Regge Couplings from Quark Models

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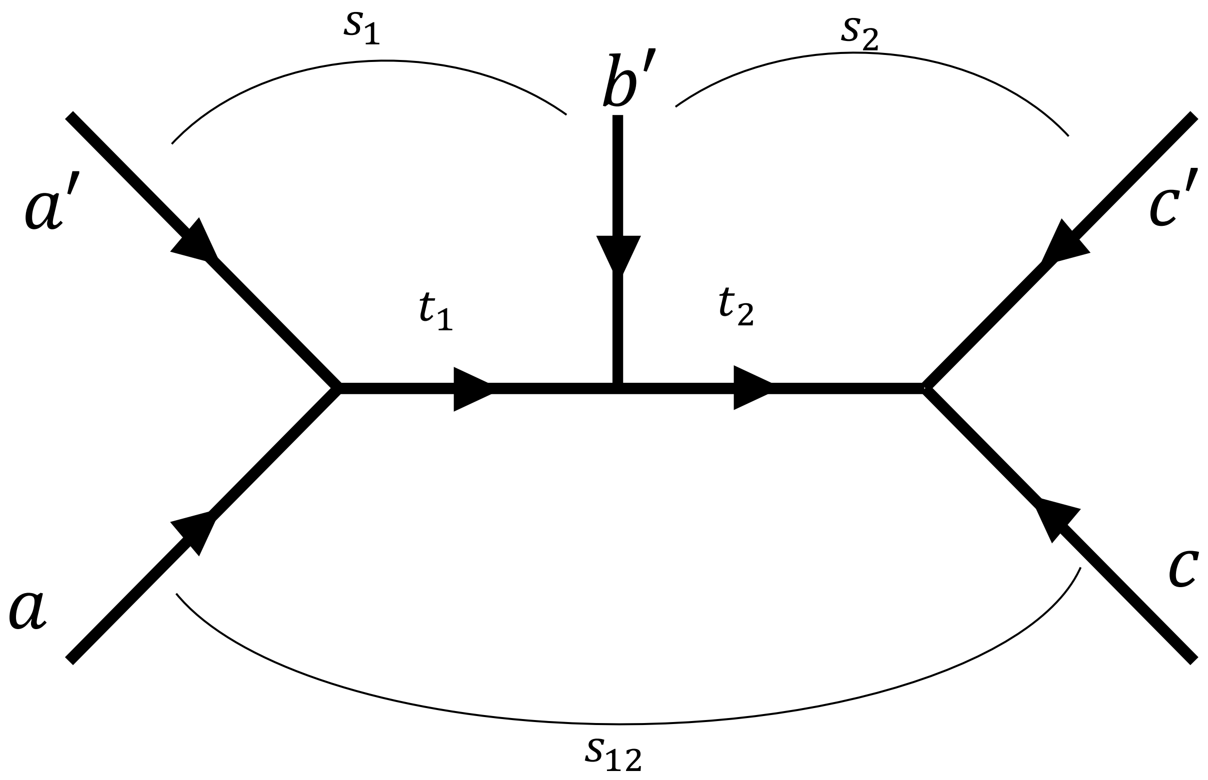
ExoHad
Exotic Hadrons Topical Collaboration

JPAC

Five-point amplitude



$$s_1 = (p_a + p_{a'})^2 , \qquad s_2 = (p_{b'} + p_{c'})^2 , \qquad s_{12} = (p_a + p_c)^2 ,$$

$$t_1 = (p_a + p_{a'})^2 , \qquad t_2 = (p_c + p_{c'})^2 .$$

Single-Regge Amplitude

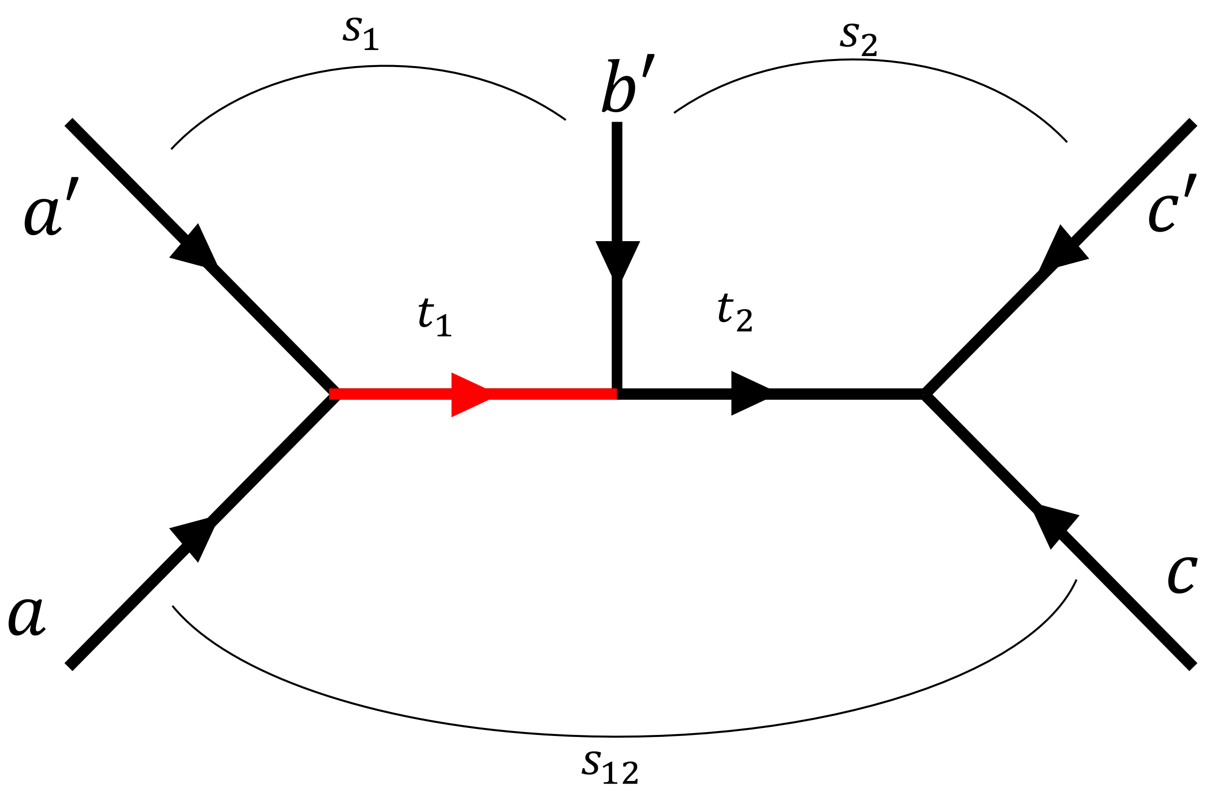
$$A = \sum_{J_1,\lambda} d^{J_1}_{0,\lambda}(\cos\theta_1)\, e^{i\omega_{12}} a(J_1,\lambda,\theta_2,t_1,t_2)$$

- Project amplitude onto t_1 partial waves
- Assume PWA contains factorizable pole in J_1

$$a(J_1,\lambda,\theta_2,t_1,t_2) \sim \frac{\beta_1(t_1)}{J_1-\alpha_1(t_1)}$$

- d-function asymptotics:

$$d^{J}_{0,\lambda} \sim (\cos\theta_1)^{J}$$



Single-Regge Amplitude

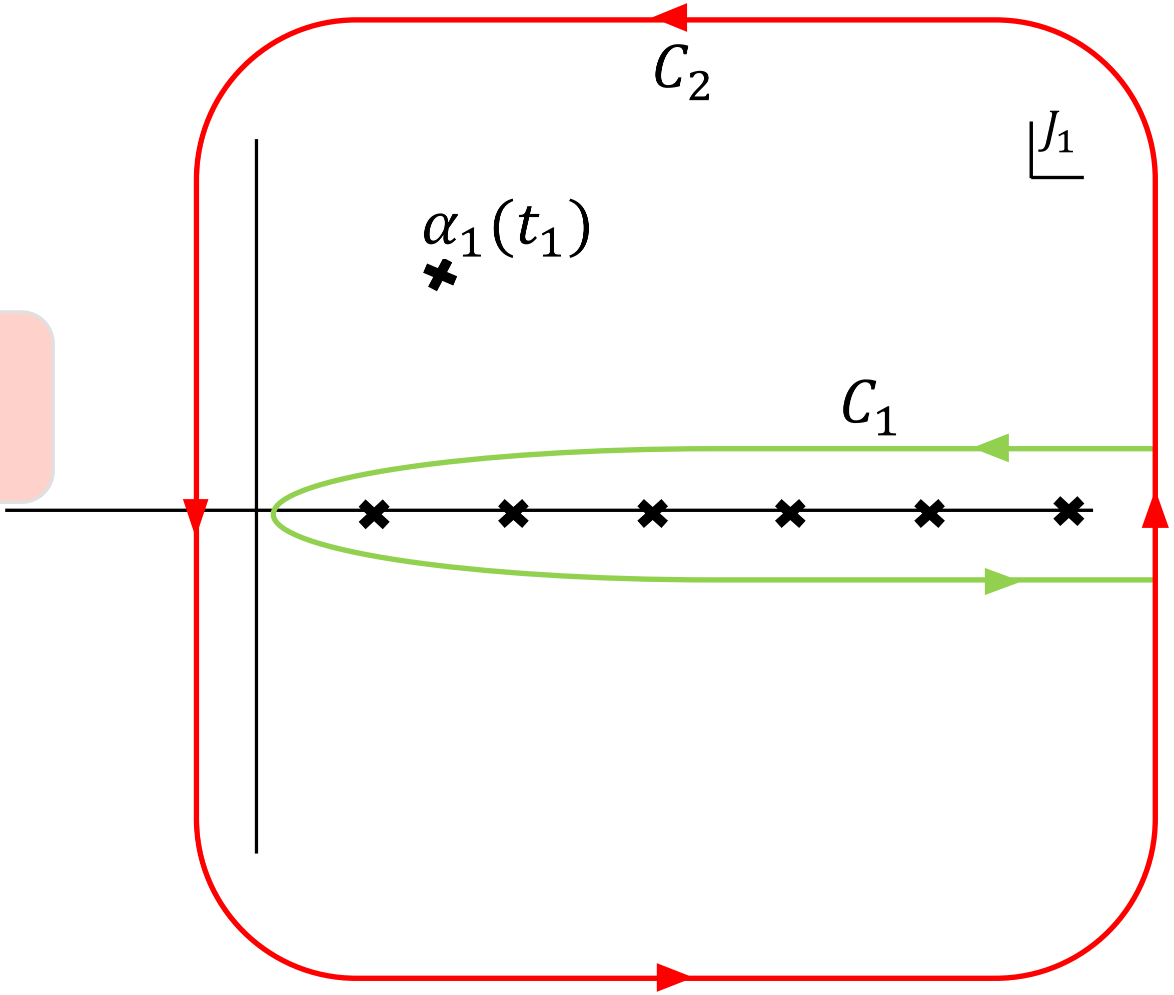
$$A \sim \sum_{J_1,\lambda} (\cos\theta_1)^{J_1} \frac{\beta_1(t)}{J_1 - \alpha_1(t_1)}$$

- **Sommerfield-Watson transform: rewrite as contour integrals**

$$A \sim \oint_{C_1} \frac{(\cos\theta_1)^{J_1}}{\sin\pi J_1} \frac{\beta_1(t)}{J_1 - \alpha_1(t_1)}$$

$C_1 \to C_2$

$$A \sim \Gamma\big(-\alpha_1(t_1)\big)\beta_1(t_1)(\cos\theta_1\,)^{\alpha_1(t_1)} R(\omega_{12}, \theta_2, t_1, t_2)$$

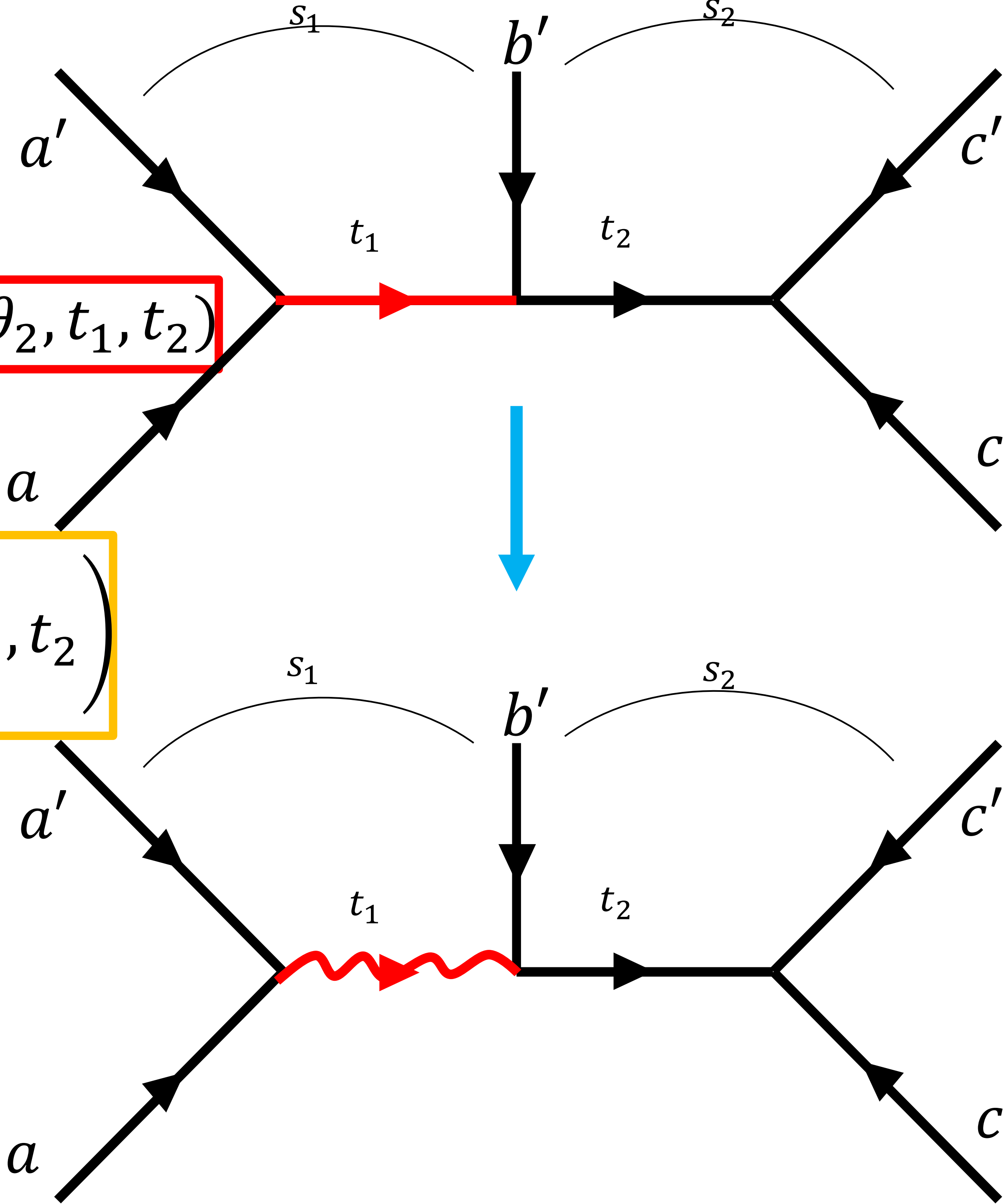


Single-Regge Limit

$s_1 \propto \cos\theta_1 \to \infty \qquad s_{12} \propto \cos\theta_1 \to \infty$

$$A \sim \Gamma(-\alpha_1(t_1))\beta_1(t_1)(\cos\theta_1)^{\alpha_1(t_1)} R(\omega_{12}, \theta_2, t_1, t_2)$$

$$A \sim \Gamma(-\alpha_1(t_1))\beta_1(t_1)(s_1)^{\alpha_1(t_1)} R\left(\frac{s_1}{s_{12}}, s_2, t_1, t_2\right)$$

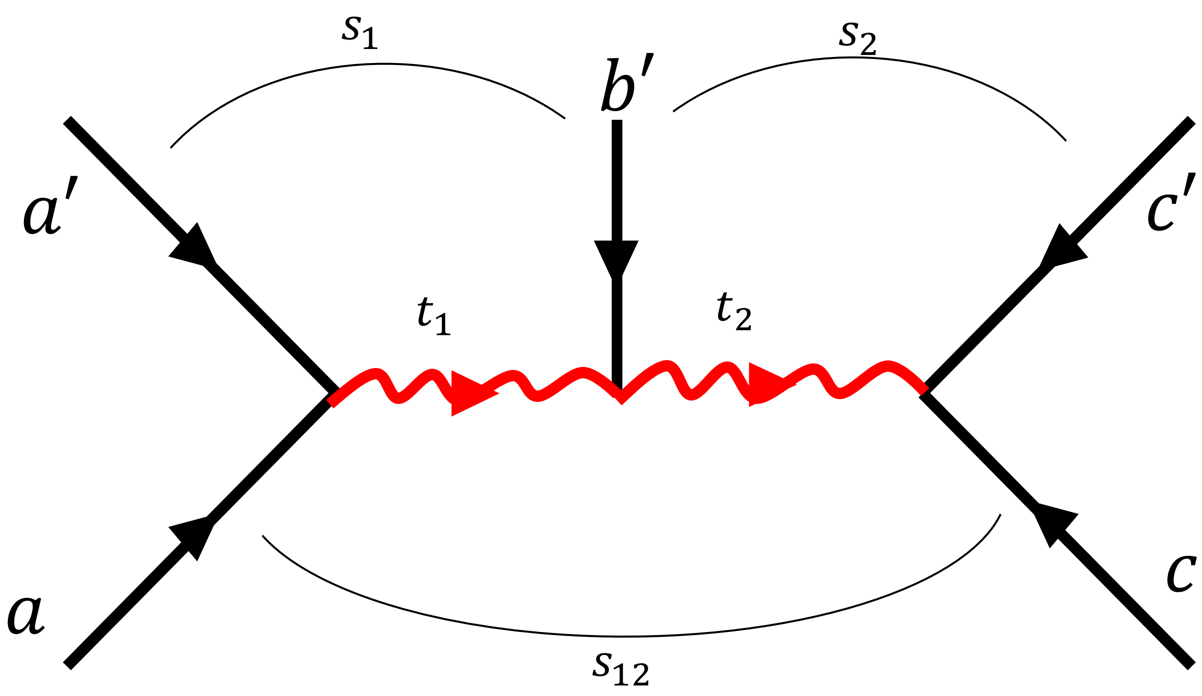


- When t_1, t_2 are large and negative, the limit gives physical result for $a + c \to \bar{a}' + \bar{b}' + \bar{c}'$

Double-Regge Limit

$$A = \sum_{J_1,J_2} \sum_{|\lambda| \leq J_1,J_2} d^{J_1}_{0,\lambda}(\cos\theta_1)\, e^{i\omega_{12}} d^{J_2}_{\lambda,0}(\cos\theta_2) a(J_1,\lambda,\theta_2,t_1,t_2)$$

$$a(J_1,\lambda,\theta_2,t_1,t_2) \sim \frac{\beta_1(t_1)}{J_1-\alpha_1(t_1)} \frac{\beta_2(t_2)}{J_2-\alpha_2(t_2)}$$

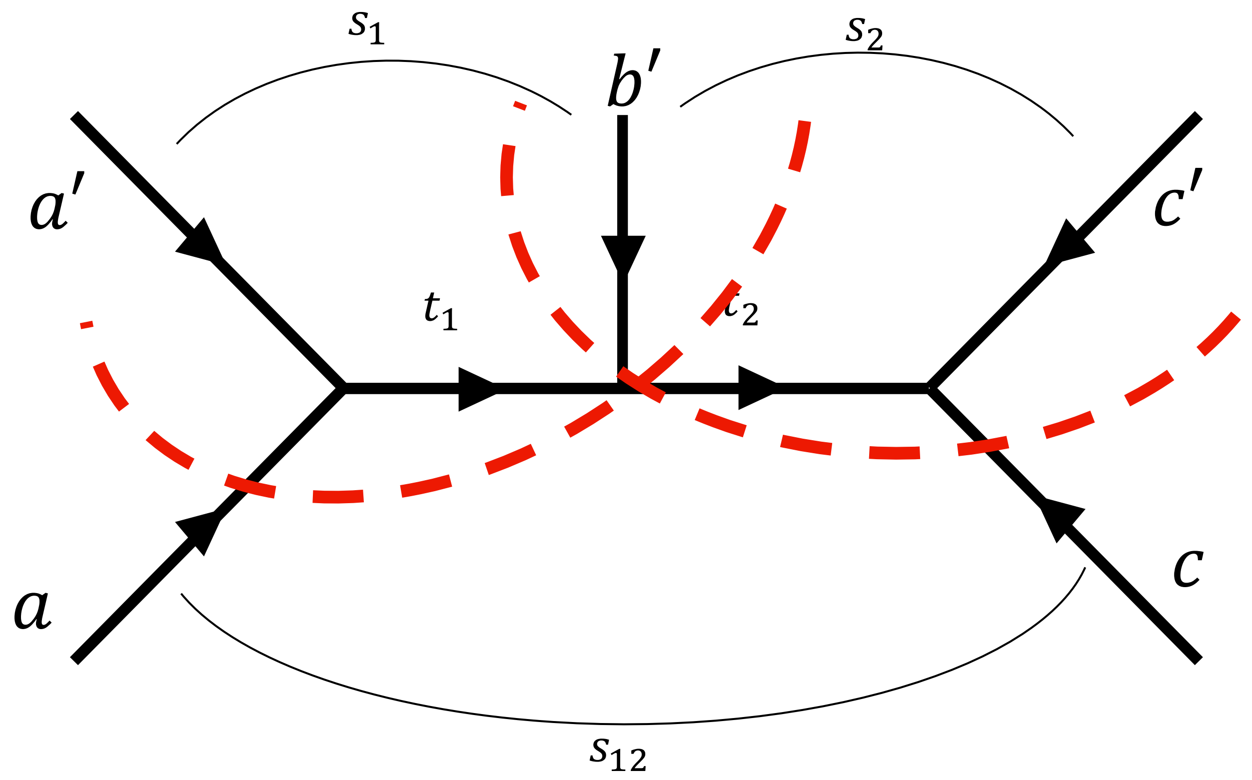


$$s_1 \propto \cos\theta_1 \to \infty\ , \qquad s_2 \propto \cos\theta_2 \to \infty\ ,$$

$$s_{12} \propto \cos\theta_1 \cos\theta_2 \to \infty\ , \qquad t_1,\ t_2,\ \eta_{12} \equiv \frac{s_{12}}{s_1 s_2}\ \text{fixed}$$

$$A \sim \Gamma\big(-\alpha_1(t_1)\big)\beta_1(t_1)(s_1)^{\alpha_1(t_1)} R(t_1,t_2,\eta_{12})(s_2)^{\alpha_2(t_2)}\Gamma\big(-\alpha_2(t_2)\big)\beta_2(t_2)$$

Constraints from cut structure



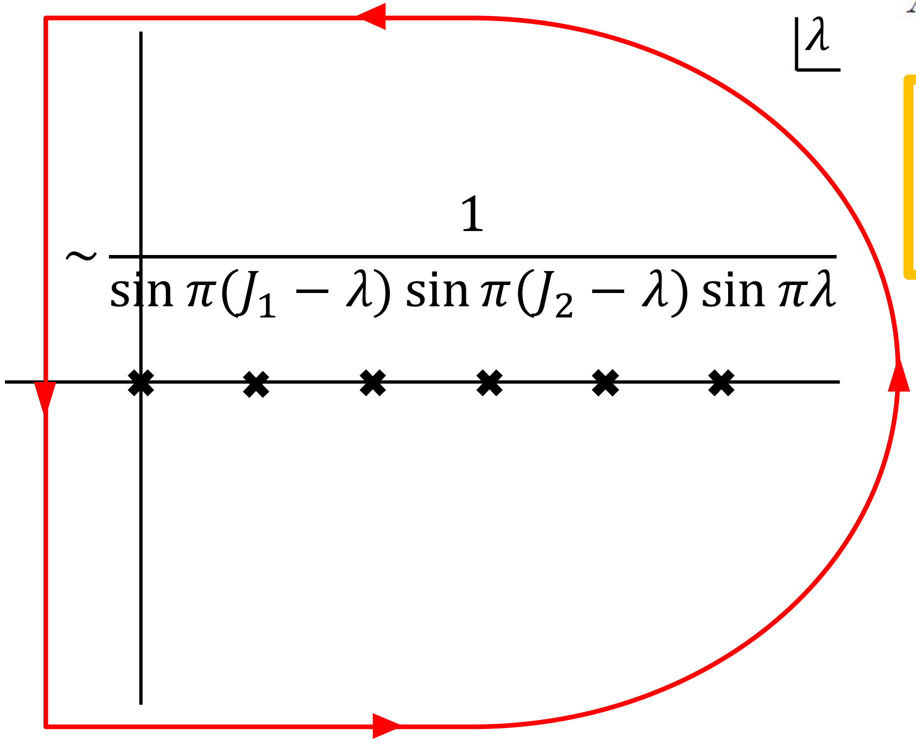
- Terms like $s_1^{\alpha_1(t_1)} s^{\alpha_2(t_2)}$ with simultaneous overlapping cuts are forbidden!

$$A \sim \Gamma(-\alpha_1)\Gamma(-\alpha_2)\left[(-s_{12})^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}V_1(\eta_{12},t_1,t_2) + (-s_{12})^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}V_2(\eta_{12},t_1,t_2)\right]$$

Shimada's Model

$$A \to \sum_{J_1, J_2, |\lambda|} \frac{s_1^{J_1}}{(J_1-\lambda)!} \frac{s_2^{J_2}}{(J_2-\lambda)!} \frac{1}{\lambda!} \frac{\beta}{(J_1-\alpha_1)(J_2-\alpha_2)}$$

- Helicity coupling choices $1/\lambda!$ ensures correct analytic behavior



$$\sim \frac{1}{\sin \pi(J_1-\lambda) \sin \pi(J_2-\lambda) \sin \pi\lambda}$$

$$A \sim \Gamma(-\alpha_1)\Gamma(-\alpha_2)\left[(-s_{12})^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1} V_1(\eta_{12}, t_1, t_2) + (-s_{12})^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2} V_2(\eta_{12}, t_1, t_2)\right]$$

$$V_1(\eta, t_1, t_2) = \beta \frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(-\alpha_2)} {}_1F_1\left(-\alpha_1, 1-\alpha_1+\alpha_2, -\frac{1}{\eta_{12}}\right)$$

Shimada's Model

$$A \to \sum_{J_1, J_2, |\lambda|} \frac{s_1^{J_1}}{(J_1-\lambda)!} \frac{s_2^{J_2}}{(J_2-\lambda)!} \frac{1}{\lambda!} \frac{\beta}{(J_1-\alpha_1)(J_2-\alpha_2)}$$

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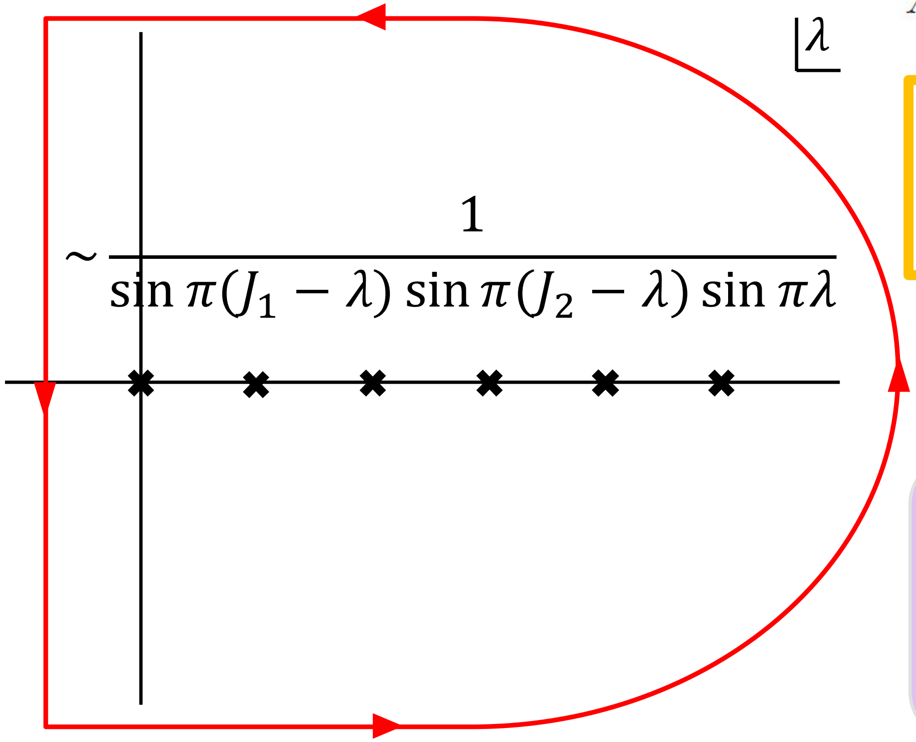
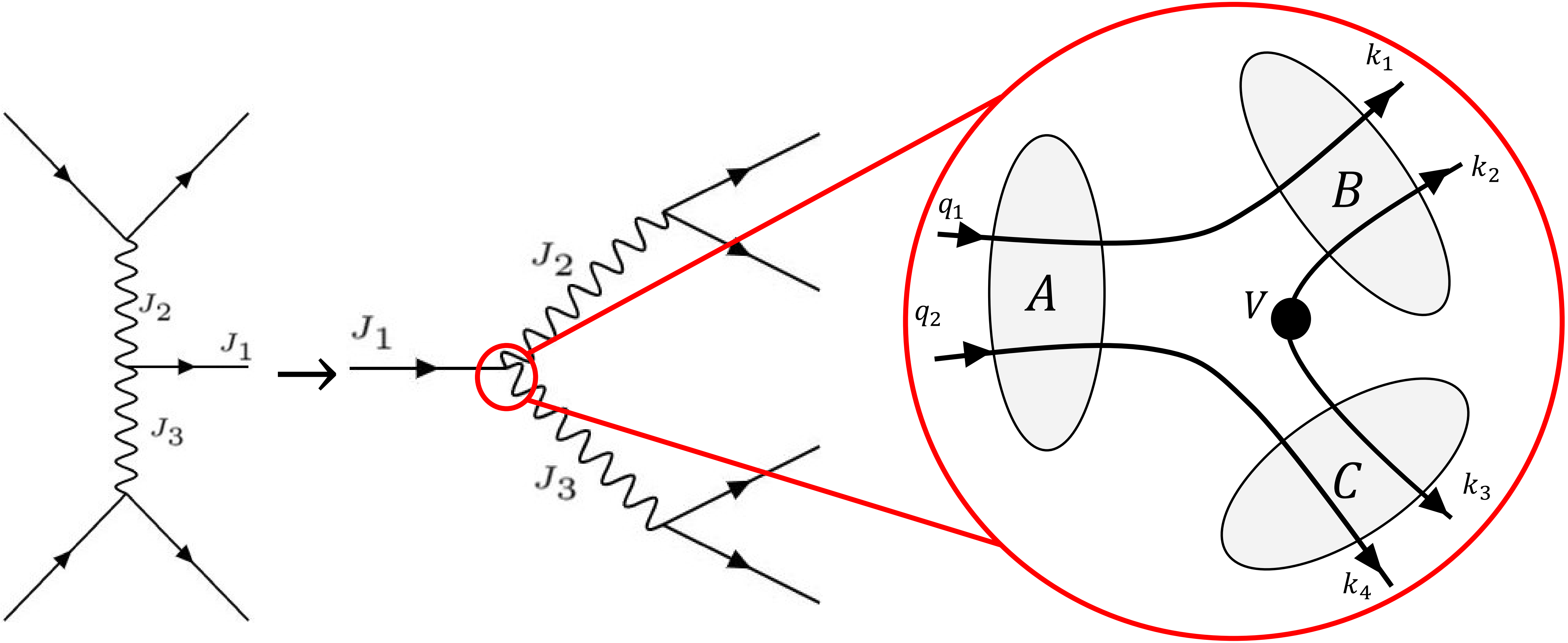


$$A \sim \Gamma(-\alpha_1)\Gamma(-\alpha_2)\left[(-s_{12})^{\alpha_1}(-s_2)^{\alpha_2-\alpha_1}V_1(\eta_{12},t_1,t_2) + (-s_{12})^{\alpha_2}(-s_1)^{\alpha_1-\alpha_2}V_2(\eta_{12},t_1,t_2)\right]$$

$$V_1(\eta,t_1,t_2) = \beta\frac{\Gamma(\alpha_1-\alpha_2)}{\Gamma(-\alpha_2)}\,{}_1F_1\left(-\alpha_1, 1-\alpha_1+\alpha_2, -\frac{1}{\eta_{12}}\right)$$

- Problem: there's no physical motivation for this amplitude form other than it 'works' !

String-Breaking Model

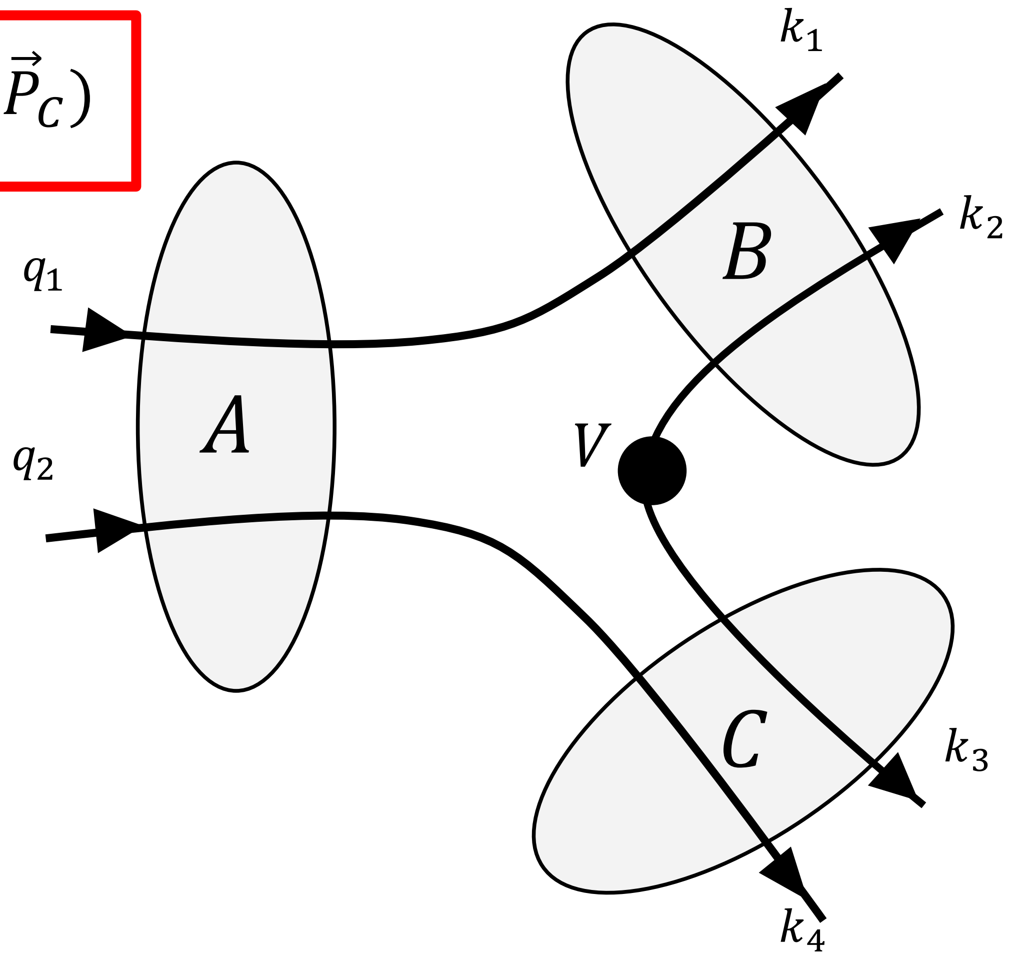


$$\langle \psi_B, \psi_C | V | \psi_A \rangle = \int d^3k \; V \, \psi_A(\vec{k}) \psi_B^*(\vec{k} + \vec{P}_c) \psi_C^*(\vec{k} + \vec{P}_C)$$

$$\psi_B(k_1, k_2) = \delta^3(P_1 - k_1 - k_2)\psi_B(k_1 - k_2)e^{iP_1}$$

$$\psi_C(k_3, k_4) = \delta^3(P_2 - k_3 - k_4)\psi_C(k_3 - k_4)e^{iP_2}$$

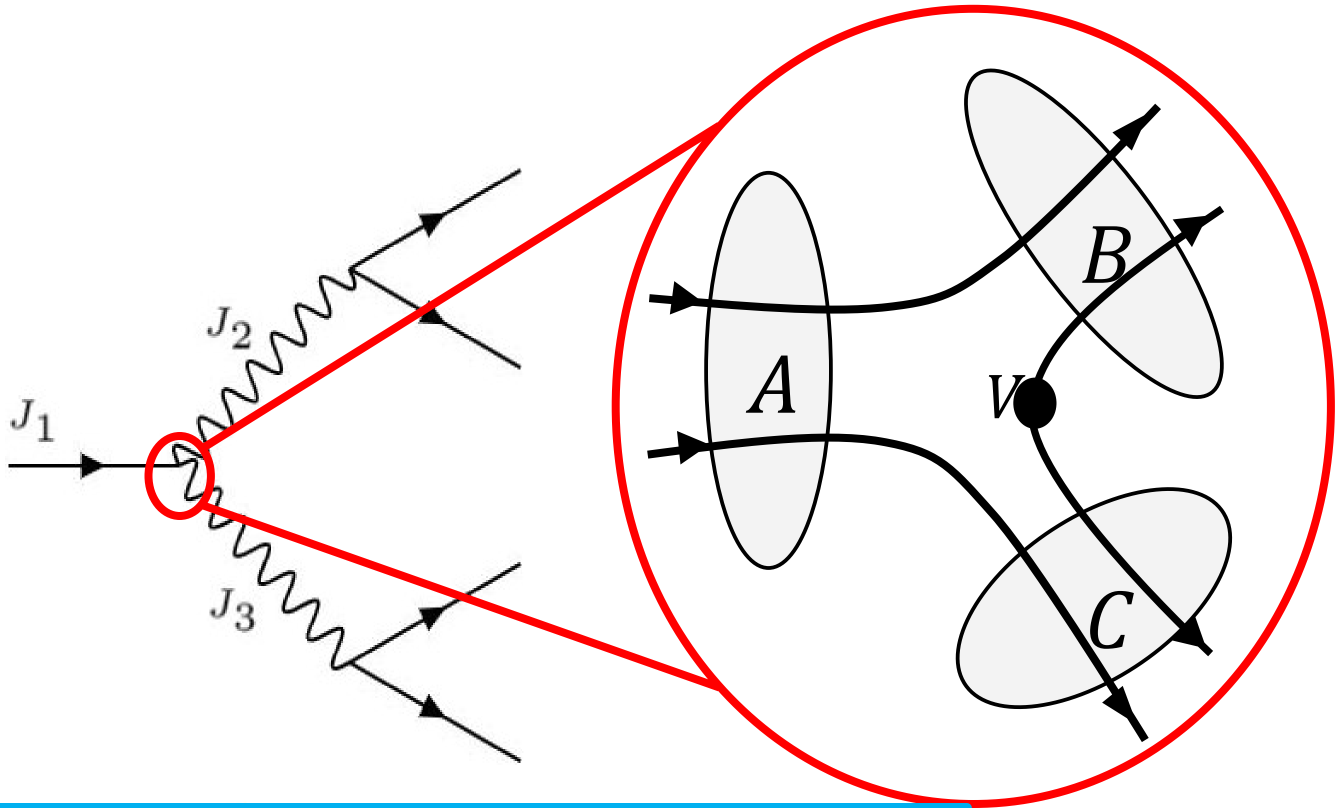
$$\psi_A(q_1, q_2) = \delta^3(P - q_1 - q_2)\psi_A(q_1 - q_2)e^{iP} \; .$$



- Insert wavefunctions for quark-antiquark pairs in mesons
- Harmonic oscillator wave functions take care of confinement
- Can combine multiple oscillators to approximate any choice function

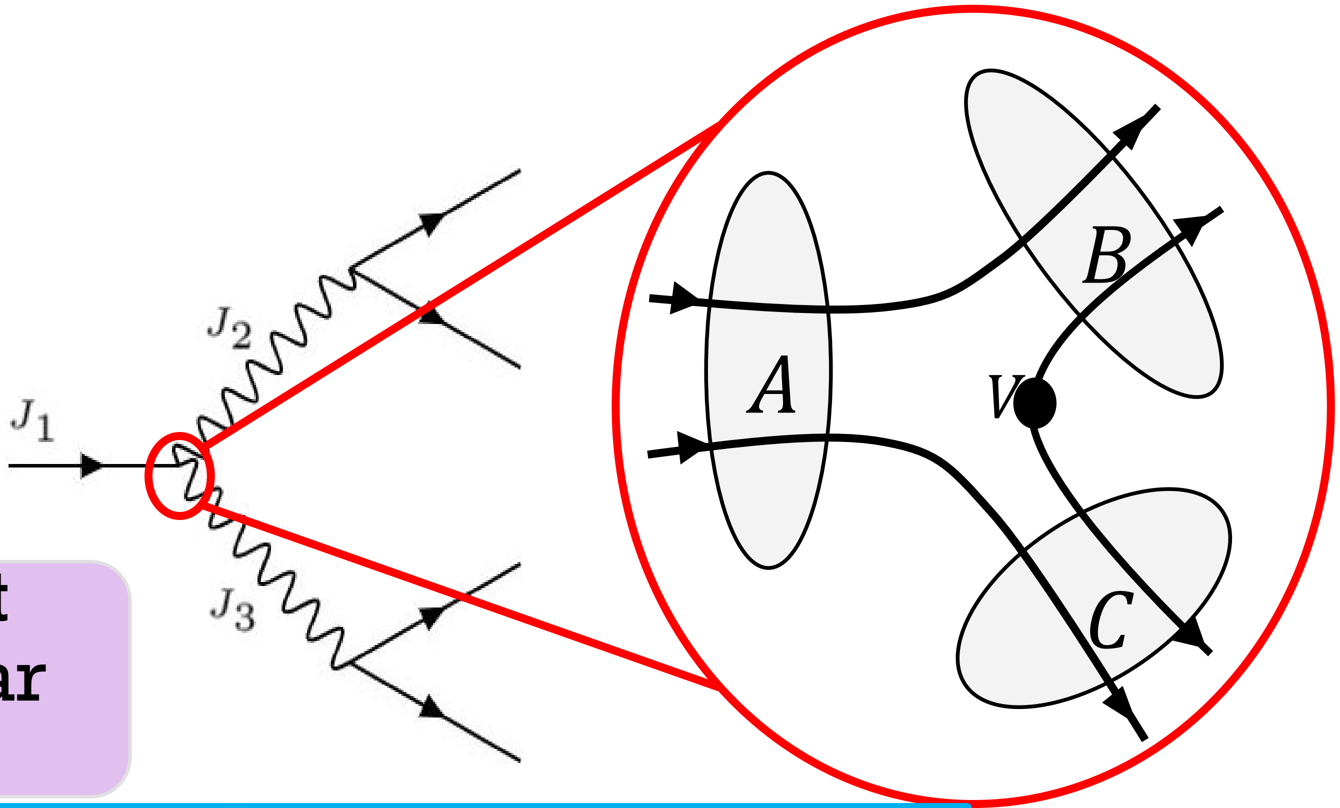
$$\psi \propto e^{\frac{-\alpha^2 k^2}{2}} k^l Y_{lm}(\Omega)$$

- Assume constant string-breaking probability
- String-breaking matrix element is calculable analytically
- Relevant matrix element: spinless A, B with spin J_1, C with spin J_2



$$
\begin{aligned}
\langle \psi_B, \psi_C | V | \psi_A \rangle = & \frac{8}{3} \pi^{\frac{5}{4}} \alpha^{\frac{3}{2}} \left(\frac{2}{3}\right)^{\frac{J_1+J_2+1}{2}} e^{-\frac{1}{3}\alpha^2 k_C^2} \sum_{n,l,m} (-1)^{n+m} \left(\frac{\alpha k_C}{\sqrt{6}}\right)^{2n+l} Y_{lm}(\hat{k_C}) \\
& \times \frac{1}{\Gamma(n+l+3/2)} \frac{1}{\sqrt{\Gamma(J_1+3/2)\Gamma(J_2+3/2)}} \frac{\Gamma(\frac{J_1+J_2+l}{2}+\frac{3}{2})\Gamma(n+\frac{l-J_1-J_2}{2})}{n!\,\Gamma(\frac{l-J_1-J_2}{2})} \qquad (4.36) \\
& \times \sqrt{\frac{(2J_1+1)(2J_2+1)}{(2l+1)}} \langle l, 0 | J_1, 0, J_2, 0 \rangle \langle l, m | J_1, M_1, J_2, M_2 \rangle \; .
\end{aligned}
$$

- Assume constant string-breaking probability
- String-breaking matrix element is calculable analytically
- Relevant matrix element: spinless A, B with spin J_1, C with spin J_2



- Problem: CG coeffs don't admit continuation to complex angular momentum!

$$\langle\psi_B,\psi_C|V|\psi_A\rangle = \frac{8}{3}\pi^{\frac{5}{4}}\alpha^{\frac{3}{2}}\left(\frac{2}{3}\right)^{\frac{J_1+J_2+1}{2}} e^{-\frac{1}{3}\alpha^2k_C^2}\sum_{n,l,m}(-1)^{n+m}\left(\frac{\alpha k_C}{\sqrt{6}}\right)^{2n+l}Y_{lm}(\hat{k_C})$$

$$\times \frac{1}{\Gamma(n+l+3/2)}\frac{1}{\sqrt{\Gamma(J_1+3/2)\Gamma(J_2+3/2)}}\frac{\Gamma(\frac{J_1+J_2+l}{2}+\frac{3}{2})\Gamma(n+\frac{l-J_1-J_2}{2})}{n!\,\Gamma(\frac{l-J_1-J_2}{2})} \tag{4.36}$$

$$\times \sqrt{\frac{(2J_1+1)(2J_2+1)}{(2l+1)}}\,\langle l,0|J_1,0,J_2,0\rangle\,\langle l,m|J_1,M_1,J_2,M_2\rangle\ .$$

- Consider the angular part of the SB matrix element integral:

$$I_\Omega = \sqrt{\frac{(2l+1)(2J_1+1)(2J_2+1)}{(4\pi^3)}} \int_{-1}^{1} dz d d^{\ell}_{00}(z) d^{J_1}_{\lambda 0}(z) d^{J_2}_{-\lambda 0}(z)$$

- Hypergeometrics and gamma functions allow analytic computation

$$d^J_{\lambda,0} = \frac{\chi_\lambda}{\Gamma(M+1)} \sqrt{\frac{\Gamma(J+M+1)}{\Gamma(J-M+1)}} \left(\frac{1-z}{2}\right)^{\frac{M}{2}} \left(\frac{1+z}{2}\right)^{\frac{M}{2}} {}_2F_1\left(-J+M, J+M+1, M+1; \frac{1-z}{2}\right)$$

$+$

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+n)} \frac{x^n}{n!}$$

$\Rightarrow$

$$I_\Omega = \sqrt{\frac{(2l+1)(2J_1+1)(2J_2+1)}{(4\pi)^3}} \chi_\lambda \chi_{-\lambda} \sqrt{\frac{\Gamma(J_1-M+1)\Gamma(J_2-M+1)}{\Gamma(J_1+M+1)\Gamma(J_2+M+1)}}$$
$$\times \sum_{m,n,k} \frac{(-1)^{m+n+k}}{m!n!k!} \frac{\Gamma(J_1+M+m+1)}{\Gamma(J_1-M-m+1)\Gamma(M+m+1)} \frac{\Gamma(J_2+M+n+1)}{\Gamma(J_2-M-n+1)\Gamma(M+n+1)}$$
$$\times \frac{\Gamma(l+k+1)}{\Gamma(l-k+1)\Gamma(k+1)} \frac{\Gamma(1+k+m+n+M)\Gamma(M+1)}{\Gamma(2M+k+m+n+2)}, \qquad (4.45)$$

$M = |\lambda|$

- Need to include helicity dependence from the original d's ($M = |\lambda|$)

$$A(t_1,\theta_1,\omega_{12},\theta_2,t_2) = \sum_{J_1,J_2}\sum_{|\lambda|\le J_1,J_2} d^{J_1}_{0,\lambda}(\cos\theta_1)e^{i\omega_{12}}d^{J_2}_{\lambda 0}(\cos\theta_2)a(J_1,J_2,\lambda,t_1,t_2)$$

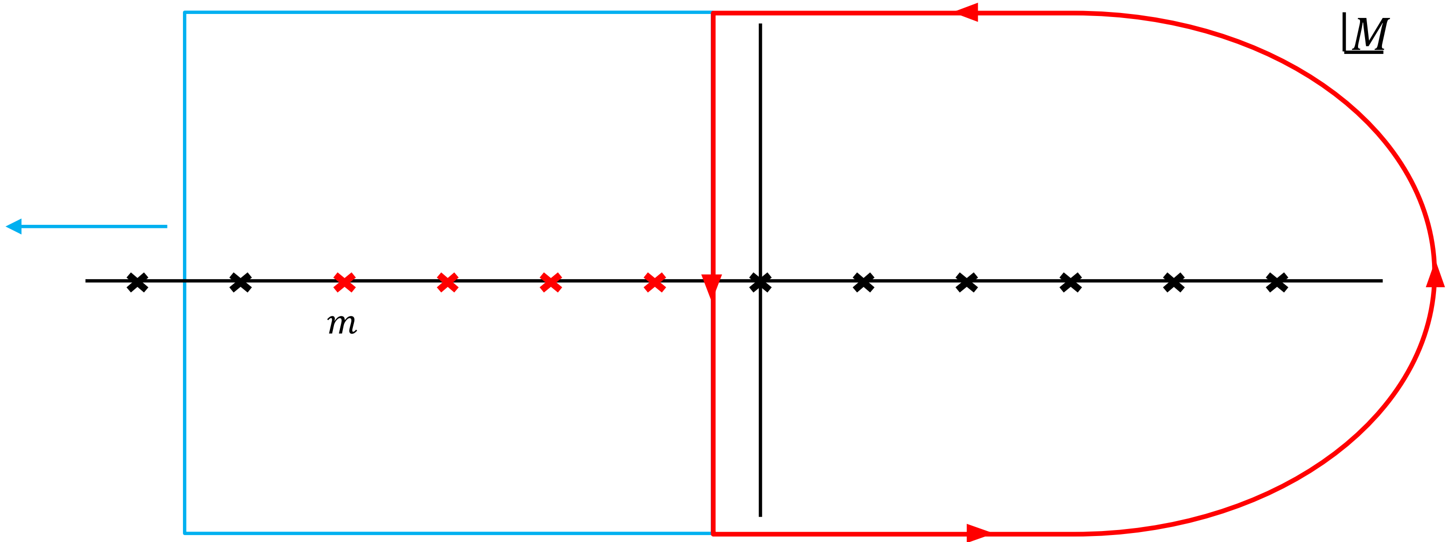
$$Y_{J\lambda} \sim \sqrt{\frac{2J+1}{4\pi}}(-1)^{\frac{\lambda}{2}}\frac{\Gamma(2J+1)}{\Gamma(J+1)}\frac{1}{\sqrt{\Gamma(J+M+1)\Gamma(J-M+1)}}2^{-J}z^J e^{i\lambda\phi}$$

$$\Rightarrow \quad g_{12}(\lambda) = 2\sqrt{\frac{(2l+1)}{4\pi}\frac{(2J_1+1)(2J_2+1)}{4\pi}}\chi_\lambda\chi_{-\lambda}\frac{\Gamma(2J_1+1)\,2^{-J_1}\Gamma(2J_2+1)\,2^{-J_2}}{\Gamma(J_1+M+1)\,\Gamma(J_2+M+1)\,\Gamma(J_1+1)\,\Gamma(J_2+1)}$$
$$\times\sum_{m,n,k}\frac{\Gamma(J_1+M+m+1)}{\Gamma(J_1-M+m+1)}\frac{\Gamma(J_2+M+n+1)}{\Gamma(J_2-M+n+1)}\frac{\Gamma(l+k+1)}{\Gamma(l-k+1)}$$
$$\times\frac{(-1)^{m+n+k}}{m!n!k!}\frac{\Gamma(M+1)}{\Gamma(M+m+1)\,\Gamma(M+n+1)\,\Gamma(k+1)}\frac{\Gamma(1+k+m+n+M)}{\Gamma(2M+k+m+n+2)}\,.$$

- We divide by sines to get the right form for analytic continuation

$$g'_{12}(\lambda) = \pi^3 \frac{I'_\Omega}{\sin(\pi(J_1 - M))\sin(\pi(J_2 - M))\sin(\pi M)}$$

$$g'_{12}(\lambda) = (-1)^M \frac{1}{(4\pi)^2}\sqrt{\frac{(2l+1)}{4\pi}}\frac{\Gamma(2J_1+2)\Gamma(2J_2+2)}{2^{J_1}\Gamma(J_1+1)2^{J_2}\Gamma(J_2+1)}\Gamma(-J_1+M)\Gamma(-J_2+M)\Gamma(-M)$$
$$\times \sum_{k,n,m=0} \frac{\Gamma(l+1+k)}{\Gamma(l+1-k)}\frac{\Gamma(J_1+M+1+m)}{\Gamma(J_1+M+1)}\frac{\Gamma(J_1-M+1)}{\Gamma(J_1-M+1-m)}\frac{\Gamma(J_2+M+1+n)}{\Gamma(J_2+M+1)}\frac{\Gamma(J_2-M+1)}{\Gamma(J_2-M+1-n)}$$
$$\times \frac{1}{k!n!m!}2(-1)^k\frac{\Gamma(1+k+m+n+M)}{\Gamma(2M+k+m+n+2)}\frac{\Gamma(M+1)^2}{\Gamma(k+1)\Gamma(M+1+n)\Gamma(M+1+m)}. \qquad (4.51)$$



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$$\begin{aligned} g'_{12}(\lambda) &= (-1)^M \frac{1}{(4\pi)^2}\sqrt{\frac{(2l+1)}{4\pi}}\frac{\Gamma(2J_1+2)\Gamma(2J_2+2)}{2^{J_1}\Gamma(J_1+1)2^{J_2}\Gamma(J_2+1)}\Gamma(-J_1+M)\Gamma(-J_2+M)\Gamma(-M) \\ &\times \sum_{k,n,m=0} \frac{\Gamma(l+1+k)}{\Gamma(l+1-k)}\frac{\Gamma(J_1+M+1+m)}{\Gamma(J_1+M+1)}\frac{\Gamma(J_1-M+1)}{\Gamma(J_1-M+1-m)}\frac{\Gamma(J_2+M+1+n)}{\Gamma(J_2+M+1)}\frac{\Gamma(J_2-M+1)}{\Gamma(J_2-M+1-n)} \\ &\times \frac{1}{k!n!m!}2(-1)^k \frac{\Gamma(1+k+m+n+M)}{\Gamma(2M+k+m+n+2)}\frac{\Gamma(M+1)^2}{\Gamma(k+1)\Gamma(M+1+n)\Gamma(M+1+m)}. \qquad (4.51) \end{aligned}$$

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Summary

- Double-Regge Amplitudes are complicated (clearly!)

- Quark models allow us to calculate a physically motivated double-Regge vertex

- Check your d-function representations

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