Black Holes and Dark Matter

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Part I - Black Holes and Dark Matter

Black Holes phenomenology:

- Study of Black Hole *inspirals*
- Accretion physics

- Gravitational Waves
- Radio waves/ X-rays/ Gamma rays/ **Neutrinos**

Dark Matter searches

- *- Can Black holes of primordial origin be a part of the Dark Matter?*
- *- Can we learn something on the nature of the Dark Matter by* **Multi-messenger astronomy** *studying Black Hole physics?*

Black Holes in the Universe

Stellar-mass black holes

Observed up to *z ~ 10* **Seed problem**

IMBHs? 100 < M < 10⁶ M_{Sun} $\sum_{i=1}^n$

Hypothetical link between stellar-mass and SMBHs IK DEIWEEN STEILAI \dots known to date, and the first stellar \sim primarily the estimate of the source location. If we assume Thass and Sivibris

- Originated by direct collapse of low-metallicity gas clouds? Primordial origin? ry and our ourselvers is and references the references that BBHs do formal and attenues the substantial and attenues that BBHs do forma a merge with with the time construction of the theory of the theory of the theory of the theory of the top the $\ln u \sim 0.1$ ibration over now-metal licity ga [−]³M[⊙] assuming perfect cali-
- Recent detection by LIGO/Virgo arXiv:[2009.01190](https://arxiv.org/abs/2009.01190) e tion by Litat Wire $U_{\rm H}$ \sim $V_{\rm H}$ \sim $V_{\rm H}$ can be relied on $V_{\rm H}$ \overline{y} parameter \overline{y} medians of the political largely unchanged. For the political largely unchanged the PDFs remains the PDFs remains the New York of the PDFs remains the New York of the PDFs remains remains the New York of the PDFs remains r <u>GUISTE COOL TIOU</u>

Black Holes as Dark Matter

Primordial Black Hole phenomenology

Credit: Bradley Kavanagh, https://github.com/bradkav/PBHbounds

Why a sub-dominant population would matter?

- A discovery of a sub-dominant population of DM in the form of (massive) PBHs could:
	- Solve the problem of the **SMBH seed**?
	- Reveal non-trivial **early universe physics**
	- Help us set stringent **upper limits** on other DM candidates

Accretion bounds $\overline{}$

• Primordial Black Holes can accrete baryonic matter

- **Astronomical environments: X-ray/** radio bounds (focus on Galactic center)
- **Cosmological bound:** for instance from Cosmic Microwave Background (focus on accretion during the Dark Ages)
- They rely on complicated accretion physics
- Comprehensive assessment of the uncertainties is very much needed!

Accretion physics under the spotlight: BHL formalism. figure 5 and has first been considered by [78]. Following [79], the accretion rate can be derived as follows. Starting the continuity equation in the starting term in the starting starting the starting Accretion physics under the spotlight: BHL formalism. sics under the spotlight: BHL formalisr ነ. luminosity and mass of black holes. n physics under the spotlight: BHL formalism.

Figure 5: Schematic of the temperature-limited case. A cloud

1)3*/*² (4)

Continuity equation for steady-state flow **Euler** equation $\bm{$ Continu The first attempt at the accretion of gas onto an object was done by Bondi, B onto an object was done by Bondi, B First, the concept of Bondiers and described accretion with a secondicity of the derived and derived a in section section 4.1. The concept of radiative experiment of radiative experiment of radiative experiment of \mathcal{L}

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\rho v\right) = 0 \qquad \qquad \rho v\frac{d}{d}
$$

Euler equation *v* = *vr*

$$
=0
$$
\n
$$
\rho v \frac{dv}{dr} = -\frac{dP}{dr} - \frac{GM\rho}{r^2}
$$

by the mass and sound accretion rate BH at rest: *Bondi* accretion rate

$$
\dot{M} = 4\pi r_s^2 \rho(r_s) c_s(r_s) = \pi \frac{(GM)^2 \rho(\infty)}{c_s^3(\infty)} \left(\frac{2}{5-3\gamma}\right)^{\frac{5-3\gamma}{2(\gamma-1)}}
$$

H. Bondi, MNRAS 112(2):195–204, 1952 T
112(2):195–201–1952

H. Bondi and F. Hoyle, MNRAS 104(5):273-282, 1944

Mov H. Bondi and F. Hoyle. MNRAS 104(5):273–282. 1944
Moving RH *Rondi-Hoyle littleton*

H. Bondi and F. Hoyle, MNRAS 104(5):273–282, 1944 **Moving BH:** *Bondi-Hoyle-Littleton* $t = t - t - t - t - t - t - t - t - t - t - t$ accretion rate $\frac{1}{\sigma}$ $\dot{M}_{\rm BHL} = 4\pi \frac{(GM)^2 \rho_{\infty}}{(m^2 + m^2)^{3/2}}$ $(v^2 + c^2)$ $\left(\frac{P\infty}{2}\right)^{3/2}$

Accretion physics under the spotlight: BHL formalism. rate, *^L* / *^M*˙ ². ACCIENOTI PHYSICS UNICE THE SPOTIFIL. DI formo

Bondi-Hoyle-Littleton formula needs to be "fudged" because of observational constraints related to local neutron stars, the SMBH at the center of the Galaxy, and constraints related to local neutron stars, the SMBH at the center of the Galaxy, and AGNs. Primordial black hole population: In order to de-

$$
\dot{M}=4\pi\lambda(GM_{BH})^2\rho\left(v_{BH}^2+c_s^2\right)^{-3/2}
$$

- Perna et al. 2003, "Bondi accretion and the problem of missing isolated neutron stars" ron-stars" distribution estate (NFW) distribution \mathcal{S}'
- S. Pellegrini 2005, "Nuclear Accretion in Galaxies of the Local Universe: Clues from Chandra
Observations" (explanation for the radiative quiescence of supermassive black boles in the lo *Observations"* (explanation for the radiative quiescence of supermassive black holes in the local Universe) Universe) ns from Chandra
Pok boles in the local
	- Wang et al. 2013, "Dissecting X-ray-emitting Gas around the Center of our Galaxy" alaxy"

temperature of the accreted gas due to radiative prehe luuge lactor takes liito account several enects, including the for $\frac{1}{2}$ the Chentral Molecular Zone (CMZ), a 300 pc wide region of \mathcal{L} The fudge factor takes into account several effects, including the role of outflows

The Park-Ricotti model

- **• Park-Ricotti model:** numerical simulations + semi-analytical parametrization in presence of radiative feedback.
- **•** Suppression of the accretion rate at low velocity, due to the formation of an ionized bubble

Torino - January 2024 **Accretion rates of interstellar gas onto a moving, i**ts velocity. Its velocity of its veloc

10

The Park-Ricotti model

- **• Park-Ricotti model:** numerical simulations + semi-analytical parametrization in presence of radiative feedback.
- **•** Peaks of accretion rate depends on ionized sound speed

Revisiting the Cosmological constraint aint behavior adopted for in

The physics behind the bound B. Relative baryon-Patient and principle in the early universe

- PBHs accrete baryonic matter. mate the relative velocity between DM and baryons to
- The accretion rate Mdot depends on ambient density and PBH - baryon relative speed. **BHL** and PR model. be of the order of the order of the theoretical baryon velocity or of the thermal baryon velocity or of the the • The accretion rate ividot depends on ambient density pt *donsity*
	- Ambient density dilutes with decreasing redshift immediately readable], **Recombination of the sound velocity drops about the sound velocity of the sound sound and the sound and the sound s** abruit were indeed with week coupling the PBH formation mechanism itself *is* a non-linear phenomenon, and peaks the PBH are likely suggests that PBH are likely suggests that PBH are likely suggests that
PBH are likely suggests that PBH are likely suggests that PBH are likely suggests that PBH are likely suggests

$$
\rho_{\infty} = m_{\rm p} \, n_{\infty} \approx m_{\rm p} \, 200 \, \text{cm}^{-3} \, \left(\frac{1+z}{1000} \right)^3 \qquad \text{Poulin+ } 1707.04206
$$

• PBH speed relative to baryons also decreases according to linear theory: • PBH speed relative to baryons also decreases according $\begin{array}{ccc} \hbox{for each } & \text{if } & \$ bination and then drops linearly with *z* [72, 73]: lar, a more meaningful background solution around which

$$
\sqrt{\langle v_{\rm L}^2 \rangle} \simeq \min \left[1, \frac{1+z}{1000} \right] \times 30 \, \rm km/s \, .
$$

Torino - January 2024 12 \mathcal{A} can shed any light on the accretion, which depends on the accr

The Visiting the Cosmological constraint thermal photons, producing an increase in the number of non-thermal particles at the expense

The physics behind the bound \mathbf{F} is a decrease in the down the particle cool down to energy. When the particles cool down to energy of the \mathbf{F} order of a keV, they start interacting strongly with atoms of hydrogen (and sub-dominantly

- Accretion disks emits *ionizing radiation* during the Dark Ages (between Recombination and Reionization)*:* to modify the equations governing the evolution of the fraction of free electrons, *x^e* © *ne/nH*, **Taking into anti-original into account of an anal collision** and continuity the contribution of the collision of \mathbb{R}^n the intergalactic medium (IGM) *T^M* a term accounting for the associated heating, which has
- IGM is heated up (alteration of *TM*) ϵ \int drops below the Lyman-*–* transition energy (10.2 eV). Then these particles are no longer able
- IGM is also partially ionized (alteration of the *freeelectron fraction* X_e) 7725

Revisiting the Cosmological constraint

The physics behind the bound

• Impact on CMB anisotropy is due to the alteration of the visibility function and the recombination optical depth ⁸

Revisiting the Cosmological constraint: Results

- Accretion rate **suppression** around PBHs is very relevant
- Dependence on the ionized sound speed
- May weaken the bound

Revisiting the Cosmological constraint

BHL vs PR: the "Unexpected robustness" of the bound
BHL vs PR: the "Unexpected robustness" of the bound

Black Holes as Portals to new Physics

- Intermediate-Mass Black Holes may exist in the Universe.
- Dark-Matter over-densities can form around them [Gondolo&Silk 9906391, Zhao&Silk 0501625, Hannuksela+ 1906.11845]. **Dain-Matter Over**

$$
\gamma_{\rm sp} = 7/3 \approx 2.333
$$
\n
$$
\rho_{\rm sp} = 200 \, M_{\odot} \, \text{pc}^{-3}
$$
\n
$$
\rho_{\rm DM}(r) = \rho_{\rm sp} \left(\frac{r_{\rm sp}}{r} \right)^{\gamma_{\rm sp}}
$$
\n
$$
\rho_{\rm sp} = 200 \, M_{\odot} \, \text{pc}^{-3}
$$
\n
$$
\rho_{\rm sp} = 0.5 \, \text{pc}
$$
\n
$$
\rho_{\rm DM}(r) = \rho_{\rm sp} \left(\frac{r_{\rm sp}}{r} \right)^{\gamma_{\rm sp}}
$$
\n
$$
\rho_{\rm sp} = 200 \, M_{\odot} \, \text{pc}^{-3}
$$
\n
$$
r_{\rm sp} = 0.5 \, \text{pc}
$$
\n
$$
\rho \sim 10^{24} \, M_{\odot} \, \text{pc}^{-3}
$$

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Black Holes as Portals to new Physics \overline{a} ET BLACK Holes as Portals to new Physics the emission of gravitation of gravitation of gravitations $\mathcal{L}_\mathbf{z}$

Trace NS/BH Assas" or "spika 1A2 C
C
d (*m*1*, m*2*, dL*) = • Stellar-mass black holes that inspiral around IMBHs can trace the extent overdensities (DM "dresses" or "spikes") by an adiabatic compact over the compact of th • Stellar-mass black holes that uspiral around IMBHs can trace the presence of either accretion disks or Dark Matter where \mathbb{R}^n is dissipated through the slow \mathbb{R}^n is slow \mathbb{R}^n on a given circular orbit to another circular orbit to another circular orbit with a second control orbit with

DM particles, it loses energy via *dynamical friction* (DF)

• Dephasing of the waveform w.r.t. GR in vacuum • Physical process: Dynamical Friction d*t* JI
. 32*G*⁴*M*(*m*1*m*2)² .i. un in vacuum
riction

Torino - January 2024 18 • Kavanagh+ 2002.12811 (PRD) • Coogan+ <u>2108.04154</u> (PRD) Cole+ [2211.01362](https://arxiv.org/abs/2211.01362) (Nature Astronomy) where the constant \mathbb{I} **m**₁ *m m* $\overline{}$ $\overline{}$ No DM $\overline{}$ $\overline{}$ With DM Halo

$$
\frac{dE_{\rm DF}}{dt} = 4\pi (Gm_2)^2 \rho_{\rm DM}(r_2) \xi(v) v^{-1} \log \Lambda
$$

3 How do DM overdensities around BHs form? *v* How do DM overde *<u>kensities</u> around BHs form*

Torino - January 2024 19 and 19 allowed to grow adiabatically, with the initial and final states in Fig. 6. Under these circumstances, the initial states in Fig. 6. Under the initial states in Fig. 6. Under these circumstances, the initial states in Fig. • Semseted *MSMS* \rightarrow 105*M*

Conclusions

- •Multiple relevant interplay between BH phenomenology and DM searches
- •**Accretion physics** is crucial to set upper limits on the PBH abundance
- Need to go beyond the textbook BHL approach
- •**The CMB bound on PBH abundance seems robust** with respect to the uncertainties associated to the accretion model!
- •**DM overdensities** around IMBHs provide a **discovery potential** thanks to GW dephasing
- •Realistic models that describe the **formation of DM overdensities** are in progress

Thank you!

The most distant quasar

Supermassive black holes at the centre of

Galaxies.

News: Observed up to *z ~ 10*

Seeds? Probably Heavy 4 *X-ray AGN in a z* ⇠ 10 *galaxy*

JWST NIRCam zoom-in on UHZ1

can in the early Universe. The early Universe are a number of Ω **Example 19 and bubble radius density perturbations, conoving Hubble radius decreases the universe toward flatness (ratio decreases toward flatness (rather than aways of primordial origin?**

RHs formed in the **early liniverse** (before RRNI) out of s*mall-scale lare* density fluctuations possibly originated during inflation BHs formed in the **early Universe** (before BBN), out of *small-scale, large-amplitude* $\overline{\mathsf{R}}$

[S. Hawking, MNRAS 152 (1971); Carr and Hawking, MNRAS 168 (1974)]

PBH mass \sim horizon mass at the time of formation the horizon mass, *M*H, at the formation epoch (e.g. Ref. [29]): Figure 7: *Left:* Evolution of the comoving Hubble radius, (*aH*)1, in the inflationary universe. The computed the Hubble sphere shows the time of formation in the set of the state in \mathcal{L}_1

$$
M_{\rm PBH} \sim M_{\rm H} \sim \frac{c^3 t}{G} \sim 10^{15} \left(\frac{t}{10^{-23} \, {\rm s}} \right) {\rm g} \, .
$$

Wide mass range for PBHs as DM candidates *bound."* \mathbf{M} ide meese were for DDL lease \mathbf{M} condidate the scale interest of cosmological interest calendary

M ~ 10¹⁶ g (10⁻¹⁷ M⊙) — 10³⁹ g (10⁵ M⊙)

 $g.$ Output $g.$ *"…it is tempting to suppose that the major part of the mass of the Universe is in the form of collapsed objects. This extra density could stabilize clusters of galaxies which, otherwise, appear mostly not to be gravitationally*

1 Introduction

 $\mathbf 1$

WIMPs and PBHs

Revisiting the Cosmological constraint: Results

- Accretion rate **suppression** around PBHs is very relevant
- Dependence on the ionized sound speed
- May weaken the bound

Uncertainties in the CMB bound

Uncertainties in the CMB pound **Role of the disc in the CMB** f_{BBH}

flow strength (*^s* ⁼ 0.4), the critical net accretion rates (*M*˙ cr*,*ADAF, *M* to the electrons -> (^δ ⁼ ¹⁰−3), (6*.*2*,* ⁷*.*1)×10−³*M*˙ Edd (^δ ⁼ ¹⁰−2), (5*.*9*,* ⁶*.*6)×10−³*M*˙ Edd dard thin disc by a factor of 4 × 10−4 × 10−4 (Yuan et al. 10−4). Because the 4 + 10−4 (Yuan et al. 2003). Bec More gravitational energy is transferred to the electrons -> more radiative efficiency -> stronger bound!

Black Holes as Portals to new Physics a
Latin **V** *c*2 aun i iuico qo i t Holes as Port

(*m*1*, m*2*, dL*) =

MBH VU NS/BH the presence of either accretion disks or Dark Matter 10°²¹ Characteristic strain
Bishop $\left(1, 1, 1\right)$ (1 MØ*,* 10°³ MØ*,* 241 Mpc) We define the *total mass* and *symmetric mass ratio* as follows: We define the *total mass* and *symmetric mass ratio* as follows: *Mtot* = *MIMBH* + *Mtest* (1.2) • Stellar-mass black holes that inspiral ardund IMBHs can trace overdensities (DM "dresses" or "spikes")

Torino - January 2024 28 • Coogan+ [2108.04154](https://arxiv.org/abs/2108.04154) (PRD) • Cole+ [2211.01362](https://arxiv.org/abs/2211.01362) (Nature where, in our case, *ln*(⇤)=3 Astronomy)

$$
\sum_{\substack{\tilde{x} \\ \tilde{y} \\ \tilde{z} \\ \tilde{z}
$$

The initial conditions are:

Black Holes as Portals to new Physics

Sizing up the dephasing • Stellar-mass black holes that inspiral around IMBHs can trace the presence of either accretion disks or Dark Matter overdensities (DM "dresses" or "spikes")

Environments Hole Environments '**Spikes**' or '**Dresses**' ⇢DM(*r*)

Particle Dark Matter 'Spikes' or 'Dresses' **D**ain Matte *m*² **b**²

- Collisionle • **Collisionless** DM overdensity • Perturbation back-
	- **Spherical** symmetry
	- **Dynamical friction** at work
	- **Feedback** on the halo is important

$$
\frac{\mathrm{d}E}{\mathrm{d}t} = m_2 v_0 \frac{\mathrm{d}v}{\mathrm{d}t} = -\frac{4\pi (G_N m_2)^2 \rho_{\rm DM}(r)\xi(v_0)}{v_0} \log l
$$

-
- Differentially rotating **baryonic** disk
	- \bullet Digk is porturbed by $\frac{3}{2}$ and the state of the state • Disk is perturbed by the inspiralling "wake"
	- **al friction** at work
 b an the bala is inepertured torques **torques** Figure 5: An accretion disk being perturbed by an inspiral with mass ratio *q* = 10³. Streams of particles from $u_{\rm eff}$ to downstream as they interact with the secondary object, as is further highlighted in the zoom-in panel. From [49]. • Perturbation back**cts and exerts**

$$
T_{\rm I}=-\Sigma(r)r^4\Omega^2q^2{\cal M}^2
$$

$$
\frac{dE}{dt} = m_2 v_0 \frac{dv}{dt} = -\frac{4\pi (G_N m_2)^2 \rho_{DM}(r)\xi(v_0)}{v_0} \log \Lambda \qquad \qquad \frac{dE_{\text{torque}}}{dt} = -\frac{1}{4} m_1 T_1 \left(\frac{G_N}{r^3 M}\right)^{1/2}
$$

30 with *r*5, and should thus be very dominant regardless of environmental e⊿ects close to the central black hole.

LISA can discriminate environmental effects Discriminability [Cole, Bertone, Coogan, Gaggero, Karydas, **BJK**, environmental effects Discriminability [Cole, Bertone, Coogan, Gaggero, Karydas, **BJK**,

IS **very hard to confuse** in 1 year of LISA data (huge Bayes factors!) Signals **very hard to confuse** in 1 year of LISA data (huge Bayes factors!)

P. Cole, G. Bertone, A. Coogan, **DG**, T. Karydas, B. Kavanagh, T. Spieksma, G. Tommaselli [2211.01362](https://arxiv.org/abs/2211.01362) (Nature Astronomy)

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Dark Dresses around IMBH as well ⇡ *r*peri *E*max(*r*) = *^f* (*r*) ¹ ⁴*r^S r*

Generalizing Gondolo and Silk, 9906391 Figure 7: (a) The density of the density α and α and α and α α α α α and **Generalizing Gondolo and Silk**, **9906391** and the compiled potential (pinch) with the BH of mass 106*M*. The BH of mass 106*M* of mass 106*M* of mass 106*M* of mass 106*M*. The BH of mass 106*M* of mass 106*M* of mass *v***din Diesses diuders by Dietail in a Gondolo and Silk, 9906391
Peralizing Gondolo and Silk, 9906391** where *r^S* = 2*G^N m*BH*/c*² the Schwarzschild radius.

and = 1*.*2 with an exponential cuto↵ at 10 kpc, and the combined potential (pink) with the BH of mass 10⁶*M*. its asymptotic behaviour. (b) The phase space of the power-law profile calculated from the Eddington-inversion •Adiabatic BH growth: • Initial density follows a "cuspy" profile $\begin{bmatrix} 10^{11} & \cdots & 10^{11} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots \end{bmatrix}^{\psi_f}$ • A BH forms and grows *adiabatically* 10^{10} $\left[\begin{array}{c} 1 & 1 \end{array}\right]$ • Eddington analysis $\begin{bmatrix} 10^1 \\ 1 \end{bmatrix}$ • Final density is computed extending the phase space is $\frac{1}{108}$ itational potential of a system. This is a tricky aspect of the calculation when dealing with composite systems, $\mathcal{N}_{\mathcal{N}_{\mathcal{N}}}$ \int attempting the Eddington phase space for a construction phase space for a component that is not actually in equilibrium will be equilibrium will be expansion of \int $\rho(r) = \int d^3{\bf v} f(\mathcal{E}, L) = 4\pi \int dv_r dv_T v_T f(\mathcal{E}, L)$, evolved adiabatically will still be in equilibrium. So as we take can restrict of our starting conditions, we can restrict on \mathbb{R} assured that the Eddington inversion for ℓ is physical results. $\pi^2 \sqrt{8} \sqrt{g} \sqrt{g} \sqrt{g^2 \sqrt{g^2 - \Psi^2}}$ (*L*^{*f*} $\sqrt{g^2 - \Psi^2}$ *fi*(*Ei*(*E^f , L^f*)). The final state density is thus explicitly given by $\rho_f(r)=4\pi$ $\int^{\mathcal{E}_{f}^{\mathrm{max}}}$ ${\cal E}^{\rm min}_f$ d*E* $\int^{L_{\rm max}}$ *L*min $dL \frac{L}{2}$ $r^2v_{r,f}$ $f_i(\mathcal{E}_i(\mathcal{E}_f, L_f), L_f)$. $\frac{1}{10^{-8}}$ $\frac{1}{10^{-5}}$ $\frac{1}{10^{-2}}$ $\frac{1}{10^{1}}$ with the sub/superscripts *i* and *f* respectively denoting the initial state of the DM halo and the final state of DM $\int_{\mathcal{C}} \mathcal{E}_t^{\max}$ in potential. The relative energy per unit mass defined as $\int_{\mathcal{C}} L_{\max}$ $(r) = 4\pi \int_0^L dE \int_0^{\max} dL \frac{L}{r^2} f_i(\mathcal{E}_i(\mathcal{E}_f, L_f), L_f).$ $\int \mathcal{E}_f^{\text{min}}$ and $\int L_{\text{min}}$ and $r^2 v_{r,f}$ • Conservation laws are applied \bullet A diabatic RH arouth: \bullet Cone even given analysis • Final density is computed $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ energy/momentum distribution is given by the phase space *f*(*E, L*), defined as $\rho(r) =$ $d^3\mathbf{v} f(\mathcal{E}, L)=4\pi$ z
Z $f_k(\mathcal{E}) = \frac{1}{\sqrt{\mathcal{E}}}\int_{0}^{\mathcal{E}}\frac{\mathrm{d}^2\rho_k}{\mathrm{d}\tau_k^2}\frac{\mathrm{d}\Psi}{\sqrt{\mathcal{E}}}} + \frac{(\mathrm{d}\rho_k/\mathrm{d}\Psi)\Psi=0}{\sqrt{\mathcal{E}}}$ $\pi^2 \sqrt{8} \left(\int_0^{\infty} d\Psi^2 \sqrt{\mathcal{E}} - \Psi \right)$ d*E*d*L* gives us our base equation for the conversion from a phase space distribution to a density \overline{E} J_{L_n} $\frac{dL}{r^2v_{r,f}}f_i$ *f*(*E, L*)*,* (3.5) **Figure 10 in terms of** α **is the straightforward, as function of** α **function of** α **is the case of** α **function o** • Conservation laws are applied $\Rightarrow 10^9$ ϵ \mathbb{R}^n are expected to be indicated to be in $\rho(r) = \int d^3{\bf v} f(\mathcal{E}, L) = 4\pi \int dv_r dv_T v_T f(\mathcal{E}, L)$, ^{IV} determined from the density profile. This is given by the Eddington inversion formula [56]: $f_k(\mathcal{E}) = \frac{1}{\pi^2}$ $\frac{1}{\pi^2\sqrt{8}}$ $\int f^{\mathcal{E}}$ 0 $\mathrm{d}^2\rho_k$ $\mathrm{d}\Psi^2$ $d\Psi$ $\sqrt{\mathcal{E}-\Psi}$ $+\frac{(\mathrm{d}\rho_k/\mathrm{d}\Psi)\Psi=0}{\sqrt{\mathcal{E}}}$ *E* !

Recall Action is Conserved

\n
$$
I_{r,x}(\mathcal{E}_x, L) = \frac{1}{\pi} \int_{r_{\text{peri}}}^{r_{\text{apo}}} dr \, v_{r,x}(r, \mathcal{E}_x, L)
$$
\nTorino - January 2024