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1) Computation of the axion potential

From a modern perspective, the basic ingredient of the PQ solution to the strong CP problem ($\delta\mathcal{L}_{QCD} = \frac{g_s^2}{32\pi^2}\theta G\tilde{G}$) is a pseudosymmetry in the Lagrangian which is conserved up to

$$\delta S = \frac{g_s^2}{32\pi^2}\alpha\int d^4x\, G\tilde{G}$$

So a global symmetry anomalous under $SU(3)_c$

Therefore, by choosing $\alpha = -\theta$ we can reabsorb the $G\tilde{G}$ operator.

There are different ways to do this (Alessandro has shown something in the lecture), but the most convenient way is to add a spin-0 field, the axion which transforms as:

$$a(x) \mapsto a(x) + \alpha f_a \qquad \text{(pseudo)shift symmetry}$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu a)^2 + \mathcal{L}(\partial_\mu a, \Psi_{SM}) + \frac{g_s^2}{32\pi^2}\frac{a}{f_a}G\tilde{G}$$

So under $a \to a + \alpha f_a$ the action changes as:

$$\delta S = \frac{g_s^2}{32\pi^2}\alpha\int d^4x\, G\tilde{G}$$

and this can be used to remove θ.

N.B. In this way we remove θ but we have to make sure that $\langle a\rangle = 0$, otherwise we would reobtain a strong CP problem.

$\langle a\rangle = 0$ is ensured by the Vafa-Witten theorem.
So now we will compute the axion potential $V(a)$ in chiral perturbation theory and check that the global minimum is at $\langle a\rangle = 0$.

Now, χPT is an effective field theory that describes the properties of strongly-interacting systems at energies far below typical hadron masses and it is consistent with the approximate chiral symmetry of QCD.

Indeed, if we focus on the QCD $\mathcal{L}$ considering only the three lightest quarks with $m_q \ll \Lambda_{QCD}$, we have $(q = u, d, s)$

$$\mathcal{L}_{QCD} = \sum_{i=1}^{3} \left(\bar{q}_{L_i} i \not{D} q_{L_i} + \bar{q}_{R_i} i \not{D} q_{R_i} - m_i \bar{q}_{R_i} q_{L_i} + h.c. \right) -$$

$$- \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \quad (+ \theta \text{ term})$$

here

$$D_\mu = \partial_\mu - i g_s T^a A^a_\mu \qquad T^a = \lambda^a/2 \qquad \lambda^a \to \text{Gell-Mann matrices}$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu \pm i f^{abc} A^b_\mu A^c_\nu \qquad [T^a, T^b] = i f^{abc} T^c$$

The Kinetic Lagrangian $\mathcal{L} \supset i\bar{q}_L \not{D} q_L + i\bar{q}_R \not{D} q_R$ is invariant under a global $U(3)_L \times U(3)_R$ transformation

$$U(3)_L \times U(3)_R = SU(3)_L \times SU(3)_R \times U(1)_L \times U(1)_R$$

or in a different basis $(V = L+R,\ A = R-L)$

$$SU(3)_V \times SU(3)_A \times U(1)_V \times U(1)_A$$

The $U(1)_V$ part is related to the baryon number $q_i \to e^{i\alpha} q_i$

The $U(1)_A$ is anomalous and it is related to $q_i \to e^{i\alpha\gamma_5} q_i$

The transformation $SU(3)_L \times SU(3)_R$ is related to the chiral symmetry

$$\cancel{L = e^{i\alpha_a T^a}}$$

$$\cancel{J^{\mu a}_{L,R} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \Psi} T^a \Psi = \bar{\Psi} \gamma^\mu T^a q}$$

$$q_L \to L q_L \qquad q_R \to R q_R$$

$$L \in SU(3)_L \qquad R \in SU(3)_R$$

And the Noether currents associated are

$$J^{\mu a}_{L,R} = \bar{q}_{L,R} \gamma^\mu \frac{\lambda^a}{2} q_{L,R} \qquad a = 1, \dots, 8 \qquad \cancel{J^\mu_{L,R} = \bar{q}_{L,R} \gamma^\mu q_{L,R}}$$

$$\cancel{J^{\mu 5} = \bar{q} \gamma^\mu \gamma_5 T^a q} \qquad \cancel{J^\mu = \bar{q} \gamma^\mu q} \qquad \cancel{J^\mu_5 = \bar{q} \gamma^\mu \gamma_5 q}$$

Note that the quark mass term is not chirally invariant

$-\bar{q}_R M^\dagger q_R + h.c.$ $\quad M = \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$

If M were a field transforming as $M \mapsto L M R^\dagger$, then the mass term would be invariant =>

=> Possible solution: promote M to a spurion field, build systematically χPT using this field and eventually set the spurion field to its actual vev

$$M \longmapsto \begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}$$

This actually will guarantee that the explicitt symmetry breaking in χPT has exactly the same structure in QCD.

The chiral symmetry, which should be approximately good in the ligh quark sector (u, d, s), is not seen in the hadronic spectrum. Although hadrons can be nicely classified in $SU(3)_V$ representations, degenerate multiplets with opposite parity do not exist.* Moreover, the octet of pseudoscalar mesons is much lighter than all the other hadronic states. To be consistent with experimental facts, the vacuum of the theory should not be symmetric under the chiral group. Indeed, the QCD vacuum breakes spontaneously:

$$SU(3)_L \times SU(3)_R \mapsto SU(3)_{L+R} \equiv SU(3)_V$$

through the quark condensate

$$\langle 0 | \bar{q}_{R_j} q_{L_i} | 0 \rangle = \tilde{\Lambda}^3 \delta_{ij}$$

$$[\tilde{\Lambda}] = \text{mass}$$

under a $SU(3)_L \times SU(3)_R$ transformation

$$q_{L_i} \to L_{ik} q_{Lk} \qquad \bar{q}_{R_j} \mapsto \bar{q}_{Re} R^\dagger_{ej}$$

* q_L and q_R are interchangeable in bound states of mesons, so an exact chiral symm. would imply parity doubling, and every state should appear in a pair of equal mass particles, "parity partners" => 0^+ meson should have same mass of 0^- mesons. However 0^- mesons (pseudoscalar) are lighter than all the other hadrons.

So:

$$\langle 0|\bar{q}_{R_j} q_{L_i}|0\rangle \to \underbrace{\langle 0|\bar{q}_{R\ell}\, q_{Lk}|0\rangle}_{\tilde{\lambda}^3 \delta_{\ell k}} R^\dagger_{\ell j} L_{ik} = \tilde{\lambda}^3 \delta_{\ell k} R^\dagger_{\ell j} L_{ik} =$$

$$= \tilde{\lambda}^3 R^\dagger_{kj} L_{ik} = \tilde{\lambda}^3 (LR^\dagger)_{ij} \equiv \tilde{\lambda}^3 \Sigma_{ij}$$

where Σ_{ij} is a $SU(3)$ matrix

So $\delta_{ij} \mapsto \Sigma_{ij}$ under $L \times R$ transformation

For $L = R \Rightarrow \Sigma_{ij} = \delta_{ij}$ and the vacuum is invariant

$$\Rightarrow \quad SU(3)_L \times SU(3)_R \mapsto SU(3)_V$$

$$8 \quad + \quad 8 \quad - \quad 8 \quad = \quad 8 \text{ GBs}$$

So Σ_{ij} can be seen as a different vacuum, degenerate with the first (δ_{ij}) in the $M=0$ limit

The 8 GBs can be parametriced as a transformation connecting degenerate vacua, ie by promoting Σ_{ij} to a field:

$$\Sigma_{ij} \to \Sigma = e^{i\frac{2\pi^a T^a}{f_\pi}} \qquad T^a = \frac{\lambda^a}{2} \qquad \mathrm{tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$$

f = parameter with mass dimension

with $\Sigma \to L \Sigma R^\dagger$ under $SU(3)_L \times SU(3)_R$ (i.e. the same transf. property of the condensate)

Explicitly one can write

$$\pi^a T^a = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & \pi^+ & K^+ \\ \pi^- & -\frac{\pi^0}{\sqrt{2}} + \frac{\eta_8}{\sqrt{6}} & K^0 \\ K^- & \bar{K}^0 & -\frac{2\eta_8}{\sqrt{6}} \end{pmatrix}$$

easy to remember thinking about the quark content

$$\pi^+ = u\bar{d} \qquad \pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d}) \qquad K^+ = u\bar{s}, \text{ etc.}$$

η_8 corresponds instead to the generator $\lambda^8 = 1/\sqrt{3} \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}$

Since there is a mass gap between the pseudoscalar octet and the rest of the hadronic spectrum, one can build an EFT containing only the Goldstone modes, valid up to $\Lambda \sim 1$ GeV

One should write the more general Lagrangian involving Σ, which is consistent with chiral symmetry and the leading order chiral Lagrangian is given by:

$$\mathcal{L}^{\chi PT} = \frac{f^2}{4} Tr\left[\partial_\mu \Sigma^+ \partial^\mu \Sigma\right] + \frac{Bf^2}{2} Tr\left[\Sigma M^+ + M\Sigma^+\right]$$

- The derivative term gives rise to an infinite series of interactions in powers of π^a
- The normalization is fixed to get the canonical Kinetic term

$$\frac{f^2}{4} Tr\left[\partial_\mu \Sigma^+ \partial^\mu \Sigma\right] > \frac{f^2}{4} Tr\left[\left(-i2\frac{\partial_\mu \pi^a}{f} T^a\right)\left(i2\frac{\partial^\mu \pi^b}{f} T^b\right)\right] =$$

$$= \frac{f^2}{4}\frac{4}{f^2} \partial_\mu \pi^a \partial^\mu \pi^b \underbrace{Tr\left[T^a T^b\right]}_{\frac{1}{2}\delta^{ab}} = \frac{1}{2}\partial_\mu \pi^a \partial^\mu \pi^a$$

- f can be identified with the pion decay constant $f_\pi \simeq 93$ MeV from the measurement of $\pi^+ \to \mu^+ \nu_\mu$

- B is a parameter with the dimension of a mass and it is related to the value of the quark condensate $\tilde{\lambda}^3 = -Bf^2$

It induces the following masses:

$m^2_{\pi^\pm} = B(m_u + m_d)$ $\qquad m^2_{K^\pm} = B(m_u + m_s)$

$m^2_{\overset{(-)}{K^0}} = B(m_d + m_s)$

and for the real fields (π^0, η_8)

$$-\frac{B}{2}\begin{pmatrix}\pi^0 & \eta_8\end{pmatrix}\begin{pmatrix} m_u + m_d & \frac{1}{\sqrt{3}}(m_u - m_d) \\ \frac{1}{\sqrt{3}}(m_u - m_d) & \frac{1}{3}(m_u + m_d + 4m_s)\end{pmatrix}\begin{pmatrix}\pi_0 \\ \eta_8\end{pmatrix}$$

This is the general χPT for QCD, now let us work in $N_f = 2$ QCD, therefore

$$q = \begin{pmatrix} u \\ d \end{pmatrix} \qquad M_q = \begin{pmatrix} m_u & \\ & m_d \end{pmatrix}$$

The QCD Lagrangian including axion can be written as:

$$\mathcal{L}_{QCD}^{axion} = \frac{g_s^2}{32\pi^2} \frac{a}{f_a} G\tilde{G} - \bar{q}_L M_q q_R + h.c. + \dots$$

The term $aG\tilde{G}$ breaks the axion shift symmetry and this will generate the axion potential $V(a)$.

So, as a first step, let's move $a(x)$ from $G\tilde{G}$ term to the phase of M_q by defining the following transformations:

$$q \mapsto e^{i\gamma_5 \frac{a(x)}{2f_a} Q_a} q \quad \Rightarrow \text{this is a field-dependent } \cancel{\text{transform}} \text{ axial transformation}$$

$$M_q \mapsto M_a = e^{i\frac{a}{2f_a} Q_a} M_q \, e^{i\frac{a}{2f_a} Q_a}$$

From the non-invariance of the path integral measure $\delta q \delta \bar{q}$ one has

$$\delta S = -2 \frac{a(x)}{2f_a} \mathrm{Tr}\, Q_a \frac{g_s^2}{32\pi^2} \int d^4x \, G\tilde{G}$$

Hence, the $aG\tilde{G}$ term is rotated away for $\mathrm{Tr}\, Q_a = 1$ and in the new basis

$$\mathcal{L}_{QCD}^{axion} = \bar{q}_L M_a q_R + h.c.$$

So the chiral axion Lagrangian can be obtained by replacing M_q with the "axion-dressed" quark mass matrix M_a:

$$\mathcal{L}_{\chi PT}^{axion} = \frac{f_\pi^2}{4} \left(\mathrm{Tr}\left[\partial_\mu \Sigma^\dagger \partial^\mu \Sigma\right] + 2B_0 \mathrm{Tr}\left[\Sigma M_a^\dagger + M_a \Sigma^\dagger\right] \right)$$

where $\quad \Sigma = \exp\left(i\pi/f_\pi\right) \quad , \quad \pi = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}$

$$f_\pi \simeq 93 \text{ MeV} \qquad m_\pi^2 = B_0 (m_u + m_d)$$

The axion potential is given by "–" the non-derivative part of $\mathcal{L}^{axion}_{\chi PT}$

$$V(a,\pi^0) = -\frac{1}{2} f_\pi^2 B_0 \, Tr\left[\Sigma M_a^\dagger + M_a \Sigma^\dagger\right]$$

To compute V we take an explicit form for Q_a.
We know $Tr\, Q_a = 1$, then we choose Q_a proportional to the identity matrix $Q_a = \frac{1}{2}\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$

With this choice the mass matrix becomes:

$$M_a = M_q e^{i \frac{a}{f_a} Q_a} = M_q e^{i \frac{a}{2f_a} \mathbb{1}}$$

$$M_a^\dagger = M_q e^{-i \frac{a}{2f_a} \mathbb{1}}$$

Since $\frac{1}{2}\left(\Sigma M_a^\dagger + M_a \Sigma^\dagger\right) = Re\left[\Sigma M_a^\dagger\right]$

we can write:

$$V(a,\pi^0) = -f_\pi^2 B_0 \, Re \, Tr\left[\Sigma M_a^\dagger\right] =$$

$$= -f_\pi^2 B_0 \, Re \, Tr\left[e^{i\pi/f_\pi} \begin{pmatrix} m_u & \\ & m_d \end{pmatrix} e^{-i\frac{a}{2f_a}\mathbb{1}}\right]$$

Due to EM conservation, only π^0 can mix with a, so we can write

$$\pi \xrightarrow[\pi^\pm \to 0]{} \begin{pmatrix} \pi^0 & 0 \\ 0 & -\pi^0 \end{pmatrix} = \sigma^3 \pi^0$$

And from properties of the Pauli matrices:

$$e^{i\pi/f_\pi}\Big|_{\pi^\pm \to 0} = \mathbb{1}\cos\frac{\pi^0}{f_\pi} + i\,\sigma^3 \sin\frac{\pi^0}{f_\pi}$$

we can write

$$V(a,\pi^0) = -f_\pi^2 B_0 \,\mathrm{Re}\,\mathrm{Tr}\left[\begin{pmatrix} m_u & \\ & m_d \end{pmatrix}\left(\mathbb{1}\cos\frac{\pi^0}{f_\pi} + i\sigma^3 \,\mathrm{sen}\frac{\pi^0}{f_\pi}\right)\left(\mathbb{1}\cos\frac{a}{2f_a} - i\,\mathbb{1}\,\mathrm{sen}\frac{a}{2f_a}\right)\right] =$$

$$= -f_\pi^2 B_0 \,\mathrm{Re}\,\mathrm{Tr}\Bigg[\begin{pmatrix} m_u & \\ & m_d \end{pmatrix}\Bigg(\mathbb{1}\cos\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} + \sigma^3 \,\mathrm{sen}\frac{\pi^0}{f_\pi}\,\mathrm{sen}\frac{a}{2f_a} +$$

$$\underbrace{+\, i\sigma^3\,\mathrm{sen}\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} - i\mathbb{1}\,\mathrm{sen}\frac{a}{2f_a}\cos\frac{\pi^0}{f_\pi}}_{\mathrm{Re}[\dots]=0}\Bigg)\Bigg] =$$

$$= -f_\pi^2 B_0 \,\mathrm{Tr}\left[\begin{pmatrix} m_u & \\ & m_d \end{pmatrix}\left(\mathbb{1}\cos\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} + \sigma^3\,\mathrm{sen}\frac{\pi^0}{f_\pi}\,\mathrm{sen}\frac{a}{2f_a}\right)\right] =$$

$$= -f_\pi^2 B_0 \left[(m_u+m_d)\cos\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} + (m_u-m_d)\,\mathrm{sen}\frac{\pi^0}{f_\pi}\,\mathrm{sen}\frac{a}{2f_a}\right] \overset{*}{=}$$

$$= -f_\pi^2 B_0 (m_u+m_d)\left[\cos\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} + \frac{m_u-m_d}{m_u+m_d}\,\mathrm{sen}\frac{\pi^0}{f_\pi}\,\mathrm{sen}\frac{a}{2f_a}\right] =$$

$$= -f_\pi^2 m_\pi^2 \left[\cos\frac{\pi^0}{f_\pi}\cos\frac{a}{2f_a} + \frac{m_u-m_d}{m_u+m_d}\,\mathrm{sen}\frac{\pi^0}{f_\pi}\,\mathrm{sen}\frac{a}{2f_a}\right]$$

From the trigonometric property:

$$\frac{\cos x\cos y + \alpha\,\mathrm{sen}\,x\,\mathrm{sen}\,y}{\sqrt{\cos^2 x + \alpha^2\,\mathrm{sen}^2 x}} = \cos\left[y - \mathrm{arctg}(\alpha\,\mathrm{tg}\,x)\right]$$

using $x = \frac{a}{2f_a}$, $y = \frac{\pi^0}{f_\pi}$, $\alpha = \frac{m_u-m_d}{m_u+m_d}$

we can obtain:

$$V(a,\pi^0) = -m_\pi^2 f_\pi^2 \sqrt{\cos^2\frac{a}{2f_a} + \left(\frac{m_u-m_d}{m_u+m_d}\right)^2\mathrm{sen}^2\frac{a}{2f_a}}\;\cos\left(\frac{\pi^0}{f_\pi} - \phi(a)\right) =$$

$$\overset{\cos^2 x = 1-\mathrm{sen}^2 x}{=} -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u+m_d)^2}\,\mathrm{sen}^2\frac{a}{2f_a}}\;\cos\left(\frac{\pi^0}{f_\pi} - \phi(a)\right) \quad (*)$$

with $\mathrm{tg}\,\phi(a) = \frac{m_u-m_d}{m_u+m_d}\,\mathrm{tg}\frac{a}{2f_a}$

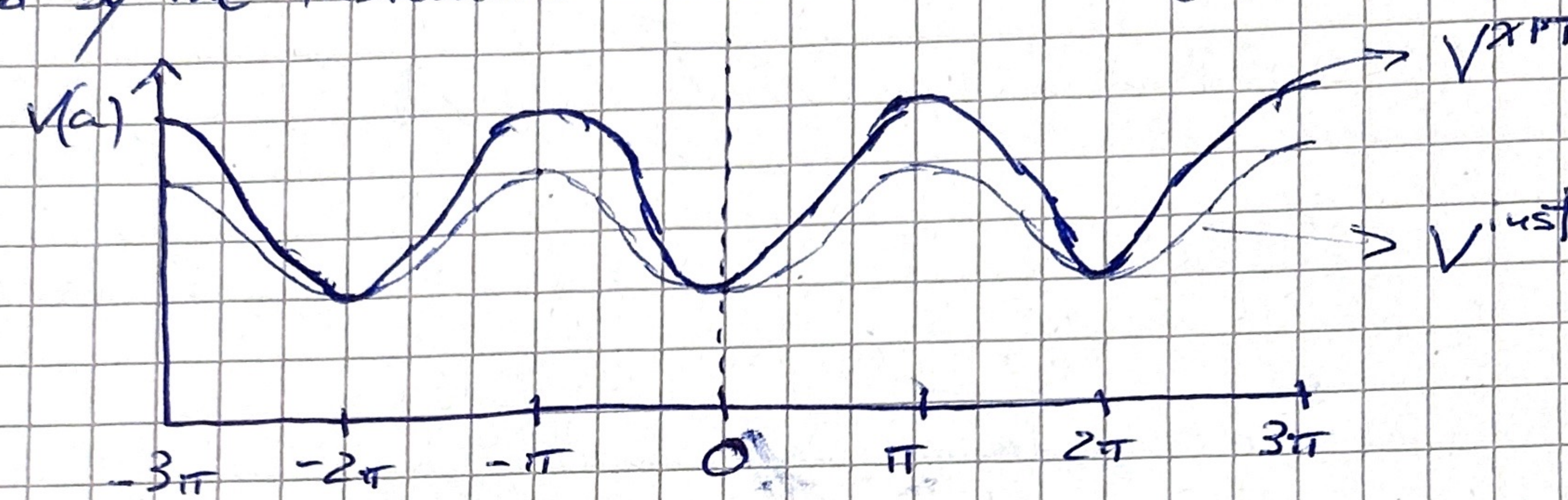
This expression has the absolute minimum for $\cos\left(\frac{\pi^0}{f_\pi} - \phi(a)\right) = 1$, so for $\langle\pi^0\rangle = f_\pi\,\phi(\langle a\rangle)$. For the solution of the strong CP problem: $\langle a\rangle = 0 \Rightarrow \phi(\langle a\rangle) = 0 \Rightarrow \langle\pi^0\rangle = 0$

On the vacuum π^0 gets a vev proportional to $\phi(a)$ to minimize the potential, the cosine in (*) is 1 and the axion effective potential is:

$$V(a) = -m_\pi^2 f_\pi^2 \sqrt{1 - \frac{4 m_u m_d}{(m_u + m_d)^2} \sin^2\left(\frac{a}{2f_a}\right)}$$

As expected the minimum is at $\langle a \rangle = 0$, solving the strong CP problem.

The chiral axion potential is nowhere close to the single cosine suggested by the instanton calculation $V^{inst}(a) = -m_a^2 f_a^2 \cos\left(\frac{a}{f_a}\right)$



In the limit $m_u = m_d$:

$$V(a)\Big|_{m_u = m_d} = -m_\pi^2 f_\pi^2 \left|\cos\frac{a}{2f_a}\right|$$

They differ $O(1)$ for $a = f_a$.

$$\overset{*}{=} -f_\pi^2 B_0 \left[m_u \cos\left(\frac{\pi^0}{f_\pi} - \frac{a}{2f_a}\right) + m_d \cos\left(\frac{\pi^0}{f_\pi} + \frac{a}{2f_a}\right)\right]$$

To read the axion mass we expand around the vacuum $\begin{cases} \langle a \rangle = 0 \\ \langle \pi^0 \rangle = 0 \end{cases}$

And from: $\sqrt{1-\alpha \sin^2 x} \simeq 1 - \frac{1}{2}\alpha \sin^2 x \simeq 1 - \frac{1}{2}\alpha x^2$

we have:

$$V(a) \underset{a \ll f_a}{\simeq} - m_\pi^2 f_\pi^2 + \frac{1}{2} \frac{m_\pi^2 f_\pi^2}{f_a^2} \frac{m_u m_d}{(m_u+m_d)^2} a^2 + O(a^4)$$

So the axion mass is:

$$m_a^2 = \frac{m_\pi^2 f_\pi^2}{f_a^2} \frac{m_u m_d}{(m_u+m_d)^2} \Rightarrow$$

$$\Rightarrow m_a \simeq 5.7\, meV \left(\frac{10^9\, GeV}{f_a} \right) \ll \Lambda_{\chi PT} \simeq 1\, GeV$$

And this justifies the use of χPT for $f_a \gg 1\, GeV$

Additionally, we can define the Compton wavelength as

$$\lambda_a = \frac{1}{m_a} = \frac{197 \cdot 10^{-4}\, \cancel{meV} \cdot cm}{\cancel{MeV}} \left(\frac{meV}{m_a} \right) =$$

$$= 1.97 \cdot 10^{-2}\, cm \left(\frac{meV}{m_a} \right)$$

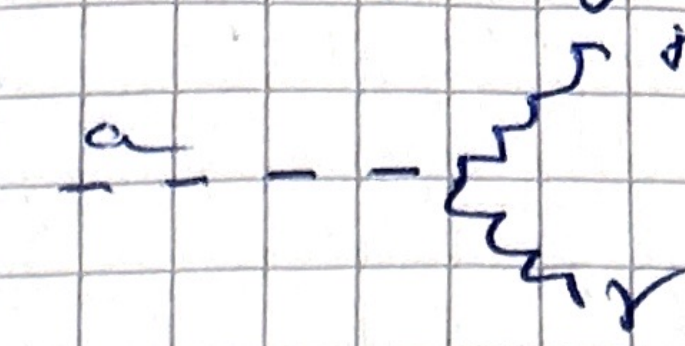
And the couplings to SM particle $g_{a x}$ scale as

$$g_{a x} \sim \frac{1}{f_a} \Rightarrow f_a \gtrsim 10^9\, GeV \quad \text{(Lab + astro bounds)}$$

So the axion is a weakly-coupled, light (sub-eV) particle, with a macroscopic wavelength

The axion-photon coupling is described by the Lagrangian

$$\mathcal{L}_{a\gamma} = -\frac{1}{4} g_{a\gamma}\, a F\tilde{F}$$



with $g_{a\gamma} = \frac{\alpha}{2\pi f_a}\left(\frac{E}{N} - 1.92\right)$

To obtain this expression for $g_{a\gamma}$ we need to introduce anomalies.
Anomalies arise from the non-invariance of the path integral measure

$$\mathcal{D}q\mathcal{D}\bar{q} \xrightarrow[q \to e^{i\alpha\gamma_5} q]{} \mathcal{D}q\mathcal{D}\bar{q}\, e^{-i\int d^4x\, \alpha\left[\frac{e^2}{16\pi^2} F\tilde{F}\right]}$$

[Fujikawa 1979]

The general form of the anomaly for a global current is:

$$\partial_\mu j^{\mu a}_{(global)} = \frac{g^2}{32\pi^2} \mathrm{Tr}\left[T^a_{(global)} \underbrace{\{t^a, t^b\}}_{\text{generators of } G}\right] \epsilon^{\mu\nu\rho\sigma} F^b_{\mu\nu} F^c_{\rho\sigma}$$

gauge coupling of G

~~Transf $e^{i\gamma_5 T^a \alpha}$, $q \to e^{i\gamma_5 T^a} q$~~

So, cases of interest in low-energy QCD are ~~$e^{i\gamma_5\alpha}$ $U(1)_A$, $e^{i\gamma_5 \frac{\lambda^a}{2}\alpha^a}$ $SU(N_f)_A$~~

$G = SU(3)_c$

$$j^\mu_5 = \bar{q}\gamma^\mu\gamma^5 q \to \partial_\mu j^\mu_5 = \frac{g_s^2}{32\pi^2} \mathrm{Tr}\left[\overset{\text{Flavour}}{\mathbb{1}} \overset{\text{Color}}{\{t^b, t^c\}}\right] \epsilon^{\mu\nu\rho\sigma} G^b_{\mu\nu} G^c_{\rho\sigma} =$$

$q \to e^{i\gamma^5\alpha} q$

$$= \frac{g_s^2}{32\pi^2} \underbrace{\mathrm{Tr}\, \mathbb{1}}_{N_f}\ 2\, \underbrace{\mathrm{Tr}\, t^b t^c}_{1/2\, \delta^{bc}} \epsilon^{\mu\nu\rho\sigma} G^b_{\mu\nu} G^c_{\rho\sigma} =$$

$$= \frac{g_s^2}{32\pi^2} N_f\, \epsilon^{\mu\nu\rho\sigma} G^a_{\mu\nu} G^a_{\rho\sigma} = \frac{g_s^2}{32\pi^2} 2N_f\, G^a_{\mu\nu} \tilde{G}^{\mu\nu a}$$

$$j^\mu_{5a} = \bar{q}\gamma^\mu\gamma^5 T^a q \to \partial_\mu j^\mu_{5a} \propto \mathrm{Tr}\left[T^a\{t^b, t^c\}\right] \propto$$

$$\propto \underbrace{\mathrm{Tr}\, T^a}_{0}\ \mathrm{Tr}\, t^b t^c = 0$$

$G = U(1)_{EM}$

$$j_5^\mu = \bar q \gamma^\mu \gamma_5 q \longmapsto \partial_\mu j_5^\mu = \frac{e^2}{32\pi^2} \mathrm{Tr}\left[\mathbb{1}\{Q,Q\}\right] \epsilon F F =$$

$$= \frac{e^2}{32\pi^2} \underbrace{N_c}_{\text{Number of colors}} \, 2\, \underbrace{\mathrm{Tr}\, Q^2}_{\mathrm{Tr}\begin{bmatrix} 4/9 & & \\ & 1/9 & \\ & & 1/9 \end{bmatrix}} \epsilon F F = \frac{e^2}{32\pi^2} N_c \frac{4}{3} F\tilde F$$

$$j_{5a}^\mu = \bar q \gamma^\mu \gamma_5 T^a q \to \partial_\mu j_{5a}^\mu = \frac{e^2}{32\pi^2} \mathrm{Tr}\left[T^a \{Q,Q\}\right] \epsilon F F =$$

$$= \frac{e^2}{32\pi^2} 2\, \mathrm{Tr}\left[T^a Q^2\right] \epsilon F F$$

E.g. $a = 3$ $\quad \mathrm{Tr}\left[T^3 Q^2\right] = N_c \, \mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\begin{pmatrix} 4/9 & & \\ & 1/9 & \\ & & 1/9 \end{pmatrix}\right] =$

$$= N_c \cdot \frac{1}{2} \cdot \frac{1}{3} = \frac{N_c}{6}$$

So $\partial_\mu j_{5,3}^\mu = \frac{e^2}{16\pi^2} \frac{N_c}{6} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \Rightarrow$ This is responsible for ~~...~~ $\pi^0 \to \gamma\gamma$

In general

$$\delta S = -\int d^4x \, \alpha \, \partial_\mu j_{5,a}^\mu$$

So now let's go back to the axion-dependent axial transform.

$$q \mapsto e^{i\gamma_5 \frac{a}{2f_a} Q_a} q$$

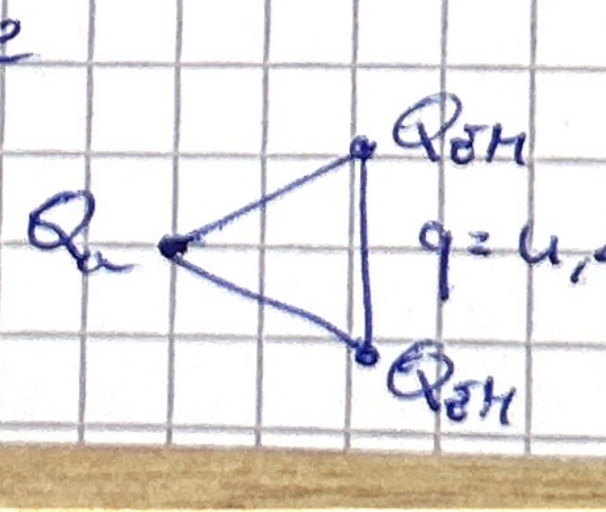
Since quarks are EM charged $Q_{EM} = \begin{pmatrix} 2/3 & \\ & -1/3 \end{pmatrix}$ [We are in $N_f = 2$!]

This transformation generates an anomalous $aF\tilde F$ term

$$\delta S = \underbrace{-2N_c \frac{e^2}{32\pi^2} \mathrm{Tr}\left[Q_a Q_{EM}^2\right] \frac{1}{f_a}}_{\equiv \frac{1}{4} g_{a\gamma}} \int d^4x \, a F \tilde F \quad , \quad \tilde F = \frac{1}{2}\epsilon F$$

So for $N_c = 3$ (number of colors) we have

$$g_{a\gamma} = -\frac{e^2}{4\pi} \frac{3}{\pi} \frac{1}{f_a} \mathrm{Tr}\left[Q_a Q_{EM}^2\right]$$



We Know $\mathrm{Tr}\,Q_a = 1$ $\Rightarrow$ In particular it is convenient to employ a choice s.t. the axion-pion mass mixing is absent, i.e. the so-called canonical basis.

Indeed if you consider the term

$$\mathcal{L}^{\chi PT}_{axion} \supset \frac{1}{2} f_\pi^2 B_0 \,\mathrm{Tr}\left[\Sigma M_a^\dagger + h.c.\right]$$

And expand to quadratic order in the fields

$$\Sigma = e^{i\pi/f_\pi} \simeq \mathbb{1} + \frac{i\pi}{f_\pi} - \frac{1}{2}\frac{\pi^2}{f_\pi^2}$$

with $\pi^2 = \begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix}\begin{pmatrix} \pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0 \end{pmatrix} = \begin{pmatrix} \pi^{0\,2}+2\pi^+\pi^- & 0 \\ 0 & \pi^{0\,2}+2\pi^+\pi^- \end{pmatrix}$

$$M_a^\dagger = e^{-i\frac{a}{2f_a}Q_a} M_q e^{-i\frac{a}{2f_a}Q_a} \simeq \left(\mathbb{1} - i\frac{a}{2f_a}Q_a\right) M_q \left(\mathbb{1} - i\frac{a}{2f_a}Q_a\right) =$$

$$= M_q - \frac{ia}{2f_a}\left(Q_a M_q + M_q Q_a\right) + \dots = M_q - \frac{ia}{2f_a}\left\{Q_a, M_q\right\}$$

We find:

$$\Sigma M_a^\dagger \simeq \Sigma M_q - \frac{ia}{2f_a}\Sigma\left\{Q_a, M_q\right\}$$

And so

$$\frac{1}{2} f_\pi^2 B_0 \,\mathrm{Tr}\left[\Sigma M_a^\dagger + h.c.\right] = f_\pi^2 B_0 (m_u + m_d) - \frac{1}{2}B_0 (m_u+m_d)\left(\pi^{0\,2} + 2\pi^+\pi^-\right) +$$

$$\underbrace{- \frac{i}{4} B_0 \frac{f_\pi^2}{f_a}\, a \,\mathrm{Tr}\left[\Sigma\left\{Q_a, M_q\right\}\right] + h.c.}_{} + \dots$$

This is the axion tadpole (e.g. axion-pion mixing)

So choosing:

$$Q_a = \frac{M_q^{-1}}{\mathrm{Tr}\, M_q^{-1}} = \begin{pmatrix} \frac{m_d}{m_u+m_d} & 0 \\ 0 & \frac{m_u}{m_u+m_d} \end{pmatrix}$$

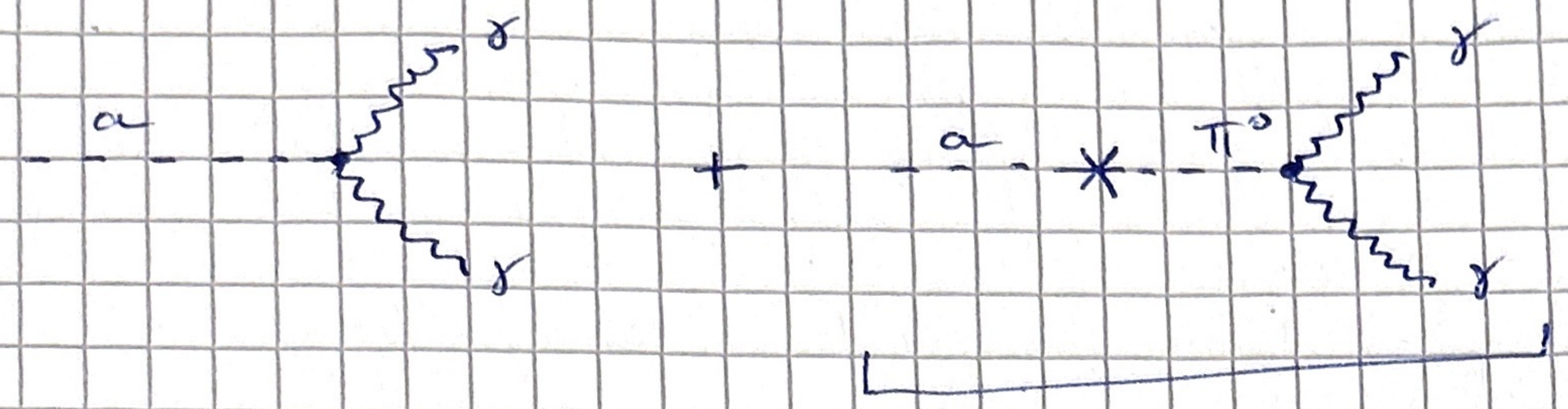
The axion tadpole terms are zero since:

$a\underbrace{\pi \dots \pi}_{odd}$ because of $\mathrm{Tr}\,\sigma^3 = 0$, $\quad a\underbrace{\pi \dots \pi}_{even}$ because of $+h.c.$

So if we consider the axion-photon coupling

$$\mathcal{L} \supset \frac{1}{4} g_{a\gamma}\, a F \tilde{F}$$

The possible interacting diagrams are



This term is not present for the choice $Q_a = \frac{M_q^{-1}}{\mathrm{Tr}\, M_q^{-1}}$

So:

$$\cancel{\tfrac{1}{4}}\, g_{a\gamma} = -\frac{e^2}{4\pi}\frac{3}{\pi}\frac{1}{f_a}\mathrm{Tr}\left[Q_a Q_{EM}^2\right] =$$

$$= -\frac{3\alpha}{\pi}\frac{1}{f_a}\mathrm{Tr}\left[Q_a Q_{EM}^2\right] =$$

$$= -\frac{3\alpha}{\pi}\frac{1}{f_a}\mathrm{Tr}\left[\begin{pmatrix} \frac{m_d}{m_u+m_d} & 0 \\ 0 & \frac{m_u}{m_u+m_d}\end{pmatrix}\begin{pmatrix} \frac{4}{9} & 0 \\ 0 & \frac{1}{9}\end{pmatrix}\right] =$$

$$= -\frac{3\alpha}{\pi}\frac{1}{f_a}\frac{4m_d + m_u}{9(m_u+m_d)} = \frac{\alpha}{2\pi f_a}\left[-\frac{2}{3}\frac{4m_d+m_u}{m_u+m_d}\right]$$

Now, since $\frac{m_u}{m_d} \simeq 0.48 \Rightarrow \frac{2}{3}\frac{4m_d+m_u}{m_u+m_d} \simeq 2.0$

Considering NLO χPT this factor becomes $1.92(4)$ [1511.02867]

More generally we can include in the axion effective Lagrangian also model-dependent coupling to photons and SM fermions

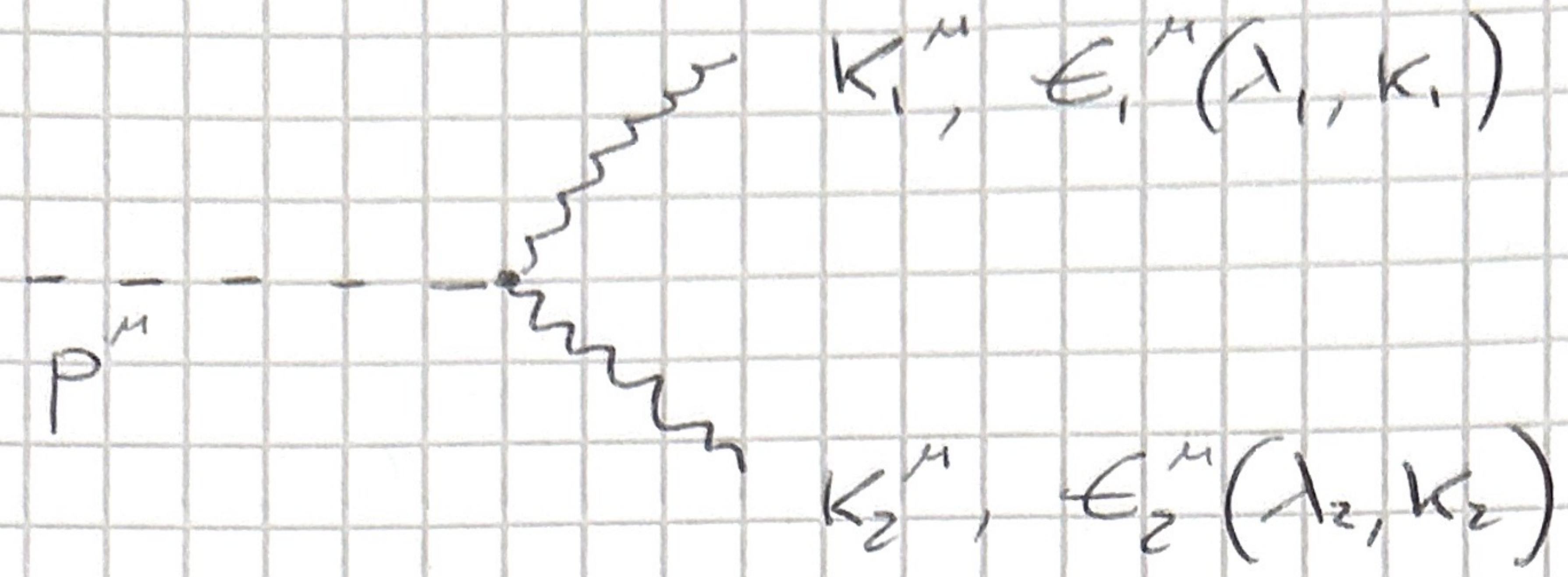
$$\mathcal{L} \supset \frac{1}{4} g^0_{a\gamma}\, a F\tilde{F} + \frac{\partial_\mu a}{2 f_a}\bar{q}\, c_q^0\, \gamma^\mu \gamma_5\, q + \ldots$$

Then the axion coupling to photons becomes

EM to QCD anomaly ratio

$$g_{a\gamma} = g^0_{a\gamma} - \frac{\alpha}{2\pi f_a} 1.92(4) = \frac{\alpha}{2\pi f_a}\left[\frac{E}{N} - 1.92(4)\right]$$

Axion decay rate



$$p^\mu = (m_a, 0)$$
$$k_1^\mu = (|\vec{k}_1|, \vec{k}_1)$$
$$k_2^\mu = (|\vec{k}_2|, \vec{k}_2)$$

We are in the rest frame of the axion, so $p^\mu = (m_a, 0)$.
Since it is a two bodies decay, we can write

$$d\Gamma = \frac{1}{2}\,\frac{1}{32\pi^2}\,\frac{|\vec{k}_1|}{m_a^2}\,|\mathcal{M}|^2 d\Omega_1$$

where $\frac{1}{2}$ comes from the fact that the decay is into two identical particles
and the term $\frac{|\mathcal{M}|^2}{32\pi^2}\frac{|\vec{k}_1|}{m_a} d\Omega_1$ comes from the general formula:

$$d\Gamma = \frac{1}{2M}\prod_f \frac{d^3 p_f}{(2\pi)^3 2E_f}\,|\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}\left(P_M - \sum_f p_f\right)$$

for the decay rate of a particle with mass M into particles with 4-momentum p_f, where $P_M^\mu = (M, 0)$.

In our case:

$$|\vec{k}_1| = |\vec{k}_2| = \frac{m_a}{2} \Rightarrow \frac{|\vec{k}_1|}{m_a} = \frac{m_a}{2m_a^2} = \frac{1}{2m_a}$$

So:

$$d\Gamma = \frac{1}{2}\,\frac{1}{32\pi^2}\,\frac{1}{2m_a}\,|\mathcal{M}|^2 d\Omega$$

$$\mathcal{M}_{fi} = \langle \gamma\gamma | a \rangle = g_{a\gamma} \varepsilon^{\mu\nu\alpha\beta} \epsilon_\mu^* \epsilon_\nu^* K_{1\alpha} K_{2\beta}$$

We need to sum over polarizations:

$$|\overline{\mathcal{M}}|^2 = \sum_{pol.} |\mathcal{M}|^2$$

So:

$$|\overline{\mathcal{M}}|^2 = g_{a\gamma}^2 \sum_{\lambda_1, \lambda_2, \lambda_1', \lambda_2'} \varepsilon^{\mu\nu\alpha\beta} \epsilon_\mu^*(\lambda_1, K_1) \epsilon_\nu^*(\lambda_2, K_2) K_{1\alpha} K_{2\beta} \cdot$$

$$\cdot \varepsilon^{\rho\sigma\gamma\delta} \epsilon_\rho(\lambda_1', K_1) \epsilon_\sigma(\lambda_2', K_2) K_{1\gamma} K_{2\delta} =$$

$$= g_{a\gamma}^2 \varepsilon^{\mu\nu\alpha\beta} \varepsilon^{\rho\sigma\gamma\delta} \sum_{\lambda_1 \lambda_1'} \epsilon_\mu^*(\lambda_1, K_1) \epsilon_\rho(\lambda_1', K_1) \cdot \sum_{\lambda_2, \lambda_2'} \epsilon_\nu^*(\lambda_2, K_2) \epsilon_\sigma(\lambda_2', K_2) \cdot$$

$$K_{1\alpha} K_{2\beta} K_{1\gamma} K_{2\delta}$$

For photons: $\sum_{\lambda, \lambda'} \epsilon_\mu^*(\lambda, K) \epsilon_\nu(\lambda', K) = -g_{\mu\nu}$, so:

$$|\overline{\mathcal{M}}|^2 = g_{a\gamma}^2 \varepsilon^{\mu\nu\alpha\beta} \varepsilon^{\rho\sigma\gamma\delta} (-g_{\mu\rho})(-g_{\nu\sigma}) K_{1\alpha} K_{2\beta} K_{1\gamma} K_{2\delta} =$$

$$= g_{a\gamma}^2 \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu}{}^{\gamma\delta} K_{1\alpha} K_{2\beta} K_{1\gamma} K_{2\delta} =$$

$$= g_{a\gamma}^2 \varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu\gamma\delta} K_{1\alpha} K_{2\beta} K_1^\gamma K_2^\delta$$

From:

$$\varepsilon^{\mu\nu\alpha\beta} \varepsilon_{\mu\nu\gamma\delta} = -2 \begin{vmatrix} \delta^\alpha_\gamma & \delta^\alpha_\delta \\ \delta^\beta_\gamma & \delta^\beta_\delta \end{vmatrix} = -2 \left(\delta^\alpha_\gamma \delta^\beta_\delta - \delta^\alpha_\delta \delta^\beta_\gamma \right)$$

we have

$$|\overline{\mathcal{M}}|^2 = g_{a\gamma}^2 \, 2 \left(\delta^\alpha_\delta \delta^\beta_\gamma - \delta^\alpha_\gamma \delta^\beta_\delta \right) K_{1\alpha} K_{2\beta} K_1^\gamma K_2^\delta =$$

$$= g_{a\gamma}^2 \, 2 \left[K_{1\alpha} K_{2\beta} K_1^\beta K_2^\alpha - K_{1\alpha} K_{2\beta} K_1^\alpha K_2^\beta \right] =$$

$$= g_{a\gamma}^2 \, 2 \left[(K_1 \cdot K_2)^2 - K_1^2 K_2^2 \right]$$

Now:

$$p = k_1 + k_2$$

$$k_1^2 = k_2^2 = 0 \Rightarrow 2k_1 \cdot k_2 = p^2 = m_a^2$$

So:

$$|\overline{\mathcal{M}}|^2 = g_{a\gamma}^2 \, 2 \left[\frac{m_a^4}{4} \right] = g_{a\gamma}^2 \frac{m_a^4}{2}$$

Thus

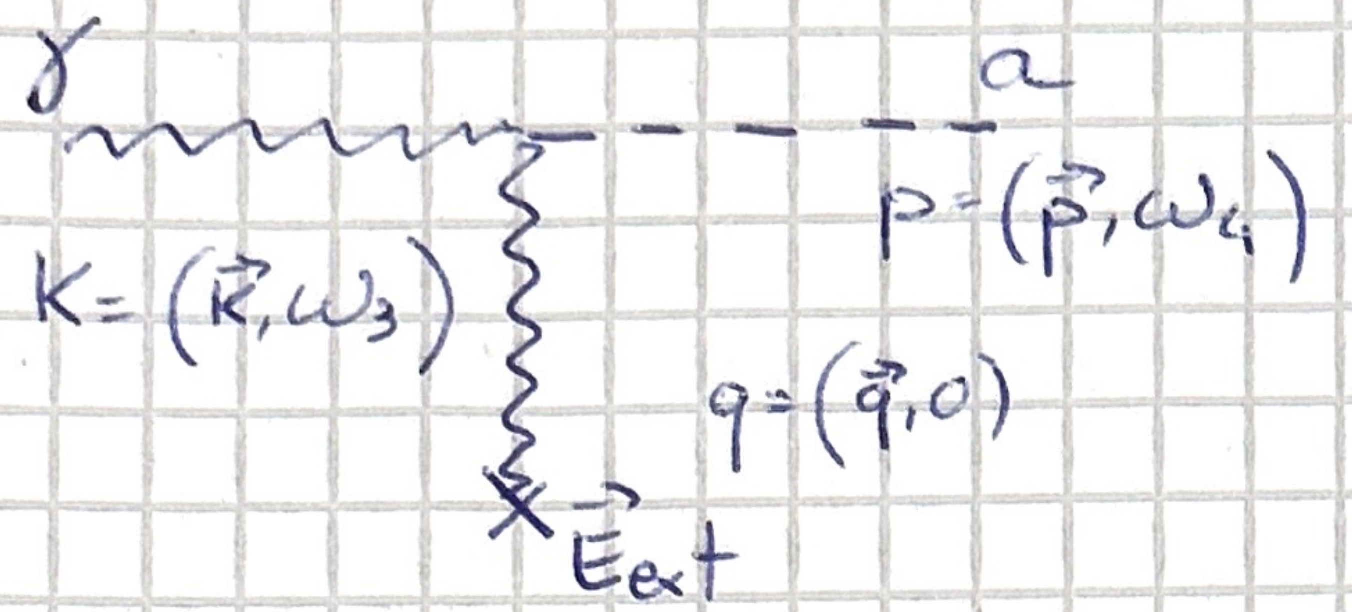
$$d\Gamma = \frac{1}{2} \cdot \frac{1}{32\pi^2} \frac{1}{2m_a} \cdot g_{a\gamma}^2 \frac{m_a^4}{2} \, d\Omega \Rightarrow$$

$$\Rightarrow \Gamma = \frac{1}{2} \frac{1}{32\pi^2} \frac{1}{2m_a} g_{a\gamma}^2 \frac{m_a^4}{2} 4\pi \Rightarrow$$

$$\Rightarrow \Gamma = \frac{g_{a\gamma}^2 m_a^3}{64\pi}$$

PRIMAKOFF EFFECT

$$\mathcal{L} = -\frac{1}{4} g_{a\gamma} F\tilde{F} = g_{a\gamma}\, \vec{E}\cdot\vec{B}$$



Since we are in an external magnetic field:

$$\mathcal{L} = g_{a\gamma}\, \vec{E}_{ext}\cdot\vec{B}_{rad} = g_{a\gamma}\, \vec{E}_{ext}\cdot\left(\vec{\nabla}\times\vec{A}\right)$$

From Feynman rules

$$\mathcal{M} = g_{a\gamma}\, \tilde{\vec{E}}_{ext}(\vec{q})\cdot\left(\vec{K}\times\varepsilon\right) = g_{a\gamma}\, \varepsilon\cdot\left(\tilde{\vec{E}}_{ext}(\vec{q})\times\vec{K}\right)$$

where ε is the photon polarization and $\tilde{\vec{E}}_{ext}$ is the Fourier transform of the external ~~magnetic~~ electric field

$$\tilde{\vec{E}}_{ext}(\vec{q}) = \int d\vec{r}\, e^{i\vec{q}\cdot\vec{r}}\, \frac{Ze\vec{r}}{r^3} = \frac{i\vec{q}\, Ze}{q^2}$$

So:

$$\mathcal{M} = iZe\frac{\cancel{1}}{|\vec{q}|^2}\, g_{a\gamma}\, \varepsilon\cdot\left(\vec{q}\times\vec{K}\right)$$

Since $\vec{q} = \vec{p}-\vec{k} \Rightarrow \vec{q}\times\vec{K} = \vec{p}\times\vec{K}$, and averaging over the initial photon polarization:

$$|\overline{\mathcal{M}}|^2 = \frac{1}{2}\left(g_{a\gamma} Ze\right)^2 \frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^4}$$

Now, the cross section is defined as the probability of interaction times density of final states divided by the incident flux. In this case the incident flux is $2\omega_3$ and the density of final states is $\frac{d^3\vec{p}}{(2\pi)^3\, 2\omega_4}$

So:

$$d\sigma = \frac{1}{2\omega_3}\frac{d^3\vec{p}}{(2\pi)^3 2\omega_4} 2\pi\delta(\omega_3-\omega_4)|\overline{\mathcal{M}}|^2 =$$

$$= \int \frac{1}{2\omega_3}\frac{d\omega_4\,\omega_4^2\,d\Omega}{(2\pi)^3 2\omega_4} 2\pi\delta(\omega_3-\omega_4)|\overline{\mathcal{M}}|^2$$

$$= \frac{1}{2\cancel{\omega}}\frac{\omega^{\cancel{2}}\,d\Omega}{(2\pi)^{\cancel{3}2} 2\cancel{\omega}} \cancel{2\pi}\frac{1}{2}(g_{a\gamma} Ze)^2\frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^4}$$

$$= \frac{1}{8}\frac{e^2}{4\pi}\frac{d\Omega}{\pi} g_{a\gamma}^2 Z^2 \frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^4} \Rightarrow$$

$$\Rightarrow \frac{d\sigma}{d\Omega} = \frac{g_{a\gamma}^2 Z^2\alpha}{8\pi}\frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^4}$$

In a plasma, the long range Coulomb potential is cut-off by screening effects and this implies the substitution in the differential cross section:

$$\frac{1}{|\vec{p}-\vec{k}|^4} \longmapsto \frac{1}{|\vec{p}-\vec{k}|^4}\frac{|\vec{p}-\vec{k}|^2}{k_s^2+|\vec{p}-\vec{k}|^2}$$

where k_s is the screening scale which is given by the Debye-Hückel formula:

$$k_s^2 = \frac{4\pi\alpha}{T}\sum_i n_i Z_i^2$$

where n_i is the number density of the nuclear species i.

So the differential cross section is:

$$\frac{d\sigma}{d\Omega} = \frac{g_{a\gamma}^2 Z^2\alpha}{8\pi}\frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^2(k_s^2+|\vec{p}-\vec{k}|^2)}$$

and the differential rate is:

$$\frac{d\Gamma}{d\Omega} = \sum_i n_i \frac{d\sigma}{d\Omega}$$

So the rate is

$$\Gamma = \frac{g_{a\gamma}^2 \alpha}{8\pi} \sum_i n_i Z_i^2 \int d\Omega \, \frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^2 \left(k_s^2 + |\vec{p}-\vec{k}|^2\right)}$$

$$\sum_i n_i Z_i^2 = \frac{T k_s^2}{4\pi\alpha}$$

So:

$$\Gamma = g_{a\gamma}^2 \frac{T k_s^2}{32\pi^2} \int d\Omega \, \frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^2 \left(k_s^2 + |\vec{p}-\vec{k}|^2\right)}$$

Now let's compute this integral for massless axions and photons:

$$|\vec{p}| = |\vec{k}| = \omega$$

$$\vec{p}\cdot\vec{k} = |\vec{p}|\,|\vec{k}|\,x = \omega^2 x$$

So

$$\int d\Omega \, \frac{|\vec{p}\times\vec{k}|^2}{|\vec{p}-\vec{k}|^2 \left(k_s^2 + |\vec{p}-\vec{k}|^2\right)} =$$

$$= 2\pi \int_{-1}^{+1} dx \, \frac{\omega^4 (1-x^2)}{(2\omega^2 - 2\omega^2 x)(k_s^2 + 2\omega^2 - 2\omega^2 x)} =$$

$$= 2\pi \int_{-1}^{+1} dx \, \frac{\omega^4 (1-x^2)}{2\omega^2 (1-x)\, 2\omega^2 \left(1 - x + \frac{k_s^2}{2\omega^2}\right)} =$$

$$= \frac{\pi}{2} \int_{-1}^{+1} dx \, \frac{1-x^2}{(1-x)\left(1-x+\frac{k_s^2}{2\omega^2}\right)} =$$

$$= \frac{\pi}{2} \left[-2 + 2\left(1 + \frac{k_s^2}{2\omega^2}\right) \ln\left(\frac{k_s^2/2\omega^2 + 2}{k_s^2/2\omega^2}\right) + \ln\left(\frac{k_s^2/2\omega^2 + 2}{k_s^2/2\omega^2}\right) \right] =$$

$$= \frac{\pi}{2} \left[-2 + 2\left(1 + \frac{k_s^2}{4\omega^2}\right) \ln\left(1 + \frac{4\omega^2}{k_s^2}\right) \right] =$$

$$= \pi \left[\left(1 + \frac{k_s^2}{4\omega^2}\right) \ln\left(1 + \frac{4\omega^2}{k_s^2}\right) - 1 \right]$$

So the production rate is:

$$\Gamma_{\gamma\to a} = g_{a\gamma}^2 \frac{T k_s^2}{32\pi}\left[\left(1+\frac{k_s^2}{4\omega^2}\right)\ln\left(1+\frac{4\omega^2}{k_s^2}\right)-1\right]$$