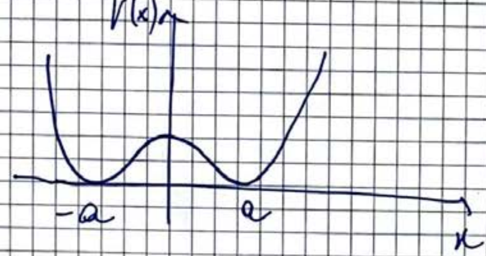
Istanton in MQ

Let us consider a particle in 1D motion with potential energy

$$V(x) = \alpha^2 (x^2 - a^2)^2$$

$\alpha(>0)$ let us determine the state of minimal energy



Classical physics: $\pm a$ $p=0$
Vacuum state is degenerate.
QM: it is not possible to measure position and momentum simultaneously. Vacuum is not degenerate
Parity of V: eigenfunctions are even or odd.

Semiclassical - approx. 2 distinct solution. Maximum probability of finding particle in $+a$ or $-a$.
However: tunnel effect → Particle can move from one minimum to the other one and none of them is eigenstate of energy.
Let us simplify $|-a\rangle$ $|+a\rangle$ wave-packet around $+a$ and $-a$.
If no tunnel were possible each of these state would be fundamental

$$H = \begin{pmatrix} E_0 & 0 \\ 0 & E_0 \end{pmatrix}$$

tunnel : off-diagonal elements
$E_0 = 0$ (up to a constant)

$$H = \begin{pmatrix} 0 & \Delta \\ \Delta & 0 \end{pmatrix}$$

$E = \pm \Delta$

Assume $\Delta < 0$

ground state $E = \Delta$

when ket

$$|0\rangle = |\Delta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$|-\Delta\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

We want estimate Δ.

Let us consider the propagator

$$U(t) = \exp\left(-\frac{iHt}{\hbar}\right)$$

$t = -i\tau$ (Assume t pure imaginary $(\tau \in \mathbb{R})$

$$U(\tau) = \exp\left(-\frac{H\tau}{\hbar}\right)$$

(Analytical prolongation of U along imaginary axis)

$$\langle +a| U(\tau) |-a\rangle = \langle +a| e^{-H\tau/\hbar} |-a\rangle$$

For small τ we can write

$$\langle a| e^{-H\tau/\hbar} |-a\rangle \simeq -\frac{\tau}{\hbar}\Delta + \dots$$

We ensure semi-classical estimation

$$\langle +a| e^{-H\tau/\hbar} |-a\rangle \simeq e^{-S_c/\hbar}$$

S_c is euclidean action calculated along the classical

Feynman path integral

$$\langle \tfrac{T}{2}, q_f | -\tfrac{T}{2}, q_i \rangle = U(q_f, t/2, q_i - t/2) \approx$$

$$\sim \int [dq]\, e^{iS[q]/\hbar} \simeq^{\text{semiclassical}} a\, e^{iS_c[x]/\hbar}$$

sum of all paths

$$iS_c = i \int_{-T/2}^{T/2} dt\, L(\dot{q}, q) \rightarrow \gamma + \int_{-T/2}^{T/2} d\tau\, L(q, \dot{q})$$

↓ Wick rotation
to get a convergent integral

$$t = -i\tau$$

$$L = \frac{m}{2}\left(\frac{dq}{dt}\right)^2 - V(q) = -\frac{m}{2}\left(\frac{dq}{d\tau}\right)^2 - V(q)$$

$$S_E = -\int d\tau \left[\frac{1}{2} m \left(\frac{dq}{d\tau}\right)^2 + V(q)\right]$$

inverted potential

$$e^{-S_E/\hbar}$$

trajectory that connect the wells

$$U(x, \tau, x') = \int [dx] \exp\left[-\frac{1}{\hbar}\int_0^\tau d\tau \, L_E\left(x, \frac{dx}{d\tau}\right)\right]$$

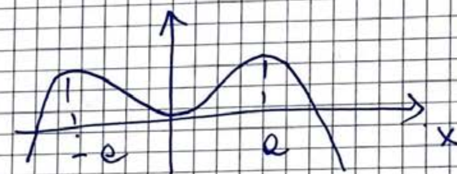
$$L_E = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)$$

Euclidean Lagrangian (sum of kinetic + potential energy)

Inverted potential

In classical mech. the particle cannot pass the eff barrier. Which is the classical trajectory!

$V'(x) = -V(x)$ Euclidean space.



$x_c(t)$: particle starts from $-a$ at $-\tau/2$ and reaches at $\tau/2$ a.

Conservation of energy

$$E = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)$$

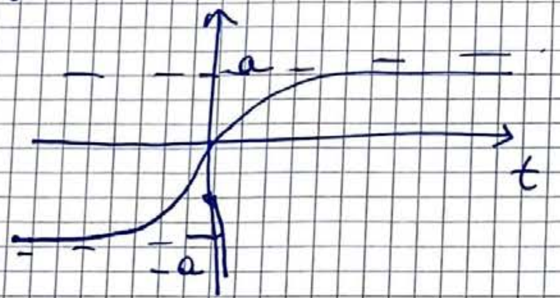
$$\int_{x_1}^{x_2} \frac{\sqrt{m}\, dx}{\sqrt{2(E + V(x))}} = \int_{t_1}^{t_2} d\tau$$

Using vacuum $E = 0$. Insert the potential energy

$$\sqrt{\frac{m}{2\lambda^2}}\int_{-a}^{a} \frac{dx}{x^2 - a^2} = \int_{-\tau/2}^{\tau/2} d\tau$$

$$x(\tau) = a \tanh \sqrt{\frac{2}{m}}\, \alpha a \tau$$

Instanton



Reaches $+a$ starting from $-a$ in infinite time

$$(-\tau/2, \tau/2) = (-\infty, +\infty)$$

$$S = \int d\tau\, (T+V) = 2\int d\tau\, T = \int_{-a}^{+a} p\, dx \overset{p = m\frac{dx}{d\tau}}{=} \int_{-a}^{+a} \sqrt{2mV(x)}\, dx$$

$$\langle +a | U(\tau) | -a \rangle \propto \exp\left[-\frac{1}{\hbar}\int_{-a}^{a} \sqrt{2mV(x)}\, dx\right]$$

Instantaneous transition ('t Hooft)

finite action configuration in euclidean space.

the transition can occur in any τ_0 in $(-\tau/2, \tau/2)$

$$\int_{-\tau/2}^{+\tau/2} d\tau_0 = \tau$$

$$\langle +a | U(\tau) | -a \rangle = \frac{\cancel{\tau}}{\hbar} \exp\left[-1/\hbar\, S_c\right]$$

$$\Delta = -\exp\left[-\frac{1}{\hbar} S_c\right].$$

$$\frac{\tau}{\hbar}\Delta = \frac{\tau}{\hbar} - \exp[-$$

$$\Delta = \frac{\hbar}{\tau} \exp[-\quad]$$

$$\Delta = \frac{\hbar}{\tau} \exp\left[-\frac{1}{\hbar} S_c\right]$$

SSB

∃ symmetry group

H symmetric

vacuum is not symmetric

2 cases in MQ

many minima separate
Infinite potential barrier. → SSB may happen at quantum level

- more minima separate by finite barrier: tunnel (instantons). Symmetry is restored.
No SSB.

Ex.

$$V(x) = 1 - \cos 2\pi x$$

minima $x = n$

$$T: \psi(x) \to \psi(x+1) \quad \text{translation symmetry}$$

$|k\rangle$ gaussian wave packet around $x = k$.

$$T|k\rangle = |k+1\rangle$$
$$[T, H] = 0$$

classically SSB

Instantons $\xrightarrow{}$ restore symmetry and remove degeneracy of vacuum

T-invariant vacuum

$$|0\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} |n\rangle$$

(e.g. solid state)

Let us consider the Hamiltonian $(\Delta > 0)$

$$H = H_0 + H_1 = \sum_{n=1}^{N} E_0 |n\rangle\langle n| - \Delta \{ |n\rangle\langle n+1| + |n+1\rangle\langle n| \}$$

$|1\rangle = |N+1\rangle$ periodic boundary translation invariant

$$T: |n\rangle \to |n+1\rangle$$

$[H, T] = 0$

$|n\rangle$ localized wave packet in $x = n$.

H_1 allows for transition $|n\rangle$ to next one.

$$H|n\rangle = E_0|n\rangle - \Delta|n-1\rangle - \Delta|n+1\rangle$$

H_1 model tunnel. (only to next one tight binding approx).

$$|\theta\rangle = \frac{1}{\sqrt{N}} \sum_{n=1}^{N} e^{in\theta} |n\rangle$$

$|\theta\rangle$ eigenket of T with eigenvalue $\exp(-i\theta)$

2) $T^N = 1$

3) $\theta = \frac{2\pi k}{N}$

4) $|\theta\rangle$ is eigenket with energy $E_0 - 2\Delta\cos\theta$

$$E_k = E_0 - 2\Delta\cos\theta = E_0 - 2\Delta\cos 2\pi k/N$$

at large N band $E_0 - 2\Delta$ a $E_0 + 2\Delta$

Brillouin zone —

What about field theory

tunneling is not possible in QFT

tunneling barrier is $V(0) - V(\pm a) \int d^3x$

integral over all space.

tunneling is shut down. Ground state ∞

concentrated in a or $-a$

SSB e.g.:

$$\mathcal{L} = \frac{1}{2}\left[(\partial\vec{\varphi})^2 + \mu^2\vec{\varphi}^{\,2}\right] - \frac{\lambda}{4}(\vec{\varphi}^{\,2})^2$$

It is impossible to pull an infinitely (spatially) extended field over a potential barrier of finite energy density.

Gauge theories loophole [lots of unphysical degrees of freedom to worry about] Topological winding about gauge degrees of freedom will lead to class of distinct gauge vacua — tunneling can occur btw these vacua (since such tunneling does not require a physical field being pulled over an energy potential) and this leads to the construction of Bloch waves (θ-vacua)

Instantons in 3+1 dim (Euclidean space)

- finite action configurations
- pure Yang-Mills for some gauge group G.

$$S = -\frac{1}{4}\int d^4x\, F^a_{\mu\nu} F^{\mu\nu a}$$

we need $F^a_{\mu\nu}$ to vanish at spatial infinity

$\frac{1}{2}\lambda_a A_a^i = A^i$ $\qquad$ $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

$F_{\mu\nu} = \sum_\alpha \frac{\lambda^\alpha}{2} F^\alpha_{\mu\nu}$

$\mathrm{Tr}(\lambda^\alpha \lambda^\beta) = 2\delta^{\alpha\beta}$ $\quad \alpha = 1 \ldots 8$ SU(3) matrices

Let us work in Weyl (= temporal) gauge $A_0 = 0$, where the theory reduces

$F_{0i}^{a} = \partial_0 A_i^a \qquad \mathcal{L} = \frac{1}{2}(\partial_0 A_i^a)^2 - \frac{1}{4} F_{ij}^a F_{ij}^a = \frac{1}{2}(\vec{E}_a^2 - \vec{B}_a^2)$

Lagrangian is difference of kinetic and potential energy in Yang Mills sense.

Action of gauge transformation Ω on a gauge field A is

$$A_\mu \to \Omega(x)\left(\frac{1}{ie}\partial_\mu + A_\mu\right)\Omega^{-1}(\vec{x})$$

Obviously the surviving invariance of gauge $A_0 = 0$ consist of time independent gauge transformations

$$\partial_t \Omega = 0 \Rightarrow \Omega(\vec{x}, t) = \Omega(\vec{x})$$

the Hamiltonian of the theory

$$H = \int d^3\vec{x}\,(E_i^a \partial_0 A_i^a - \mathcal{L}) = \frac{1}{2}\int d^3\vec{x}\,(\vec{E}_a^2 + \vec{B}_a^2)$$

commutes with gauge transformation

(sum of kinetic potential)

$$[H, \Omega(\vec{x})] = 0$$

We can diagonalize both operators simultaneously

$H|\psi\rangle = E|\psi\rangle \quad , \quad \Omega(\vec{x})|\psi\rangle = \omega(\vec{x})|\psi\rangle$

ω constant of motion.

* Infinitesimal gauge transformation give rise to eigenvalues λ

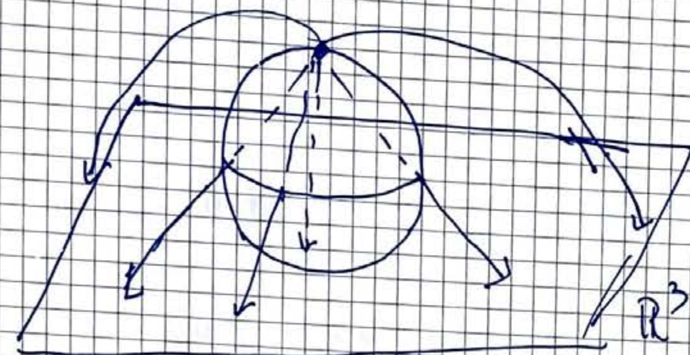
$\Omega(\vec{x}) = \mathbb{1} + i\varepsilon\Lambda(\vec{x}) \qquad \Lambda(\vec{x})|\psi\rangle = \lambda(\vec{x})|\psi\rangle$

the only values of λ consistent with unbroken special Lorentz transformation is

$\boxed{\lambda(\vec{x}) = 0}$

However, a class of $\Omega(\vec{x})$ exists that cannot be obtained from infinitesimal gauge rotations

E.g. stereographic projection



Identifies the three-space $\mathbb{R}^3$ compactified at spatial infinity with the three-sphere S^3.
If Ω has the same limit when going to spatial infinity in any direction, it can be regarded as a function on $\mathbb{R}^3 \cup \{\infty\} \simeq S^3$
Since $SU(2)$ is again a 3sphere we have

$$\Omega : S^3 \to S^3$$

these mapping can be classified by homotopy group.

Maps of S^n into a manifold M are classified by homotophy group $\pi_n(M)$ counts number of topologically inequivalent maps

$\Pi_n(S_m)$ $\quad S_n \to S_m$

Mapping circles onto circles

S_1 (ϑ defined mod 2π) mapped onto circle S_1 (described by Λ)

a) $\Lambda_0(\vartheta) = 0 \quad \forall \vartheta$:

b) $\Lambda_0^1(\vartheta) = \begin{cases} t\,\vartheta & 0 \le \vartheta < \pi \\ t(2\pi - \vartheta) & \pi \le \vartheta < 2\pi \end{cases}$

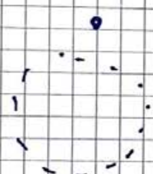
$t \in [0, 1]$

varying t the second map can be mapped onto the first continuously to 0

(a)

Map circle (continuous line) onto another circle (dashed line)

(b)



trivial map ($\Lambda = 0$) first circle is mapped onto a point of second circle

(c)

$\Lambda_0^1(\vartheta) = t\,\vartheta \qquad 0 \le \vartheta < \pi$

$t(2\pi - \vartheta) \qquad \pi \le \vartheta < 2\pi$

(b) (c) can be continuously deformed shrinking the loop.

They belong to same class of homotopy

$\Lambda_1(\theta) = \theta \qquad \forall\, \theta$

It is a continuous map and ...
It cannot be deformed in the previous one.

$\Rightarrow$ winding number: number of times map wraps around the other.

$$\tilde{Q} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\Lambda}{d\theta}\, d\theta$$

$Q = 0$ (a) (S)

$Q = 1$ (d)

$\Lambda_n(\theta) = n\theta \qquad \tilde{Q} = \cancel{\theta}\ n$

$\pi_1(S_1) = \mathbb{Z}$

$\pi_n(S^n) = \mathbb{Z}$

Coming back to our case for $SU(2)$

$$\pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$$

Again the one to one mapping Ω_1 is distinct from the trivial mapping $\Omega_0(\vec{x}) = 1$ and has [illegible] winding number one.

[illegible] of winding [illegible]

$\Lambda_1(\theta) = \theta \quad \forall \theta$

It is a continuous map not zero. It cannot be deformed in the previous one.

$\Rightarrow$ winding number: number of times map wraps around the other.

$$\tilde{Q} = \frac{1}{2\pi}\int_0^{2\pi} \frac{d\Lambda}{d\theta}\, d\theta \qquad Q = 0 \quad (a)\ (b) \qquad Q = 1 \quad (c)$$

$$\Lambda_n(\theta) = n\theta \qquad \tilde{Q} = n$$

$$\Pi_1(S_1) = \mathbb{Z}$$

$$\Pi_n(S^n) = \mathbb{Z}$$

Coming back to our case for $SU(2)$

$$\Pi_3(SU(2)) = \Pi_3(S^3) = \mathbb{Z}$$

Again the one to one mapping Ω_1 is distinct from the trivial mapping $\Omega_0(\vec{x}) = 1$ and has winding number one.
Representatives of higher windings are derived by

$$\Omega_n(\vec{x}) = (\Omega_1(\vec{x}))^n$$

Still these operators can be diagonalized together with the Hamiltonian. Since they are unitary their constant of motion are characterized by an angle θ

$$\Omega_1(\vec{x})|\psi\rangle = e^{i\theta}|\psi\rangle, \quad \Omega_n(\vec{x})|\psi\rangle = e^{in\theta}|\psi\rangle \quad 0 \le \theta \le 2\pi$$

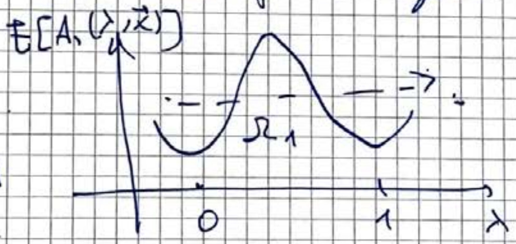
θ is a Lorentz invariant   Instanton angle
fundamental parameter of the theory
$\Omega_n(\vec{x})$ form topologically distinct gauge transformations
but they act on the space $\{A_i(\vec{x})\}$ which
is topologically trivial. Consider a
continuous line of gauge fields connecting two
gauge equivalent A's

$$A_i(\vec{x}) \to A_i(\lambda, \vec{x}) \qquad A_i(1, \vec{x}) = {}^{\Omega_1}A_i(0, \vec{x})$$

But at fractional λ is not a gauge transformation
these gauge fields lie on different orbits,
i.e. they are physically different!
So do their energies, i.e. the expectation
values

$$E[A_i(\lambda, \vec{x})] = \langle \{A_i(\lambda, \vec{x})\} | H | \{A_i(\lambda, \vec{x})\} \rangle$$

$\lambda = 0$, $\lambda = 1$ are vacua. The energy is higher in between



The system may tunnel through the gauge
transformation Ω_1

Finite action of a field at infinity
Euclidean space time $S^3 \subset \mathbb{R}^4$
Finite action requires Lagrangian density falls off quickly on this surface as $r \to \infty$.
$F \sim r^{-3}$ (boundary is a 3-sphere)
$\Rightarrow A_\mu \sim r^{-2}$
We are tempted to write
$$A_\mu(x)\big|_{|x|=R} = 0$$
this statement is not gauge invariant, the correct gauge-invariant statement is that ~~$A_\mu(R) \to$ ... ~~ $\Omega \frac{1}{g} \partial_\mu \Omega$
$$A_\mu(x)\big|_{|x|=R} \to -\frac{1}{g}\Omega^{-1}\partial_\mu \Omega$$
where $\Omega(x)$ is element of our gauge algebra $\Omega \in G$
Field must be <u>pure gauge</u>
configuration at ∞ given by maps from S^3 to G in the Euclidean space-time

E.g $G = SU(2)$ S^3 $\pi_3(S^3) = \mathbb{Z}$
↙ instanton number.

What about <u>Abelian case</u>?
~~[illegible]~~
$G = U(1)$ $\pi_3(U(1)) = 0$ no non trivial maps that cannot be smoothly deformed

no finite action configuration.

Next group is $SU(2)$. The key point is that any non-abelian group has $SU(2)$ as subgroup

$SU(2)$ generators: isospin

$G = SU(2)$

$\Omega = n^0 \mathbb{1} + i n^i \frac{\sigma_i}{2}$ $\quad (n^0)^2 + n^i n^i = 1 \rightarrow$ 3-sphere

$\tau = i\sigma/2$

Maps from $S^3_\infty \rightarrow SU(2)$

$$\Omega_1(x) = \frac{x_4 \mathbb{1} + i x_i \tau_i}{|x|} \qquad |x| = \sqrt{x_\mu x^\mu}$$

$S^3_\infty \rightarrow SU(2)$

$(\varphi, \theta_1, \theta_2) \rightarrow \Omega(\varphi, \theta_1, \theta_2)$

$\Pi_3(S^3) = \mathbb{Z}$ $\quad \Omega_n \rightarrow e^{i 2\pi n}$ $\quad r \rightarrow \infty$

$\Omega^n = (\Omega_1)^n$ $\quad n = 0, \pm 1, \pm 2$

Map from unit circle S_1 onto $U(1)$

Unit circle by α, $0 \leq \alpha < 2\pi$.

$$U(\alpha) = e^{i(n\alpha + \beta)} \qquad n \in \mathbb{N}$$

Winding number $\quad \Lambda(\alpha) = \vartheta = n\alpha + \beta$

$$n = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\Lambda(\alpha)}{d\alpha} d\alpha$$

Using the group element

$$n = -\frac{i}{2\pi} \int_0^{2\pi} \left(U^{-1} \frac{d}{d\alpha} U \right) d\alpha$$

$$n = \frac{1}{24\pi^2} \int d^3\theta\, \varepsilon^{ijk}\, \mathrm{tr}\left[(\Omega^{-1}\partial_i \Omega)(\Omega^{-1}\partial_j \Omega)(\Omega^{-1}\partial_k \Omega)\right]$$

The funny factor of $24\pi^2$ cancel terms from the angular integral so that n really is an integer. The winding number is related to the iteration of $S_3 \to S_3$ map.

Analogously one can write

$$n = \frac{i\, g_s^3}{24\pi^2} \int d^3r\, \mathrm{tr}\left(\varepsilon_{ijk} A^i_m A^j_n A^k_m\right)$$

"Pontryagin index"

If we introduce Chern-Simons current

$K^\mu = \varepsilon^{\mu\alpha\beta\gamma}$

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma} A^a_\alpha \left(G^a_{\beta\gamma} - \frac{g_s}{3} f_{abc} A^b_\beta A^c_\gamma\right)$$

that satisfies

$$\partial_\mu K^\mu = G_a^{\mu\nu} \tilde{G}_a^{\mu\nu}$$

where

$$\tilde{G}^a_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} G^{a\,\alpha\beta}$$

One finds

$$K^0 = -\frac{g_s}{3} \varepsilon_{ijk}\, \varepsilon_{abc} A^i_a A^j_b A^k_c =$$

$$= \frac{4}{3} i g_s\, \varepsilon_{ijk}\, \mathrm{Tr}(A^i A^j A^k)$$

$$n = \frac{g_s^2}{32\pi^2} \ldots$$

$$= \frac{g_s^2}{32\pi^2} \ldots$$

Instanton

$$S_E = \frac{1}{4} \ldots$$

$$= \frac{8\pi^2 n}{g^2}$$

$S_E \geq$

→ instan…

tunnel…

$x_4 \to +\infty$

$A = 0$

$x_4 \to -\infty$

In the … is trivi… then … vacuu…

$$n = \frac{g_s^2}{32\pi^2}\int K^\mu d\sigma_\mu = \frac{g_s^2}{32\pi^2}\int \partial_\mu K^\mu d^4x =$$

$$= \frac{g_s^2}{32\pi^2}\int G^a_{\mu\nu}\tilde{G}^{a\,\mu\nu} d^4x = \frac{g_s^2}{16\pi^2}\int \mathrm{tr}\left[G_{\mu\nu}\tilde{G}^{\mu\nu}\right] d^4x$$

Instanton

$$S_E = \frac{1}{4}\int d^4x\, F^a_{\mu\nu}F^{a\,\mu\nu} = \frac{1}{4}\int d^4x\left[F\tilde{F} + \frac{1}{2}(F-\tilde{F})^2\right] =$$

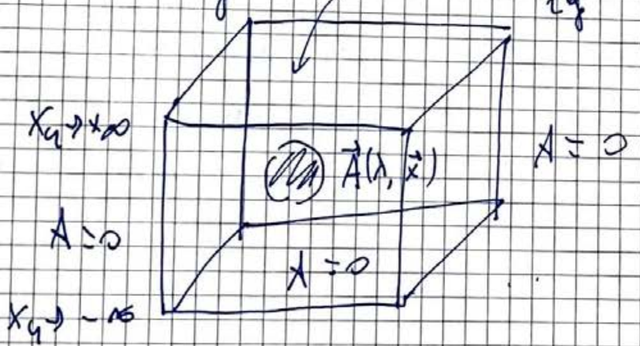
$$= \frac{8\pi^2 n}{g^2} + \frac{1}{8}\int d^4x\,(F-\tilde{F})^2$$

$$S_E \geq \frac{8\pi^2 n}{g^2}$$ saturated for $F = \pm\tilde{F}$

e^{-S_E}

→ <u>instanton</u>

tunneling $\vec{A} = \Omega_1(\vec{x})\frac{1}{ig}\vec{\partial}\Omega_1^{-1}(\vec{x})$



In the infinite (Euclidean) past the gauge field is trivial $A=0$

then it evolves and arrives at first non-trivial vacuum $\vec{A} = \Omega_1(\vec{x})\frac{1}{ig}\vec{\partial}\Omega_1^{-1}(\vec{x})$ in infinite future

Pure gauge

$$\vec{A} \to \Omega_1(\vec{x}) \frac{1}{ig} \vec{\partial}\, \Omega_1^{-1}(x) \qquad \Omega_1(x) = \begin{cases} \Omega_1(\vec{x}) & x_4 \to \infty \\ \text{const elsewhere} \end{cases}$$

Gauge equivalent (now we have $A_4 = 0$) but more symmetric way

$$A_\mu \to \Omega_1(x) \frac{1}{ig} \partial_\mu \Omega_1^{-1}(x)$$

$$\Omega_1(x) = \frac{x_4 \mathbb{1} + i x_i \sigma_i}{|x|} \qquad |x| = \sqrt{x_\mu x^\mu}$$

Ω_1 lives on the boundary of $\mathbb{R}^4$ which is a three-sph.

$$A_\mu(-T/2, \vec{x}) \to \frac{1}{ig} \Omega^{(0)} \partial_\mu \Omega^{(0)^{-1}}$$

$$A_\mu(T/2, \vec{x}) \to \Omega^{(1)} \partial_\mu \Omega_1^{-1}(x)$$

this suggest that we have a term J in our effective Lagrangian encoding the instanton and so that the tunnelling btw two vacua asymptotic value

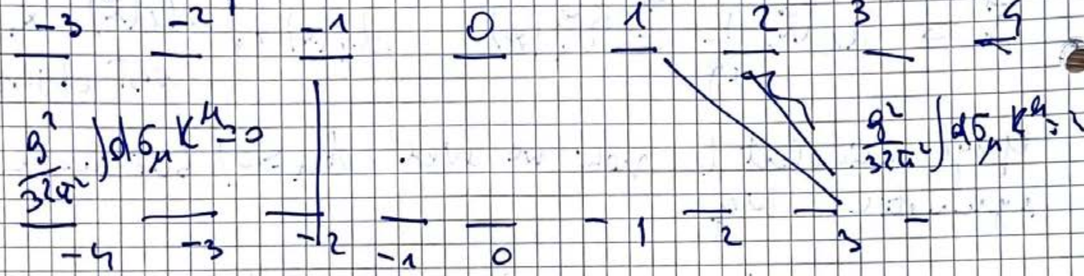
$$\langle n | e^{-iHt} | m \rangle_J = \int [dA]_{(n-m)}\, e^{-i\int d^4x\, \mathcal{L} + J I(x)}$$

J is the source of instanton $I(x)$.

θ- vacua

$|n\rangle$ vacua are not true vacua. Our quantum mec. analogy serves to construct Yang-Mills equivalent of $|\theta\rangle$ states, $|\theta\rangle$ vacua.

Pictorially one has



Correspon
u Q
Ana
Waves

Vacu
ectin =

⟨Ω
tan

⟨θ'q
⟨θ'|

$= \sum_{m,}$

This

Δ_J

$= \sum$

$\nu =$

Self

17.

Correspondence btw $|n\rangle$ vacua of nonmassive potential in QM and n-winding vacuum state of YM.
Analog of "translation invariance leads to "Bloch waves"

$$|\theta\rangle = \sum_n e^{-in\theta}|n\rangle$$

Vacuum-to-vacuum amplitudes using effective action, $|\theta\rangle$ vacua should be orthogonal

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \delta(\theta-\theta')\Delta_J(\theta).$$

task: determine Δ_J.

~~$\langle\theta'|$~~ learn about Lagrangian in contributions

$$\langle\theta'|e^{-iHt}|\theta\rangle_J = \sum_{m,n} e^{im\theta'}e^{-in\theta}\langle m|e^{-iHt}|n\rangle_J =$$

$$= \sum_{m,n} e^{-i(n-m)\theta}e^{im(\theta'-\theta)}\int[dA]_{n-m}\, e^{i\int d^4x\,\mathcal{L}+J\,\mathcal{I}}$$

J: external source

This tells us (writing $\nu = (n-m)$)

$$\Delta_J(\theta) = \sum_\nu e^{-i\nu\theta}\int[dA]_\nu\, e^{-i\int d^4x\,\mathcal{L}+J\,]} =$$

$$= \sum_\nu \int[dA]_\nu\, e^{-i\int d^4x\,\mathcal{L}_{eff}+J\,]}$$

$$\nu = n-m = \frac{g^2}{32\pi^2}\int d\sigma_\mu K^\mu\Big|_{t=-\infty}^{t=+\infty} = \frac{g^2}{32\pi^2}\int d^4x\, F_a^{\mu\nu}\tilde{F}_{a\,\mu\nu}$$

$$S_{eff} = S_{QCD} + \frac{\theta\, g^2}{32\pi^2}\int d^4x\, F_a^{\mu\nu}\tilde{F}_{a\,\mu\nu}$$

The $U(1)_A$ Problem of QCD

- In the 1970's strong interactions had a puzzling problem, which became particularly clear with the development of QCD.

QCD Lagrangian for N flavor

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F_{a\mu\nu} - \sum_f \bar{q} (-i\gamma^\mu D_\mu + m_f) q_f .$$

in the limit $m_f \to 0$ has a large global symmetry $U(N)_V \times U(N)_A$

$$q_f \to [e^{i\alpha_a T_a/2}]_{ff'} q_{f'}$$

Vector

$$q_f \to [e^{i\alpha_a T_a \gamma_5/2}]_{ff'} q_{f'}$$

- Since $m_u, m_d \ll \Lambda_{QCD}$, for these quarks $m_f \to 0$ limit is sensible. thus expect strong interactions to be approximately $U(2)_V \otimes U(2)_A$ invariant.
  - Indeed, experimentally known that $U(2)_V = SU(2)_V \otimes U(1)_V \equiv$ Isospin $\otimes$ Baryon# is a good approximate symmetry of nature $\Rightarrow (p, n)$ and $(\pi^\pm, \pi^0)$ multiplets in nature

$\to$ For axial symmetries, things are different. Dynamically quark condensates $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle \neq 0$ break $SU(2)_A$ down spontaneously: no mixed parity multiplets: E.g. no particle with $J^P = \frac{1}{2}^-$

approximately degenerate in mass with proton which has $J^P = \frac{1}{2}^+$

- However, because $U(2)_A$ is SSB, expect appearance in the spectrum of approximate Nambu-Goldstone bosons, with $m \simeq 0$ [$m \to 0$ as $m_u, m_d \to 0$]

For $U(2)_A$

generators $(T^1, T^2, T^3, T^4) = (\frac{1}{2}\sigma_1, \frac{1}{2}\sigma_2, \frac{1}{2}\sigma_3, \frac{1}{2}\mathbb{1})$

$[T^a, T^b] = i f_{abc} T^c$.

4 generators

$\rightarrow$ 4 such bosons (π, η)

Although pions are light, $m_\pi \simeq 0$, see no sign of another light state in the hadron spectrum

Mass formula suggests $m_\eta < \sqrt{3}\, m_\pi$

η' $m_{\eta'} = 958$ MeV cannot be η!

$m_\eta^2 \gg m_\pi^2$.

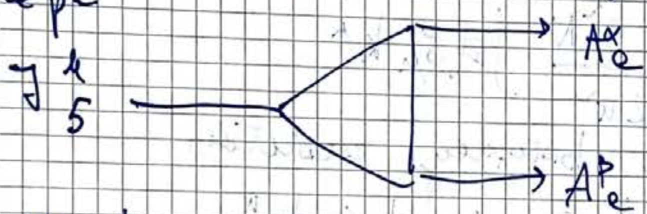
- Weinberg dubbed this the $U(1)_A$ problem and suggested that, somehow, there was no $U(1)_A$ symmetry in strong interactions and suggested that there was no $U(1)_A$ symmetry in the strong interactions

QCD vacuum.

- Resolution of $U(1)_A$ problem came through realization that QCD vacuum is more complicated ['t Hooft].
- Complexity → $U(1)_A$ not a symmetry of QCD even though it is an apparent symmetry of $\mathcal{L}_{QCD}$ in the limit $m_f \to 0$.

Chiral anomaly for axial current [Adler, Bell, Jackiw]

- the divergence of axial current get quantum corrections from the triangle graph



with fermions going around the loop

this anomaly gives a non-zero divergence

$$\partial_\mu J^a_5 = \frac{g^2 N}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\mu\nu}$$

N colour anomaly

$\neq$

even in symmetry limit

- Hence, in $m_f \to 0$ limit, although formally QCD is invariant under a $U(1)_A$ transformation

$$q_f \to e^{i\alpha/2\,\gamma_5} q_f$$

The chiral anomaly affect the action

$$\delta W = \alpha \int d^4x\, \partial_\mu J^\mu_5 = \alpha \frac{g^2 N}{32\pi^2} \int d^4x\, F_a^{\mu\nu} \tilde{F}_{a\,\mu\nu}$$

However, matters are not that simple!

- This is because the pseudoscalar density entering the anomaly, is in fact, a total divergence

$$F_a^{\mu\nu} \tilde{F}_{a\,\mu\nu} = \partial_\mu K^\mu$$

where

$$K^\mu = \varepsilon^{\mu\alpha\beta\gamma} A_{a\alpha} \left[F_{a\beta\gamma} - \frac{g_s}{3} f_{abc} A_{b\beta} A_{c\gamma} \right]$$

- This makes δW a pure surface integral

$$\delta W = \alpha \frac{g^2 N}{32\pi^2} \int d\sigma_\mu K^\mu$$

Using naive boundary conditions

$A_a^\mu = 0$ at $\infty \Rightarrow \int d\sigma_\mu K^\mu = 0$

$U(1)_A$ appears to be a symmetry again!

- What 't Hooft showed, however, is that the correct boundary condition to use is that A_a^μ be a pure gauge at ∞

i.e either $A_a^\mu = 0$ or gauge transformation of 0.

- It turns out that with these boundary conditions there are gauge configurations for which

$$\int d\sigma_\mu K^\mu \neq 0$$

and thus $U(1)_A$ is not a symmetry !.

- Resolution of $U(1)_A$ problem, by recognizing complicated nature of QCD's vacuum, effectively adds a extra-term to $\mathcal{L}_{QCD}$

$$\mathcal{L}_\theta = \theta \frac{g^2}{32\pi^2} F_a^{\mu\nu} \tilde{F}_{a\,\mu\nu} =$$

$$= \frac{g^2\theta}{16\pi^2} \operatorname{tr}\left[F_{\mu\nu} \tilde{F}^{\mu\nu}\right]$$

For: $\tilde{F}_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$

In electrodynamics $\tilde{F}$ is F with $\vec{B} \to -\vec{E}$, $\vec{E} \to \vec{B}$

$\tilde{F}$ is called dual field strength tensor

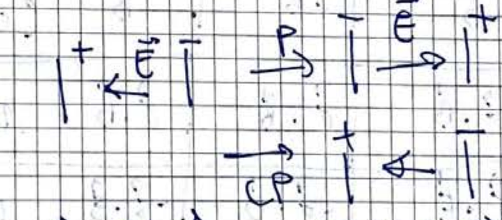
Most general Lagrangian

$$\mathcal{L} = -\frac{1}{2} \operatorname{tr}\left[F^{\mu\nu} F_{\mu\nu}\right] - \frac{g^2\theta}{16\pi^2} \operatorname{tr}\left[F_{\mu\nu} \tilde{F}^{\mu\nu}\right]$$

$$= -\frac{1}{2}\left(\vec{B}^2 - \vec{E}^2\right) + \frac{e^2\theta}{8\pi^2} \vec{B}\cdot\vec{E}$$

Let us talk about the new term

It is CP violating

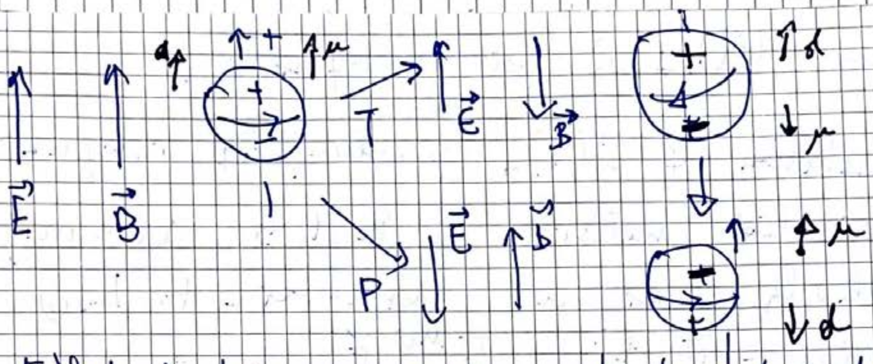


$$\vec{B} \propto \vec{r} \times \vec{J} \xrightarrow{P} (-\vec{r}) \times (-\vec{J}) \propto \vec{B}$$

$$\xrightarrow{CP} (-\vec{r}) \times (\vec{J}) \propto -\vec{B}$$

$$\Rightarrow F^{\mu\nu} \tilde{F}_{\mu\nu} \propto \vec{E}\cdot\vec{B} \xrightarrow{P} -F^{\mu\nu} \tilde{F}_{\mu\nu}$$

$$\xrightarrow{CP} -F^{\mu\nu} \tilde{F}_{\mu\nu}$$



Effect of time and parity transformation on electric or magnetic dipole moment ($\propto$ to spin) on an E or B field

$\vec{d} = q\,\vec{x}$ (negative to positive)

$\vec{\mu} = \vec{I}A$

Quantity	T	P	
$\vec{r}$	$\vec{r}$	$-\vec{r}$	
$\vec{p}$	$-\vec{p}$	$-\vec{p}$	polar
$\vec{\sigma}$	$-\vec{\sigma}$	$\vec{\sigma}$	(Axial $\vec{r}\wedge\vec{p}$)
$\vec{E}$	$\vec{E}$	$-\vec{E}$	($\vec{E} = -\frac{\partial\psi}{\partial r}$)
$\vec{B}$	$-\vec{B}$	$\vec{B}$	(consider a ring current)
$\vec{\sigma}\cdot\vec{B}$	$\vec{\sigma}\cdot\vec{B}$	$\vec{\sigma}\cdot\vec{B}$	Magnetic dipole moment
$\vec{\sigma}\cdot\vec{E}$	$-\vec{\sigma}\cdot\vec{B}$	$-\vec{\sigma}\cdot\vec{B}$	Electric dipole moment

CP and P violation.

Where could be observe this?

Electric dipole moment (EDM) $\vec{d}$

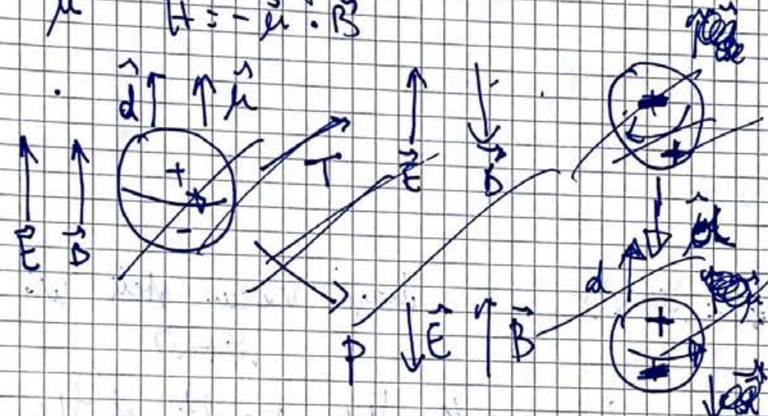
energy associated to electric field $\vec{E}$

$$H = -\vec{d}\cdot\vec{E} \qquad -d_n \vec{E}\cdot\vec{S}$$

Lower energy $\vec{d}$ aligned with $\vec{E}$

Neutron magnetic dipole moment (MDM)

μ $\quad H = -\vec{\mu}\cdot\vec{B}$

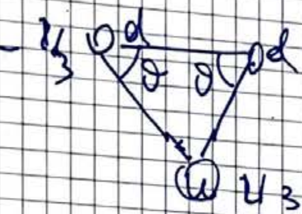


Existence of EDM would violate ~~CP, because it separately behaves differently from MDM under P and T~~

Both T and P symmetry as spin transform with opposite sign to the electric field under both transformations. Conversely, a magnetic dipole moment does not violate P or T symmetry as spin transforms in the same way as B field.

CPT is conserved → CP is violated

θ-term of QCD induces n EDM $\vec{d} \propto \theta$



smallness of the angles $\theta \simeq 10^{-10}$,

all quarks lie on almost perfect straight line

Dim analysis

EDM has unit charge × distance

quark charge $e/3$ $d \sim 10^{-15}$ m size neutron

product number 3×10^{-14} e m.

QFT

$$d_n \simeq e\frac{\theta m_q}{M_N^2} = 5\times 10^{-14}\,\theta\, e\, m.$$

Strong experimental bound

$$|d_n| \le 3\times 10^{-26}\ e\,cm \Rightarrow$$

$$|\theta| \le 10^{-10} \Rightarrow \text{strong CP problem}$$

this is actually worse if one considers the effect of chiral transformation of θ-vacuum. Chiral transformations, because of anomaly, actually can change that vacuum

$$e^{i\alpha Q_5}|\theta\rangle = |\theta+\alpha\rangle$$

If, besides QCD, one includes weak interactions the quark mass matrix is in general complex

$$L_{mass} = \bar{q}_{iR} M_{ij} q_{jL} + h.c.$$

to go on physical basis one must diagonalize the mass matrix and when one does one performs a chiral transformation that changes θ by arg det M, so that in full theory, the coefficient of $F\tilde{F}$ is

$$\bar{\theta} = \theta + \arg\det M$$

the strong CP problem is why the $\bar{\theta}$ angle, coming from strong and weak interactions, is so small?

Approaches to the strong CP problem

there are three possible "solutions" to the strong CP problem

1. there is really no CP problem.
2. Imposing CP symmetry (Spont. broken CP)
3. An additional chiral symmetry.

1) Is there a strong CP problem?

CP problem could be solved simply not taking the topological structure of QCD vacuum seriously. However, what about $U(1)_A$ problem? ~~It~~ it one cannot simply get rid of QCD topology, without providing an alternative explanation of why the η meson is not light! There is a way to preserve the feature of θ vacuum and yet have no strong CP problem.

If some quark really had zero mass, then, although $U(1)_A$ problem is solved by the existence of θ-vacua, all these vacua are equivalent and there is no CP violation. The most natural quark to have vanishing mass is u quark.
However pion masses and current algebra says no for physical up-quark mass.

Could bare up-quark mass be zero?
$\bar{m}_{ud} = (3.364 \pm 0.041)$ MeV
this idea is ruled out

2) Imposing CP symmetry (CP symmetry all SM also work -ish)
• Must have spontaneous CP violation to get observed CP breaking.
Then one can set $\theta = 0$ at the Lagrangian level.
However, θ gets induced back at the loop-level and to get $\theta < 10^{-9}$ one needs, in general, also to insure that $\theta_{1-loop} = 0$
• Although models exists where this is so, theories with SB CP need complex Higgs' VEV, leading to FCNC.
There is also the domain wall problem. CP is a discrete symmetry and if spontaneously broken by the vacuum, as the Universe cools

down to temperature below where the symmetry is SB, different domains of different CP phases form. Zeldovich Kobzarev and Okun showed that the surface energy density in the walls separating the various domains is intolerable. Furthermore, because these walls are 2D objects, the rate of total energy decreases as the universe cools down only like T. Hence, eventually, the energy density in the domain walls exceeds far the closure density of the universe unless process exist to annihilate the domain walls.

If no such process exist, typically, one expects

$$\rho_{wall} \sim \langle\phi\rangle^3 T \sim 10^{-7}\,\mathrm{GeV}^4$$

$\gg$ closure density of the universe today ($\rho_{clos} \sim 10^{-46}\,\mathrm{GeV}^4$)

the only sure cure against domain wall problem, if CP is SB, is to imagine CP phase transition occurs before an inflationary period. In that case the domains get enormously stretched and the domain wall energy is irrelevant, since we live in one domain. Solution of strong CP problem only if SB occurs at very high energy scale. (E.g. GUT)

Even if CP is broken sp at GUT scale it is not immediately obvious that CP problem is solved. Still loop contribution to Arg det M.

Moreover experimental data are in excellent agreement with CKM model, where CP is explicitly not spontaneously broken

3) Introducing an additional chiral symmetry

$e^{-i\theta Q} : |\theta\rangle = |0\rangle$

two suggestions for the chiral symmetry:

(i) the u-quark has no mass, $m_u = 0$

(ii) SM had an additional global $U(1)$ chiral symmetry [Peccei Quinn]

- $m_u = 0$ disfavoured by current algebra analysis. Difficult to understand $\operatorname{Arg}\det M = 0$. Which is the origin of this chiral symmetry?

Adjusting $\bar{\theta}$ to zero Dynamically

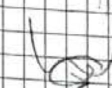
~~The SM~~ If SM has an additional global chiral symmetry, and this symmetry is unbroken, there is never any strong CP problem. This is like having a massless quark, because by a chiral transformation one can rotate $\bar{\theta}$ away.

The interesting physical case to consider, though, is the case when the SM is augmented by a $U(1)_{PQ}$ symmetry and this symmetry is SB. If the $U(1)_{PQ}$ symmetry is SB, there must exist in the theory a spin 0 excitation, with zero mass at the Lagrangian level. This excitation, the Goldstone boson of the broken $U(1)_{PQ}$ symmetry → axion discussed by Weinberg and Wilczek.

Under $U(1)_{PQ}$ symmetry, the axion field, being the field of a Goldstone boson, translates:

$$\cancel{a(x) \to a(x) + \delta\alpha f_{PQ}}$$

where
$$a(x) \to a(x) + \alpha f_a$$



shift symmetry

where f_a is the order parameter associated with the breaking of $U(1)_{PQ}$.

For a normal SB symmetry the fact that Goldstone bosons translate under the symmetry, implies that the effective Lagrangian containing the Goldstone bosons can only involve derivatives of these fields. The $U(1)_{PQ}$ symmetry, however, because it is chiral suffers from Adler Bell Jackiw anomaly. This anomalous behaviour of the Lagrangian of the theory is reproduced, simply, by having a term linear in the axion coupled to the anomaly.

Hence, the SM is augmented by a SB $U(1)_{PQ}$ symmetry, it will be described by

$$\mathcal{L}_{tot} = \mathcal{L}_{SM} + \bar{\theta}\frac{g_s^2}{32\pi^2} G_a^{\mu\nu}\tilde{G}_{a\mu\nu} - \frac{1}{2}\partial_\mu a\, \partial^\mu a + \mathcal{L}_{int}[\partial^\mu a/f_a, \psi] + \xi \frac{a}{f_a}\frac{g_s^2}{32\pi^2} G_b^{\mu\nu}\tilde{G}_{b\mu\nu}$$

where ξ and $\mathcal{L}_{int}$ are model-dependent quantities related to how one assigns $U(1)_{PQ}$ transformation on the fermions of the theory.

The last term is needed to ensure that $U(1)_{PQ}$ current indeed has a chiral anomaly

$$\partial_\mu J^\mu_{PQ} = \xi \frac{g_s^2}{32\pi^2} G_b^{\mu\nu}\tilde{G}_{b\mu\nu}$$

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the existence of U(1) breaking interaction $a F\tilde{F}$, due to chiral anomaly, actually provides a potential for the axion field. It is no longer true that in vacuum all values of $\langle a \rangle$ are allowed, reflecting the naive $U(1)_{PQ}$ symmetry of the Lagrangian. Including the anomaly contribution, one finds that the vacuum expectation value of the field is

$$\langle \bar{\theta} | a | \bar{\theta} \rangle = -\bar{\theta} \frac{1}{\xi} f_a$$

the physical axion field, of course, is the excitation with this vacuum expectation removed

$$a_{phys} = a - \langle \bar{\theta} | a | \theta \rangle$$

thus, in terms of this field one has

$$\mathcal{L}_{eff} = \mathcal{L}_{SM} + \mathcal{L}(\partial_\mu a_{phys}, \psi) - \frac{1}{2} \partial^\mu a_{phys} \partial_\mu a_{phys} + \frac{a_{phys}}{f_{PQ}} \xi \frac{g_s^2}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{a\,\mu\nu}$$

the presence of the extra $U(1)_{PQ}$ symmetry has eliminated the offending P, T and CP violating $\bar{\theta}$ term, replacing it by a dynamical field: the axion!

How can one understand $\langle \bar{\theta} | a | \bar{\theta} \rangle = -\bar{\theta} \frac{f_{PQ}}{\xi}$

this is easily seen by examining the e.o.m of the axion field. From (a) one has

$$-\partial^2 a + \partial^\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu a} = \frac{\xi}{f_{PQ}} \frac{g^2}{32\pi^2} F^{\mu\nu}_a \tilde{F}_{a\,\mu\nu}$$

If it were not for the anomaly term, it is clear that any constant value for the axion field in the vacuum is allowed. Eq of motion informs us however that the axion field settles in the vacuum at the value where

$$\left\langle \frac{\partial V_{eff}}{\partial a} \right\rangle = - \frac{\xi}{f_{PQ}} \frac{g^2}{32\pi^2} \langle F^a_{\mu\nu} \tilde{F}_a^{\mu\nu} \rangle \Big|_{\langle a \rangle = -\bar{\theta} f_{PQ}/\xi} = 0$$

The expectation value of the P, T, and CP violating density $F\tilde{F}$ is periodic in the relevant θ-parameter of the theory. From (a) this is simply $\bar{\theta} + \frac{\langle a \rangle}{f_{PQ}} \xi$ and the expectation of $F\tilde{F}$ in vacuum vanishes precisely when $\langle 0 | a | 0 \rangle = -\bar{\theta} \frac{1}{\xi} f_{PQ}$ is fulfilled (For instance, in one instanton approx, $\langle F^{\mu\nu}_a \tilde{F}_{a\mu\nu} \rangle$ is proportional to $\sin\left(\bar{\theta} + \frac{\langle a \rangle \xi}{f_{PQ}}\right)$)

Although the introduction of a SB $U(1)_{PQ}$ symmetry solves the strong CP problem, it necessitates the presence of a new dynamical field of the theory: the axion. The excitation which is nominally because it is a Goldstone boson, acquires a mass as a result of the chiral anomaly. Indeed,

$$m_a^2 = \left\langle \frac{\partial V_{eff}}{\partial a^2} \right\rangle = - \frac{\xi}{f_{PQ}} \frac{g^2}{32\pi^2} \frac{\partial}{\partial a} \langle F\tilde{F} \rangle \Big|_{\langle a \rangle = -\frac{\bar{\theta}}{\xi} f_{PQ}}$$

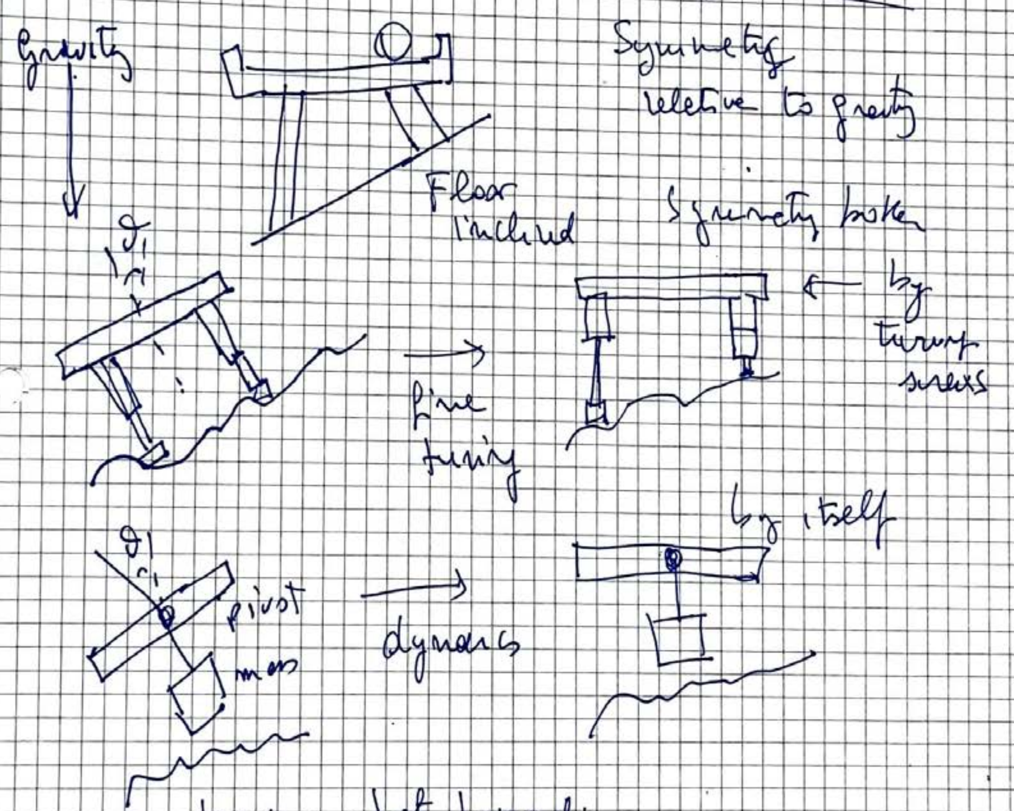
The axion mass is proportional to the curvature of the effective potential induced by the anomaly.

We ~~stick~~ also
On a pure dimensional ground

$$m_\phi^2 \sim \frac{\Lambda^4_{QCD}}{f_{PQ}^2}$$

where Λ^4_{QCD} typifies the scale of expectation of $\langle F\tilde{F}\rangle$.
If $f_{PQ} \gg \Lambda_{QCD}$, one sees that one predicts a very light boson, as the price for resolving the strong CP problem dynamically.

Pool table top analogy Sikivie hep-ph/9506229



Here's what happened:
Adds a potential $V = mg(1-\cos\theta)$
Pool table attached to T

Axion dynamics.

Standard and variant visible axion models

Simplest axion model. (Peccei and Quinn)

$U(1)_{PQ}$ to SM → Replace the ordinary Higgs doublet Φ and its charge conjugate $\tilde{\Phi} = i\tau_2\phi^*$ by two separate doublets Φ_1 and Φ_2. These fields have the same hypercharge as Φ and $\tilde{\Phi}$ but, since they are independent fields, they can now carry another U(1) charge. The excitation associated with this $U(1)_{PQ}$ charge the axion. Since this is all we are really interested in, it is convenient to isolate this overall phase field in Φ_1 and Φ_2 and drop the other excitations. In so doing, however, we must make sure not to mix pieces of axions with the hypercharge phase field, which eventually gets eaten by the Z_0. If the fields Φ_i have vacuum expectation values $\frac{1}{\sqrt{2}} v_i$, it is to see that one should write

$$\Phi_1 = \frac{1}{\sqrt{2}} v_1 e^{i a x / v_F} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \cancel{\Phi_2 = \frac{1}{\sqrt{2}} v_2 e^{i a/x}}$$

$$\Phi_2 = \frac{1}{\sqrt{2}} v_2 e^{i a / x v_F} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where $x = v_2/v_1$, $v_F = \sqrt{v_1^2 + v_2^2}$ ≈ 246 GeV. Orthogonal to weak hypercharge.

In this simple model, $v_{PQ} = v$, so that

$$a \to a + \alpha v_F$$

Having fixed the $U(1)_{PQ}$ transformation of the Higgs field 33
the transformation law of fermions are now determined by demanding that the Yuk. interactions be also $U(1)_{PQ}$ invariant. A particularly convenient definition for PQ symmetry transformation laws for fermions is to ensure that all l.h. doublet fields are PQ singlets. then PQ charge of r.h. fermions are fixed directly by specifying to which Higgs field they couple. In PQ model all charge 2/3 r.h. quark fields couple to Φ_1 and all $-\frac{1}{3}$ r.h. quark fields ~~couple~~ and r.h. leptons couple to Φ_2

$$L_{yuk} = \Gamma^u_{ij} \bar{Q}_{Li} \Phi_1 u_{Rj} + \Gamma^d_{ij} \bar{Q}_{Li} \Phi_2 d_{Rj} +$$
$$+ \Gamma^e_{ij} \bar{L}_{ei} \Phi_2 e_{Rj} + h.c.$$

thus their $U(1)_{PQ}$ transformations are simply

$$u_{Ri} \to e^{-i\alpha x} u_{Ri}$$
$$d_{Rj} \to e^{-i\alpha/x} d_{Rj}$$
$$l_{Ri} \to e^{-i\alpha/x} l_{Ri}$$

~~It is however~~

From these transformation the symmetry current transforms as

$$J^\mu_{PQ} = -v \partial^\mu a + x \sum_i \bar{u}_{Ri} \gamma^\mu u_{Ri} + \qquad \text{check}$$
$$+ \frac{1}{x} \sum_i \bar{d}_{Ri} \gamma^\mu d_{Ri} + \frac{1}{x} \sum_i \bar{l}_{Ri} \gamma^\mu l_{Ri}$$

this current has a chiral color anomaly which is proportional to the number of fermion families N_g and depends on x
$$\partial_\mu J^\mu_{PQ} = \underbrace{\left[N\left(x+\frac{1}{x}\right)\right]}\frac{g^2}{32\pi^2}F_a^{\mu\nu}\tilde{F}_{a\mu\nu}$$
$\frac{v_2}{v_1}+\frac{v_1}{v_2} = \frac{v_1^2+v_2^2}{v_1v_2}$

to compute the axion mass, it is useful to separate the effect of axion interaction with light quarks from the rest. → Effective chiral lagrangian. Heavy quarks accounted for through their contributions to chiral anomaly of J^μ_{PQ}.

For two light quarks u and d, $U(2)\times U(2)$ invariant chiral Lagrangian, describing strong interaction of pions and η, plus axion kinetic energy
$$\mathcal{L}_{chiral} = -\frac{1}{4}F_\pi^2\, \mathrm{Tr}\, \partial_\mu \Sigma^\dagger \partial^\mu \Sigma - \frac{1}{2}\partial_\mu a\, \partial^\mu a$$
Here the chiral field Σ is given by
$$\Sigma = \exp\left(i\,\frac{\vec{\tau}\cdot\vec{\pi}+\eta}{f_\pi}\right)$$
$SU(2)$ symmetry

f_π pion decay constant.

The second contribution is the effective Lagrangian mimics the effect of Yukawa interactions, which give masses to the quarks as a result of the $SU(2)\otimes U(1)$ breakdown.

At meson level, the presence of quark mass terms breaks the $U(2)\otimes U(2)$ symmetry explicitly. However, since the Yukawa interactions preserves the PQ symmetry, one must ensure that the mass breaking term one writes down for the mesons also preserves this symmetry. It is easy to see that the desired term is

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$$\mathcal{L}_{\text{mass breaking}} = \frac{1}{2}(f_\pi m_\pi^0)^2 \, \text{tr}\left[\Sigma A M + (\Sigma A M)^\dagger\right]$$

where

$$A = \begin{pmatrix} e^{-iax/v_F} & 0 \\ 0 & e^{-ia/xv_F} \end{pmatrix} \qquad M = \begin{pmatrix} \frac{m_u}{m_u+m_d} & 0 \\ 0 & \frac{m_d}{m_u+m_d} \end{pmatrix}$$

Note that PQ invariance is preserved since the shift $a \to a + \alpha v$ is compensated by the (right-handed) $U(2)$ transformation

$$\Sigma \to \Sigma \begin{pmatrix} e^{i\alpha x} & 0 \\ 0 & e^{i\alpha/x} \end{pmatrix}$$ to make u d quark

[Axion-pion Lagrangian]

L_{mass}, however, only gives part of the physics associated with the symmetry breakdown of $U(2)_A$. In fact, the quadratic terms, u mass involving neutral fields

expansion of $\cos(\ )$

$$L^{(2)}_{\text{mass}} = -\frac{(m_\pi^0)^2}{2}\left[\frac{m_u}{m_u+m_d}\left(\pi^0 + \eta - \frac{x f_\pi}{v_F} a\right)^2 + \frac{m_d}{m_u+m_d}\left(\eta - \pi^0 - \frac{f_\pi}{x v_F} a\right)^2\right]$$

give $\frac{m_\eta^2}{m_\pi^2} = \frac{m_d}{m_u} \simeq 1.6$

which contradicts experiments. Indeed, if L_{mass} was all that there was, we would have raised in the effective Lagrangian leaving the $U(1)_A$ problem!

Furthermore, with only this term the axion is still massless.

The resolution of $U(1)_A$ problem in the effective Lagrangian theory is achieved adding a further mass term that takes into account of the anomaly in both $U(1)_A$ and $U(1)_{PQ}$. This mass term gives the η the right mass and produces a mass for the axion. This term has the form

$$\mathcal{L}_{anomaly} = -\frac{(m_\eta^0)^2}{2}\left[\eta + \frac{f_\pi}{v_1}\frac{(N_g-1)(x+1/x)\,a}{2}\right]^2$$

where $(m_\eta^0)^2 \simeq m_\eta^2 \gg m_\pi^2$ decouples from low-energy dynamics.

The coefficient in front of the axion field in $\mathcal{L}_{anomaly}$ reflects the relative strength of the couplings of the axion and the η to $G\tilde{G}$ as the result of the anomalies in $U(1)_{PQ}$ and $U(1)_A$. Namely the ratio of these couplings is just $f_\pi/2v_1\xi$.

However, the reason that $N_g - 1$ appears in above, rather than N_g, is that $\mathcal{L}_{mass}$ already includes the light quark interaction of axions, so only the contribution of heavy quarks to PQ anomaly should be taken into account in $\mathcal{L}_{anomaly}$.

The presence of a mass breaking term and anomaly term gives mass to all excitations: π, η, a.

In the charged pion ~~term~~ sector the mass term

$$\mathcal{L}_{charged} = -\frac{(m_u+m_d)}{f_\pi^2}\, C_0\, \pi^+\pi^-$$

identifies $C_0 = \frac{f_\pi^2\, m_\pi^2}{(m_u+m_d)}$

In the neutral boson sector, the calculation of the mass eigenstates and physical fields is facilitated by the fact that $m_0^2 >> m_\pi^2$. One has

$$\mathcal{L}_{neutral} = -\frac{1}{2} m_\pi^2 \left\{ \frac{m_u}{(m_u+m_d)} \left[\pi^0 + \eta - \frac{a\, x\, f_\pi}{v}\right]^2 + \frac{m_d}{(m_u+m_d)} \left[-\pi^0 + \eta - \frac{a\, f_\pi}{x\, v}\right]^2 \right\} -$$

$$-\frac{1}{2} m_0^2 \left[\eta + \frac{f_\pi}{v}\left\{\frac{(N_g-1)}{2}\left(x+\frac{1}{x}\right)\right\} a\right]^2$$

For $m_0^2 >> m_\pi^2$ one easily reads, diagonalizing mass matrix, the axion mass

$$m_a^2 = \frac{m_\pi^2\, f_\pi^2}{v^2}\, \underbrace{N_g^2\left(x+\frac{1}{x}\right)^2}_{\lambda_m}\, \frac{m_u\, m_d}{(m_u+m_d)^2}$$

and that the field a contains a small admixture of π^0 and η fields:

$$a = a_{phys} - \xi_{a\pi}\, \pi^0_{phys} - \xi_{a\eta}\, \eta_{phys}$$

$$\xi_{a\pi} = \lambda_3 \frac{f_\pi}{v}, \quad \xi_{a\eta} = \lambda_0 \frac{f_\pi}{v}$$

$$\lambda_3 = \frac{1}{2}\left[\left(x - \frac{1}{x}\right) - N_g\left(x+\frac{1}{x}\right)\frac{m_d - m_u}{m_u+m_d}\right]$$

$$\lambda_0 = \frac{1}{2}(1-N_g)\left(x+\frac{1}{x}\right)$$

for ceric role

Remarks

1) $m_e \to 0$ when m_u or/and $m_d \to 0$.

Using $\frac{(m_u - m_u)}{(m_d + m_u)} \simeq 0.26$

one has $m_a \simeq 25\, K_g \left(x + \frac{1}{x}\right)$ keV.

In addition to the parameters above, all axion models are characterized also by how axions couple to two photons. Writing the interaction Lagrangian describing the coupling as

$$\mathcal{L}_{a\gamma\gamma} = \frac{\alpha}{4\pi} K_{a\gamma\gamma}\, \underline{a_{phys}}\, F_{\mu\nu} \tilde{F}^{\mu\nu}$$

f_a

one needs to find the coupling $K_{a\gamma\gamma}$ for the PQ model. This coupling follows from the electromagnetic anomaly of the PQ current

$$\partial_\mu J^\mu_{PQ} = \frac{\alpha}{4\pi} \xi_\gamma F_{\mu\nu} \tilde{F}^{\mu\nu}$$

where ξ_γ gets contributions from both quarks and leptons

$$\xi_\gamma = N_g \left\{ \left[3\left(\tfrac{2}{3}\right)^2\right] x + \left[3\left(-\tfrac{1}{3}\right)^2 + (-1)^2\right] \frac{1}{x} \right\}$$

$$= \frac{4}{3} N_g \left(x + \frac{1}{x}\right)$$

Separating light quark contribution of axions to the anomaly, $\xi_\gamma^{eff} = \frac{4}{3} N_g \left(x + \frac{1}{x}\right) - \frac{4}{3} x - \frac{1}{3}\frac{1}{x}$

this gives $K_{a\gamma\gamma} = N_g \left(x + \frac{1}{x}\right) \frac{m_u}{m_u + m_d} \doteq$

... + for cenic role

Unfortunately, a multitude of experiments cancel out / ruled out visible axion models.

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The strong bound obtained at KEK on the process
$BR(K^+ \to \pi^+ + \text{Nothing}) \leq 3.8 \times 10^{-8}$.

Estimation

$$BR(K^+ \to \pi^+ + a) \simeq 3 \times 10^{-5} \lambda_0^2 = 3 \times 10^{-5} \left(x + \frac{1}{x}\right)^2$$

However, invisible axion models, where $f_a \gg v_F$ are still viable.

Invisible axion models

All invisible axion models make use of some complex scalar field σ which:

- carry a PQ charge and possesses a large vacuum expectation value $\langle \sigma \rangle = \frac{v_{PQ}}{\sqrt{2}} \gg v$
- is an $SU(2) \times U(1)$ invariant.

Both these conditions are obviously necessary since one wants to split the scales of the PQ breaking from that of e.w. scale. Where invisible axion models differ is in the transformation properties of ordinary quarks (and leptons) under extra chiral symmetry. Broadly speaking two options have been considered

1. the known quarks and leptons do not feel the PQ symmetry, but there exist some (presumably very heavy) new quarks which carry a PQ charge. Prototype models of this type were first discussed by Kim, Shifman, Vainshtein, Zakharov (KSVZ axions)

... for their role

2. Ordinary quarks and leptons carry a PQ charge.
In that one necessitates again two Higgs doublets in the theory. Since the fermions in the theory do not couple directly to σ, they feel the PQ breaking only through the Higgs potential. Invisible axion models of this sort were first suggested by Dine, Fischler and Srednicki and by Zhitnitskii (DFSZ axions)

Both KSVZ and DFSZ axion models solve the strong CP problem, since vacuum expectation value of the axion field cancels the CP violating $\bar{\theta}$ term in the Lagrangian. The resulting axions, however, have somewhat different couplings to the ordinary fermions and to photons. These couplings can be obtained by following steps analogous to what used for standard axions.

The KSVZ Axion

The KSVZ model introduces a heavy quark X which carries a PQ charge and therefore can couple to σ

$$\mathcal{L} = -h\,\bar{X}_L \sigma X_R - h^*\,\bar{X}_R \sigma^\dagger X_L$$

It is again convenient to adopt a definition of the PQ transformation in which only X_R carries a PQ charge and to concentrate on the axion content of σ, by freezing the radial component of the field

$$\sigma = \frac{v_{PQ}}{\sqrt{2}}\, e^{i a / v_{PQ}}$$

KSVZ: new complex field Φ do not participate in weak interactions, i.e. it is an $SU(2)\otimes U(1)$ singlet. Massless fermion ψ and one consider a Lagrangian

$$\mathcal{L} = \left(\frac{i}{2}\bar\psi\, \partial_\mu \gamma^\mu \psi + h.c.\right) + \partial_\mu \phi^\dagger \partial^\mu \phi - V(|\phi|)$$

$$- h\left(\bar\psi_L \psi_R \Phi + h.c.\right)$$

h Yukawa coupling

Lagrangian invariant under chiral phase transformation

$$\Phi \to e^{i\alpha}\Phi, \quad \psi_L \to e^{i\alpha/2}\psi_L, \quad \psi_R \to e^{-i\alpha/2}\psi_R$$

Peccei - Quinn symmetry $U(1)_{PQ}$

$V(|\phi|)$ Mex. hat, $|\phi| = f_{PQ}/\sqrt{2}$ minimum

$$\langle\phi\rangle = (f_{PQ}/\sqrt{2})\, e^{i\varphi}$$

$$\Phi = \frac{f_{PQ}+\rho}{\sqrt{2}}\, e^{ia/f_{PQ}} \qquad a \text{ angular excitation}$$

V provides large mass for ρ, which will be of no further interest.

$$\mathcal{L} = \left(\frac{i}{2}\bar\psi\gamma_\mu\gamma^\mu\psi + h.c.\right) + \frac{1}{2}(\partial_\mu a)^2 - m\bar\psi e^{i\gamma_5 a/f_{PQ}}\psi$$

$m = h f_{PQ}/\sqrt{2}$. Variation of fermion field under the transformation $a \to a + \alpha f_{PQ}$. Shift symmetry [$U(1)$ sym]

Massless Goldstone-boson

Expanding the last term of $\mathcal{L}$ in powers of a/f_{PQ}

the zeroth order term $m\bar\psi\psi$ plays a role of effective fermion mass. Higher order describe interaction of a with ψ

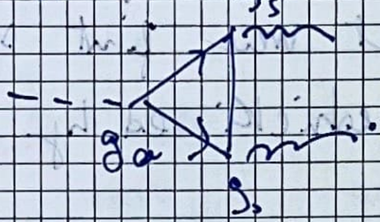
$$\mathcal{L}_{int} = -i\frac{m}{f_{PQ}}\, a\,\bar\psi\gamma_5\psi + \frac{m}{2f_{PQ}^2}\, a^2\bar\psi\psi$$

Yukawa coupling $g_a = m/f_{PQ}$.

ψ: exotic heavy quark, with usual strong interaction i.e. $SU_c(3)$ triplet. Lowest order interaction of a with gluons

$$\mathcal{L}_{aG} = -\frac{g_a}{m} \cdot \frac{\alpha_s}{8\pi}\, a\, G\tilde{G} \quad (*)$$

$$\alpha_s = \frac{g_s^2}{4\pi}$$



In more general models, several conventional or exotic quark fields may participate in this scheme. The transformation of each field under $U(1)_{PQ}$ is characterized by its PQ charge X_j:

$$\psi_L^j \to e^{i X_j \alpha/2}\, \psi_L^j$$

The total $aG\tilde{G}$ interaction is obtained as a sum of $(*)$ for all X_j. Because $g_{aj} = X_j m_j / f_{PQ}$ the fermion masses drop out. With

$N = \sum_j X_j$ and $f_a = f_{PQ}/N$

one has the required coupling $\dfrac{\alpha_s}{8\pi f_a}\, a G\tilde{G}$

The potential $V(a)$ is periodic with $2\pi f_a = 2\pi f_{PQ}/N$. The interpretation of a as phase of Φ, on the other hand, implies a periodicity with $2\pi f_{PQ}$ so that N must be a nonzero integer. This requirement also restricts assignment to the possible values of PQ charges. It also implies that there are N different equivalent ground states for axion fields, each of which sets $\bar{\theta} = 0$ and thus solves CP problem.

... for cleric ...

the DFSZ axion

In this model quarks and leptons have the same PQ symmetry properties as the SM axion, the two Higgs fields Φ_1 and Φ_2 necessary to allow the implementation of a PQ symmetry at the quark level are coupled to the Higgs field σ via a quartic term in the Higgs potential

$$\mathcal{L}_{quart} = \lambda \left[\Phi_1^T \epsilon (\sigma^*)^2 \Phi_2 \right] + h.c. \quad (*)$$

which fixes the PQ properties of σ relative to those of the Higgs fields and fermions. In the limit $v_{PQ} \gg v$ the axion content of the three Higgs fields is easily seen as

$$\sigma = \frac{v_{PQ}}{\sqrt{2}} e^{i a / v_{PQ}}$$

$$\Phi_1 = \frac{v_1}{\sqrt{2}} e^{i \frac{2 v_2^2}{v^2 v_{PQ}} a} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{v_1}{\sqrt{2}} e^{i \frac{X_1}{v_{PQ}} a} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\Phi_2 = \frac{v_2}{\sqrt{2}} e^{i \frac{2 v_1^2}{v^2 v_{PQ}} a} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{v_2}{\sqrt{2}} e^{i \frac{X_2}{v_{PQ}} a} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$X_1 = \frac{2 v_2^2}{v^2}, \quad X_2 = \frac{2 v_1^2}{v^2} \qquad v = \sqrt{v_1^2 + v_2^2}$$

that is, the hypercharge zero excitation of $\Phi(2\beta)$ only contains a fraction $\alpha = \frac{2 v_1 v_2}{v\, v_{PQ}}$ of the invisible axion field

thereby guaranteeing that Lagrangian (**) is invariant under PQ shift

$$a \to a + \kappa v_{PQ}$$

DFSZ $N = 3$ standard families

only known fermions carry PQ charges.

$f_a = f_{PQ}/N$, $x = \frac{v_1}{v_2}$ $x = \cot\beta$

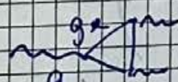
$\cos^2\beta = \frac{x^2}{x^2+1}$

Axion mass current algebra chiral perturbation

$$m_a = 5.70(6)(4)\,\mu eV \left(\frac{10^{12}\,GeV}{f_a}\right)$$

$$\mathcal{L}_{int} = -\frac{1}{4} g_{a\gamma} F_{\mu\nu}\tilde{F}^{\mu\nu} a = g_{a\gamma} \vec{E}\cdot\vec{B}\, a$$

In models where the quarks and leptons which carry PQ charges also carry electric charges, there is a contribution from a single loop diagram coupling



$Q_j \to Q_j e$ electric charge of lepton

It yields and axion-photon coupling proport to

$E = 2\sum_j X_j Q_j^2 D_j$ $D_j = 3$ for color triplet (quarks)

1 color singlet (charged leptons)

The total axion-photon coupling

$$g_{a\gamma} = -\frac{\alpha}{2\pi f_a}\frac{3}{4}\xi$$

$$\xi = \frac{4}{3}\left(\frac{E}{N} - \frac{2}{3}\,\frac{4+z+w}{1+z+w}\right) = \frac{4}{3}\left(\frac{E}{N} - 1.92 \pm 0.08\right)$$

$z = m_u/m_d \simeq 0.56$

$w = m_u/m_s = 0.029$

DFSZ one has a given family of quarks and leptons $\frac{E}{N} = 8/3$

$\xi \neq 1$

$$g_{a\gamma} = \frac{\alpha_{em}}{2\pi f_a}\left(\frac{E}{N} - 1.92(4)\right)$$

In original KSVZ, the new fermion has no charge and the ratio $E/N = 0$. In the new heavy quark has hypercharge similar to d quark (up) $E/N = 8/3$ $(2/3)$. KSVZ can be easily generalized to include more colored fermions and scalars, allowing for other values of E/N. However, under certain requirement of stability, values are $E/N \in [5/3, 44/3]$ yellow band.

DFSZ, this depends on which of the Higgs are coupled in Yukawa term of leptons, two new ones are possible, DFSZ-I, DFSZ-II

$\frac{E}{N} = 8/3$ DFSZ-I — compatible with GUT.

$2/3$ DFSZ-II

$f_a = \frac{v_a}{2N_G}$ $\quad N_{DW} = 2N$

number of inequivalent degenerate minima of axion potential, that corresponds to $\theta = 2\pi k/N_{DW}$ $\quad k \in \{0, 1, \dots, N_{DW}-1\}$, degeneracy in vacuum

to also compute
the axions across
scope, allowing the
models.

$\bar{q}\, c_q \gamma^\mu \gamma_5 q$

free ψ_R

$\bar{\psi}\gamma_\mu \gamma^5 \psi$

Neutra, linear

... for generic scale

Axion-like-particles (ALPs)

Share with QCD axion the two-photon vertex
The mass is nearly arbitrary. No relation btw m_a–$g_{a\gamma\gamma}$

String theory: Attempt to unify SM with gravity
Quantum gravity

- Spectrum of low-energy effective theory in (3+1)D is supersymmetric and possibly contains several kinds of very weakly interacting slim particles (WISPs)

An axiverse – QCD axions + many ultra-light ALPs whose mass spectrum is log hierarchical, may naturally arise from strings [Arvanitaki et al 0905.4720]

- String needs extra D → must compactify.
- Shape and size deformation of extra D correspond to fields: Moduli, and Axions, connected to fundamental string scale.

- Gauge field term

$$\mathcal{L} = -\frac{1}{4g^2}F^2 - \frac{\theta}{32\pi^2}F\tilde{F}$$

+ susy $\mathcal{L} = \mathrm{Re}\,[f(\Phi)]\,F^2 + \mathrm{Im}\,[f(\Phi)]\,F\tilde{F}$

Scalar moduli. Pseudoscalar ALP

Axion moduli always present