

Photoproduction of $b_1\Delta$

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Introduction

- QCD demands that the physically observable states must be color singlets.
 - Mesons ($q\bar{q}$), and baryons (qqq) are the most common combinations.
- Are other combinations allowed? Yes!
 - Hybrids ($q\bar{q}g$), tetra/pentaquarks, glueballs, molecular states, etc.
- Quark models, effective QFT models, sum rules, BSE/DSE, etc.
- Quark model says one hybrid nonet ($m \lesssim 2$ GeV), Lattice shows one hybrid isovector.
- Experimental evidence?
 - Two known states: $\rho(1600)$ (known since 30 years) and the recently observed $\rho(1855)$ (BESIII).

Where are the hybrids?

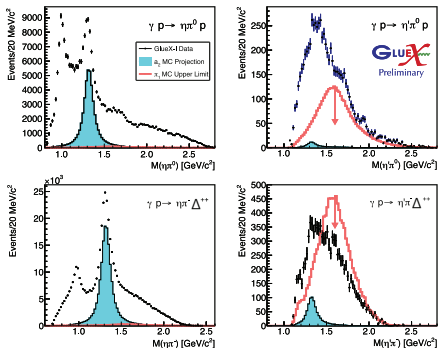
- The $\pi_1(1600)$:
 - First discovered light hybrid state.
 - Originally, two light resonances: $\pi_1(1600)$ and the lighter $\pi_1(1400)$
 - Same quantum numbers: $J^{PC} = 1^{-+}$.
 - Complimentary decay channels.
 - JPAC analysis of COMPASS data shows only one pole [1].
 - Lattice simulations by HadSpec shows various decay channels [2].
 - Possible π - η state interactions
- The $\eta_1(1855)$:
 - Recently seen by BESIII in the η channel; $J^{PC} = 1^{-+}$.
 - Mass ($1855 \pm 9^{+6}_1$ MeV) and total width ($188 \pm 18^{+3}_8$) known; partial width unknown [3].
 - Nature of the state is up for interpretation ! various models (for example, [4] and refs within).
- Where are the remaining hybrids and other exotics?

GlueX Experiment

- Photoproduction of mesons up to $\sqrt{s} = 3$ GeV.
- Linearly polarised photon beam:
 - $E = 8.2 - 8.8$ GeV, $P = 0.35$.
- Probe the production mechanisms of the mesons via spin density matrix elements (SDMEs).
 - Photoproduction of $\rho^0, b_1, \omega, \pi^0, a_2, \dots$, etc
 - Complexities: $b_1 \rightarrow \rho^0 + (\pi^0), \omega \rightarrow \rho^0 + (\pi^0), \dots$, etc.
 - Complexities: Line shapes, decays, spin, etc.
- Why use $\pi_1 \Delta$?

Why $b_1\Delta$?

- $\pi_1\Delta$ has a larger upper limit for the xsection than $\pi_1\rho$ [5].
- $\pi_1 \rightarrow b_1\pi$ is the dominant decay channel [2, 4]



Plot taken from [5].

- We start with $\pi\Delta$ (simplest process involving Δ).
- $b_1\Delta$ is the next level of complexity + important background for $\pi_1\rho$.

The $b_1\Delta$ photoproduction

- Intensity, beam spin asymmetry (BSA), SDMEs.
- The process: $\gamma p \rightarrow b_1\Delta \rightarrow (\omega\pi)(\pi p) \rightarrow ((3\pi)\pi)(\pi p)$.
 - Decays at each stage model using line shapes and Wigner-D's.

$$A_{\lambda_1 \lambda_2} = \sum_{\Lambda=1}^1 \sum_{\Delta=\frac{3}{2}}^{\frac{3}{2}} V_{\gamma \Lambda; \lambda_1 \Delta}(s; t) \sum_{\lambda=1}^1 F D_{\Lambda; \lambda}^J(\Omega) Y^1(\theta) G F_{\Delta; \lambda_2}^{\frac{3}{2}}(\rho) \quad (1)$$

λ 's are the helicities of the various states, Ω 's are the decay angles, F , G are the line shapes.

Some properties and relations: cross section

The intensity is proportional to

$$\begin{aligned}
 I(\Omega_I, \Omega_p, \Phi) &= \kappa \sum_{\substack{(\gamma) \\ \gamma; 1; 2}} A_{\gamma; 1; 2} \hat{p}_{\gamma; 0; \gamma}(\Phi) A_{\gamma; 0; 1; 2} \\
 &= I^0(\Omega_I, \Omega_p) + I(\Omega_I, \Omega_p) P(\Phi)
 \end{aligned} \tag{2}$$

giving us the intensity for $\gamma p \rightarrow b_1 \Delta$ as

$$I = \kappa \left(\frac{8\pi^3}{3} \right) \sum_{\gamma; 0; \gamma} \sum_{\Delta; \Lambda; 1; 1} V_{\gamma; \Lambda; 1; \Delta}(s, t) \sigma_{\gamma; 0; \gamma} V_{\gamma; \Lambda; 0; 1; \Delta}(s, t) \tag{3}$$

Some properties: SDMEs

The Δ -SDMEs are defined as,

$${}^0_{\Delta; \Delta} = \frac{1}{2N} \sum_{\gamma; \Lambda; 1} V_{\gamma; \Lambda; 1; \Delta} V_{\gamma; \Lambda; 1; \Delta} \quad (4)$$

$${}^1_{\Delta; \Delta} = \frac{1}{2N} \sum_{\gamma; \Lambda; 1} V_{\gamma; \Lambda; 1; \Delta} V_{\gamma; \Lambda; 1; \Delta} \quad (5)$$

$${}^2_{\Delta; \Delta} = \frac{i}{2N} \sum_{\gamma; \Lambda; 1} V_{\gamma; \Lambda; 1; \Delta} V_{\gamma; \Lambda; 1; \Delta} \quad (6)$$

Shifting to the reflectivity basis, we get,

$${}^0_{\Delta; \Delta} = \frac{1}{N} \sum_{\Lambda; 1} \left[(+) V_{\Lambda; 1; \Delta} (+) V_{\Lambda; 1; \Delta} + (-) V_{\Lambda; 1; \Delta} (-) V_{\Lambda; 1; \Delta} \right] \quad (7)$$

$${}^1_{\Delta; \Delta} = \frac{1}{N} \sum_{\Lambda; 1} (-1)^\Lambda \left[(+) V_{\Lambda; 1; \Delta} (+) V_{\Lambda; 1; \Delta} - (-) V_{\Lambda; 1; \Delta} (-) V_{\Lambda; 1; \Delta} \right] \quad (8)$$

$${}^2_{\Delta; \Delta} = \frac{i}{N} \sum_{\Lambda; 1} (-1)^\Lambda \left[(+) V_{\Lambda; 1; \Delta} (-) V_{\Lambda; 1; \Delta} - (-) V_{\Lambda; 1; \Delta} (+) V_{\Lambda; 1; \Delta} \right] \quad (9)$$

Theoretical model for photoproduction of $b_1\Delta$

- Only t -channel process involved.
- Upper and lower vertices factorize.
- Number of interaction terms in a vertex dependent on the number of L values involved.
- Simplified (preliminary) model involves only two exchanges: π and ρ .
- Model can be expanded to include any or every J^P state.

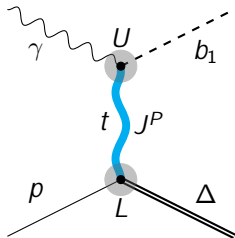


Figure: The t -channel photoproduction process of b_1 . U and L are the upper and lower vertices.

Vertex factors

- Lorentz, parity, and charge conjugation invariance.
- Minimal number of terms.
- The J^P exchanges must couple to these L -channels:
 $L = jJ - 2j, J, J + 2$ unnatural J^P , and $L = jJ - 1j, J + 1$ for natural J^P .
- Same L -values are valid for the $p - \Delta$ vertex.
- The general form of the upper vertex is,

$$hJ^P(M)j\gamma(\lambda) b_1^+(\Lambda)i = a_{\gamma;\Lambda}^J d_{M; \gamma \Lambda}^J(\omega). \quad (10)$$

- The general form of the lower vertex is,

$$hJ^P(M)jp(\lambda_1)\Delta(\lambda_\Delta)i = b_{\gamma;\Lambda}^J d_{M; 1 \Delta}^J(\omega). \quad (11)$$

- The full amplitude is given by,

$$V_{\gamma;\Lambda; 1; \Delta}^{(J)} = a_{\gamma;\Lambda}^J b_{\Delta; 1}^J d_{\gamma \Lambda; 1 \Delta}^J(\theta_t) P_J(t) \quad (12)$$

Vertex factors

- | | |
|--|---|
| <ol style="list-style-type: none"> 1. $g^{\mu_1\alpha} p_N^{\mu_2} \dots p_N^{\mu_J}$ 2. $\gamma_\alpha p_N^{\mu_1} \dots p_N^{\mu_J} \rightarrow 0$ (Rarita-Schwinger framework) 3. $p_\alpha^J p_N^{\mu_1} \dots p_N^{\mu_J}$ 4. $p_\alpha^J \gamma^{\mu_1} p_N^{\mu_2} \dots p_N^{\mu_J}$ 5. $\sigma^{\mu_1\mu_2} p_\alpha^J p_N^{\mu_3} \dots p_N^{\mu_J} \rightarrow 0$ (symmetry of the indices μ_1, μ_2) 6. $\sigma^{\mu_1\alpha} p_N^{\mu_2} \dots p_N^{\mu_J}$ same as Eq. 57 upto a factor of i. 7. $\sigma^{\mu_1\mu_2} p_N^\alpha p_N^{\mu_3} \dots p_N^{\mu_J} \rightarrow 0$ (symmetry of the indices μ_1, μ_2) 8. $g^{\mu_1\alpha} \gamma^{\mu_2} p_N^{\mu_3} \dots p_N^{\mu_J}$ | <ol style="list-style-type: none"> 1. $g^{\mu_1\alpha} \gamma_5 p_N^{\mu_2} \dots p_N^{\mu_J}$ 3. $p_\alpha^J \gamma_5 p_N^{\mu_1} \dots p_N^{\mu_J}$ 4. $p_\alpha^J \gamma^{\mu_1} \gamma_5 p_N^{\mu_2} \dots p_N^{\mu_J}$ 6. $\sigma^{\mu_1\alpha} \gamma_5 p_N^{\mu_2} \dots p_N^{\mu_J}$ 8. $g^{\mu_1\alpha} \gamma^{\mu_2} \gamma_5 p_N^{\mu_3} \dots p_N^{\mu_J}$ |
|--|---|

Figure: Vertex factors: Unnatural (left) and natural (right) exchanges

Some fun with bases

Reflectivity basis:

$$(-) V_{\Lambda; 1; \Delta}^{(J)} = \frac{1}{2} \left[\left(a_{+1; \Lambda}^J d_{1; \Lambda; 1; \Delta}^J(t) + (-1)^\Lambda a_{1; \Lambda}^J d_{1+ \Lambda; 1; \Delta}^J(t) \right) b_{1; \Delta}^J P_J(t) \right]; \quad (13)$$

For (un)natural exchange,

$$(-) V_{\Lambda; 1; \Delta}^{(J)} = \frac{1}{2} \left[d_{1; \Lambda; 1; \Delta}^J(t) + (-1)^\Lambda d_{1+ \Lambda; 1; \Delta}^J(t) \right] a_{+1; \Lambda}^J b_{1; \Delta}^J P_J(t); \quad (14)$$

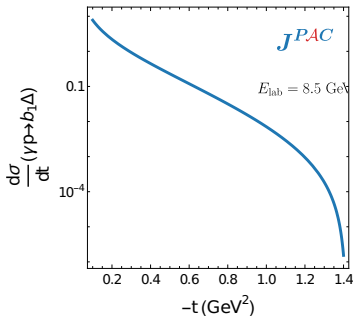
Further, making use of the $s! = 1$ property $d_{1; 2}^J = (-1) d_{1; 2}^J$, we get,

$$(-) V_{\Lambda; 1; \Delta}^{(J)} \stackrel{s! = 1}{=} \frac{1}{2} \left[d_{1; \Lambda; 1; \Delta}^J(\theta_t) a_{+1; \Lambda}^J b_{1; \Delta}^J P_J(t) \right]; \quad J \geq \text{Unnatural} \quad (15)$$

$$(+) V_{\Lambda; 1; \Delta}^{(J)} \stackrel{s! = 1}{=} \frac{1}{2} \left[d_{1; \Lambda; 1; \Delta}^J(\theta_t) a_{+1; \Lambda}^J b_{1; \Delta}^J P_J(t) \right]; \quad J \geq \text{Natural}. \quad (16)$$

Results

- Simple model: only π and ρ exchange
- BSA is approximately zero.



Results: Amplitudes

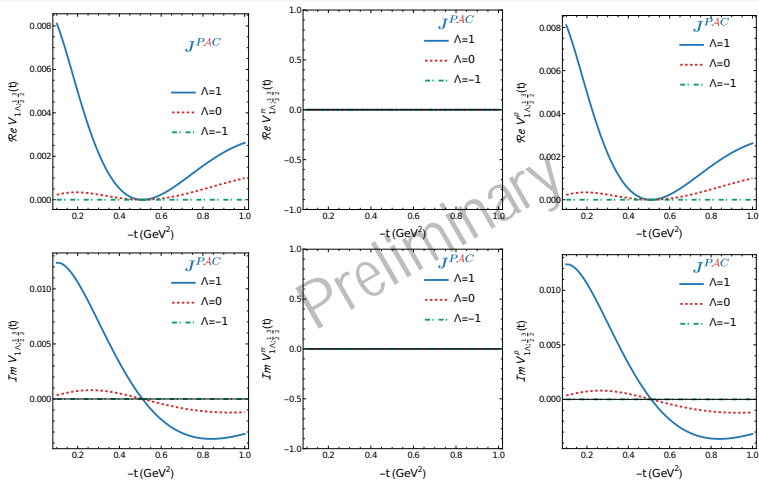


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Results: Amplitudes

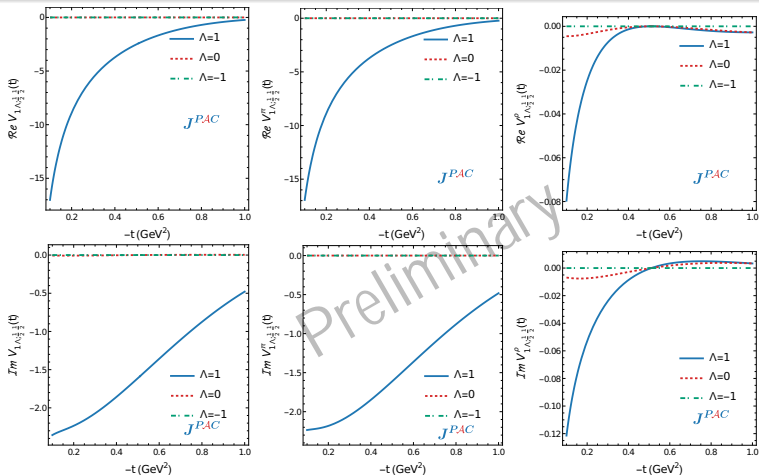


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Results: Amplitudes

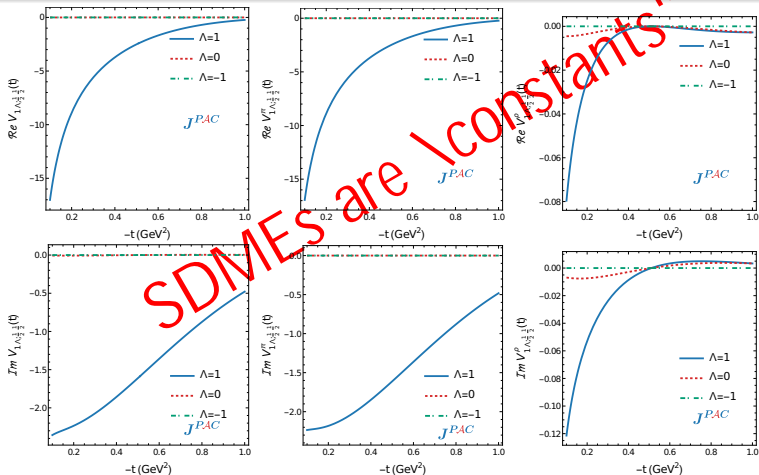


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Summing up. . .

- Preliminary model has pion dominance
- More sophistication **is** needed.
- Experimental inputs.

Work in progress...

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Some properties: cross section

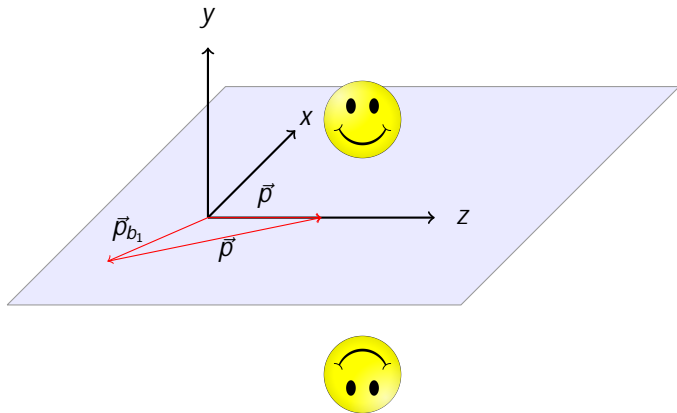
$$\begin{aligned}
 I = 2\kappa \sum_{\lambda_2} (1 - P_\gamma) & \left| \sum_{j, \Lambda, \lambda_\Delta} \tilde{F} D_{\lambda_\Delta, \lambda_2}^{\text{via} *}(\Omega_p) \left([J^P]_{\Lambda, \lambda_\Delta}^{(+)} U_\Lambda^j + [J^P]_{\Lambda, \lambda_\Delta}^{(-)} \tilde{U}_\Lambda^j \right) \right|^2 \\
 + (1 - P_\gamma) & \left| \sum_{j, \Lambda, \lambda_\Delta} \tilde{F} D_{\lambda_\Delta, \lambda_2}^{\text{via}}(\Omega_p) \left([J^P]_{\Lambda, \lambda_\Delta}^{(+)} U_\Lambda^j - [J^P]_{\Lambda, \lambda_\Delta}^{(-)} \tilde{U}_\Lambda^j \right) \right|^2 \\
 + (1 + P_\gamma) & \left| \sum_{j, \Lambda, \lambda_\Delta} \tilde{F} D_{\lambda_\Delta, \lambda_2}^{\text{via} *}(\Omega_p) \left([J^P]_{\Lambda, \lambda_\Delta}^{(+)} \tilde{U}_\Lambda^j + [J^P]_{\Lambda, \lambda_\Delta}^{(-)} U_\Lambda^j \right) \right|^2 \\
 + (1 + P_\gamma) & \left| \sum_{j, \Lambda, \lambda_\Delta} \tilde{F} D_{\lambda_\Delta, \lambda_2}^{\text{via}}(\Omega_p) \left([J^P]_{\Lambda, \lambda_\Delta}^{(+)} \tilde{U}_\Lambda^j - [J^P]_{\Lambda, \lambda_\Delta}^{(-)} U_\Lambda^j \right) \right|^2
 \end{aligned}$$

$$U_\Lambda^j = G \sum_{\lambda} F_\lambda^j \text{Re } Z_{\Lambda, \lambda}$$

$$\tilde{U}_\Lambda^j = iG \sum_{\lambda} F_\lambda^j \text{Im } Z_{\Lambda, \lambda}$$

$$[J^P]_{\Lambda, \lambda_\Delta}^{(\epsilon)} = {}^{(\epsilon)}V_{\Lambda; +, \lambda_\Delta}^j$$

Reflectivity operation



Reflectivity basis - any J^P

- Reflectivity operation involves 180° rotation about the “y-axis” + parity inversion) inversion of the “y-axis”.
- The amplitude in the reflectivity basis is defined as:

$$T_{1; \Lambda; \Delta}^{(\pm)}(s, t) = \frac{1}{2} (T_{1; \Lambda; \Delta}(s, t) + \epsilon P(-1)^{J+\Lambda} T_{1; \Lambda; \Delta}(s, t)) \quad (17)$$

- The SDMEs take the form,

$$\rho_{\Delta}^0 = \frac{1}{N} \sum_1 \left[T_{1; \Lambda; \Delta}^{(+)} T_{1; \Lambda; \Delta}^{(+)} + T_{1; \Lambda; \Delta}^{(-)} T_{1; \Lambda; \Delta}^{(-)} \right] \quad (18)$$

$$\rho_{\Delta}^1 = \frac{1}{N} \sum_1 (-1)^{\Lambda} \left[T_{1; \Lambda; \Delta}^{(+)} T_{1; \Lambda; \Delta}^{(+)} - T_{1; \Lambda; \Delta}^{(-)} T_{1; \Lambda; \Delta}^{(-)} \right] \quad (19)$$

- The $\epsilon = (\pm)$ amplitudes are dominated by (un)natural parity meson exchange.

Vertex factors: Upper vertex

The general form of the vertex is,

$$hJ^P(M)j\gamma(\lambda) b_1^+(\Lambda)i = a_{\gamma;\Lambda}^J d_{M;\gamma}^J(\omega) \quad (20)$$

The $b_1\gamma J^P$ vertex with natural J^P is,

$$hJ^P j\gamma b_1^+ i = i\varepsilon \left[g_1 p_J \epsilon_{b_1} (\epsilon_{p;2} p_{\epsilon;2}) + g_2 p_{\epsilon_{b_1}} \epsilon_{p;2} + g_3 p_J p_{\epsilon_{b_1}^2} \right] \epsilon_J^{2\cdots J} p_3 \cdots p_J. \quad (21)$$

We then move to the t -channel rest frame where $p_J = (\vec{p}_t, \vec{0})$ to get,

$$hJ^P j b_1^+ i = g_1 \left\{ \begin{aligned} & (1)^{\Lambda P_t} \left(\frac{E_{b_1}^t}{m_{b_1}} \right)^{\Lambda,0} (p)^J {}_1 C_{J-1}^J \gamma \Lambda_{\Lambda;J} 10 \\ & (p)^J {}_1 P_t C_{J-2}^J \gamma \Lambda_{\Lambda;J} 1 \gamma C_{J-1}^J \gamma \Lambda_{\Lambda;J} 20 \end{aligned} \right\} \\ + g_2 (p)^J \left(\frac{p_t}{m_{b_1}} \right)^{\Lambda,0} C_{J-1}^J \gamma \Lambda_{\Lambda;J} 10 \\ + g_3 (p)^J {}_1 P_t C_{J-2}^J \gamma \Lambda_{\Lambda;J} 1 \gamma C_{J-1}^J \gamma \Lambda_{\Lambda;J} 20 \Big] d_{M;\gamma}^J(\omega) \quad (22)$$

Vertex factors: Upper vertex

The same for unnatural J^P is,

$$\begin{aligned}
 h^{J^P} b_1^+ i = i & \left[g_1(p \quad p \quad) b_1 p + g_2 \bar{p} \quad p \quad g p_J \quad b_1 \right] J \quad 3 \cdots J p \quad 3 \cdots p \quad J \\
 & + i [g_3(p \quad p \quad) b_1 p_J p \quad 1 + g_4(p \quad 1 \quad p \quad 1) b_1 p_J p \\
 & + g_5(p \quad 1 \quad p \quad 1) b_1 p_J p] J^{1 \cdots J} p \quad 2 \cdots p \quad J \quad (23)
 \end{aligned}$$

which gives us,

$$\begin{aligned}
 = (p)^J \quad 2 C_J \quad 2 \lambda^2 & \left[g_1 p^2 \frac{C_J \quad 1}{C_J \quad 2} C_J^J \quad \gamma \quad 10;1 \quad \gamma \quad \delta_{\Lambda,0} + g_3 p^2 \frac{C_J \quad 1}{C_J \quad 2} \sqrt{\frac{J}{2J \quad 1}} \delta_{\Lambda; \quad \gamma} \right. \\
 & \left. + g_2 \left(\frac{E_{b_1}}{2m_{b_1}} \right)^{\Lambda,0} E \quad C_J^J \left(\begin{matrix} \gamma & \Lambda \\ 1 & \Lambda \end{matrix} \right); 1 \quad \gamma \quad C_J^J \left(\begin{matrix} 1 & \Lambda \\ 20;1 & \Lambda \end{matrix} \right) \right] d_{M; \quad \gamma \quad \Lambda}^J(\omega). \quad (24)
 \end{aligned}$$

Vertex factors: Lower vertex

For the unnatural J^P ,

$$hJ^P j p \quad i = b_{\Delta; \bar{N}} d_{M; \Delta}^J (!) \quad (25)$$

$$\begin{aligned}
 &= (p)^J_{\Delta} \left(\frac{g_1}{p} C_{J-1}(\Delta) C_J^{J(\Delta, \bar{N})}_{10;1(\Delta, \bar{N})} C_{\frac{3}{2}\Delta}^{\frac{3}{2}\Delta}_{\bar{N};1(\Delta, \bar{N})} \left(\frac{E_{\Delta}}{m_{\Delta}} \right)^{\lambda_{\Delta}, \lambda_{\bar{N}}} \right. \\
 &+ g_2 C_J \frac{m_J}{m_{\Delta}} p \sqrt{\frac{2}{3}} \frac{1}{2; \frac{1}{4}}(\Delta) (\text{sgn}(\Delta) \text{sgn}(\bar{N}) \Delta)_{\Delta; \bar{N}} \\
 &+ g_3 C_{J-1} \frac{m_J}{m_{\Delta}} p^{\frac{1}{2}} \frac{1}{2; \frac{1}{4}}(\text{sgn}(\Delta) \Delta \text{sgn}(\bar{N})) C_{\frac{1}{2}\Delta}^{\frac{1}{2}\Delta}_{\bar{N};1(\Delta, \bar{N})} C_J^{J(\Delta, \bar{N})}_{10;1(\Delta, \bar{N})} \\
 &+ \frac{g_4}{p^2} C_{J-2} p^{\frac{1}{2}} \frac{1}{3} \sum_2 \left[C_J^{J(\Delta, \bar{N})}_{1-2;1(\Delta, \bar{N}-2)} C_J^{J-1-2}_{20;1-2} C_{\frac{3}{2}(\Delta-2);1-2}^{\frac{3}{2}\Delta} C_{\frac{1}{2}\bar{N};1(\Delta-2, \bar{N})}^{\frac{1}{2}(\Delta-2)} \right. \\
 &\left. (\text{sgn}(\bar{N}) \Delta \text{sgn}(\Delta-2)) \left(\frac{E_{\Delta}}{m_{\Delta}} \right)^{\lambda_{2,0}} \right] \left. \right) d_{M; \bar{N} \Delta}^J (!): \quad (26)
 \end{aligned}$$

Vertex factors: Lower vertex

For natural J^P ,

$$\begin{aligned}
 hJ^P j p \Delta i = & (p)^J \Delta \left(\frac{g_1}{p} C_{J-1}(\text{sgn}(\bar{N}) + \text{sgn}(\Delta)) C_J^{J(\Delta, \bar{N})} \right. \\
 & C_{\frac{1}{2} \Delta}^{\frac{3}{2} \Delta} \left(\frac{E_\Delta}{m_\Delta} \right)^{\lambda_\Delta, \lambda_{\bar{N}}} + g_2 C_J \frac{m_J}{m_\Delta} p \sqrt{\frac{2}{3}} \left. C_{\Delta, \frac{1}{4}}^{\frac{2}{\Delta}, \frac{1}{4}}(\text{sgn}(\bar{N}) + \text{sgn}(\Delta)) \right. \\
 & + g_3 C_{J-1} \frac{m_J}{m_\Delta} p^{\frac{1}{2}} C_{\Delta, \frac{1}{4}}^{\frac{2}{\Delta}, \frac{1}{4}}(\text{sgn}(\Delta) \Delta \text{sgn}(\bar{N})) C_{\frac{1}{2} \Delta}^{\frac{1}{2} \Delta} \left. C_J^{J(\Delta, \bar{N})} \right. \\
 & + \frac{g_4}{p^2} C_{J-2} \frac{p}{3} \sum_2 \left[C_{J-1, 2; 1}^{J(\Delta, \bar{N})} C_{J-2, 0; 1}^J C_{\frac{1}{2}(\Delta, 2); 1}^{\frac{3}{2} \Delta} C_{\frac{1}{2} \Delta}^{\frac{1}{2}(\Delta, 2)} \right. \\
 & \left. (\text{sgn}(\bar{N}) \Delta \text{sgn}(\Delta, 2)) \left(\frac{E_\Delta}{m_\Delta} \right)^{\lambda_{2,0}} \right] d_{M; \Delta, \bar{N}}^J (!) \quad (27)
 \end{aligned}$$