Fracton gravity from spacetime dipole symmetry

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Outline

- Introduction to fractons
- Fracton theories with ordinary gauge fields
- Spin-two field from covariant fracton theory
- Outlook

Outline

- Introduction to fractons

Introduction to fractons

What is a fracton?

It is an excitation characterised by restricted mobility

Discovered/devised on the lattice [Chamon '05, Haah '11]

Fracton models exhibit exotic properties:

- Subsystem or multipole symmetries
- Vacuum degeneracy sensitive to the lattice size
- IR/UV mixing challenging renormalisation group paradigm

Interesting from different points of view: hydrodynamics, quantum information, pure QFT (See [Pretko, Chen, You '20, Nandkishore, Hermele '19, Grosvenor, Hoyos, Peña-Benitez, Surowka '22] for some reviews)

Model with subsystem symmetry

Let us consider the non-relativistic model [Seiberg, Shao '20]

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \partial_y \phi)^2$$

Subsystem symmetry

$$\phi(t, x, y) \to \phi(t, x, y) + f_1(x) + f_2(y)$$

with arbitrary functions f_1 and f_2: infinite vacuum degeneration!

Dispersion relation exhibiting IR/UV mixing:

$$\omega^2 = p_x^2 p_y^2$$

Models with dipole symmetry

Fractons may appear in theories with dipole symmetry [Pretko '17, Seiberg '20]

A charged particle propagates by changing its dipole moment:

dipole symmetry → charged particles are immobile!

Monopole and dipole charges:

$$Q = \int d^{d-1}x\, \rho \qquad Q^i = \int d^{d-1}x\, x^i \rho$$

Conservation law

$$\partial_t \rho + \partial_i \partial_j J^{ij} = 0$$

Monopole-Dipole-Momentum Algebra (MDMA!) [Gromov '18, Peña-Benitez '21]

$$i[P_i, Q^j] = \delta^i_j Q \qquad \text{Heisenberg algebra}$$

Models with dipole symmetry

Pretko devised a way to construct dipole-conserving Lagrangians.

Given a charged scalar field $\Phi \to e^{i\alpha(x)}\Phi$, covariant derivative

$$D_{ij}(\Phi,\Phi) = \Phi\partial_j\partial_j\Phi - \partial_i\Phi\partial_j\Phi \qquad D_{ij} \to e^{i\alpha(x)}\Big(D_{ij} - \partial_i\partial_j\alpha(x)\Big)$$

Model with dipole symmetry (non-Gaussian!)

$$\mathcal{L} = \frac{1}{2}|\partial_t\Phi|^2 - \lambda_1\partial_i|\Phi|^2\partial_i|\Phi|^2 - \lambda_2|D_{ij}(\Phi,\Phi)|^2 - \Big[\lambda_3\overline{\Phi}^2 D_{ii}(\Phi,\Phi) + \text{h.c}\Big]$$

Low-energy dispersion relation

$$\omega = 0$$

Models with dipole symmetry

$$\delta S = \int \left(\rho\,\partial_t\alpha(x) - J^{ij}\partial_i\partial_j\alpha(x)\right) \quad \to \quad \partial_t\rho + \partial_i\partial_j J^{ij} = 0$$

We can gauge the symmetry by introducing A_t and A_{ij}

$$A_t \to A_t + \partial_t\alpha(x) \qquad A_{ij} \to A_{ij} + \partial_i\partial_j\alpha(x)$$

The spatial gauge field is a symmetric rank-two tensor!

Gauge theory with A_t, A_{ij} dual to elasticity theory [Pretko, Radzihovsky '18]

Issues with coupling to curved geometry [Slagle, Prem, Pretko '18, Jain, Jensen '21, Bidussi, Hartong, Have, Musaeus, Prohazka '21]

$$\delta A_{ij} = \nabla_i\nabla_j\alpha\ , \qquad [\nabla_i, \nabla_j] \neq 0$$

Are we able to write fracton theory with ordinary gauge fields?

Outline

- Fracton theories with ordinary gauge fields

Gauging the MDMA algebra

We can construct fracton theories with ordinary gauge fields by realising the MDMA algebra in an internal space [Peña-Benitez '21, AC, Hoyos, Musso '22]

$$i[P_a, Q_1^b] = \delta_a^b Q \qquad a, b \text{ spatial internal indices}$$

We introduce ordinary connection (one-form in spacetime)

$$\mathcal{A}_\mu = e_\mu^a P_a + a_\mu Q_0 + b_{\mu a} Q_1^a$$

Gauge transformation fixed by algebra

$$\begin{aligned} \delta e_\mu^a &= \partial_\mu \xi^a \\ \delta a_\mu &= \partial_\mu \lambda_0 + e_\mu^a \lambda_{1\,a} - b_{\mu a} \xi^a \\ \delta b_{\mu a} &= \partial_\mu \lambda_{1\,a} \end{aligned}$$

Gauging the MDMA algebra

We take a_μ and $b_{\mu a}$ as dynamical fields, whereas ${e_\mu}^a$ is background

$$e_\mu^a = \delta_\mu^a \qquad \text{identification of spaces}$$

Coupling to currents

$$\mathcal{L}_J = -a_\mu J^\mu - b_{\mu a} J^{\mu a}$$

Ward identities given by the algebra

$$\partial_\mu J^\mu = 0 \qquad \partial_\mu J^{\mu a} = J^\nu \delta_\nu^a$$

It is possible to choose improvement terms such that $J^{ta} = J^{[ia]} = 0$:

$$\partial_t J^t + \partial_i \partial_j J^{ij} = 0 \qquad \text{dipole conservation law}$$

MDMA, Heisenberg, Crystals, Factons, Breaking Bad

We previously applied the MDMA algebra approach in two contexts:

Fracton/elasticity duality [AC, Hoyos, Musso '22]:

- recovered fracton/elasticity duality
- description of incommensurate materials (quasi-crystals and moiré lattices)
- mobility restriction of elasticity defects from MDMA gauge invariance

Breaking of dipole symmetry in 1+1 dimensions [Afxonidis, AC, Hoyos, Musso '23]:

- model with classical potential yielding linear dispersion for dipole symmetry Nambu-Goldstone bosons
- model with dipole symmetry breaking and charge symmetry preservation at one-loop
- avoidance of Hohenberg-Coleman-Mermin-Wagner theorem

Outline

- Spin-two field from covariant fracton theory

Motivations for fracton gravity

A symmetric rank-two tensor emerges naturally in fracton gauge theories

Relation between dipole conservation and Mach principle [Pretko '17]

In ordinary gravity, the spin-two field coupling is universal:

$$\mathcal{L}_{\text{coupling}} = -h_{\mu\nu} T^{\mu\nu} \qquad \partial_\mu T^{\mu\nu} = 0$$

Fracton conservation law is less constraining:

$$\partial_\mu \partial_\nu J^{\mu\nu} = 0$$ Non-universal couplings?

Our approach is easily generalised to higher multipoles:

higher-spin theories from a fresher perspective?

Covariant MDMA algebra

We generalise the MDMA algebra approach considering an internal space with the number of dimensions as that of spacetime

$$i[P_A, Q_1^B] = \delta_A^B Q \qquad A, B \text{ internal indices}$$

We introduce an ordinary connection (one-form in spacetime)

$$\mathcal{A}_\mu = e_\mu^A P_A + a_\mu Q_0 + b_{\mu A} Q_1^A$$

Gauge transformation fixed by the algebra

$$\begin{aligned}
\delta e_\mu^A &= \partial_\mu \xi^A \\
\delta a_\mu &= \partial_\mu \lambda_0 + e_\mu^A \lambda_{1\,A} - b_{\mu A} \xi^A \\
\delta b_{\mu A} &= \partial_\mu \lambda_{1\,A}
\end{aligned}$$

Covariant MDMA algebra

Two invariant curvatures (two-forms in spacetime)

$$T^A_{\mu\nu} = \partial_\mu e_\nu{}^A - \partial_\nu e_\mu{}^A \qquad H_{\mu\nu A} = \partial_\mu b_{\nu A} - \partial_\nu b_{\mu A}$$

and a covariant one (here $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu$)

$$B_{\mu\nu} = b_{\mu A} e_\nu{}^A - b_{\nu A} e_\mu{}^A - f_{\mu\nu} \qquad \delta B_{\mu\nu} = -T^A_{\mu\nu}\lambda_{1\,A} + H_{\mu\nu\,A}\xi^A$$

We take a_μ and $b_{\mu A}$ dynamical and e^A_μ background chosen as

$$e_\mu{}^A = \delta_\mu{}^A \qquad \text{breaking to diagonal Lorentz}$$

We allow for explicit breaking of internal translations. Other invariant

$$\Gamma_{\mu\nu\lambda} = 2\left[\partial_{(\mu} B_{\nu)\lambda} - \frac{1}{2}\left(H_{\lambda\mu A} e_\nu{}^A + H_{\lambda\nu A} e_\mu{}^A\right)\right]$$

All the invariants can be written in terms of $b_{\mu\nu} - \partial_\mu a_\nu$.

Quadratic action

From now on: $d+1=4$. Most general action with at most two derivatives of $b_{\mu\nu}$

$$\begin{aligned}\mathcal{L} &= -\frac{\alpha_1}{4}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{\alpha_2}{4}B_{\mu\nu}B^{\mu\nu} - \frac{\alpha_3}{4}\Gamma_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda} - \frac{\alpha_4}{4}\Gamma_{\mu}^{\ \mu\lambda}\Gamma^{\nu}_{\ \nu\lambda} \\ &- \frac{\alpha_5}{4}H_{\mu\nu\lambda}\Gamma^{\mu\lambda\nu} + b_{\mu\nu}J^{\mu\nu} + a_\mu J^\mu\end{aligned}$$

From the algebra

$$\partial_\mu J^\mu = 0 \qquad \partial_\mu J^{\mu\nu} = J^\nu \qquad \to \quad \partial_\mu\partial_\nu J^{\mu\nu} = 0$$

Also the coupling with currents depends on $b_{\mu\nu} - \partial_\mu a_\nu$. Redefinition

$$b_{\mu\nu} = \partial_\mu a_\nu + \frac{1}{2}\left(B_{\mu\nu} + h_{\mu\nu}\right)$$

where $h_{\mu\nu}$ is a symmetric field transforming as

$$\delta h_{\mu\nu} = \partial_\mu\partial_\nu\lambda_0 \qquad \text{longitudinal “linearised diffeom.”}$$

Quadratic action

Choosing the parameters

$$\alpha_1 = -2g_2 \qquad \alpha_3 = \frac{2}{3}(4g_1 - g_2) \qquad \alpha_4 = -4g_1 \qquad \alpha_5 = -\frac{8}{3}g_2$$

$h_{\mu\nu}$ and $B_{\mu\nu}$ decouple, $\mathcal{L}_{B^2}$ is the action of a two-form and

$$\mathcal{L}_{h^2} = (g_1 - g_2)h_{\mu\nu}G^{\mu\nu} + g_2 H_\mu H^\mu$$

is the fracton gravity action studied also in [Blasi, Maggiore '22, Bertolini, Blasi, Damonte, Maggiore '23]

$$\begin{aligned} H^\mu &= \partial_\sigma h^{\sigma\mu} - \partial^\mu h \\ G^{\mu\nu} &= \partial^2 h^{\mu\nu} - \partial^\mu \partial^\nu h - (\partial^\mu H^\nu + \partial^\nu H^\mu) + \eta^{\mu\nu}\partial_\sigma H^\sigma \end{aligned}$$

Solution for the symmetric field

Let's look at the massless degrees of freedom.

We decouple the two-form. Equations of motion for $h_{\mu\nu}$

$$2(g_1 - g_2)\, G^{\mu\nu} - g_2 \left(\partial^\mu H^\nu + \partial^\nu H^\mu - 2\eta^{\mu\nu}\partial_\sigma H^\sigma\right) + \frac{1}{2} J^{(\mu\nu)} = 0$$

Mode content:

helicity A	$\varepsilon^A_{\mu\nu}$	pure gauge
2σ	$e^\sigma_\mu e^\sigma_\nu$	$g_1 = g_2$
σ	$ik_{(\mu} e^\sigma_{\nu)}$	$g_2 = 0$
0	$g_2 \eta_{\mu\nu} - 2(2g_1 + g_2) ik_{(\mu} q_{\nu)}$	$g_2 = 0,\ g_2 = -2g_1$

Same spectrum when $h_{\mu\nu}$ and $B_{\mu\nu}$ are coupled.

Gauge fixing and propagator for $h_{\mu\nu}$

To find the propagator of $h_{\mu\nu}$ we need to gauge-fix.

We use ordinary BRST formalism: ghost, anti-ghost and auxiliary fields $\left\{c, \bar{c}, b\right\}$ for monopole and $\left\{c_A, \bar{c}^A, b^A\right\}$ for dipole

$$
\begin{array}{lcllcl}
\mathfrak{s}\, a_\mu & = & \partial_\mu c + \delta_\mu^A c_A & \qquad \mathfrak{s}\, b_{\mu A} & = & \partial_\mu c_A \\
\mathfrak{s}\, c & = & 0 & \qquad \mathfrak{s}\, c_A & = & 0, \\
\mathfrak{s}\, \bar{c} & = & b & \qquad \mathfrak{s}\, \bar{c}^A & = & b^A \\
\mathfrak{s}\, b & = & 0 & \qquad \mathfrak{s}\, b^A & = & 0
\end{array}
$$

and add a BRST-exact term in the action

$$
\mathcal{L}_{\text{g.f.}} = -\mathfrak{s}\, W
$$

Gauge fixing and propagator for $h_{\mu\nu}$

The most convenient W is a scalar one

$$W_s = \bar{c}\left[\partial^\mu\partial^\nu(b_{\mu A}\delta^A_\nu - \partial_\mu a_\nu) - \frac{\xi}{2}b\right] + \bar{c}^A\left[a_\mu\delta^\mu_A - \frac{\kappa}{2}\eta_{AB}b^B\right]$$

The Lagrangian after integration of auxiliary fields is

$$\mathcal{L}' = \mathcal{L}(b_{\mu\nu} - \partial_\mu a_\nu) - \frac{1}{2\xi}\left[\partial^\mu\partial^\nu(b_{\mu\nu} - \partial_\mu a_\nu)\right]^2 - \frac{1}{2\kappa}a_\mu a^\mu + \text{ghosts}$$

The $a_\mu a^\mu$ term imposes $a_\mu = 0$:

$$[\text{E.O.M. for } a_\mu]^\mu = \partial_\nu\,[\text{E.O.M. for } b_{\mu\nu}]^{\mu\nu} - \frac{1}{\kappa}a^\mu \quad = \quad 0$$

The ξ-dependent term kills pure-gauge mode $k_\mu k_\nu$ except for $k^2 = 0$.

Gauge fixing and propagator for $h_{\mu\nu}$

Eq. of motion for $h_{\mu\nu}$ in Fourier space:

$$\Delta^{\mu\nu,\alpha\beta}\varepsilon_{\alpha\beta} = \frac{1}{2}\tilde{J}^{\mu\nu} \qquad \Delta^{\mu\nu,\alpha\beta}G_{\alpha\beta,\sigma\rho} = \frac{1}{4}\left(\delta^\mu_\sigma\delta^\nu_\rho + \delta^\mu_\rho\delta^\nu_\sigma\right)$$

Calling

$$\begin{aligned}
I^{\mu\nu,\alpha\beta} &= \eta^{\mu\alpha}\eta^{\nu\beta} + \eta^{\mu\beta}\eta^{\nu\alpha} \ , \\
K_1^{\mu\nu,\alpha\beta} &= k^\mu k^\alpha \eta^{\nu\beta} + k^\mu k^\beta \eta^{\nu\alpha} + k^\nu k^\alpha \eta^{\mu\beta} + k^\nu k^\beta \eta^{\mu\alpha} \\
K_2^{\mu\nu,\alpha\beta} &= k^2\eta^{\mu\nu}\eta^{\alpha\beta} - k^\alpha k^\beta \eta^{\mu\nu} - k^\mu k^\nu \eta^{\alpha\beta} \ .
\end{aligned}$$

the $h_{\mu\nu}$ propagator reads

$$\begin{aligned}
G_{\alpha\beta,\sigma\rho} \quad &= \quad \frac{1}{8(g_1-g_2)}\frac{1}{k^2}\left[I_{\alpha\beta,\sigma\rho} + \frac{g_2-2g_1}{g_2}\frac{1}{k^2}K_{1\,\alpha\beta,\sigma\rho} - \frac{2g_1}{2g_1+g_2}\frac{1}{k^2}K_{2\,\alpha\beta,\sigma\rho}\right] \\
&+ \quad \frac{1}{k^2}\left(\frac{1}{g_2} + \frac{g_1+g_2}{4(g_1-g_2)(2g_1+g_2)} + \frac{2\xi}{k^2}\right)\frac{k_\alpha k_\beta k_\sigma k_\rho}{(k^2)^2}
\end{aligned}$$

Interactions

In ordinary gravity, the theory couples to $T_{\mu\nu}$. Linearised gravity becomes inconsistent, one is forced to add non-linear terms with Einstein gravity as unique solution with two derivatives [Kraichnan '55, Gupta '52, Deser '70]

Fracton gravity is less constraining: $h_{\mu\nu}$ can couple to

$$J^{\mu\nu} = c_V \left(\partial^\mu V^\nu + \partial^\nu V^\mu - 2\eta^{\mu\nu} \partial_\sigma V^\sigma \right)$$

V^μ arbitrary vector with one derivative at most. However, the helicity-two modes are not involved:

$$\frac{1}{2} h_{\mu\nu} J^{\mu\nu} = -c_V H_\mu V^\mu + \text{total derivative}$$

And with invariant curvatures? Asking for two derivatives, $B_{\mu\nu}$ must enter:

There are no gauge-invariant self-coupling terms for $h_{\mu\nu}$

Interactions

There could be cubic terms $h_{\mu\nu}J^{\mu\nu}$ invariant up to total derivatives

$$J^{\mu\nu} = J^{\mu\nu}(\partial^2, h^2_{\mu\nu}) \qquad \partial_\mu\partial_\nu J^{\mu\nu} = 0$$

Interactions

There could be cubic terms $h_{\mu\nu}J^{\mu\nu}$ invariant up to total derivatives

$$J^{\mu\nu} = J^{\mu\nu}(\partial^2, h^2_{\mu\nu}) \qquad \partial_\mu\partial_\nu J^{\mu\nu} = 0$$

There are no such terms!

For the future: one can generalise the $h_{\mu\nu}$ gauge transformation:

$$\delta h_{\mu\nu} = \partial_\mu \xi_\nu + \chi\, \xi^\alpha \partial_\alpha h_{\mu\nu}$$

Is there an all-order approach like in [Deser '70]?

May self-interactions terms be achieved through condensation of dipole-charged fields of [Afxonidis, AC, Hoyos, Musso '23]? If $D_\mu\phi_a = \partial_\mu\phi_a - ib_{\mu A}d^A_a\phi_a$

$$\Gamma_{\mu\nu}{}^{\sigma}\Gamma^{\mu\nu\rho}\sum_a \eta_{BC}\, e^B_\sigma d^C_a \left(i\phi^*_a D_\rho\phi_a - iD_\rho\phi^*_a\phi_a\right) \to 2d\, v^2\, \Gamma_{\mu\nu}{}^{\sigma}\Gamma^{\mu\nu\rho} b_{\sigma\rho}$$

Map to linearized gravity solutions

Recalling the $h_{\mu\nu}$ equation of motion

$$2(g_1 - g_2)\, G^{\mu\nu} - g_2 \left(\partial^\mu H^\nu + \partial^\nu H^\mu - 2\eta^{\mu\nu}\partial_\sigma H^\sigma\right) + \frac{1}{2} J^{(\mu\nu)} = 0$$

we can map solutions of linearised gravity $G_{\mu\nu} = 0$ to solutions of fracton gravity through:

$$J^{\mu\nu} = c_V \left(\partial^\mu V^\nu + \partial^\nu V^\mu - 2\eta^{\mu\nu}\partial_\sigma V^\sigma\right) \qquad V^\mu = \frac{2g_2}{c_V} H^\mu$$

We study solutions of linearised gravity that are solutions of full Einstein's theory. Solutions in Kerr-Schild form $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$:

$$h_{\mu\nu} = V \ell_\mu \ell_\nu \qquad \ell_\mu \ell^\mu = 0 \qquad H_\mu H^\mu = \partial_\mu H^\mu = 0$$

Map to linearized gravity solutions

Complex scalar ϕ and a gauge fields A_μ coupled to fracton gravity:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - D_\mu\phi^* D^\mu\phi - \frac{\lambda}{2}(|\phi|^2 - v^2)^2 - c_V H_\mu J^\mu - c_{BF} B_{\mu\nu}F^{\mu\nu}$$

If $B_{\mu\nu}$ is a massive two-form ($\mathcal{H} = \mathrm{d}B$), the e.o.m. are

$$\begin{aligned} \frac{g_1 - 2g_2}{3}\,\partial_\sigma \mathcal{H}^{\sigma\mu\nu} - \frac{\alpha_2}{2}B^{\mu\nu} - c_{BF}F^{\mu\nu} &= 0 \\ \partial_\sigma F^{\sigma\mu} + 2c_{BF}\,\partial_\sigma B^{\sigma\mu} - 2qJ^\mu - qc_V|\phi|^2 H^\mu &= 0 \\ D_\mu D^\mu\phi - ic_V H^\mu D_\mu\phi - \frac{i}{2}c_V\partial_\mu H^\mu\phi - \lambda(|\phi|^2 - v^2)\phi &= 0 \end{aligned}$$

Imposing $\alpha_2 = 4c_{BF}^2$, we find solution

$$\phi = v\ , \qquad A_\mu = -\frac{c_V}{2q}H_\mu\ , \qquad B_{\mu\nu} = -\frac{1}{2c_{BF}}F_{\mu\nu}$$

Coupling to curved background

Issues with usual formulation of fracton theories in curved background
[Slagle, Prem, Pretko '18, Jain, Jensen '21, Bidussi, Hartong, Have, Musaeus, Prohazka '21]

With ordinary gauge fields, coupling to $\gamma_{\mu\nu}$ is straightforward.

Under background diffeo $\delta\gamma_{\mu\nu} = \nabla_\mu\zeta_\nu + \nabla_\nu\zeta_\mu$ and a_μ, $b_{\mu A}$ and ${e_\mu}^A$ transform as their Lie derivative.

The two-forms $B_{\mu\nu}$ and $H_{\mu\nu A}$ don't depend on the metric, whereas $\Gamma_{\mu\nu\rho}$ need to be covariantised as

$$\Gamma_{\mu\nu\lambda} = 2\left[\nabla_{(\mu}B_{\nu)\lambda} - \frac{1}{2}\left(H_{\lambda\mu A}{e_\nu}^A + H_{\lambda\nu A}{e_\mu}^A\right)\right]$$

All the curvatures are gauge invariant also in curved geometry.

Conclusions

- Fracton theories with ordinary gauge fields
- Covariant extension of fracton gauge theories
- Degrees of freedom, gauge fixing and propagator
- Chart of the allowed cubic interactions
- Map between solutions of linearised gravity and fracton gravity
- Consistent coupling to curved background

Outlook

- To address stability of fracton gravity
- Canonical quantisation, unitarity, Lorentz invariance
- To include higher multipoles: new interactions for spin-two field?
- To include higher multipoles: higher-spin?
- All-order approach to field-dependent gauge transformations of $h_{\mu\nu}$
- To study fracton models in curved backgrounds

Thanks for your attention!

Spacetime dipole symmetry

For spacetime dipole symmetry, the dipole is not conserved

$$\partial_t Q^i \equiv \partial_t \int d^3x \, x^i J^t = \partial_t \int d^3x \, J^{ti}$$

However, if J^{ti} is thought as intrinsic dipole moment

$$Q^i_{\text{tot}} = \int d^3x \left(x^i J^t - J^{ti} \right) \qquad \partial_t Q^i_{\text{tot}} = 0$$

Covariant version. Hypersurface Σ with normal and tangent vectors n_μ, ϵ^a_μ,

$$Q^a_\Sigma = \int_\Sigma d^3x \, e^a_\mu \hat{n}_\nu \left(x^\mu J^\nu - J^{\nu\mu} \right) \qquad \hat{n}^\mu \partial_\mu Q^a_\Sigma = 0$$

Spacetime dipole symmetry

What configuration is sourced by a fracton?

$$J^t = q\,\delta^{(3)}(\boldsymbol{x}) \qquad\qquad J^{ti} = J^{it} = -\frac{q}{4\pi}\partial^i\frac{1}{r} = \frac{q}{4\pi}\frac{x^i}{r^3}$$

Then, a configuration sourced by a fracton (in the absence of other matter fields) is

$$h_{it} = \frac{q}{16\pi g_2}\partial_i r = \frac{q}{16\pi g_2}\frac{x_i}{r}$$

The fracton carries spacetime dipole charge. Take the hyperplane Σ_i located at $x^i = x_0^i \neq 0$ for a fixed value of i. Then,

$$Q^t_{\Sigma_i} = \int dt \int_{\parallel} d^2x \left(tJ^i - J^{it}\right) = -\frac{q}{2}\,\mathrm{sign}(x_0^i)\int dt$$

A "true" fracton should be an instanton!

Schwarzschild solution

The Schwarzschild solution in Kerr-Schild form is

$$\Phi = \frac{2m}{r} \qquad \ell_\mu dx^\mu = -dt - \frac{x^i dx^i}{r} \qquad r^2 = \sum_{i=1}^{3} (x^i)^2$$

When $V^\mu = J^\mu$, the current of a complex scalar fields ϕ, there is a solution which is an spherical wave of a massless field

$$\phi(x) = A\frac{e^{\mp i\omega(t+r)}}{r} \qquad \partial^2\phi = 0 \qquad E = \omega|A|^2 = \pm\frac{4mg_2}{c_V}$$

If we add to the action of a massless scalar the coupling $-h_{\mu\nu}J^{\mu\nu}/2$, the conserved $U(1)$ current would be given by

$$J^\mu_{\text{tot}} = J^\mu + \frac{c_V}{2}H_\mu\phi^*\phi$$

Similarly, the equations of motion will be modified

$$\partial^2\phi - ic_V H^\mu\partial_\mu\phi - \frac{i}{2}c_V\partial_\mu H^\mu\phi = 0$$

Schwarzschild solution

The spherical wave is not a solution to the modified equations,

One could use a perturbative expansion in c_V to systematically find corrections.

Alternatively, one could add a term to the scalar action

$$\Delta\mathcal{L} = \frac{c_V^2}{4g_2} J_\mu J^\mu \quad \rightarrow \quad \mathcal{L}_{H^2} = g_2 \left(H_\mu - \frac{c_V}{2g_2} J_\mu \right)^2$$

This cancels the contributions proportional to H^μ in the equations of motion of the scalar field and in J^μ_{tot}.

The scalar spherical wave and Schwarzschild metric are exact solutions of the coupled scalar field and spacetime dipole theory.