Photoproduction of $b_1 \Delta$

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Contents

Introduction

- QCD demands that the physically observable states must be color singlets.
  - Mesons ($\bar{q}q$), and baryons (qqq) are the most common combinations.
- Are other combinations allowed? Yes!
  - Hybrids ($\bar{q}qg$), tetra/pentaquarks, glueballs, molecular states, etc.
- Quark models, effective QFT models, sum rules, BSE/DSE, etc.
- Quark model says one hybrid nonet ($m \lesssim 2$ GeV), Lattice shows one hybrid isovector.
- Experimental evidence?
  - Two known states: $\pi_1(1600)$ (known since ~ 30 years) and the recently observed $\eta_1(1855)$ (BESIII).

Where are the hybrids?

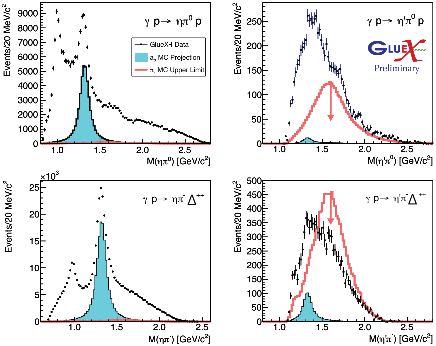
- The $\pi_1(1600)$:
  - First discovered light hybrid state.
  - Originally, two light resonances: $\pi_1(1600)$ and the lighter $\pi_1(1400)$
  - Same quantum numbers: $J^{PC} = 1^{-+}$.
  - Complimentary decay channels.
  - JPAC analysis of COMPASS data shows only one pole [1].
  - Lattice simulations by HadSpec shows various decay channels [2].
  - Possible final state interactions
- The $\eta_1(1855)$:
  - Recently seen by BESIII in the $\eta\eta'$ channel; $J^{PC} = 1^{-+}$.
  - Mass ($1855 \pm 9^{+6}_{-1}$ MeV) and total width ($188 \pm 18^{+3}_{-8}$) known; partial width unknown [3].
  - Nature of the state is up for interpretation $\rightarrow$ various models (for example, [4] and refs within).
- Where are the remaining hybrids and other exotics?

GlueX Experiment

- Photoproduction of mesons up to ~ 3 GeV.
- Linearly polarised photon beam:
- $E_\gamma \sim 8.2 - 8.8$ GeV, $P_\gamma \sim 0.35$.
- Probe the production mechanisms of the mesons via spin density matrix elements (SDMEs).
  - Photoproduction of $\pi\Delta$, $b_1\Delta$, $\pi_1\Delta$, $a_2\Delta$, etc
  - Complexities: $b_1\Delta \rightarrow (\omega\pi) + (p\pi)$, $\pi_1\Delta \rightarrow (\omega\pi\pi) + (p\pi)$, etc.
  - Complexities: Line shapes, decays, spin, etc.
- Why use $\pi_1\Delta$?

Why $b_1\Delta$?

- $\pi_1\Delta$ has a larger upper limit for the xsection than $\pi_1 p$ [5].
- $\pi_1 \to b_1\pi$ is the dominant decay channel [2, 4]



Plot taken from [5].

- We start with $\pi\Delta$ (simplest process involving Δ).
- $b_1\Delta$ is the next level of complexity + important background for $\pi_1 p$.

The $b_1\Delta$ photoproduction

- Intensity, beam spin asymmetry (BSA), SDMEs.
- The process: $\gamma p \to b_1\Delta \to (\omega\pi)(\pi p) \to ((3\pi)\pi)(\pi p)$.
    - Decays at each stage model using line shapes and Wigner-D's.

$$A_{\lambda_\gamma,\lambda_1,\lambda_2} = \sum_{\Lambda=-1}^{1} \sum_{\lambda_\Delta=-\frac{3}{2}}^{\frac{3}{2}} V_{\lambda_\gamma,\Lambda;\lambda_1,\lambda_\Delta}(s,t) \sum_{\lambda=-1}^{1} F_\lambda D^{J*}_{\Lambda,\lambda}(\Omega_\omega) Y^1_\lambda(\Omega_H) G \tilde{F}_{\lambda_2} D^{\frac{3}{2}*}_{\lambda_\Delta,\lambda_2}(\Omega_p) \tag{1}$$

λ's are the helicities of the various states, Ω's are the decay angles, F, G are the line shapes.

Some properties and relations: cross section

The intensity is proportional to

$$I(\Omega_\omega, \Omega_p, \Phi) = \kappa \sum_{\lambda_\gamma^{(\prime)}, \lambda_1, \lambda_2} A_{\lambda_\gamma, \lambda_1, \lambda_2} \hat{\rho}_{\lambda_\gamma, \lambda'_\gamma}(\Phi) A^*_{\lambda'_\gamma, \lambda_1, \lambda_2}$$
$$= I^0(\Omega_\omega, \Omega_p) + \boldsymbol{I}(\Omega_\omega, \Omega_p) \cdot \boldsymbol{P}(\Phi) \quad (2)$$

giving us the intensity for $\gamma p \to b_1 \Delta$ as

$$I^\alpha = \kappa \left(\frac{8\pi^3}{3}\right) \sum_{\lambda_\gamma, \lambda'_\gamma} \sum_{\lambda_\Delta, \Lambda, \lambda_1, \lambda} V_{\lambda_\gamma, \Lambda; \lambda_1, \lambda_\Delta}(s,t) \sigma^\alpha_{\lambda_\gamma, \lambda'_\gamma} V^*_{\lambda'_\gamma, \Lambda'; \lambda_1, \lambda_\Delta}(s,t) \quad (3)$$

Some properties: SDMEs

The Δ-SDMEs are defined as,

$$\rho^0_{\lambda_\Delta,\lambda'_\Delta} = \frac{1}{2N} \sum_{\lambda_\gamma,\Lambda,\lambda_1} V_{\lambda_\gamma,\Lambda;\lambda_1,\lambda_\Delta} V^*_{\lambda_\gamma,\Lambda;\lambda_1,\lambda'_\Delta} \quad (4)$$

$$\rho^1_{\lambda_\Delta,\lambda'_\Delta} = \frac{1}{2N} \sum_{\lambda_\gamma,\Lambda,\lambda_1} V_{-\lambda_\gamma,\Lambda;\lambda_1,\lambda_\Delta} V^*_{\lambda_\gamma,\Lambda;\lambda_1,\lambda'_\Delta} \quad (5)$$

$$\rho^2_{\lambda_\Delta,\lambda'_\Delta} = \frac{i}{2N} \sum_{\lambda_\gamma,\Lambda,\lambda_1} \lambda_\gamma V_{-\lambda_\gamma,\Lambda;\lambda_1,\lambda_\Delta} V^*_{\lambda_\gamma,\Lambda;\lambda_1,\lambda'_\Delta}. \quad (6)$$

Shifting to the reflectivity basis, we get,

$$\rho^0_{\lambda_\Delta,\lambda'_\Delta} = \frac{1}{N} \sum_{\Lambda,\lambda_1} \left[{}^{(+)}V_{\Lambda;\lambda_1,\lambda_\Delta} {}^{(+)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} + {}^{(-)}V_{\Lambda;\lambda_1,\lambda_\Delta} {}^{(-)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} \right] \quad (7)$$

$$\rho^1_{\lambda_\Delta,\lambda'_\Delta} = \frac{1}{N} \sum_{\Lambda,\lambda_1} (-1)^\Lambda \left[{}^{(+)}V_{-\Lambda;\lambda_1,\lambda_\Delta} {}^{(+)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} - {}^{(-)}V_{-\Lambda;\lambda_1,\lambda_\Delta} {}^{(-)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} \right] \quad (8)$$

$$\rho^2_{\lambda_\Delta,\lambda'_\Delta} = \frac{i}{N} \sum_{\Lambda,\lambda_1} (-1)^\Lambda \left[{}^{(+)}V_{-\Lambda;\lambda_1,\lambda_\Delta} {}^{(-)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} - {}^{(-)}V_{-\Lambda;\lambda_1,\lambda_\Delta} {}^{(+)}V^*_{\Lambda;\lambda_1,\lambda'_\Delta} \right] \quad (9)$$

Theoretical model for photoproduction of $b_1\Delta$

- Only t-channel process involved.
- Upper and lower vertices factorize.
- Number of interaction terms in a vertex dependent on the number of L values involved.
- Simplified (preliminary) model involves only two exchanges: π and ρ.
- Model can be expanded to include any or every J^P state.

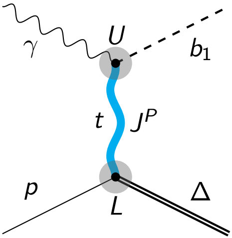


Figure: The t-channel photoproduction process of $b_1\Delta$. U and L are the upper and lower vertices.

Vertex factors

- Lorentz, parity, and charge conjugation invariance.
- Minimal number of terms.
- The J^P exchanges must couple to these L-channels: $L = |J-2|,\ J,\ J+2$ unnatural J^P, and $L = |J-1|,\ J+1$ for natural J^P.
- Same L-values are valid for the $p - \Delta$ vertex.
- The general form of the upper vertex is,

$$\langle J^P(M)|\gamma(\lambda_\gamma) b_1^+(\Lambda)\rangle = a^J_{\lambda_\gamma,\Lambda} d^J_{M,\lambda_\gamma-\Lambda}(\omega). \tag{10}$$

- The general form of the lower vertex is,

$$\langle J^P(M)|p(\lambda_1)\Delta(\lambda_\Delta)\rangle = b^J_{\lambda_\gamma,\Lambda} d^J_{M,\lambda_1-\lambda_\Delta}(\omega). \tag{11}$$

- The full amplitude is given by,

$$V^{(J)}_{\lambda_\gamma,\Lambda;\lambda_1,\lambda_\Delta} = a^J_{\lambda_\gamma,\Lambda} b^J_{\lambda_\Delta,\lambda_1} d^J_{\lambda_\gamma-\Lambda,\lambda_1-\lambda_\Delta}(\theta_t)\mathcal{P}_J(t) \tag{12}$$

Vertex factors

1. $g^{\mu_1\alpha}p_N^{\mu_2}\dots p_N^{\mu_J}$
2. $\gamma_\alpha p_N^{\mu_1}\dots p_N^{\mu_J} \to 0$ (Rarita-Schwinger framework)
3. $p_\alpha^J p_N^{\mu_1}\dots p_N^{\mu_J}$
4. $p_\alpha^J \gamma^{\mu_1} p_N^{\mu_2}\dots p_N^{\mu_J}$
5. $\sigma^{\mu_1\mu_2} p_\alpha^J p_N^{\mu_3}\dots p_N^{\mu_J} \to 0$ (symmetry of the indices μ_1, μ_2)
6. $\sigma^{\mu_1\alpha} p_N^{\mu_2}\dots p_N^{\mu_J}$ same as Eq. 57 upto a factor of i.
7. $\sigma^{\mu_1\mu_2} p_N^{\alpha} p_N^{\mu_3}\dots p_N^{\mu_J} \to 0$ (symmetry of the indices μ_1, μ_2)
8. $g^{\mu_1\alpha}\gamma^{\mu_2} p_N^{\mu_3}\dots p_N^{\mu_J}$

1. $g^{\mu_1\alpha}\gamma_5 p_N^{\mu_2}\dots p_N^{\mu_J}$
3. $p_\alpha^J \gamma_5 p_N^{\mu_1}\dots p_N^{\mu_J}$
4. $p_\alpha^J \gamma^{\mu_1}\gamma_5 p_N^{\mu_2}\dots p_N^{\mu_J}$
6. $\sigma^{\mu_1\alpha}\gamma_5 p_N^{\mu_2}\dots p_N^{\mu_J}$
8. $g^{\mu_1\alpha}\gamma^{\mu_2}\gamma_5 p_N^{\mu_3}\dots p_N^{\mu_J}$

Figure: Vertex factors: Unnatural (left) and natural (right) exchanges

Some fun with bases

Reflectivity basis:

$$^{(\epsilon)}V^{(J)}_{\Lambda;\lambda_1,\lambda_\Delta} = \frac{1}{2}\left[\left(a^J_{+1,\Lambda} d^J_{1-\Lambda,\lambda_1-\lambda_\Delta}(\theta_t) + \epsilon(-1)^\Lambda a^J_{-1,-\Lambda} d^J_{-1+\Lambda,\lambda_1-\lambda_\Delta}(\theta_t)\right) b^J_{\lambda_1,\lambda_\Delta} \mathcal{P}_J(t)\right]. \tag{13}$$

For (un)natural exchange,

$$^{(\epsilon)}V^{(J)}_{\Lambda;\lambda_1,\lambda_\Delta} = \frac{1}{2}\left[d^J_{1-\Lambda,\lambda_1-\lambda_\Delta}(\theta_t) \mp \epsilon(-1)^\Lambda d^J_{-1+\Lambda,\lambda_1-\lambda_\Delta}(\theta_t)\right] a^J_{+1,\Lambda} b^J_{\lambda_1,\lambda_\Delta} \mathcal{P}_J(t). \tag{14}$$

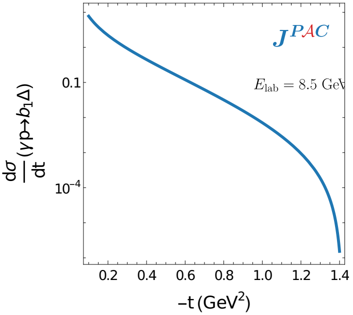
Further, making use of the $s \to \infty$ property $d^J_{-\mu_1,\mu_2} \simeq (-1)^{\mu_1} d^J_{\mu_1,\mu_2}$, we get,

$$^{(-)}V^{(J)}_{\Lambda;\lambda_1,\lambda_\Delta} \overset{s\to\infty}{=\!=} \frac{1}{2}\left[d^J_{1-\Lambda,\lambda_1-\lambda_\Delta}(\theta_t) a^J_{+1,\Lambda} b^J_{\lambda_1,\lambda_\Delta} \mathcal{P}_J(t)\right];\ J \in \text{Unnatural} \tag{15}$$

$$^{(+)}V^{(J)}_{\Lambda;\lambda_1,\lambda_\Delta} \overset{s\to\infty}{=\!=} \frac{1}{2}\left[d^J_{1-\Lambda,\lambda_1-\lambda_\Delta}(\theta_t) a^J_{+1,\Lambda} b^J_{\lambda_1,\lambda_\Delta} \mathcal{P}_J(t)\right];\ \ J \in \text{Natural}. \tag{16}$$

Results

- Simple model: only π and ρ exchange
- BSA is approximately zero.



Results: Amplitudes

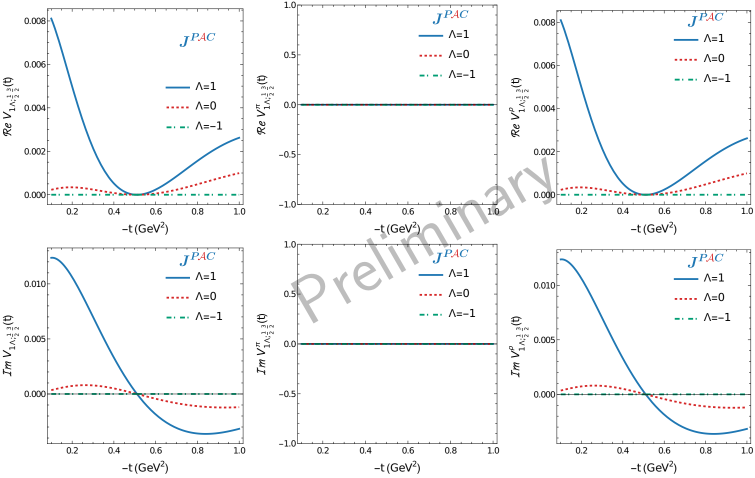


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Results: Amplitudes

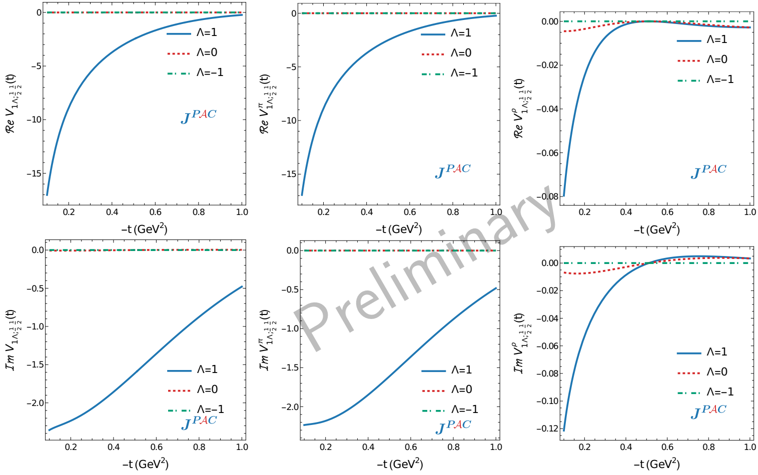


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Results: Amplitudes

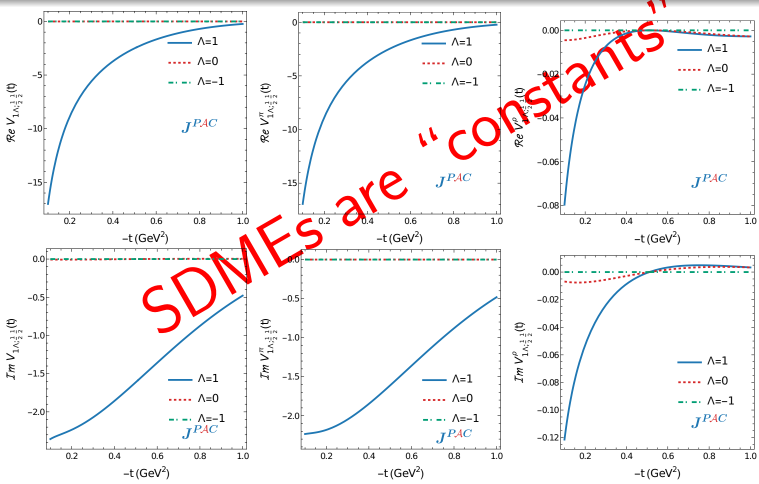


Figure: Amplitudes: real (upper row) and imaginary parts (lower row).

Summing up. . .

- Preliminary model has pion dominance
- More sophistication **is** needed.
- Experimental inputs.

Work in progress...

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Some properties: cross section

$$I = 2\kappa \sum_{\lambda_2} (1 - P_\gamma) \left| \sum_{j,\Lambda,\lambda_\Delta} \tilde{F} D^{\frac{3}{2}*}_{\lambda_\Delta,\lambda_2}(\Omega_p) \left([J^P]^{(+)}_{\Lambda,\lambda_\Delta} U^j_\Lambda + [J^P]^{(-)}_{\Lambda,\lambda_\Delta} \tilde{U}^j_\Lambda \right) \right|^2$$
$$+ (1 - P_\gamma) \left| \sum_{j,\Lambda,\lambda_\Delta} \tilde{F} D^{\frac{3}{2}}_{\lambda_\Delta,\lambda_2}(\Omega_p) \left([J^P]^{(+)}_{\Lambda,\lambda_\Delta} U^j_\Lambda - [J^P]^{(-)}_{\Lambda,\lambda_\Delta} \tilde{U}^j_\Lambda \right) \right|^2$$
$$+ (1 + P_\gamma) \left| \sum_{j,\Lambda,\lambda_\Delta} \tilde{F} D^{\frac{3}{2}*}_{\lambda_\Delta,\lambda_2}(\Omega_p) \left([J^P]^{(+)}_{\Lambda,\lambda_\Delta} \tilde{U}^j_\Lambda + [J^P]^{(-)}_{\Lambda,\lambda_\Delta} U^j_\Lambda \right) \right|^2$$
$$+ (1 + P_\gamma) \left| \sum_{j,\Lambda,\lambda_\Delta} \tilde{F} D^{\frac{3}{2}}_{\lambda_\Delta,\lambda_2}(\Omega_p) \left([J^P]^{(+)}_{\Lambda,\lambda_\Delta} \tilde{U}^j_\Lambda - [J^P]^{(-)}_{\Lambda,\lambda_\Delta} U^j_\Lambda \right) \right|^2$$

$$U^j_\Lambda = G \sum_\lambda F^j_\lambda \,\mathrm{Re}\, Z_{\Lambda,\lambda} \qquad \tilde{U}^j_\Lambda = iG \sum_\lambda F^j_\lambda \,\mathrm{Im}\, Z_{\Lambda,\lambda} \qquad [J^P]^{(\epsilon)}_{\Lambda,\lambda_\Delta} = {}^{(\epsilon)}V^j_{\Lambda;+,\lambda_\Delta}$$

Ambiguities in the cross section

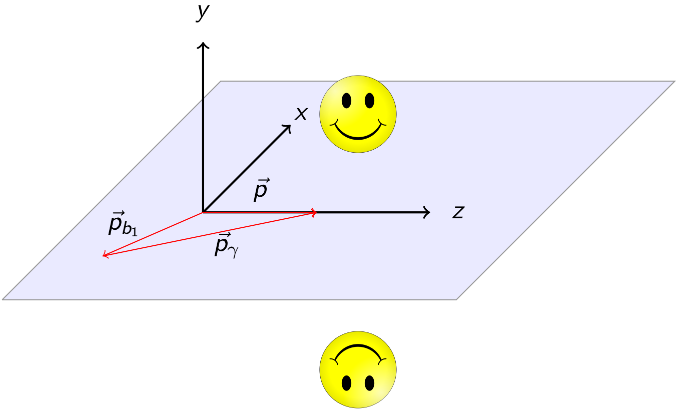
The full intensity is,

$$I = 2\kappa|\tilde{F}|^2\frac{1}{4}(5+3\cos(2\theta_p))\left[\left\{\sum_{\lambda_\Delta=\pm\frac{1}{2}}\left|[1^+]^{(+)}_{1,\lambda_\Delta}\right|^2\right\}\left((1-P_\gamma)|U^1_1|^2+(1+P_\gamma)|\tilde{U}^1_1|^2\right)\right.$$

$$\left.+\left\{\sum_{\lambda_\Delta=\pm\frac{1}{2}}\left|[1^+]^{(-)}_{1,\lambda_\Delta}\right|^2\right\}\left((1+P_\gamma)|U^1_1|^2+(1-P_\gamma)|\tilde{U}^1_1|^2\right)\right]$$

for a restricted case of $\epsilon = \pm,\ \Lambda = 1,\ \lambda_\Delta = \pm\frac{1}{2}$.

Reflectivity operation



Reflectivity basis - any J^P

- Reflectivity operation involves 180° rotation about the "y-axis" + parity inversion ⇒ inversion of the "y-axis".
- The amplitude in the reflectivity basis is defined as:

$$T^{(\epsilon)}_{\lambda_1,\Lambda,\lambda_\Delta}(s,t) = \frac{1}{2}\left(T_{1,\lambda_1,\Lambda,\lambda_\Delta}(s,t) + \epsilon P(-1)^{J+\Lambda} T_{-1,\lambda_1,\Lambda,\lambda_\Delta}(s,t)\right) \tag{17}$$

- The SDMEs take the form,

$$\rho^0_{\lambda_\Delta\lambda'_\Delta} = \frac{1}{N}\sum_{\lambda_1}\left[T^{(+)}_{\lambda_1\Lambda\lambda_\Delta} T^{(+)^*}_{\lambda_1,\Lambda,\lambda'_\Delta} + T^{(-)}_{\lambda_1,\Lambda,\lambda_\Delta} T^{(-)*}_{\lambda_1,\Lambda,\lambda'_\Delta}\right] \tag{18}$$

$$\rho^1_{\lambda_\Delta\lambda'_\Delta} = \frac{1}{N}\sum_{\lambda_1}(-1)^\Lambda\left[T^{(+)}_{\lambda_1,\Lambda,\lambda_\Delta} T^{(+)^*}_{\lambda_1,\Lambda,\lambda'_\Delta} - T^{(-)}_{\lambda_1,\Lambda,\lambda_\Delta} T^{(-)*}_{\lambda_1,\Lambda,\lambda'_\Delta}\right] \tag{19}$$

- The $\epsilon = (-)+$ amplitudes are dominated by (un)natural parity meson exchange.

Vertex factors: Upper vertex

The general form of the vertex is,

$$\langle J^P(M)|\gamma(\lambda_\gamma)b_1^+(\Lambda)\rangle = a^J_{\lambda_\gamma,\Lambda} d^J_{M,\lambda_\gamma-\Lambda}(\omega) \tag{20}$$

The $b_1\gamma J^P$ vertex with natural J^P is,

$$\langle J^P|\gamma b_1^+\rangle = i\varepsilon_{\mu\nu\alpha\beta}\left[g_1 p_J^\mu \epsilon_{b_1}^{\alpha*}\left(\epsilon_\gamma^{\beta*} p_{\gamma,\nu_2} - p_\gamma^\beta \epsilon^*_{\gamma,\nu_2}\right) + g_2 p_\gamma^\mu \epsilon_{b_1}^{\alpha*}\epsilon_\gamma^{\beta*} p_{\gamma,\nu_2} + g_3 p_J^\mu p_\gamma^\alpha \epsilon_\gamma^{\beta*}\epsilon_{b_1}^{\nu_2*}\right]\epsilon_J^{\nu\nu_2\ldots\nu_J} p^\gamma_{\nu_3}\ldots p^\gamma_{\nu_J}. \tag{21}$$

We then move to the t-channel rest frame where $p_J = (\sqrt{t},\vec{0})$ to get,

$$\begin{aligned}\langle J^P|\gamma b_1^+\rangle = g_1\Bigg[\Bigg\{&\lambda_\gamma(-1)^\Lambda\sqrt{t}\left(\frac{E^t_{b_1}}{m_{b_1}}\right)^{\delta_{\Lambda,0}}(-p)^{J-1}c_{J-1}C^{J\lambda_\gamma-\Lambda}_{1\lambda_\gamma-\Lambda;J-10} \\ &-(-p)^{J-1}\sqrt{t}\Lambda c_{J-2}C^{J\lambda_\gamma-\Lambda}_{1-\Lambda;J-1\lambda_\gamma}C^{J-1\lambda_\gamma}_{1\lambda_\gamma;J-20}\Bigg\} \\ &+g_2\lambda_\gamma(-p)^J\left(-\frac{\sqrt{t}}{m_{b_1}}\right)^{\delta_{\Lambda,0}}c_{J-1}C^{J\lambda_\gamma-\Lambda}_{1\lambda_\gamma-\Lambda;J-10} \\ &+g_3(-p)^{J-1}\sqrt{t}\lambda_\gamma c_{J-2}C^{J\lambda_\gamma-\Lambda}_{1\lambda_\gamma;J-1-\Lambda}C^{J-1-\Lambda}_{1-\Lambda;J-20}\Bigg]d^J_{M,\lambda_\gamma-\Lambda}(\omega)\end{aligned} \tag{22}$$

Vertex factors: Upper vertex

The same for unnatural J^P is,

$$\langle J^P|\gamma b_1^+\rangle = i\left[g_1(p^\gamma_\lambda\epsilon^{\gamma *}_\mu - p^\gamma_\mu\epsilon^{\gamma *}_\lambda)\epsilon^{*\lambda}_{b_1}p^\gamma_\nu + g_2\{p^\gamma_\lambda\epsilon^{*\gamma}_\mu - p^\gamma_\mu\epsilon^{*\gamma}_\lambda\}p^\lambda_J\epsilon^{*b_1}_\nu\right]\epsilon_J^{\mu\nu\mu_3\ldots\mu_J}p^\gamma_{\mu_3}\ldots p^\gamma_{\mu_J}$$
$$+ i[g_3(p^\gamma_\lambda\epsilon^{\gamma *}_\nu - p^\gamma_\nu\epsilon^{\gamma *}_\lambda)\epsilon^{*\lambda}_{b_1}p^\nu_J p^\gamma_{\mu_1} + g_4(p^\gamma_{\mu_1}\epsilon^{\gamma *}_\nu - p^\gamma_\nu\epsilon^{\gamma *}_{\mu_1})\epsilon^{*\nu}_{b_1}p^\mu_J p^\gamma_\mu$$
$$+ g_5(p^\gamma_{\mu_1}\epsilon^{\gamma *}_\mu - p^\gamma_\mu\epsilon^{\gamma *}_{\mu_1})\epsilon^{*\alpha}_{b_1}p^\mu_J p^\gamma_\alpha]\epsilon_J^{\mu_1\ldots\mu_J}p^\gamma_{\mu_2}\ldots p^\gamma_{\mu_J} \tag{23}$$

which gives us,

$$= (-p)^{J-2}c_{J-2}\lambda_\gamma^2\left[g_1p^2\frac{c_{J-1}}{c_{J-2}}C^{J\lambda_\gamma}_{J-10;1\lambda_\gamma}\delta_{\Lambda,0} + g_3p^2\frac{c_{J-1}}{c_{J-2}}\sqrt{\frac{J}{2J-1}}\delta_{\Lambda,\lambda_\gamma}\right.$$
$$\left.+g_2\left(\frac{E_{b_1}}{\sqrt{2}m_{b_1}}\right)^{\delta_{\Lambda,0}}E_\gamma C^{J(\lambda_\gamma-\Lambda)}_{J-1(-\Lambda);1\lambda_\gamma}C^{J-1(-\Lambda)}_{J-20;1(-\Lambda)}\right]d^J_{M,\lambda_\gamma-\Lambda}(\omega). \tag{24}$$

Vertex factors: Lower vertex

For the unnatural J^P,

$$\langle J^P|p\Delta\rangle = b_{\lambda_\Delta,\lambda_{\bar{N}}} d^J_{M,\lambda_\Delta-\lambda_{\bar{N}}}(\omega) \tag{25}$$

$$= (-p)^J \beta\beta_\Delta \left(\frac{g_1}{p} c_{J-1}(\alpha+\alpha_\Delta) C^{J(\lambda_\Delta-\lambda_{\bar{N}})}_{J-10;1(\lambda_\Delta-\lambda_{\bar{N}})} C^{\frac{3}{2}\lambda_\Delta}_{\frac{1}{2}\lambda_{\bar{N}};1(\lambda_\Delta-\lambda_{\bar{N}})} \left(\frac{E_\Delta}{m_\Delta}\right)^{\delta_{\lambda_\Delta,\lambda_{\bar{N}}}} \right.$$

$$+ g_2\ c_J \frac{m_J}{m_\Delta} p \sqrt{\frac{2}{3}} \delta_{\lambda^2_\Delta,\frac{1}{4}} (\alpha + \text{sgn}(\lambda_\Delta)\text{sgn}(\lambda_{\bar{N}})\alpha_\Delta) \delta_{\lambda_\Delta,\lambda_{\bar{N}}}$$

$$+ g_3\ c_{J-1} \frac{m_J}{m_\Delta} \sqrt{2} \delta_{\lambda^2_\Delta,\frac{1}{4}} (\text{sgn}(\lambda_\Delta)\alpha_\Delta\alpha - \text{sgn}(\lambda_{\bar{N}})) C^{\frac{1}{2}\lambda_\Delta}_{\frac{1}{2}\lambda_{\bar{N}};1(\lambda_\Delta-\lambda_{\bar{N}}))} C^{J(\lambda_\Delta-\lambda_{\bar{N}})}_{J-10;1(\lambda_\Delta-\lambda_{\bar{N}})}$$

$$+ \frac{g_4}{p^2} c_{J-2} \sqrt{3} \sum_{\lambda_2} \left[C^{J(\lambda_\Delta-\lambda_{\bar{N}})}_{J-1\lambda_2;1(\lambda_\Delta-\lambda_{\bar{N}}-\lambda_2))} C^{J-1\lambda_2}_{J-20;1\lambda_2} C^{\frac{3}{2}\lambda_\Delta}_{\frac{1}{2}(\lambda_\Delta-\lambda_2);1\lambda_2} C^{\frac{1}{2}(\lambda_\Delta-\lambda_2)}_{\frac{1}{2}\lambda_{\bar{N}};1(\lambda_\Delta-\lambda_2-\lambda_{\bar{N}})} \right.$$

$$\left.\left. (\text{sgn}(\lambda_{\bar{N}}) - \alpha\alpha_\Delta \text{sgn}(\lambda_\Delta - \lambda_2)) \left(\frac{E_\Delta}{m_\Delta}\right)^{\delta_{\lambda_2,0}} \right] \right) d^J_{M,\lambda_{\bar{N}}-\lambda_\Delta}(\omega). \tag{26}$$

Vertex factors: Lower vertex

For natural J^P,

$$\langle J^P|p\Delta\rangle = (-p)^J\beta\beta_\Delta\left(\frac{g_1}{p}c_{J-1}(\mathrm{sgn}(\lambda_{\bar N})+\mathrm{sgn}(\lambda_{\bar N})\alpha\alpha_\Delta)C^{J(\lambda_\Delta-\lambda_{\bar N})}_{J-10;1(\lambda_\Delta-\lambda_{\bar N})}\right.$$
$$C^{\frac{3}{2}\lambda_\Delta}_{\frac{1}{2}\lambda_{\bar N};1(\lambda_\Delta-\lambda_{\bar N})}\left(\frac{E_\Delta}{m_\Delta}\right)^{\delta_{\lambda_\Delta,\lambda_{\bar N}}}+g_2c_J\frac{m_J}{m_\Delta}p\sqrt{\frac{2}{3}}\delta_{\lambda^2_\Delta,\frac{1}{4}}(\mathrm{sgn}(\lambda_{\bar N})+\mathrm{sgn}(\lambda_\Delta)\alpha\alpha_\Delta)\delta_{\lambda_\Delta,\lambda_{\bar N}}$$
$$+g_3\ c_{J-1}\frac{m_J}{m_\Delta}\sqrt{2}\delta_{\lambda^2_\Delta,\frac{1}{4}}(\mathrm{sgn}(\lambda_\Delta)\alpha_\Delta\mathrm{sgn}(\lambda_{\bar N})-\alpha)C^{\frac{1}{2}\lambda_\Delta}_{\frac{1}{2}\lambda_{\bar N};1(\lambda_\Delta-\lambda_{\bar N}))}C^{J(\lambda_\Delta-\lambda_{\bar N})}_{J-10;1(\lambda_\Delta-\lambda_{\bar N})}$$
$$+\frac{g_4}{p^2}c_{J-2}\sqrt{3}\sum_{\lambda_2}\left[C^{J(\lambda_\Delta-\lambda_{\bar N})}_{J-1\lambda_2;1(\lambda_\Delta-\lambda_{\bar N}-\lambda_2))}C^{J-1\lambda_2}_{J-20;1\lambda_2}C^{\frac{3}{2}\lambda_\Delta}_{\frac{1}{2}(\lambda_\Delta-\lambda_2);1\lambda_2}C^{\frac{1}{2}(\lambda_\Delta-\lambda_2)}_{\frac{1}{2}\lambda_{\bar N};1(\lambda_\Delta-\lambda_2-\lambda_{\bar N})}\right.$$
$$\left.\left.(\mathrm{sgn}(\lambda_{\bar N})\alpha_\Delta\mathrm{sgn}(\lambda_\Delta-\lambda_2)-\alpha)\left(\frac{E_\Delta}{m_\Delta}\right)^{\delta_{\lambda_2,0}}\right]\right)d^J_{M,\lambda_\Delta-\lambda_{\bar N}}(\omega) \qquad (27)$$