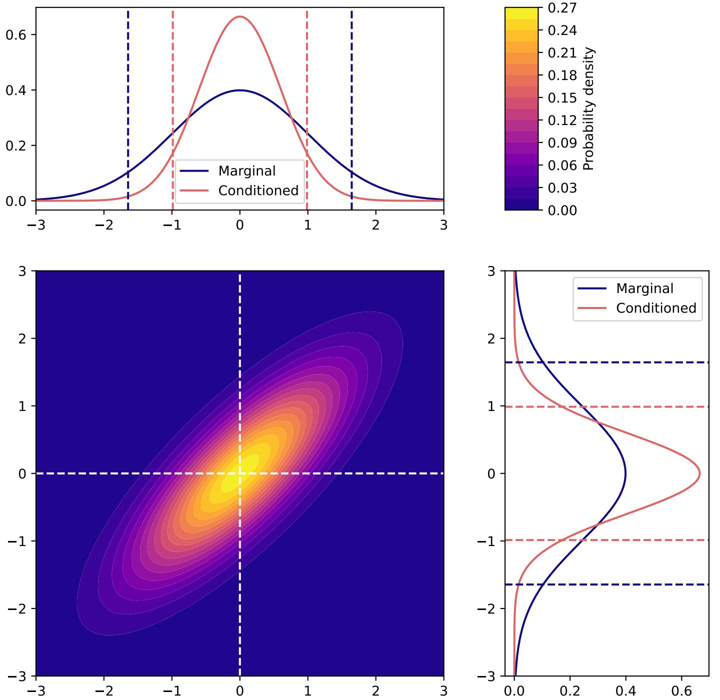
$$p(\theta|d) = \mathcal{L}(d|\theta)\pi(\theta) = \mathcal{N}\exp\left((d|h_\theta) - \frac{1}{2}(h_\theta|h_\theta)\right)\pi(\theta)\,,$$

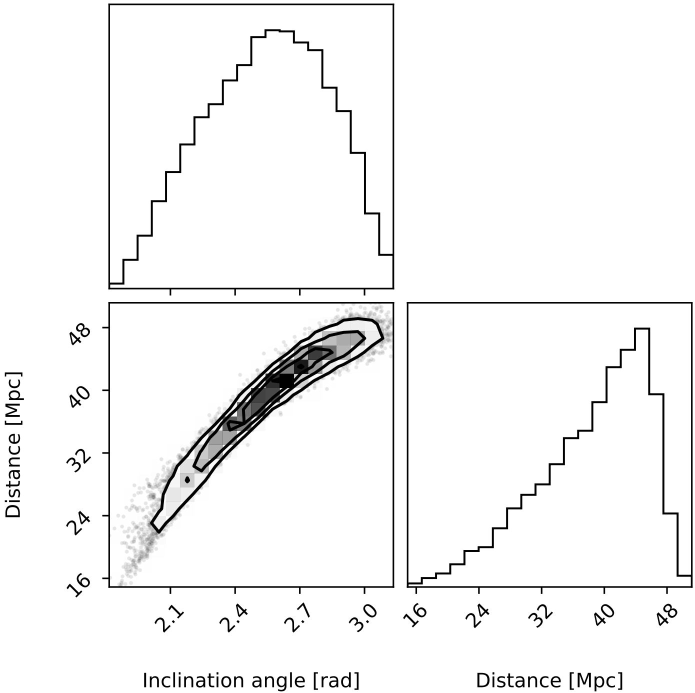
where $\pi(\theta)$ is our **prior distribution** on the parameters. We are neglecting non-Gaussianities in the noise, and assuming its spectral density is known!

The posterior is explored **stochastically** (with MCMC, nested sampling...) yielding many samples θ_i distributed according to $p(\theta|d)$, with which can compute summary statistics:

- mean $\langle\theta_i\rangle$,
- variance $\sigma_i^2 = \langle(\theta_i - \langle\theta_i\rangle)^2\rangle$,
- covariance $\mathcal{C}_{ij} = \langle(\theta_i - \langle\theta_i\rangle)(\theta_j - \langle\theta_j\rangle)\rangle$.

At this stage, we are not making any approximation, and the covariance matrix is just a **summary** tool - the full posterior is still available.


Marginal
Conditioned
Probability density
Marginal
Conditioned



48
40
32
24
16
Distance [Mpc]
2.1
2.4
2.7
3.0
Inclination angle [rad]
16
24
32
40
48
Distance [Mpc]

Parameter dependence of CBC signals

A list of the parameters a BNS signal depends on, with **relative error** (σ_x/x) values computed from the parameter estimation of **GW170817**.

Intrinsic parameters

- **masses** m_1 and m_2: $\sigma_x/x \sim 10\%$,
- **chirp mass** $\mathcal{M}$: $\sigma_x/x \sim 0.1\%$.

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$

- **mass ratio** $q = m_1/m_2$: $\sigma_x/x \sim 20\%$.

We are measuring the *detector-frame* mass:

$$\mathcal{M} = \mathcal{M}_{\textsf{source}}(1+z)$$

Alternative parametrization:

- **symmetric mass ratio** $\nu = \mu/M = q/(1+q)^2$: $\sigma_x/x \approx 4\%$
- **total mass** $M = m_1 + m_2$: $\sigma_x/x \approx 3\%$

- **aligned spin**: χ_{1z} and χ_{2z}: $\sigma_x/x \sim 3$ and 10 respectively,
- **effective aligned spin** $\chi_{\text{eff}} = (m_1\chi_{1z} + m_2\chi_{2z})/(m_1 + m_2)$: $\sigma_x/x \sim 1$ (compatible with zero)
- **precessing spin** χ_p: compatible with zero,

- **tidal polarizability** Λ_1 and Λ_2: $\sigma_x/x \sim 1.5$,
- **effective tidal parameter** $\widetilde{\Lambda}$: $\sigma_x/x \sim 0.6$.

$$\Lambda_i = \frac{2}{3}\kappa_2 \left(\frac{R_i c^2}{G m_i}\right)^5$$

Extrinsic parameters

- **distance** d_L $\sigma_x/x \sim 20\%$,
- degeneracy with the **inclination** of the source, ι: $\sigma_x/x \sim 10\%$,
- **arrival time** at geocenter $t_\oplus$,
- **phase** ϕ,
- **polarization** angle ψ: $\sigma_x \sim 0.3\text{rad}$,
- **sky position** (ra, dec): $\sigma_x \sim 2\text{deg}$ and 9deg.

The 1σ **sky area** in steradians can be written in the Gaussian case as:

$$\Delta\Omega_{1\sigma} = 2\pi |\cos(\text{dec})| \sqrt{\sigma^2_{\text{ra}}\sigma^2_{\text{dec}} - \text{cov}^2_{\text{ra, dec}}}$$

and it satisfies $p(\text{source within } \Delta\Omega) = 1 - \exp(-\Delta\Omega/\Delta\Omega_{1\sigma})$. With this we can compute the 90 sky area in square degrees:

$$\Delta\Omega_{90\%} = -\log(1 - 0.9)\Delta\Omega_{1\sigma} \left(\frac{180 \text{ deg}}{\pi \text{ rad}}\right)^2$$

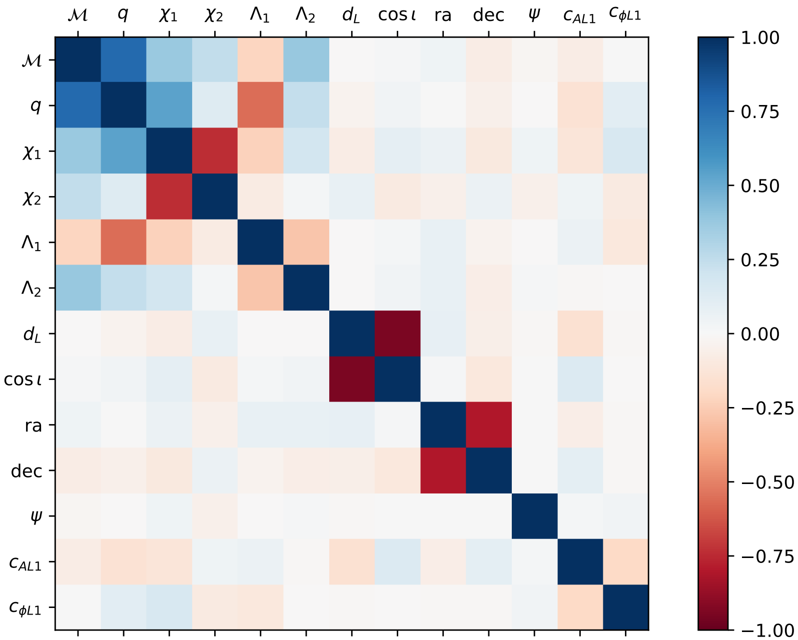
GW150914 comparison

- $\sigma_{\mathcal{M}}/\mathcal{M} = \sigma_M/M \approx 3\%$: not so many cycles
- two-detector event: sky area was 600deg^2, but the Gaussian approximation gives 1800deg^2.

Correlation structure

We can use Pearson correlation coefficients to visualize the (linear) **correlation structure** of the posterior:

$$\rho_{ij} = \frac{\mathrm{cov}(\theta_i, \theta_j)}{\sigma_i \sigma_j}$$



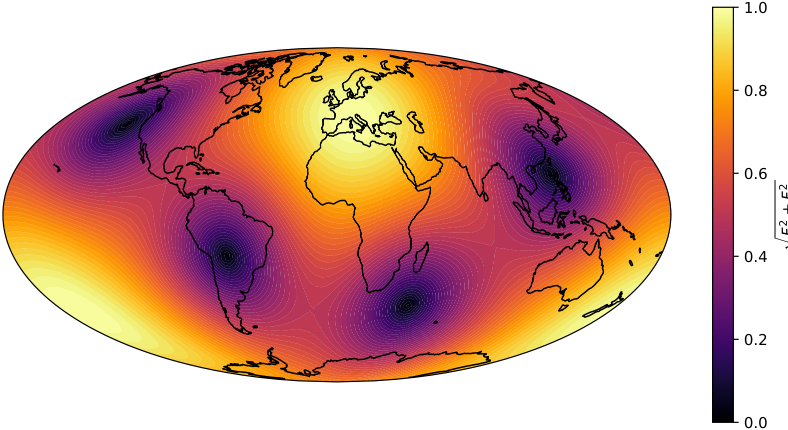
$\mathcal{M}$
q
χ_1
χ_2
Λ_1
Λ_2
d_L
$\cos\iota$
ra
dec
ψ
c_{AL1}
$c_{\phi L1}$
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
−0.75
−1.00

Antenna pattern

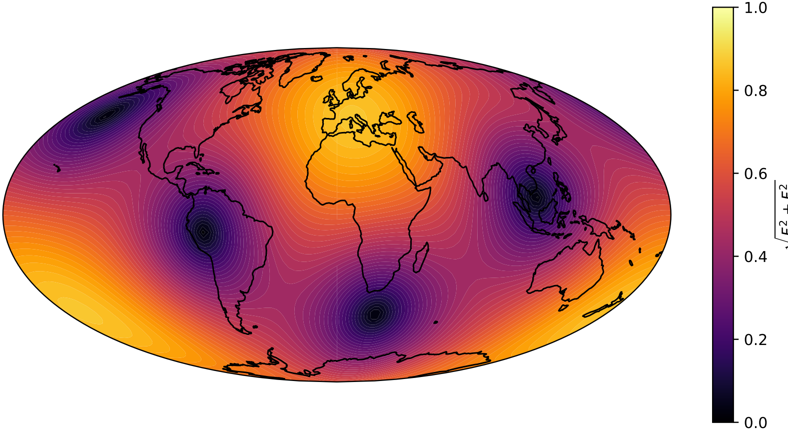
The strain at the detector depends on the antenna pattern:

$$h(t) = h_{ij}(t) D_{ij}(t) = h_+(t) F_+(t) + h_\times(t) F_\times(t)\,.$$

Virgo antenna pattern



One ET antenna pattern



Fisher matrix

In the Fisher matrix approximation, we are approximating the likelihood as

$$\mathcal{L}(d|\theta) \approx \mathcal{N} \exp\left(-\frac{1}{2}\Delta\theta^i \mathcal{F}_{ij} \Delta\theta^j\right)$$

where $\Delta\theta^i = \theta^i - \langle\theta^i\rangle$.

A **multivariate normal distribution**, with covariance matrix $\mathcal{C}_{ij} = \mathcal{F}^{-1}_{ij}$. This is a good approximation for the posterior in the high-SNR limit, since the prior matters less then.

The Fisher matrix $\mathscr{F}_{ij}$ can be computed as the scalar product of the derivatives of waveforms:

$$\mathscr{F}_{ij} = \left\langle \partial_i \partial_j \mathcal{L} \right\rangle \Big|_{\theta = \langle\theta\rangle} = (\partial_i h | \partial_j h) = 4\Re \int_0^\infty \frac{1}{S_n(f)} \frac{\partial h}{\partial \theta_i} \frac{\partial h^*}{\partial \theta_j} \mathsf{d} f \, .$$

For N detectors,

$$\mathcal{F}_{ij} = \sum_{k=1}^{N} \mathcal{F}_{ij}^{(k)}$$

The covariance matrix can be evaluated in seconds, while full parameter estimation takes hours to weeks.

Also, it is easy in the Fisher approach to account for new effects such as **the rotation of the Earth**.

Tricky step computationally: **inverting** $\mathcal{F}_{ij}$ to get $\mathcal{C}_{ij}$.