The hadronic contribution to the muon $g-2$

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CERN

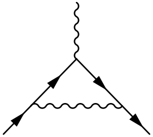
The muon $g-2$: A probe for new physics

- Magnetic moment of charged leptons $l \in \{e, \mu, \tau\}$:

$$\vec{\mu}_l = g_l \cdot \frac{e}{2m_l} \cdot \vec{s}$$

- Quantum corrections lead to deviations from the classical value $g = 2$ (Dirac), the anomalous magnetic moment,

$$a_l = \frac{g_l - 2}{2} = \frac{\alpha}{2\pi} + \mathrm{O}(\alpha^2) \quad \text{(Schwinger)}$$

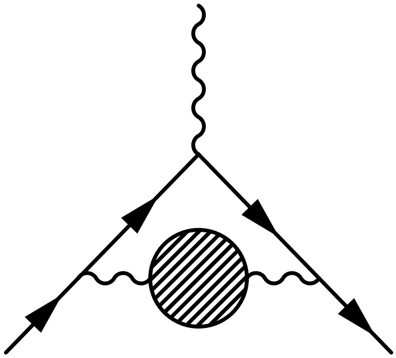


- Contributions from new physics at the scale Λ_{NP} enter a_l via

$$a_l - a_l^{\mathrm{SM}} \propto \frac{m_l^2}{\Lambda_{\mathrm{NP}}^2}$$

with $m_\mu / m_e \approx 207$.

The muon $g-2$: A probe for new physics



(leading order)
hadronic vacuum polarization

- Standard Model prediction from QED, electroweak and hadronic contributions:

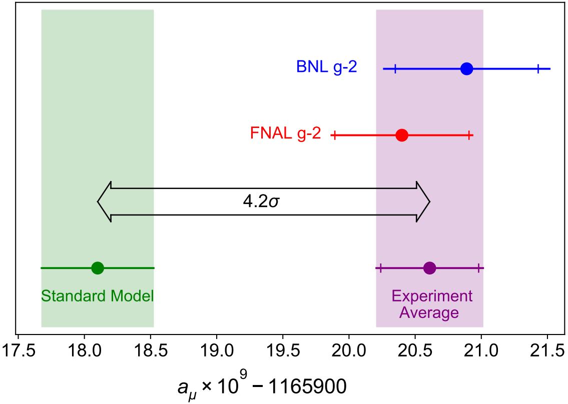
$$a_l^{\mathrm{SM}} = a_l^{\mathrm{QED}} + a_l^{\mathrm{EW}} + a_l^{\mathrm{had}}$$

where $a_l^{\mathrm{had}} = a_l^{\mathrm{hvp}} + a_l^{\mathrm{hlbl}}$.

- $\Delta a_\mu^{\mathrm{SM}}$ is fully dominated by $\Delta a_\mu^{\mathrm{hvp}}$.

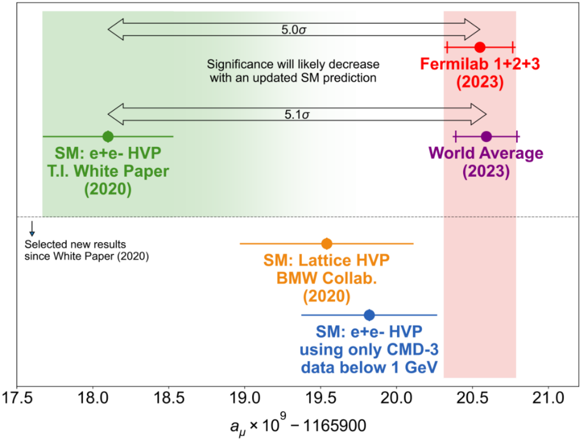
Compute the hadronic contributions to a_μ^{hvp} from lattice QCD.

The muon $g-2$: A probe for new physics



- Comparison of Standard Model prediction and experimental average [Muon $g-2$, 2104.03281]
- After Run-1 results of the Fermilab $g-2$ experiment.
- Standard Model prediction based on the White Paper of the Muon g-2 Theory Initiative [Aoyama et al., 2006.04822]

THE MUON $g-2$: A PROBE FOR NEW PHYSICS



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- After Run-1 results of the Fermilab $g-2$ experiment.
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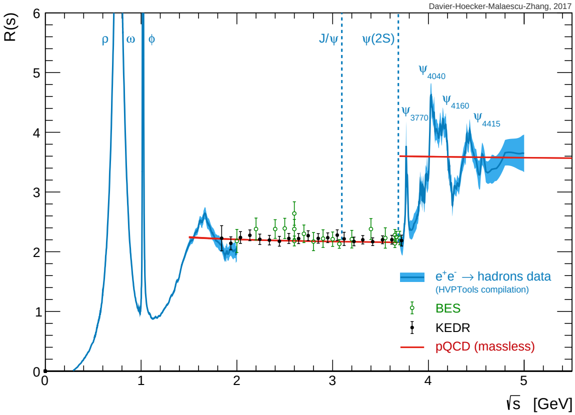
← Everything is much more complicated now [Venanzoni] [Muon $g-2$, 2308.06230].

R-ratio: $$R(s) = \frac{\sigma^0(e^+e^- \to \text{hadrons}(+\gamma))}{\sigma_{\text{pt}}}, \qquad \sigma_{\text{pt}} = \frac{4\pi\alpha^2}{3s}$$



- Data-driven extraction of the HVP contribution via dispersion integral

$$a_\mu^{\text{HVP,LO}} = \frac{\alpha^2}{3\pi^2}\int_{M_\pi^2}^{\infty}\frac{K(s)}{s}R(s)\mathrm{d}s$$

[Davier et al., 1706.09436]

a_μ^{hvp}: THE DISPERSIVE APPROACH

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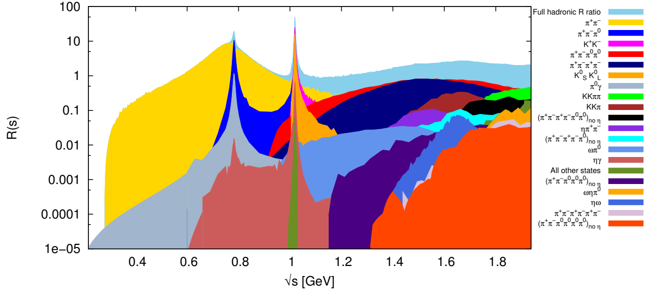

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- R-ratio constructed from exclusive channels
  $\rightarrow$ source of systematic uncertainty.

[Davier et al., 1706.09436] [Keshavarzi et al., 1802.06229]

a_μ^{hvp}: The dispersive approach

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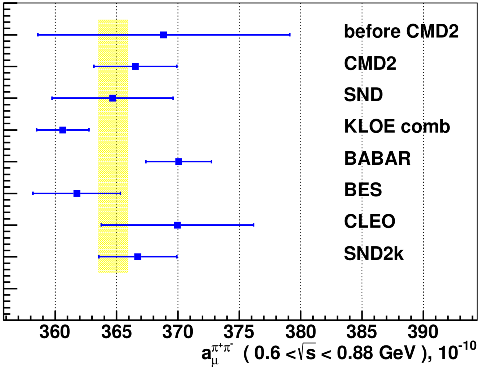

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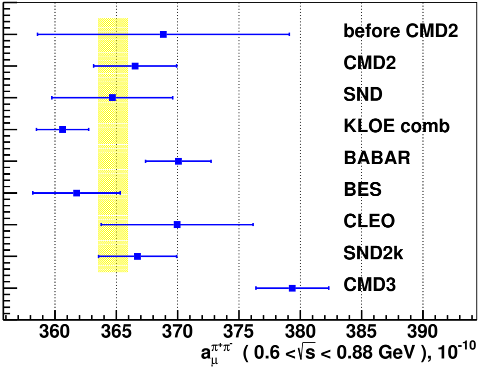

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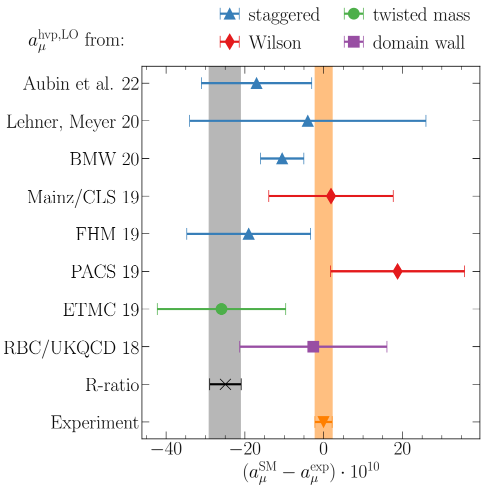
- R-ratio constructed from exclusive channels
  $\to$ source of systematic uncertainty.
- The discrepancies are **not understood**.

[Davier et al., 1706.09436] [Keshavarzi et al., 1802.06229] [Ignatov et al., 2302.08834]

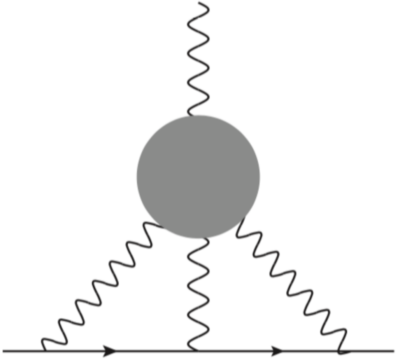


[BNL $g-2$, hep-ex/0602035]
[FNAL $g-2$, 2104.03281, 2308.06230]

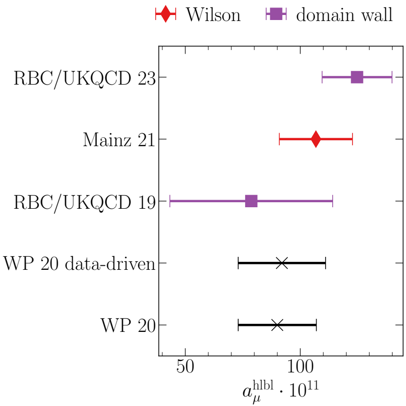
- $5.1\,\sigma$ discrepancy between the current experimental average and the White Paper average [2006.04822] (**pre CMD-3**).
- Average based on data-driven evaluation of the LO HVP contribution ("R-ratio") with 0.6% precision.
- One sub-percent determination of $a_\mu^{\rm hvp}$ from the lattice [BMWc, 2002.12347]:
  In tension with the dispersive result.

Goal

Several lattice results at $< 0.5\%$ precision.



- Hadronic light-by-light scattering: $\mathrm{O}(\alpha^3)$, target precision: 10%.

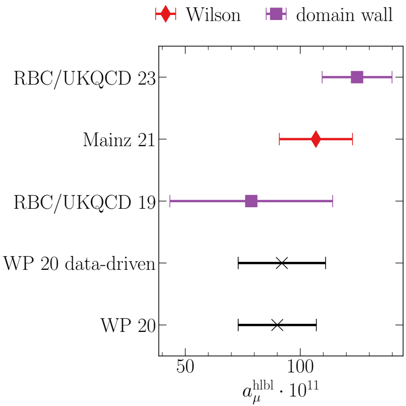


- Hadronic light-by-light scattering: $O(\alpha^3)$, target precision: 10%.
- White paper recommended value:

$$a_\mu^{\text{hlbl}} = (92 \pm 18) \cdot 10^{-11}$$

- Two lattice calculations since then, [Mainz 21, 2104.02632, 2204.08844] and [RBC/UKQCD 23, 2304.04423].
- Lattice and data-driven computations are an outstanding success.

Hadronic light-by-light scattering



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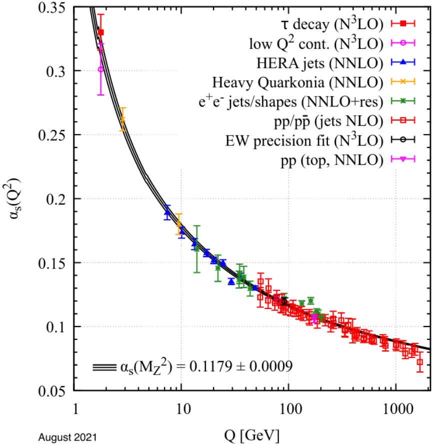
- Lattice and data-driven computations are an outstanding success.

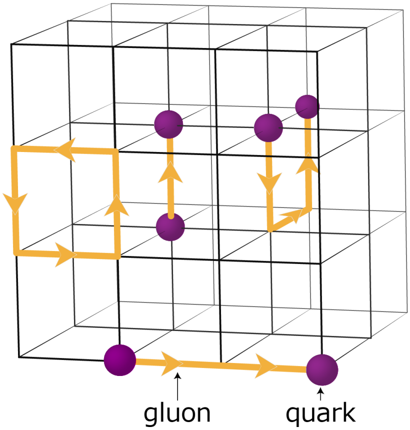
- Probably not the reason for tensions between SM and experiment.
- Data-driven and lattice predictions are compatible.

a_μ^{hvp} ON THE LATTICE (LEADING ORDER)



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.

[1][PDG, PTEP **2022** (2022), 083C01]



- QCD is a strongly coupled theory in the hadronic regime at $Q \sim 300\,\mathrm{MeV}$.
- Perturbative expansion fails below $1\,\mathrm{GeV}$.
- Formulate the theory
  - on a finite grid $\rightarrow$ regulator Λ_{UV}.
  - in finite volume $\rightarrow \Lambda_{\mathrm{IR}}$.
  - in Euclidean space-time
  - as a Boltzmann distribution
- Compute expectation values $\langle \mathrm{O} \rangle$ by sampling the QCD path integral with Markov Chain Monte Carlo methods.

[2] http://www.jicfus.jp/en/promotion/pr/mj/guido-cossu/

The QCD Lagrange density

$$\mathcal{L}_{\mathrm{QCD}} = \sum_{f=1}^{N_f} \bar{\psi}_f(\not{D} + m_f)\psi_f + \frac{1}{4}\mathrm{tr}F_{\mu\nu}F^{\mu\nu}$$

- Contains $N_f + 1$ **bare** parameters (gauge coupling and N_f quark masses)
- Renormalize the theory from hadronic input, e.g., m_Ω, m_π, m_K, m_{D_s}, m_{B_s}.
  $\rightarrow$ All other observables are **predictions**.
- Freedom of choice on how to discretize $\mathcal{L}_{\mathrm{QCD}}$: Wilson, twisted mass, staggered, domain wall, overlap, ...
- *Ab initio* predictions after lifting the cutoffs:
  - Λ_{IR}: Infinite-volume limit.
  - Λ_{UV}: Continuum limit.

$a_\mu^{\rm hvp}$ ON THE LATTICE

- Compute $a_\mu^{\rm hvp}$ via [Laurup et al.] [Blum, hep-lat/0212018]

$$a_\mu^{\rm hvp} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty \mathrm{d}Q^2 f(Q^2)\hat{\Pi}(Q^2)\,, \qquad \text{with} \quad \hat{\Pi}(Q^2) = 4\pi^2\left[\Pi(Q^2) - \Pi(0)\right]$$

  from a known QED kernel function $f(Q^2)$ and the polarization tensor

$$\Pi_{\mu\nu}(Q) = \int \mathrm{d}^4x\, \mathrm{e}^{iQ\cdot x}\langle j_\mu^{\rm em}(x)\, j_\nu^{\rm em}(0)\rangle = (Q_\mu Q_\nu - \delta_{\mu\nu}Q^2)\Pi(Q^2)\,.$$

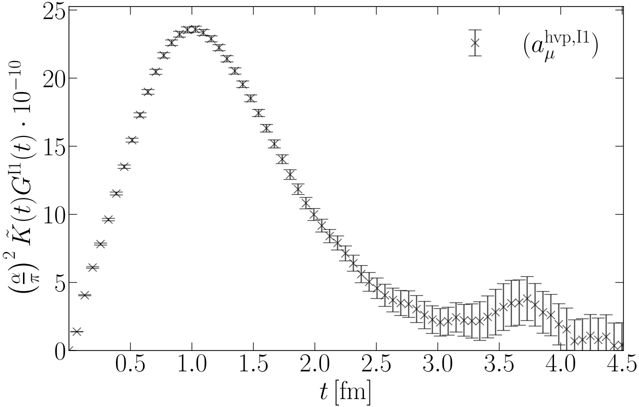
- $a_\mu^{\rm hvp}$ in the time-momentum representation (TMR) [Bernecker, Meyer, 1107.4388],

$$a_\mu^{\rm hvp} := \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t) \quad \text{with the known QED kernel function } \widetilde{K}(t)\,,$$

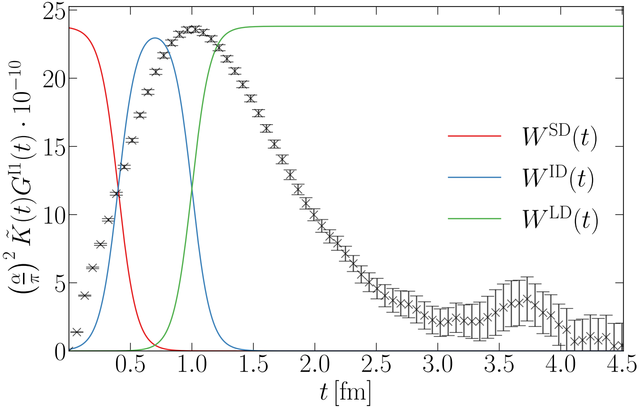
  in terms of the zero-momentum vector correlator $G(t)$ (de facto standard).

- Alternative: coordinate space method [Meyer, 1706.01139] [Chao et al., 2211.15581].

$$(a_\mu^{\rm hvp}) = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t)\widetilde{K}(t), \qquad G(t) = -\frac{a^3}{3}\sum_{k=1}^{3}\sum_{\vec{x}} \langle j_k^{\rm em}(t,\vec{x})\, j_k^{\rm em}(0)\rangle$$



$$(a_\mu^{\mathrm{hvp}})^i = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt\, G(t) \widetilde{K}(t)\; W^i(t; t_0; t_1)\,, \qquad G(t) = -\frac{a^3}{3} \sum_{k=1}^{3} \sum_{\vec{x}} \langle j_k^{\mathrm{em}}(t,\vec{x})\, j_k^{\mathrm{em}}(0)\rangle$$



- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].

a_μ^{hvp} ON THE LATTICE: EUCLIDEAN TIME WINDOWS

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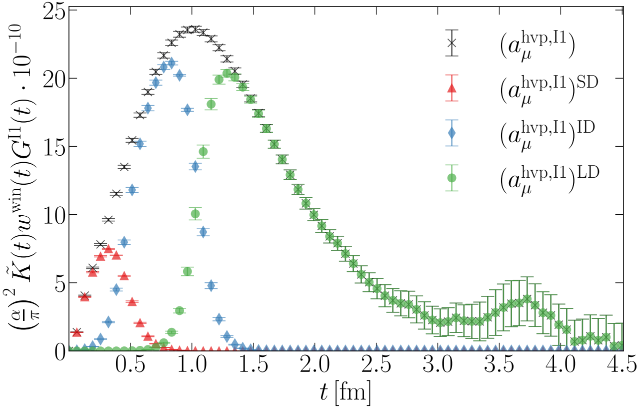


- Windows in the TMR: separate short- from long-distance effects [RBC/UKQCD, 1801.07224].
- Intermediate window a_μ^{win}:
  - Cutoff effects suppressed.
  - No signal-to-noise problem.
  - Finite-volume effects small.

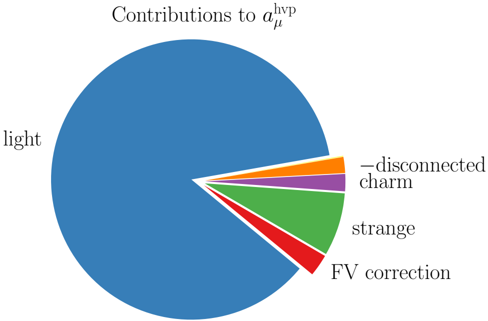
a_μ^{hvp} ON THE LATTICE: CONTRIBUTIONS

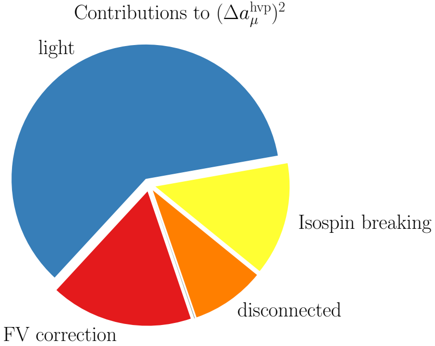
The electromagnetic current

$$j_\mu^{\text{em}} = \tfrac{2}{3}\bar{u}\gamma_\mu u - \tfrac{1}{3}\bar{d}\gamma_\mu d - \tfrac{1}{3}\bar{s}\gamma_\mu s + \tfrac{2}{3}\bar{c}\gamma_\mu c + \ldots = j_\mu^{I=1} + j_\mu^{I=0}$$

from zero-momentum vector-vector correlation functions

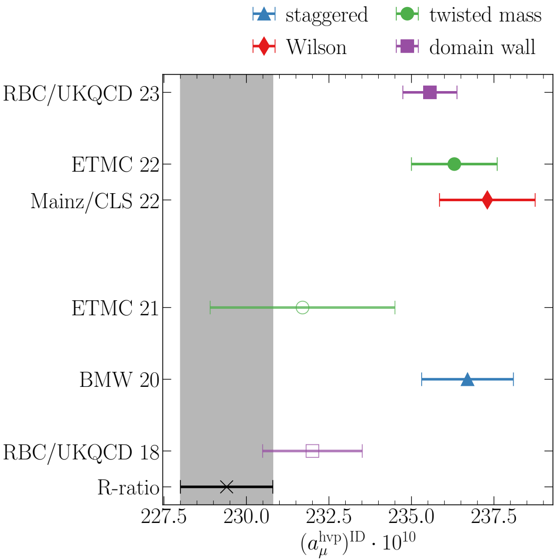
$$G^{\text{isoQCD}}(t) = \tfrac{5}{9}G^{\text{light}}(t) + \tfrac{1}{9}G^{\text{strange}}(t) + \tfrac{4}{9}G^{\text{charm}}(t) + G^{\text{disc}}(t) + \ldots$$



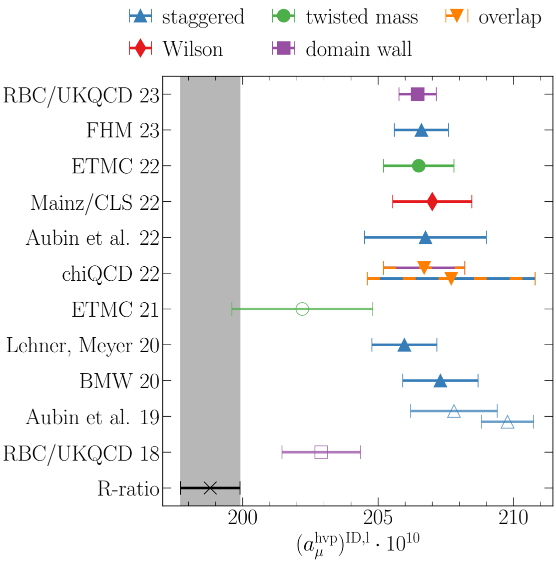


Based on [BMWc, 2002.12347]: $a_\mu^{\text{hvp}} = 707.5\,(5.5)\cdot 10^{-10}$

WINDOW OBSERVABLES

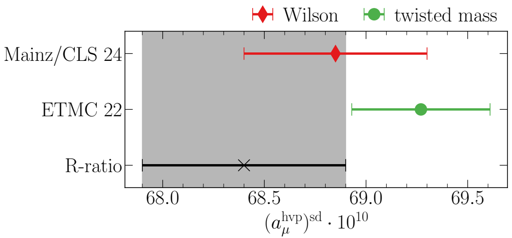


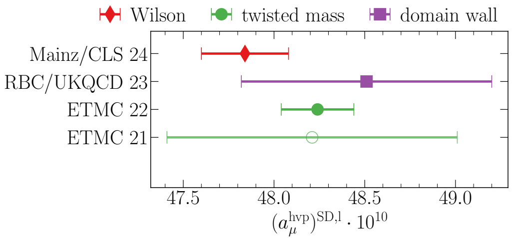
- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% **of the difference** between BMW 20 and the White Paper average for a_μ^{hvp}.



- $3.8\,\sigma$ tension between lattice QCD and data-driven evaluation [Colangelo et al., 2205.12963].
- This accounts for 50% **of the difference** between BMW 20 and the White Paper average for a_μ^{hvp}.
- Agreement across many actions for the light-connected contribution (87% of $(a_\mu^{\mathrm{hvp}})^{\mathrm{ID}}$).
- Data-driven estimate: [Benton et al., 2306.16808]

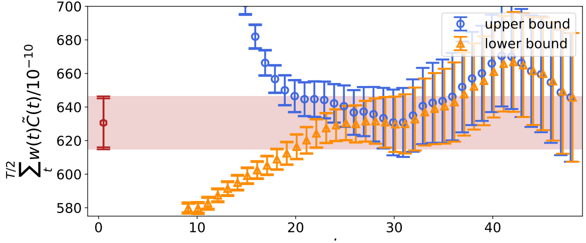
The short-distance window





- Continuum extrapolation is the major difficulty for the short-distance window.
- However: Small uncertainties w.r.t. the full HVP.
- No significant difference between lattice and R-ratio - could expect about 1 unit (1.44%) based on what is seen in the intermediate window [SK at al., 2401.11895].

DOMINANT SOURCES OF UNCERTAINTY FOR a_μ^{hvp}



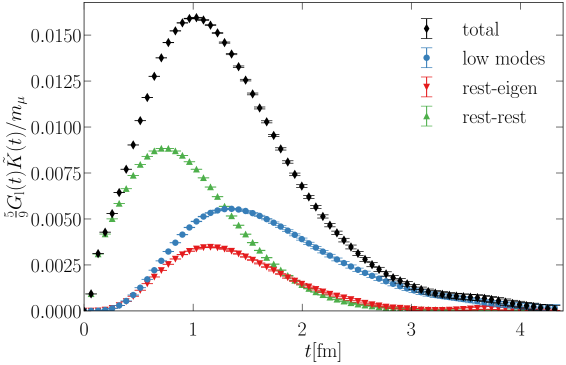
Exponential deterioration of the signal-to-noise ratio.

Improve the signal at large t via:

- Bounds on the correlator.
- Noise reduction methods:
  - Truncated Solver Method
  - Low Mode Averaging
  - All Mode Averaging
- Spectral reconstruction of the $\pi\pi$ contributions.
- Multi-level integration. [Dalla Brida et al., 2007.02973]

[RBC/UKQCD, 1910.11745]

Controlling the long-distance tail



[Mainz, Lattice 2022]

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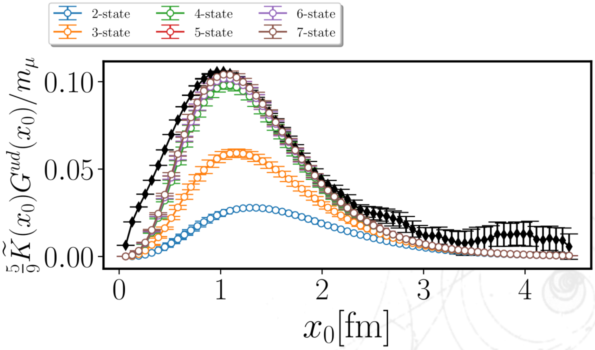

[Mainz]

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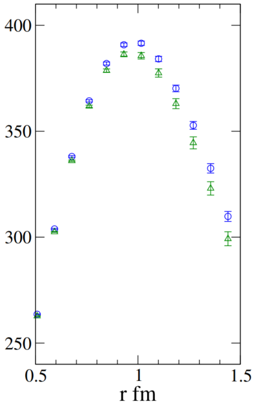
Finite-volume effects

3% finite-L corrections for $a_\mu^{\rm hvp}$ at $m_\pi L = 4$, mostly in the **isovector channel**.

- EFT and model calculations.
  - NNLO χPT
  - Two-pion spectrum in finite-volume and the timelike pion form factor [Meyer, 1105.1892] [Lellouch and Lüscher, hep-lat/0003023] [Giusti et al., 1808.00887].
  - Pions winding around the torus and the electromagnetic pion form factor [Hansen, Patella, 1904.10010, 2004.03935].
  - Rho-pion-gamma model [Sakurai] [Jegerlehner, Szafron, 1101.2872] [HPQCD, 1601.03071].
- Simulations at $L > 10\,\mathrm{fm}$ [PACS, 1902.00885] [BMWc, 2002.12347].
  - Uncertainty statistics dominated.



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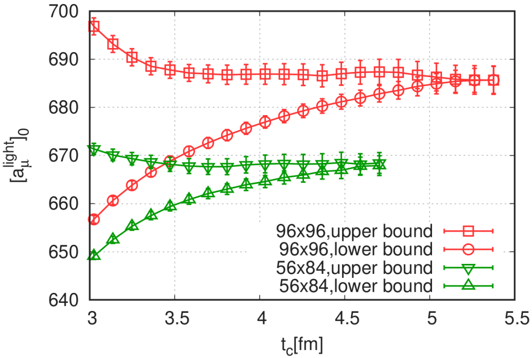

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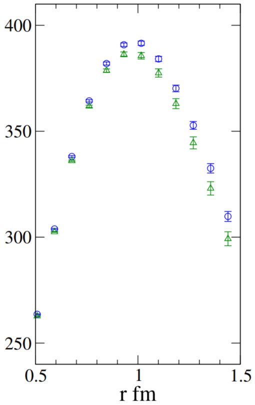

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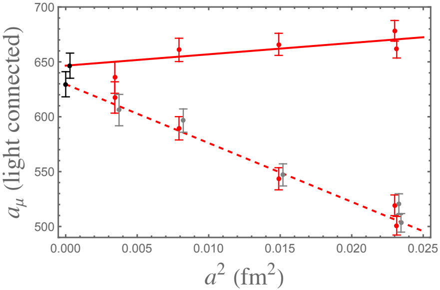
Cutoff effects

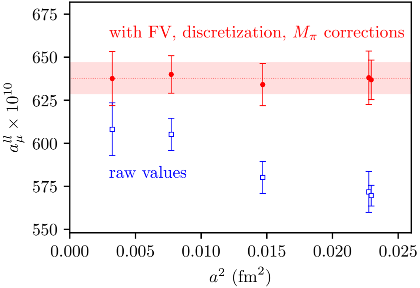
Systematic uncertainties from the continuum extrapolation may be dominant.

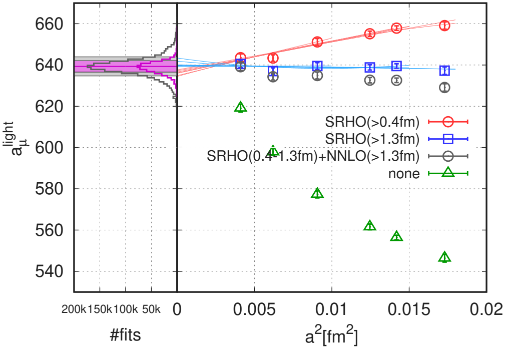
- Extrapolation to the continuum limit guided by Symanzik effective theory.
- Cutoff effects start at $\mathrm{O}(a^2)$ in modern lattice calculations.
- Mandatory to
  - include ≥ 4 resolutions to constrain higher order cutoff effects.
  - include fine resolutions $a \leq 0.05\,\mathrm{fm}$ for per-mil uncertainties.
- Staggered quarks: taste violations distort the pion spectrum.
  - This is a cutoff effect: Vanishes in the continuum limit.
  - Taste breaking may introduce non-linear effects (in a^2).

  $\rightarrow$ Corrections applied at finite lattice spacing.

The continuum limit: staggered quarks



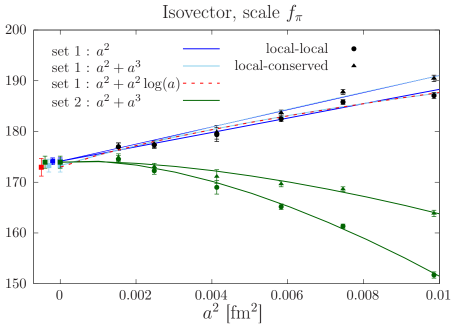


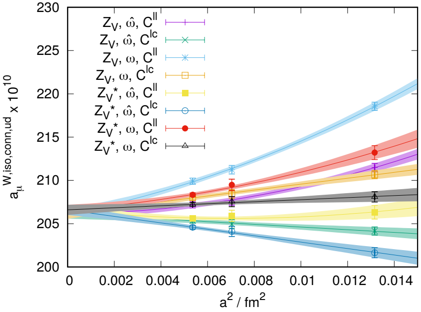


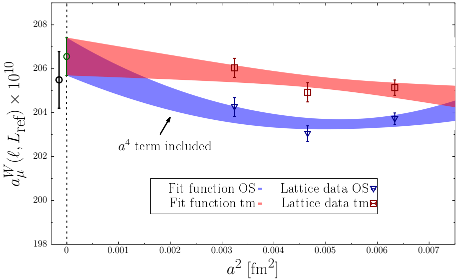
- Continuum extrapolations of $a_\mu^{\rm hvp}$ computed with staggered quarks.
- Compare raw and corrected data.

[Aubin et al., 2204.12256] [BMWc, 2002.12347] [Fermilab, HPQCD, MILC, 1902.04223]

THE CONTINUUM LIMIT: INTERMEDIATE WINDOW







- Different discretization prescriptions have to agree in the continuum.
- Strong cross-check for **valence** cutoff effects.

[Mainz, 2206.06582] [RBC/UKQCD, 2301.08696] [ETMC, 2206.15084]

QED AND STRONG ISOSPIN BREAKING

Need to include $\mathrm{O}(\frac{m_\mathrm{u}-m_\mathrm{d}}{\Lambda_\mathrm{QCD}})$ and $\mathrm{O}(\alpha)$ effects for per-mil precision.

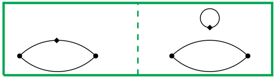
- Various ways to compute isospin breaking corrections:
  - **Perturbative expansion around isospin symmetric QCD** [RM123, 1303.4896].
  - Simulation of dynamical QCD+QED [CSSM/QCDSF/UKQCD] [RC*, 2212.11551].
  - Infinite volume QED [RBC/UKQCD, 1801.07224] [Biloshytskyi et al., 2209.02149]

- Major challenge: Formulation of QED in a finite box.

- QED_L: Finite-volume corrections scale as $\mathrm{O}(1/L^3)$ [Bijnens et al., 1903.10591]
  $\rightarrow$ sufficient for the precision goal.

QED and strong isospin breaking: Results

Overview of published results - contributions to $a_\mu \times 10^{10}$



6.60(63)(53)		BMW
10.6(4.3)(6.8)		RBC/UKQCD
6.0(2.3)		ETM
7.7(3.7)	9.0(2.3)	FHM
9.0(0.8)(1.2)		LM

- Strong isospin breaking: Five groups agree within $1\,\sigma$.

BMW [Nature 593 (2021) 7857, 51-55]
RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
ETM [Phys. Rev. D 99, 114502 (2019)]
FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

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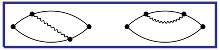
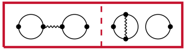
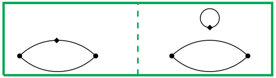


BMW $-1.23(40)(31)$
RBC/UKQCD $5.9(5.7)(1.7)$
ETM $1.1(1.0)$



$-0.55(15)(10)$ BMW
$-6.9(2.1)(2.0)$ RBC/UKQCD



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- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.

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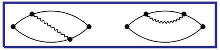

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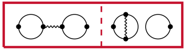

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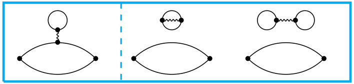
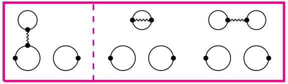
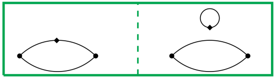
−6.9(2.1)(2.0) RBC/UKQCD



−0.0093(86)(95) 0.37(21)(24) BMW



0.011(24)(14) −0.040(33)(21) BMW



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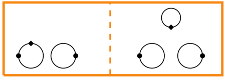

−4.67(54)(69) BMW

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RBC/UKQCD [Phys.Rev.Lett. 121 (2018) 2, 022003]
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FHM [Phys.Rev.Lett. 120 (2018) 15, 152001]
LM [Phys.Rev.D 101 (2020) 074515]

- Strong isospin breaking: Five groups agree within $1\,\sigma$.
- QED: agreement on the total valence contribution.
- One complete calculation [BMWc, 2002.12347]: $\delta a_\mu^{\text{hvp}} = 0.5(1.4) \cdot 10^{-10}$
- Work in progress:
  [Mainz, 2206.06582]
  [RBC/UKQCD, Lattice 2022]
  [BMWc, Lattice 2022]
  [FHM, 2212.12031]
  [Harris et al., 2301.03995]

Adapted from [V. Gülpers @ Lattice HVP workshop 2020]

Conclusions: Tensions

- The discrepancy between lattice and data-driven calculations (pre CMD3) in the **intermediate window** is firmly established.
- Further checks via $a_\mu^{\text{hvp,SD}}$ ($\checkmark$) and $a_\mu^{\text{hvp,LD}}$ (to come).
- Other windows have been computed for further scrutiny [Lehner et al., 2003.04177] [Colangelo et al., 2205.12963] [FHM, 2207.04765] [Boito et al., 2210.13677]
- More insights from direct comparison with the smeared R-ratio? [EMTC, 2212.08467].
- Similar tension in $\Delta\alpha_{\text{had}}$ [BMWc, 1711.04980, 2002.12347] [Mainz, 2203.08676] [Davier et al., 2308.04221].
- $a_\mu^{\text{hvp,ID}}$ from τ data shows less tension with the lattice results [Masjuan et al., 2305.20005] [Davier et al., 2312.02053].

Conclusions: The way ahead

- Lattice QCD can and will provide SM predictions with sub-percent precision.
- More and more precise lattice results for a_μ^{hvp} urgently needed (and expected).
- Improvements: In the last years and ongoing
  - Isovector contribution with sub-percent precision.
  - EFT and data based finite-size corrections.
  - Finer lattices, more lattice spacings.
  - **More precise scale setting**.
  - Isospin breaking effects (beyond the electroquenched approximation).
  - **Blinded analyses**.