MAXWELL EQUATIONS

From the Lagrangian

$$\mathcal{L}_{a\gamma} = -\frac{1}{4} g_{a\gamma}\, a\, F_{\mu\nu}\tilde{F}^{\mu\nu}$$

we want to obtain the Maxwell equations in presence of axions, in the vacuum.

So let's write the Lagrangian for the axion-photon system.

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{4} g_{a\gamma}\, a\, F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\partial_\mu a\, \partial^\mu a - \frac{1}{2} m_a^2 a^2$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,, \quad \tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$$

with $\varepsilon^{\mu\nu\alpha\beta}$ the Levi-Civita tensor.

So first, let's derive the equations of motion for the A_μ field, so the Euler-Lagrange equation

$$\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0$$

We notice that $\frac{\partial \mathcal{L}}{\partial A_\nu} = 0$, so we need to evaluate only $\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right)$.

We need to compute $\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(F_{\alpha\beta}F^{\alpha\beta}\right)$ and $\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right)$

So:

$$\frac{\partial}{\partial(\partial_\mu A_\nu)} F_{\alpha\beta}F^{\alpha\beta} = 2F^{\alpha\beta}\frac{\partial}{\partial(\partial_\mu A_\nu)} F_{\alpha\beta} = 2F^{\alpha\beta}\frac{\partial}{\partial(\partial_\mu A_\nu)}\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right) =$$

$$= 2F^{\alpha\beta}\left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha\right) =$$

$$= 2F^{\mu\nu} - 2F^{\nu\mu} = 4F^{\mu\nu}$$

since $F^{\mu\nu} = -F^{\nu\mu}$

Moreover

$$\frac{\partial}{\partial_\mu A_\nu}\left(F_{\alpha\beta}\tilde{F}^{\alpha\beta}\right) = \frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}\frac{\partial}{\partial_\mu A_\nu}\left(F_{\alpha\beta}F_{\rho\sigma}\right) =$$

$$= \frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}\left[F_{\rho\sigma}\frac{\partial}{\partial_\mu A_\nu}F_{\alpha\beta} + F_{\alpha\beta}\frac{\partial}{\partial_\mu A_\nu}F_{\rho\sigma}\right] =$$

$$= \frac{1}{2}\varepsilon^{\alpha\beta\rho\sigma}\left[F_{\rho\sigma}\left(\delta^\mu_\alpha\delta^\nu_\beta - \delta^\nu_\alpha\delta^\mu_\beta\right) + F_{\alpha\beta}\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\nu_\rho\delta^\mu_\sigma\right)\right] =$$

$$= \frac{1}{2}\left[\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} - \varepsilon^{\nu\mu\rho\sigma}F_{\rho\sigma} + \varepsilon^{\alpha\beta\mu\nu}F_{\alpha\beta} - \varepsilon^{\alpha\beta\nu\mu}F_{\alpha\beta}\right] =$$

$$= \frac{1}{2}\left[\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} + \varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} + \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} + \varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}\right] = 4\tilde{F}^{\mu\nu}$$

So:

$$\partial_\mu\frac{\partial}{\partial_\mu A_\nu}\mathcal{L} = 0 \Rightarrow$$

$$\Rightarrow \partial_\mu\frac{\partial}{\partial_\mu A_\nu}\left[-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{4}g_{a\gamma}aF_{\alpha\beta}\tilde{F}^{\alpha\beta}\right] = 0 \Rightarrow$$

$$\Rightarrow \partial_\mu\left[-F^{\mu\nu} - g_{a\gamma}a\tilde{F}^{\mu\nu}\right] = 0 \Rightarrow$$

$$\Rightarrow \partial_\mu F^{\mu\nu} + g_{a\gamma}\partial_\mu\left(a\tilde{F}^{\mu\nu}\right) = 0$$

This represents the first set of Maxwell equations.
The second set comes from the Bianchi identity which can be written as:

$$\partial_\mu\tilde{F}^{\mu\nu} = 0$$

So the full set of Maxwell equations can be written as:

$$\partial_\mu F^{\mu\nu} + g_{a\gamma}\tilde{F}^{\mu\nu}\partial_\mu a = 0$$
$$\partial_\mu\tilde{F}^{\mu\nu} = 0$$

where in the first set we applied $\partial_\mu\left(a\tilde{F}^{\mu\nu}\right) = \tilde{F}^{\mu\nu}\partial_\mu a$

Now we want to express the equations in terms of the electric and magnetic fields, so let us explicitly write the components of the electromagnetic field tensor.

$$F^{i0} = E^i \quad , \qquad F^{ij} = -\epsilon^{ijk} B_k$$

and lowering indices

$$F_{i0} = -E^i \quad , \qquad F_{ij} = -\epsilon_{ijk} B^k$$

where ϵ_{ijk} is the Euclidean Levi-Civita symbol.

For the dual field let us apply the definition

$$\tilde{F}^{i0} = \frac{1}{2} \varepsilon^{i0jk} F_{jk} = \frac{1}{2} \left(-\epsilon_{ijk}\right)\left(-\epsilon_{jk\ell} B^{\ell}\right) =$$

$$= \frac{1}{2} \epsilon_{ijk} \epsilon_{jk\ell} B^{\ell} = \frac{1}{2} 2 \delta_{i\ell} B_{\ell} = B_i$$

since $\epsilon_{ijk} \epsilon_{jk\ell} = 2\delta_{i\ell}$

while

$$\tilde{F}^{ij} = \frac{1}{2} \varepsilon^{ij0k} F_{0k} + \frac{1}{2} \varepsilon^{ijk0} F_{k0} = \epsilon_{ijk} E_k$$

So, to summarize

$$F^{i0} = E^i \quad , \qquad F^{ij} = -\epsilon^{ijk} B_k$$

$$\tilde{F}^{i0} = B^i \quad , \qquad \tilde{F}^{ij} = \epsilon^{ijk} E_k$$

So now we can rewrite the Maxwell equations.

Let's start from $\partial_\mu \tilde{F}^{\mu\nu} = 0$

$\nu = 0$

$$\partial_i \tilde{F}^{i0} = \partial_i B_i = 0 \Rightarrow \vec{\nabla} \cdot \vec{B} = 0$$

This is the Gauss law for the magnetic field, which tells us there is no magnetic monopole.

$\nu = i$

$$\partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} = 0 \Rightarrow$$

$$\Rightarrow \quad -\partial_0 B_i + \partial_j \epsilon_{jik} E_k = 0 \Rightarrow$$

$$\Rightarrow \quad \partial_0 B_i + \epsilon_{ijk} \partial_j E_k = 0 \Rightarrow$$

$$\Rightarrow \quad \vec{\nabla} \times \vec{E} = -\dot{\vec{B}} \qquad \underline{\text{Faraday law}}$$

Now let us consider

$$\partial_\mu F^{\mu\nu} + g_{a\gamma} \tilde{F}^{\mu\nu} \partial_\mu a = 0$$

$\nu = 0$

$$\partial_i F^{i0} + g_{a\gamma} \tilde{F}^{i0} \partial_i a = 0$$

$$\partial_i E_i + g_{a\gamma} B_i \partial_i a = 0$$

$$\vec{\nabla} \cdot \vec{E} = -g_{a\gamma} \vec{B} \cdot \vec{\nabla} a \qquad \underline{\text{Gauss law for the electric field}}$$

$\nu = i$

$$\partial_0 F^{0i} + \partial_j F^{ji} + g_{a\gamma} \tilde{F}^{ji} \partial_j a + g_{a\gamma} \tilde{F}^{0i} \partial_0 a = 0$$

$$-\partial_0 E_i + \partial_j \epsilon_{jik} B_k + g_{a\gamma} (-\epsilon_{ijk} E_k) \partial_j a + g_{a\gamma} (-B_i) \partial_0 a = 0$$

$$-\partial_0 E_i + \epsilon_{ijk} \partial_j B_k - g_{a\gamma} \epsilon_{ijk} \partial_j a E_k - g_{a\gamma} B_i \partial_0 a = 0$$

$$\vec{\nabla} \times \vec{B} = \dot{\vec{E}} + g_{a\gamma} (\vec{\nabla} a) \times \vec{E} + g_{a\gamma} \dot{a} \vec{B} \qquad \underline{\text{Ampere-Maxwell law}}$$

So

$$\begin{cases} \vec{\nabla} \cdot \vec{B} = 0 & \text{Gauss law for } \vec{B} \\ \vec{\nabla} \times \vec{E} = -\dot{\vec{B}} & \text{Faraday law} \\ \vec{\nabla} \cdot \vec{E} = -g_{a\gamma} \vec{B} \cdot \vec{\nabla} a & \text{Gauss law for } \vec{E} \\ \vec{\nabla} \times \vec{B} = \dot{\vec{E}} + g_{a\gamma} (\vec{\nabla} a) \times \vec{E} + g_{a\gamma} \dot{a} \vec{B} & \text{Ampere-Maxwell law} \end{cases}$$

So the effect of the axion field on the Maxwell equations is that it induces some effective polarization and magnetization which otherwise in vacuum is zero, so it acts as some kind of medium.

So one of the example would be the detection of axion DM where we know that axions behave as an oscillating field inside the galaxy and we know that the spatial gradient term will be suppressed because of the velocity of DM, so if a is DM:

$$\vec{\nabla} a \sim \vec{K} a \sim V_{DM} \sim 10^{-3} \text{ in the galaxy}$$

Then $g_{a\gamma}(\vec{\nabla} a) \times \vec{E}$ is suppressed while the term $g_{a\gamma} \dot{a} \vec{B}$ is not suppressed because it is of the order of the axion oscillation frequency which is of the order of the axion mass.

Therefore, in haloscope experiments they create strong magnetic fields in the microwave cavity and they search for the extra magnetic field which originates from $g_{a\gamma} \dot{a} \vec{B}$.

So, it is convenient to express $\vec{E}$ and $\vec{B}$ as the sum of a background contribution and an axion-induced contribution

$$\vec{E} = \vec{E}_0 + \vec{E}_a \qquad \vec{B} = \vec{B}_0 + \vec{B}_a$$

Now, the axion-induced contributions are small since they are suppressed by $g_{a\gamma} \sim \frac{1}{f_a} \ll 1$

The background fields satisfy the free Maxwell equations and we can write now:

$$\vec{\nabla} \cdot \vec{B}_a = 0$$

$$\vec{\nabla} \times \vec{E}_a = -\dot{\vec{B}}_a$$

$$\vec{\nabla} \cdot \vec{E}_a = -g_{a\gamma} \vec{B}_0 \cdot \vec{\nabla} a$$

$$\vec{\nabla} \times \vec{B}_a = \dot{\vec{E}}_a + g_{a\gamma} (\vec{\nabla} a) \times \vec{E}_0 + g_{a\gamma} \dot{a} \vec{B}_0$$

since terms containing $g_{a\gamma} \vec{E}_a$ or $g_{a\gamma} \vec{B}_a$ will be very small and we leave only terms which are first order in $g_{a\gamma}$.

Now we want to express the Maxwell equations in terms of the effective polarization and magnetization induced by axions. So we write:

$$\vec{\nabla}\cdot\vec{E}_a = -\vec{\nabla}\cdot\vec{P}$$

And from $\vec{\nabla}\cdot\vec{E}_a = -g_{a\gamma}\vec{B}_0\cdot\vec{\nabla}a$

we obtain:

$$\vec{P} = g_{a\gamma}\, a\,\vec{B}_0 \qquad \text{since } \vec{\nabla}\cdot\vec{B}_0 = 0$$

To find the magnetization, let's rewrite the Ampere-Maxwell equation as

$$\vec{\nabla}\times\vec{B}_a = \dot{\vec{E}}_a + \frac{\partial\vec{P}}{\partial t} + \vec{\nabla}\times\vec{M}$$

We know

$$\frac{\partial\vec{P}}{\partial t} = g_{a\gamma}\,\dot{a}\,\vec{B}_0 + g_{a\gamma}\, a\,\dot{\vec{B}}_0$$

and from Faraday law $\dot{\vec{B}}_0 = -\vec{\nabla}\times\vec{E}_0$ so:

$$\frac{\partial\vec{P}}{\partial t} = g_{a\gamma}\,\dot{a}\,\vec{B}_0 - g_{a\gamma}\, a(\vec{\nabla}\times\vec{E})$$

So from:

$$\vec{\nabla}\times\vec{B}_a = \dot{\vec{E}}_a + g_{a\gamma}(\vec{\nabla}a)\times\vec{E} + g_{a\gamma}\,\dot{a}\,\vec{B}_0$$

since $g_{a\gamma}\,\dot{a}\,\vec{B}_0 = \frac{\partial\vec{P}}{\partial t} + g_{a\gamma}\, a\,(\vec{\nabla}\times\vec{E})$

we can write

$$\vec{\nabla}\times\vec{B}_a = \dot{\vec{E}}_a + \frac{\partial\vec{P}}{\partial t} + g_{a\gamma}\, a\,(\vec{\nabla}\times\vec{E}_0) + g_{a\gamma}(\vec{\nabla}a)\times\vec{E}_0$$

and so

$$\vec{\nabla}\times\vec{B}_a = \dot{\vec{E}} + \frac{\partial\vec{P}}{\partial t} + g_{a\gamma}\vec{\nabla}\times(a\vec{E}_0)$$

and we can identify $\vec{M} = g_{a\gamma}\, a\,\vec{E}_0$

So to summarize:

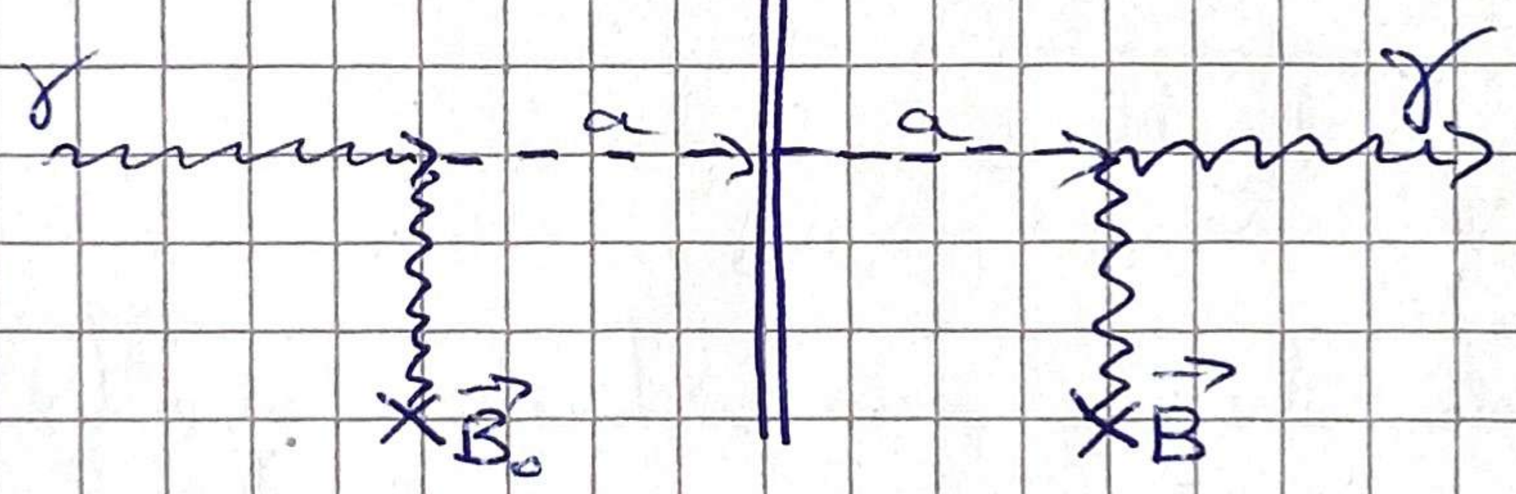
$$\vec{P} = g_{a\gamma}\, a\,\vec{B}_0\,, \qquad \vec{M} = g_{a\gamma}\, a\,\vec{E}_0$$

and all the axion electrodynamics can be understood through these two equations. So haloscope experiments measure $\dot{\vec{P}}$.

Now experimental techniques to detect axions are based on Maxwell equations since they exploit the axion interaction with photons. Besides haloscopes, another common class of experiments is the so-called "light-shining-through-wall" experiment, known as LSW experiments. These are laboratory experiments which do not require that axions are DM but they can create axions in laboratory.

The basic idea is that a photon originating from a laser can convert into an axion if an external magnetic field ~~exper~~ and viceversa.

So if you shine a laser through an external magnetic field toward a wall, the laser would be absorbed by the barrier. However if axions are generated due to the interaction of the photons from the laser with the external magnetic field, axions will pass through the wall. And then, if a strong magnetic field is placed after the wall, axions can reconvert into photons that can be detected. So the final result would be a laser shining through a wall.



So in this way one can probe axions without assuming they are DM.

Now, in order to study this set-up we need to consider the Maxwell equations as well as the propagation of the axion so we need to compute the Euler-Lagrange equations for the axion field.

So from

$$\mathcal{L} \supset -\frac{1}{4} g_{a\gamma} a F_{\mu\nu}\tilde{F}^{\mu\nu} + \frac{1}{2}\partial_\mu a \partial^\mu a - \frac{1}{2} m_a^2 a^2$$

We compute

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu a)} - \frac{\partial \mathcal{L}}{\partial a} = 0$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial_\mu a} = \partial_\mu \partial^\mu a = \Box a$$

$$\frac{\partial \mathcal{L}}{\partial a} = -\frac{1}{4} g_{a\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu} - m_a^2 a$$

So:

$$(\Box + m_a^2) a + \frac{1}{4} g_{a\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$$

Now

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = F_{0i}\tilde{F}^{0i} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij} = 2F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij} =$$

$$= -2E_i B_i + (-\epsilon_{ijk} B_k)(\epsilon_{ijl} E_l) =$$

$$= -2\delta_i B_i - 2\delta_{kl} B_k E_l = -4 E_i B_i$$

So:

$$(\Box + m_a^2) a = g_{a\gamma} \vec{E}_a \cdot \vec{B}_0$$

where we assume there is an external magnetic field $\vec{B}_0$ and an axion-induced $\vec{E}_a$.

So at this point let us consider a laser beam propagating in the direction orthogonal to the external magnetic field, so the generated axion gradient $\vec{\nabla} a$ will be orthogonal to $\vec{B}_0$ and we can write the full set of Maxwell equations and the propagation equation for the axion field.

$\vec{\nabla} \cdot \vec{B}_a = 0$

$\vec{\nabla} \cdot \vec{E}_a = 0 \quad (\text{since } \vec{B}_0 \perp \vec{\nabla} a)$

$\vec{\nabla} \times \vec{E}_a = -\dot{\vec{B}}_a$

$\vec{\nabla} \times \vec{B}_a = \dot{\vec{E}}_a + g_{a\gamma} \dot{a} \vec{B}_0 \quad (\text{since } \vec{E}_0 = 0)$

From $\vec{\nabla} \cdot \vec{B}_a = 0 \Rightarrow \vec{B}_a = \vec{\nabla} \times \vec{A}_a$ potential vector

Then, from $\vec{\nabla} \times \vec{E}_a = -\dot{\vec{B}}_a$ we can write

$\vec{E}_a = -\dot{\vec{A}}_a + \vec{\nabla} A_0$ and we fix the gauge so that $\vec{\nabla} A_0 = 0$ to consider plane electromagnetic waves and so from

$$\vec{B}_a = \vec{\nabla} \times \vec{A}_a \quad , \quad \vec{E}_a = -\dot{\vec{A}}_a$$

we substitute in the Ampere Maxwell law

$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_a) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}_a) - \Delta \vec{A}_a$

and we put $\vec{\nabla} \cdot \vec{A}_a = 0$ in the Coulomb gauge

So:

$-\Delta \vec{A}_a = -\ddot{\vec{A}}_a + g_{a\gamma} \dot{a} \vec{B}_0 \Rightarrow \Box \vec{A}_a - g_{a\gamma} \dot{a} \vec{B}_0 = 0$

And for the axion, since $\vec{E}_a = -\dot{\vec{A}}_a$ we have

$(\Box + m_a^2) a = -g_{a\gamma} \dot{\vec{A}}_a \cdot \vec{B}_0$

So we have a system of two differential equations

$$\begin{cases} \Box \vec{A}_a - g_{a\gamma} \dot{a} \vec{B}_0 = 0 \\ (\Box + m_a^2) a + g_{a\gamma} \dot{\vec{A}} \cdot \vec{B}_0 = 0 \end{cases}$$

And this tells us that in an external magnetic field the axion field will generate a photon field $\vec{A}_a$ and viceversa and only the parallel component of $\vec{A}_a \parallel \vec{B}_0$ mixes with the axion, so we will denote $\vec{A}_a \equiv A_\parallel$

Now we can solve this system of differential equations by substituting A, a with the plane wave ansatz, so

we set $A_{\parallel}, a \sim e^{-i(\omega t - kz)}$

for a plane wave propagating in the z direction

So we can write

$$\begin{cases} (-\omega^2 - \partial_z^2) A_{\parallel} + i g_{a\gamma} \omega B_0 a = 0 \\ (-\omega^2 - \partial_z^2 + m_a^2) a - i g_{a\gamma} \omega B_0 A_{\parallel} = 0 \end{cases}$$

So we can write

$$-(\omega^2 + \partial_z^2)\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} + \begin{pmatrix} 0 & i g_{a\gamma} \omega B_0 \\ -i g_{a\gamma} \omega B_0 & m_a^2 \end{pmatrix}\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} = 0$$

By redefining $a \to +ia = a e^{+i\pi/2}$ we absorb a relative phase between $A_{\parallel}$ and a in order to make the matrix real and we obtain:

$$-(\omega^2 + \partial_z^2)\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} + \begin{pmatrix} 0 & -g_{a\gamma} \omega B_0 \\ -g_{a\gamma} \omega B_0 & m_a^2 \end{pmatrix}\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} = 0$$

So this expression can be written as

$$(\omega^2 + \partial_z^2 - M^2)\mathcal{A} = 0 \quad \text{with}$$

$$M^2 = \begin{pmatrix} 0 & g_{a\gamma} \omega B_0 \\ g_{a\gamma} \omega B_0 & -m_a^2 \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} A_{\parallel} \\ a \end{pmatrix}$$

Now we want to linearize this matrix, so we rewrite

$$\omega^2 + \partial_z^2 = (\omega + i\partial_z)(\omega - i\partial_z) \simeq 2\omega(\omega + i\partial_z)$$

where we have used $i\partial_z \to \pm k$ and then

$\omega = nk \simeq k$ since $|n-1| \ll 1$, with n the refractive index.

So we can write

$$\left(\omega + i\partial_z - \frac{M^2}{2\omega}\right)A = 0$$

or

$$i\partial_z A = \left(-\omega + \frac{M^2}{2\omega}\right)A$$

Ignoring the mass term the equation is

$$\partial_z A = i\omega A$$

with solution

$$A(z) = e^{i\omega z} A(0)$$

The sign in the exponential gives us the direction in which the wave is running (left-moving or right moving). The sign in the linearization picks out the direction of motion.

The $e^{i\omega z}$ term is just a common phase to all components of A and is irrelevant for probabilities, so for our purposes we may always drop a common phase.

So the physical relevant equation of motion is:

$$i\partial_z A = HA \quad \text{where} \quad H = \frac{M^2}{2\omega}$$

This is the same for all relativistic mixing phenomena, e.g. also for neutrino flavour oscillations where M is the matrix of neutrino masses.

This equation looks like an ordinary Schrödinger equation with the Hamiltonian operator H, except that the evolution is along the z axis, not a time evolution.

So for M^2 constant, i.e. independent of z, so in an homogeneous medium and B-field, the equation is solved:

$$A(z) = U(z)A(0), \qquad U(z) = e^{-iHz} = e^{-i\frac{M^2}{2\omega}z}$$

So we can write

$$\left(\omega + i\partial_z - H\right)\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} = 0$$

with $H = \frac{M^2}{2\omega} = \begin{pmatrix} 0 & \Delta \\ \Delta & +\Delta_a \end{pmatrix}$

$$\Delta = \frac{g_{\gamma} B}{2} \qquad \Delta_a = -\frac{m_a^2}{2\omega}$$

The equation of motion implies that $A_{\parallel}, a$ are not propagation eigenstates but interaction eigenstates and the propagation eigenstates $A'_{\parallel}, a'$ ~~are~~ a superposition of $A_{\parallel}, a$ and can be obtained through a rotation

$$\begin{pmatrix} A'_{\parallel} \\ a' \end{pmatrix} = \begin{pmatrix} \cos\theta & \operatorname{sen}\theta \\ -\operatorname{sen}\theta & \cos\theta \end{pmatrix}\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix} = R(\theta)\begin{pmatrix} A_{\parallel} \\ a \end{pmatrix}$$

Now, the interaction eigenstates will evolve as:

$$\begin{pmatrix} A_{\parallel}(z) \\ a(z) \end{pmatrix} = \mathcal{M}(z)\begin{pmatrix} A_{\parallel}(0) \\ a(0) \end{pmatrix}$$

while the ~~interaction~~ propagation eigenstates

$$\begin{pmatrix} A'_{\parallel}(z) \\ a'(z) \end{pmatrix} = \tilde{\mathcal{M}}(z)\begin{pmatrix} A'_{\parallel}(0) \\ a'(0) \end{pmatrix} = \begin{pmatrix} e^{-i\Delta'_{\parallel} z} & 0 \\ 0 & e^{-i\Delta'_a z} \end{pmatrix}\begin{pmatrix} A'_{\parallel}(0) \\ a'(0) \end{pmatrix}$$

where $\tilde{\mathcal{M}}(z)$ is diagonal and therefore the mixing angle θ is chosen such that the "mass matrix" H becomes diagonal in the propagation base.

So we can write

$$\begin{pmatrix} \Delta'_{\parallel} & 0 \\ 0 & \Delta'_{a} \end{pmatrix} = R(\theta) \begin{pmatrix} 0 & \Delta \\ \Delta & \Delta_a \end{pmatrix} R^{-1}(\theta) =$$

$$= \begin{pmatrix} \cos\vartheta & \operatorname{sen}\vartheta \\ -\operatorname{sen}\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} 0 & \Delta \\ \Delta & \Delta_a \end{pmatrix} \begin{pmatrix} \cos\vartheta & -\operatorname{sen}\vartheta \\ \operatorname{sen}\vartheta & \cos\vartheta \end{pmatrix} =$$

$$= \begin{pmatrix} \cos\vartheta & \operatorname{sen}\vartheta \\ -\operatorname{sen}\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \Delta \operatorname{sen}\vartheta & \Delta\cos\vartheta \\ \Delta\cos\vartheta + \Delta_a \operatorname{sen}\vartheta & -\Delta\operatorname{sen}\vartheta + \Delta_a\cos\vartheta \end{pmatrix} =$$

$$= \begin{pmatrix} 2\Delta \operatorname{sen}\vartheta\cos\vartheta + \Delta_a \operatorname{sen}^2\vartheta & \Delta(\cos^2\vartheta - \operatorname{sen}^2\vartheta) + \Delta_a \operatorname{sen}\vartheta\cos\vartheta \\ \Delta(\cos^2\vartheta - \operatorname{sen}^2\vartheta) + \Delta_a \operatorname{sen}\vartheta\cos\vartheta & -2\Delta\operatorname{sen}\vartheta\cos\vartheta + \Delta_a\cos^2\vartheta \end{pmatrix}$$

So:

$$0 = \Delta(\cos^2\vartheta - \operatorname{sen}^2\vartheta) + \Delta_a \operatorname{sen}\vartheta\cos\vartheta =$$

$$= \Delta\cos 2\vartheta + \Delta_a \frac{\operatorname{sen} 2\vartheta}{2} \Rightarrow$$

$$\Rightarrow \Delta + \frac{1}{2}\Delta_a \operatorname{tg} 2\vartheta = 0 \Rightarrow \frac{1}{2}\operatorname{tg} 2\vartheta = -\frac{\Delta}{\Delta_a}$$

Moreover, $\Delta'_{\parallel}$ and Δ'_a will be given by the eigenvalues of H, so:

$$\begin{vmatrix} 0-\lambda & \Delta \\ \Delta & \Delta_a - \lambda \end{vmatrix} = -\lambda(\Delta_a - \lambda) - \Delta^2 = \lambda^2 - \Delta_a\lambda - \Delta^2 = 0$$

$$\lambda_{1/2} = \frac{\Delta_a \pm \sqrt{\Delta_a^2 + 4\Delta^2}}{2}$$

So:

$$\Delta'_{\parallel} = \frac{\Delta_a}{2} + \frac{\sqrt{\Delta_a^2 + 4\Delta^2}}{2}$$

$$\Delta'_a = \frac{\Delta_a}{2} - \frac{\sqrt{\Delta_a^2 + 4\Delta^2}}{2}$$

Therefore:

$$\Delta'_{11} - \Delta'_a = \sqrt{\Delta_a^2 + 4\Delta^2}$$

and from the diagonalization of H

$$\Delta'_{11} - \Delta'_a = 4\Delta \operatorname{sen}\vartheta \cos\vartheta - \Delta_a(\cos^2\vartheta - \operatorname{sen}^2\vartheta) =$$

$$= 2\Delta \operatorname{sen}2\vartheta - \Delta_a \cos 2\vartheta$$

From $\frac{1}{2} \operatorname{tg} 2\vartheta = -\frac{\Delta}{\Delta_a} \Rightarrow -\Delta_a = \frac{2\Delta}{\operatorname{tg} 2\vartheta}$

$$\Delta'_{11} - \Delta'_a = 2\Delta \operatorname{sen}2\vartheta + 2\Delta \frac{\cos^2 2\vartheta}{\operatorname{sen}2\vartheta} = \frac{2\Delta}{\operatorname{sen}2\vartheta} = \sqrt{\Delta_a^2 + 4\Delta^2} \Rightarrow$$

$$\Rightarrow \operatorname{sen}2\vartheta = \frac{2\Delta}{\sqrt{\Delta_a^2 + 4\Delta^2}}$$

And then

$$\frac{1}{2} \frac{\operatorname{sen}2\vartheta}{\cos 2\vartheta} = -\frac{\Delta}{\Delta_a} \Rightarrow \cos 2\vartheta = -\frac{1}{2} \operatorname{sen}2\vartheta \frac{\Delta_a}{\Delta} = -\frac{1}{2} \frac{\Delta_a}{\Delta} \frac{2\Delta}{\sqrt{\Delta_a^2 + 4\Delta^2}} \Rightarrow$$

$$\Rightarrow \cos 2\vartheta = -\frac{\Delta_a}{\sqrt{\Delta_a^2 + 4\Delta^2}}$$

So:

$$\Delta'_{11} = \frac{\Delta_a}{2} - \frac{\Delta_a}{2\cos 2\vartheta}$$

$$\Delta'_a = \frac{\Delta_a}{2} + \frac{\Delta_a}{2\cos 2\vartheta}$$

So for an initial pure photon state (propagating laser) propagating in the external $\vec{B}$ of a LSW experiment, we need to:

- Rotate the initial state in the propagation state
- Evolve the propagation state
- Rotate back to the interaction basis to find the amplitude of the produced axion

So at a location z we find:

$$\begin{pmatrix} A_\parallel(z) \\ a(z) \end{pmatrix} = \begin{pmatrix} \cos\vartheta & -\text{sen}\,\vartheta \\ \text{sen}\,\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} e^{-i\Delta'_\parallel z} & 0 \\ 0 & e^{-i\Delta'_a z} \end{pmatrix} \begin{pmatrix} \cos\vartheta & \text{sen}\,\vartheta \\ -\text{sen}\,\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} A_0 \\ a_0 \end{pmatrix}$$

$$= \begin{pmatrix} \cos\vartheta & -\text{sen}\,\vartheta \\ \text{sen}\,\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \cos\vartheta\, e^{-i\Delta'_\parallel z} & \text{sen}\,\vartheta\, e^{-i\Delta'_\parallel z} \\ -\text{sen}\,\vartheta\, e^{-i\Delta'_a z} & \cos\vartheta\, e^{-i\Delta'_a z} \end{pmatrix} \begin{pmatrix} A_0 \\ a_0 \end{pmatrix} =$$

$$= \begin{pmatrix} \cos^2\vartheta\, e^{-i\Delta'_\parallel z} + \text{sen}^2\vartheta\, e^{-i\Delta'_a z} & \text{sen}\,\vartheta\cos\vartheta\left(e^{-i\Delta'_\parallel z} - e^{-i\Delta'_a z}\right) \\ \text{sen}\,\vartheta\cos\vartheta\left(e^{-i\Delta'_\parallel z} - e^{-i\Delta'_a z}\right) & \text{sen}^2\vartheta\, e^{-i\Delta'_\parallel z} + \cos^2\vartheta\, e^{-i\Delta'_a z} \end{pmatrix} \begin{pmatrix} A_0 \\ a_0 \end{pmatrix}$$

So for a pure initial photon state $\begin{pmatrix} A_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the probability to produce a is:

$$|a(z)|^2 = \text{sen}^2\vartheta\cos^2\vartheta \left| e^{-i\Delta'_\parallel z} - e^{-i\Delta'_a z} \right|^2 =$$

$$= \frac{1}{4}\,\text{sen}^2 2\vartheta\; 4\,\text{sen}^2\left(\frac{\Delta'_\parallel - \Delta'_a}{2} z\right) =$$

$$= \text{sen}^2 2\vartheta\;\text{sen}^2\left(\frac{\sqrt{\Delta_a^2 + 4\Delta^2}\,z}{2}\right) = \frac{4\Delta^2}{\Delta_a^2 + 4\Delta^2}\,\text{sen}^2\left(\frac{\sqrt{\Delta_a^2 + 4\Delta^2}\,z}{2}\right)$$

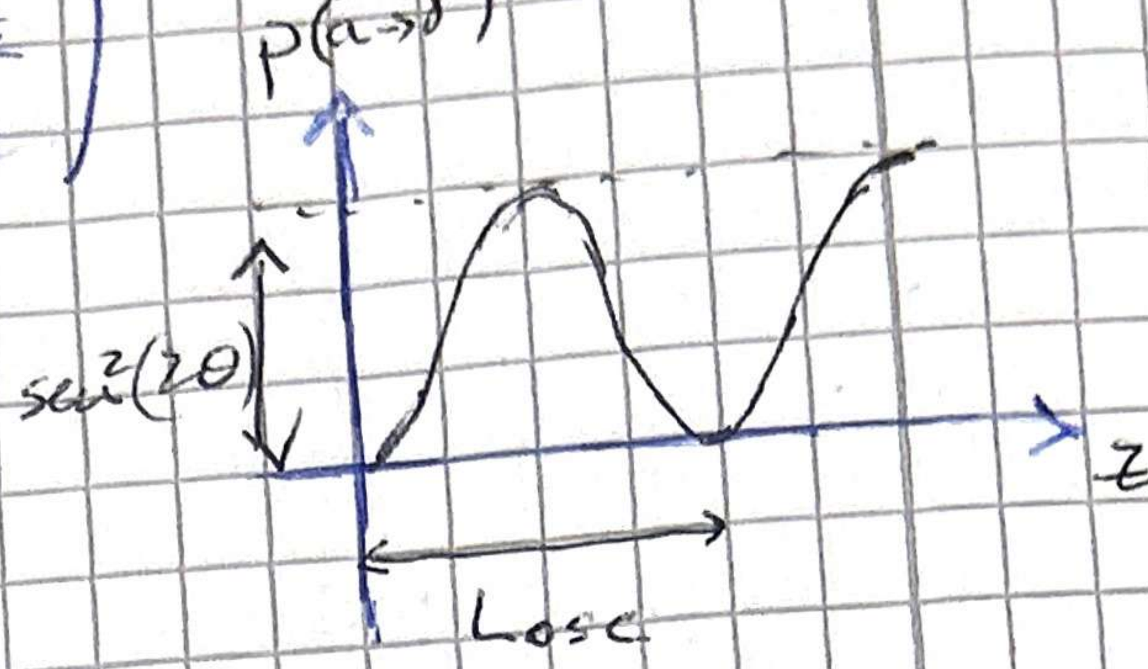
Notice that for a pure initial axion $\begin{pmatrix} A_0 \\ a_0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and the probability to produce photon is the same:

$$|A_\parallel(z)|^2 = \frac{4\Delta^2}{\Delta_a^2 + 4\Delta^2}\,\text{sen}^2\left(\frac{\sqrt{\Delta_a^2 + 4\Delta^2}\,z}{2}\right)$$

So we can write:

$$p(a\to\gamma) = p(\gamma\to a) = \frac{4\Delta^2}{\Delta_a^2+4\Delta^2}\sin^2\left(\frac{\sqrt{\Delta_a^2+4\Delta^2}\,z}{2}\right)$$

By rewriting $\sin^2\left(\frac{\sqrt{\Delta_a^2+4\Delta^2}\,z}{2}\right) = \sin^2\left(\frac{\pi z}{L_{osc}}\right)$



we can define the oscillation length

$$L_{osc} = \frac{2\pi}{\sqrt{\Delta_a^2+4\Delta^2}}$$

If the travelling distance L is much shorter than L_{osc}, we can expand the sine and we obtain (coherent conversions)

$$p(a\to\gamma) = p(\gamma\to a) = \frac{4\Delta^2}{\Delta_a^2+4\Delta^2}\,\frac{\Delta_a^2+4\Delta^2}{4}L^2 = \Delta^2 L^2 =$$

$$= \frac{1}{4}g_{a\gamma}^2 B^2 L^2$$

Moreover, for $\Delta_a \gg \Delta$:

$$p(a\to\gamma) = \frac{4\Delta^2}{\Delta_a^2}\sin^2\left(\frac{|\Delta_a|\,z}{2}\right)$$

and defining $q = |\Delta| = \frac{m_a^2}{2\omega}$ the a-γ momentum transfer in vacuum:

$$p(a\to\gamma) = \frac{g_{a\gamma}^2 B^2}{q^2}\sin^2\left(\frac{qz}{2}\right)$$

and in the massless limit $q\to 0$, $z=L$

$$p(a\to\gamma) = \frac{1}{4}g_{a\gamma}^2 B^2 L^2$$

we reobtain the previous expression

Exercise 3

- The scattering rate of a photon in a proton gas of density n_p is

$$\Gamma = \sigma n_p$$

where we have used that the relative velocity is the speed of light and we have assumed that protons are non relativistic. In each collision, the energy ω_γ of a photon is lost in form of axions and the density photons is n_γ. Since the approximate cross section is independent of energy, the energy loss rate per unit volume is

$$Q = \sigma n_p \rho_\gamma$$

where $\rho_\gamma = \frac{\pi^2}{15} T^4$ is the thermal photon energy density in the solar interior. For pure hydrogen, the solar mass density is $\rho = n_p m_p$ so:

$$Q = \sigma \frac{\rho}{m_p} \rho_\gamma$$

Therefore, the energy loss rate per unit mass is:

$$\epsilon = \frac{Q}{\rho} = \frac{\sigma \rho_\gamma}{m_p} \sim \frac{\alpha g_{a\gamma}^2}{8\pi} \frac{1}{m_p} \frac{\pi^2}{15} T^4 = \frac{\pi}{120} \frac{\alpha g_{a\gamma}^2 T^4}{m_p}$$

With $T = 1\,\text{keV}$ and $m_p = 938\,\text{MeV}$ one finds

$$\epsilon \sim g_{10}^2 \cdot 3 \cdot 10^{-3}\ \text{erg}\,g^{-1} s^{-1}$$

or

$$L_a = \epsilon \cdot M_\odot \simeq g_{10}^2\, 1.5 \cdot 10^{-3}\, L_\odot$$

- The Sun is halfway through its normal lifetime, so it would have burnt out already if it had lost twice the usual amount of energy all along. With the more exact energy loss rate we have the criterion

$$L_a = g_{10}^2 \, 1.85 \cdot 10^{-3} L_\odot \lesssim L_\odot$$

implying:

$$g_{10} \lesssim 20$$

Using globular clusters, one obtains $g_{10} \lesssim 0.66$.

- Approximate the Sun as a homogeneous body with mass $M_\odot$, consisting purely of hydrogen, with radius $R_\odot$. Therefore, the average proton density is:

$$n_p = \frac{M_\odot}{m_p} \frac{1}{(4\pi/3) R_\odot^3} = \frac{2 \cdot 10^{33} \, g}{1.661 \cdot 10^{-24} \, g} \frac{1}{(4\pi/3)(6.96 \cdot 10^{10} \, cm)^3} = 85 \cdot 10^{23} \, cm^{-3}.$$

The cross section for axion-photon conversion is

$$\sigma \sim \frac{g_{a\gamma}^2}{8\pi} \alpha = g_{10}^2 \cdot 1.13 \cdot 10^{-51} \, cm^2$$

Therefore the mfp is

$$\lambda \sim \frac{1}{\sigma n_p} = g_{10}^2 \, 1 \cdot 10^{27} \, cm = g_{10}^2 \, 1.5 \cdot 10^{16} \, R_\odot$$

This is much larger than the solar radius, so axions escape freely once produced.

Exercise 4

- Since $p(a\to\gamma) \sim \sin^2\left(\frac{qL}{2}\right)$, the oscillation length is

$$L_{osc} = \frac{2\pi}{q} \quad \left(\text{from } \frac{q}{2} = \frac{\pi}{L_{osc}}\right)$$

From $q = \frac{m_a^2}{2E}$

$$L_{osc} = \frac{4\pi E}{m_a^2}$$

The coherence is lost when $L > L_{osc}$, so

$$L > \frac{4\pi E}{m_a^2} \Rightarrow m_a > \sqrt{\frac{4\pi E}{L}}$$

Using $E \sim 1\,keV$ and $L = 9.26\,m = 4.7 \cdot 10^7\,eV^{-1}$ one has:

$$m_a^c = \sqrt{\frac{4\pi E}{L}} = 0.016\,eV \simeq 0.02\,eV$$

- For $m_a \ll m_a^c$, the conversion probability is

$$p(a\to\gamma) = \frac{1}{4} g_{a\gamma}^2 B^2 L^2 \simeq 1.7 \cdot 10^{-17}\, g_{10}^2$$

So at the end of the CAST magnet one expects a X-ray flux of

$$F_\gamma = p(a\to\gamma) \cdot F_a = 6.375 \cdot 10^{-6}\, g_{10}^4\ cm^{-2}\,s^{-1}$$

Since the cross-sectional area of the two pipes is $A = 2 \cdot 14.5\,cm^2$ and in $1\,h$ there are $3600\,s$, one expects in CAST:

$$N_\gamma = F_\gamma \cdot A \cdot \tau = 6.375 \cdot 10^{-6}\, g_{10}^4 \cdot 2 \cdot 14.5 \cdot 3600 \simeq 0.66\, g_{10}^4\ \frac{photons}{h}$$

So for $g_{10} = 0.66$ one has

$$N_\gamma \simeq 0.13\ \frac{photons}{h}$$

Exercise 5

Useful conversions:

$1\,W = 6.242 \cdot 10^{18}\, eV/s$

$1\,T = 10^4\, G = 1.953 \cdot 10^2\, eV^2$

$1\,cm = 0.507 \cdot 10^5\, eV^{-1}$

A laser at 1064 nm has an energy of

$$\omega = \frac{2\pi}{\lambda} \simeq 1.2\, eV$$

So it will produce $\frac{P_{in}}{\omega} = \frac{30\, W}{1.2\, eV} \simeq 1.6 \cdot 10^{20}$ photons/s

The expected number of photons from reconversion per second is:

$$\dot{N}_\gamma = \frac{P_{in}}{\omega}\, p(\gamma \to a \to \gamma) = \frac{P_{in}}{\omega} \frac{1}{16} \beta_{PC} \cdot \beta_{RC} \left(g_{a\gamma} B L\right)^4 \simeq 2.2 \cdot 10^{-4}\, s^{-1}$$