Revisiting evolution of GPDs

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Based on:

V. Bertone, R.F. del Castillo, M.G. Echevarria, O. del Rio, and S. Rodini,
[Phys.Rev.D 109 (2024) 3, 034023]

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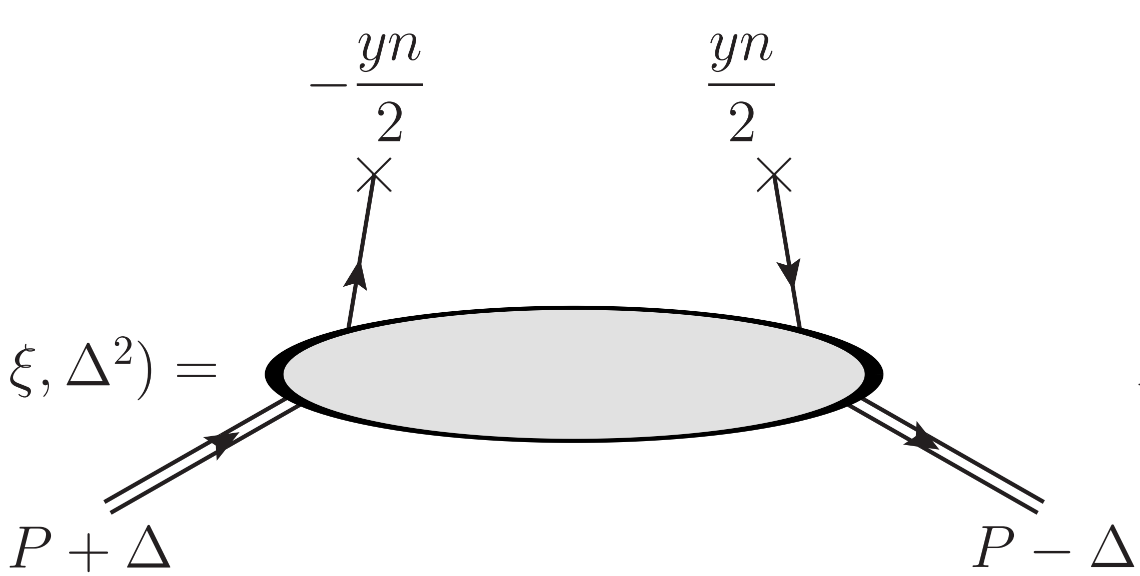
GPD definition

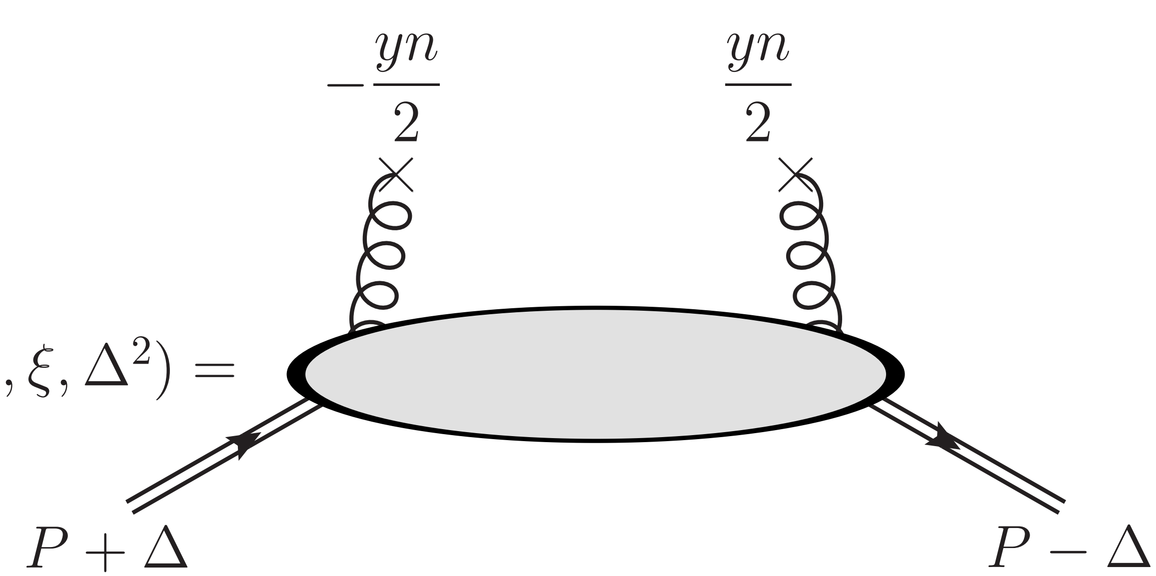
- Generalised parton distributions (GPDs) are a "byproduct" of factorisation of *amplitudes* for **exclusive** processes such as deeply-virtual Compton scattering.
  [Collins, Freund, *Phys.Rev.D* 59 (1999) 074009] [Ji, *Phys.Rev.D* 55 (1997) 7114-7125]

- An operator definition of the GPDs in the **light-cone gauge** ($n \cdot A = 0$) reads:

$$\hat{F}^{ij}_{q/H}(x, \xi, \Delta^2) = \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P - \Delta \left| \overline{\psi}^i_q \left(\frac{yn}{2}\right) \psi^j_q \left(-\frac{yn}{2}\right) \right| P + \Delta \right\rangle \qquad \xi = \frac{\Delta^+}{P^+}$$

$$\hat{F}^{\mu\nu}_{g/H}(x, \xi, \Delta^2) = \frac{n_\alpha n_\beta}{x(n \cdot P)} \int \frac{dy}{2\pi} e^{-ix(n\cdot P)y} \left\langle P - \Delta \left| F^{\mu\alpha}_a \left(\frac{yn}{2}\right) F^{\nu\beta}_a \left(-\frac{yn}{2}\right) \right| P + \Delta \right\rangle$$

$\hat{F}_{q/H}(x, \xi, \Delta^2) =$ 

$\hat{F}_{g/H}(x, \xi, \Delta^2) =$ 

GPD definition

- GPD correlators are obtained by projection:

$$\hat{F}^{[\Gamma]}_{q/H}(x,\xi,\Delta^2) = \frac{1}{2}\Gamma_q^{ij}\hat{F}^{ij}_{q/H}(x,\xi,\Delta^2)$$

$$\hat{F}^{[\Gamma]}_{g/H}(x,\xi,\Delta^2) = \Gamma_{g,\mu\nu}\hat{F}^{\mu\nu}_{g/H}(x,\xi,\Delta^2)$$

- Projectors are parameterised in terms of a basis of four four-vectors:
  - n and $\overline{n}$ parameterise the **longitudinal** directions,
  - R and L parameterise the **transverse** directions,
  - all scalar products are zero except: $(n\overline{n}) = -(RL) = 1$.
  - A typical realisation in Sudakov decomposition is:

$$n^\mu = (0,1,\mathbf{0}_T)\,,\quad \overline{n}^\mu = (1,0,\mathbf{0}_T)\,,\quad R^\mu = \left(0,0,-\frac{1}{\sqrt{2}}(1,i)\right)\,,\quad L^\mu = \left(0,0,-\frac{1}{\sqrt{2}}(1,-i)\right)$$

- The relevant **twist-2** projectors are:

$$\Gamma_q \in \{\not{n}, \not{n}\gamma_5, i\sigma^{\alpha+}\gamma_5\}$$

$$\Gamma_g^{\mu\nu} \in \left\{-g_T^{\mu\nu} \equiv -g^{\mu\nu} + n^\mu\overline{n}^\nu + \overline{n}^\mu n^\nu,\ -i\epsilon_T^{\mu\nu} \equiv -i\epsilon^{\alpha\beta\mu\nu}\overline{n}_\alpha n_\beta,\ -R^\mu R^\nu - L^\mu L^\nu\right\}$$

GPD definition

- GPD correlators are typically parameterised in terms of **eight** independent GPDs for quarks ($i = q$) and as many for gluons ($i = g$):
  - labelling $\Gamma_{q/g} \in \{U, L, T\}$.

$$\hat{F}^{[U]}_{i/H}(x,\xi,\Delta^2) = \frac{1}{n\cdot P}\overline{u}(P-\Delta)\left[\hat{H}_{i/H}(x,\xi,\Delta^2)\frac{\not{n}}{2} + \hat{E}_{i/H}(x,\xi,\Delta^2)\frac{i\sigma^{\mu\nu}n_\mu\Delta_\nu}{4M}\right]u(P+\Delta)$$

$$\hat{F}^{[L]}_{i/H}(x,\xi,\Delta^2) = \frac{1}{n\cdot P}\overline{u}(P-\Delta)\left[\hat{\tilde{H}}_{i/H}(x,\xi,\Delta^2)\frac{\not{n}\gamma^5}{2} + \hat{\tilde{E}}_{i/H}(x,\xi,\Delta^2)\frac{n^\mu\Delta_\mu\gamma^5}{4M}\right]u(P+\Delta)$$

$$\begin{aligned}\hat{F}^{[T]}_{i/H}(x,\xi,\Delta^2) &= \frac{1}{n\cdot P}\overline{u}(P-\Delta)\Bigg[\hat{H}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\sigma^{\mu j}\gamma^5}{2} + \hat{\tilde{H}}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}\Delta_\alpha P_\beta}{2M^2}\\ &+ \hat{E}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}\Delta_\alpha\gamma_\beta}{4M} + \hat{\tilde{E}}^{[T]}_{i/H}(x,\xi,\Delta^2)\frac{n_\mu\epsilon^{\mu j\alpha\beta}P_\alpha\gamma_\beta}{4M}\Bigg]u(P+\Delta)\end{aligned}$$

[Diehl, *Eur.Phys.J.C* 19 (2001) 485-492]

- All GPDs with the same polarisation label evolve in the same way.

GPD evolution

- Using dimensional regularisation in $4-2\varepsilon$ dimensions, the **UV** renormalisation of GPDs can be implemented in a multiplicative fashion:

$$F_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu) = \lim_{\varepsilon\to 0}\sum_{j=q,g}\int_{-1}^{1}\frac{dy}{|y|}Z_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s(\mu),\varepsilon\right)\hat{F}_{j/H}^{[\Gamma]}(y,\xi,\Delta^2;\varepsilon,\mu^{-\varepsilon})$$

- In the $\overline{\mathrm{MS}}$ scheme, renormalisation constants have the following structure:

$$Z_{ij}^{[\Gamma]}(z,\kappa,\alpha_s,\varepsilon) = \delta_{ij}\delta(1-z) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\sum_{p=1}^{n}\frac{1}{\varepsilon^p}Z_{ij}^{[\Gamma],[n,p]}(z,\kappa)$$

- Exploiting the independence of the bare GPDs on μ (for $\varepsilon\to 0$), one can derive a **RGE** governing the evolution of renormalised GPDs w.r.t. μ:

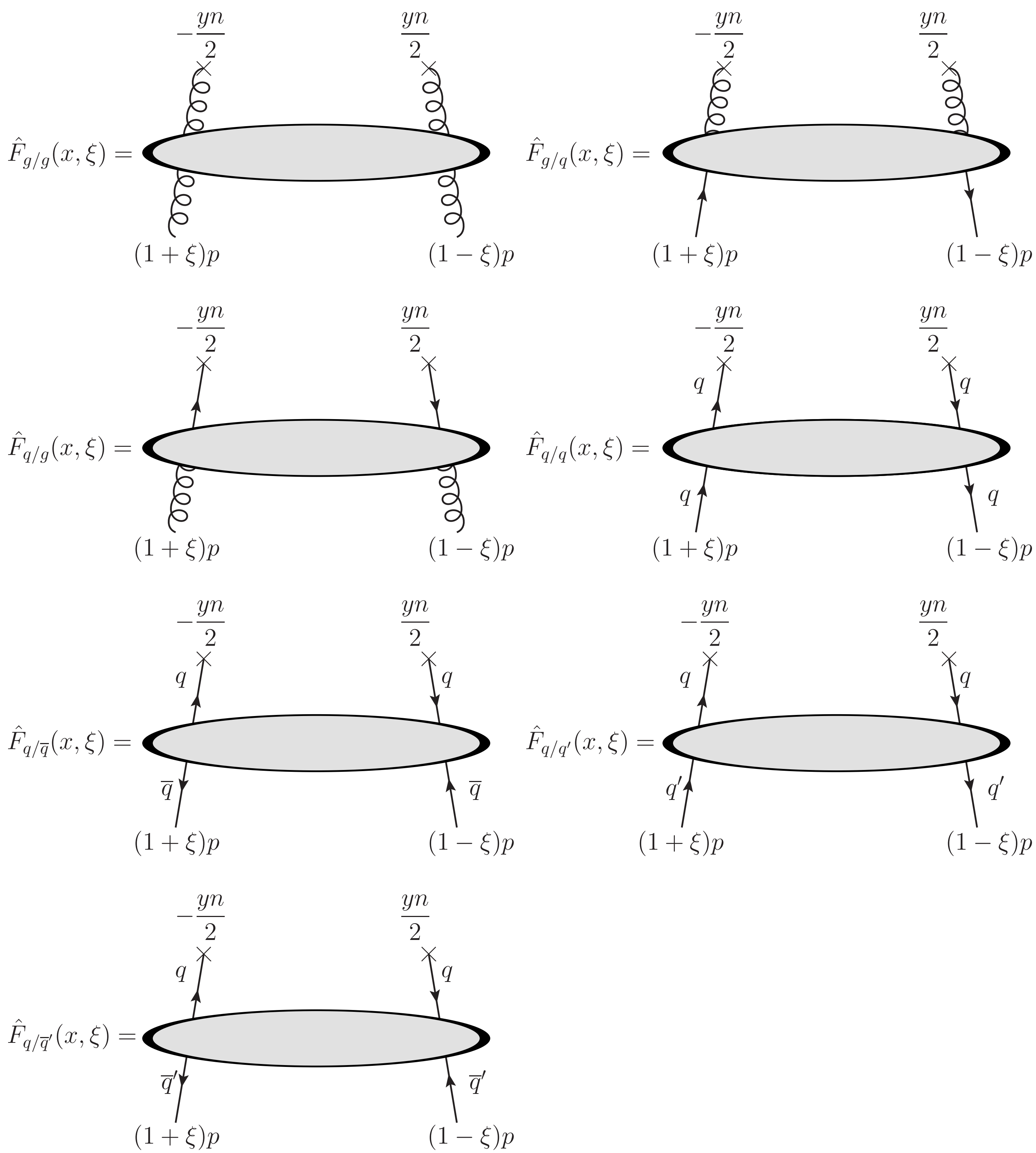
$$\frac{dF_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu)}{d\ln\mu^2} = \sum_{k=q,g}\int_{-1}^{1}\frac{dz}{|z|}\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s(\mu)\right)F_{k/H}^{[\Gamma]}(z,\xi,\Delta^2;\mu)$$

- The evolution kernels $\mathscr{P}$ are related to the normalisation constants Z:

$$\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s\right) = \lim_{\varepsilon\to 0}\sum_{j}\int_{-1}^{1}\frac{dy}{|y|}\frac{dZ_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s,\varepsilon\right)}{d\ln\mu^2}Z_{jk}^{[\Gamma]-1}\left(\frac{y}{z},\frac{\xi}{y},\alpha_s,\varepsilon\right)$$

Parton-in-parton GPDs

- The ren. constants Z can be extracted by means of **parton-in-parton** GPDs, *i.e.* GPDs where the *hadronic* states are replaced by *partonic* states.



- Dependence on Δ^2 can be neglected at twist-2.

Parton-in-parton GPDs

- In light-cone gauge:

$$\hat{F}^{[\Gamma]}_{g/g,q}(x,\xi) = \frac{(n\cdot p)(x^2-\xi^2)}{2(N_c^2-1)x}\int \frac{dy}{2\pi} e^{-ix(n\cdot p)y}\left\langle (1-\xi)p, s' \left| A^\mu_a\left(\frac{yn}{2}\right)\Gamma_{g,\mu\nu}A^\nu_a\left(-\frac{yn}{2}\right)\right| (1+\xi)p, s\right\rangle_{g,q} \Lambda^{[\Gamma]}_{s's}$$

$$\hat{F}^{[\Gamma]}_{q/g,q,\bar{q},q',\bar{q}'}(x,\xi) = \frac{1}{2N_c}\int \frac{dy}{2\pi} e^{-ix(n\cdot p)y}\left\langle (1-\xi)p, s' \left| \overline{\psi}^i_q\left(\frac{yn}{2}\right)\Gamma^{ij}_q\psi^j_q\left(-\frac{yn}{2}\right)\right| (1+\xi)p, s\right\rangle_{g,q,\bar{q},q',\bar{q}'} \Lambda^{[\Gamma]}_{s's}$$

- The projectors $\Lambda_{s's}$ are introduced for convenience to project out the physical partonic spin/helicity states that contribute to the amplitude:

$$\Lambda^{[\Gamma]}_{s's}\overline{u}_{q,s'}((1-\xi)p)u_{q,s}((1+\xi)p) = \Lambda^{[\Gamma]}_q = \sqrt{1-\xi^2}\,\{\not{p}, \not{p}\gamma^5, i\sigma^{\mu\nu}P_\nu\gamma^5\}$$

$$\Lambda^{[\Gamma]}_{s's}e^{\mu *}_{s'}((1-\xi)p)e^\nu_s((1+\xi)p) = \Lambda^{[\Gamma]\mu\nu}_g = \{-g^{\mu\nu}_T, -i\varepsilon^{\mu\nu}_T, -R^\mu R^\nu - L^\mu L^\nu\}$$

$$\Gamma \in \{U, L, T\}$$

Evolution kernels at one loop

- The general structure is for **all channels**:

$$\mathcal{P}_{ij}^{[\Gamma],[0]}(y,\kappa) = \theta(1-y)\left[\theta(1+\kappa)p^{\Gamma}_{i/j}(y,\kappa)+\theta(1-\kappa)p^{\Gamma}_{i/j}(y,-\kappa)\right]$$

$$+ \quad \delta_{ij}\delta(1-y)C_i\left[K_i-\ln\left(\left|1-\kappa^2\right|\right)-2\int_0^1\frac{dz}{1-z}\right] \qquad \kappa=\frac{\xi}{x}$$

- with $C_q = C_F$ and $C_g = C_A$, and:

$$K_q=\frac{3}{2} \qquad K_g=\frac{11C_A-4n_fT_R}{6C_A}$$

- In [Eur. Phys. J. C 82 (2022) 10, 888] we have computed the full set of $p^U_{i/j}$:

$$p^U_{q/q}\left(x,\frac{\xi}{x}\right) = C_F\frac{(x+\xi)(1-x+2\xi)}{\xi(1+\xi)(1-x)}$$

$$p^U_{q/g}\left(x,\frac{\xi}{x}\right) = T_R\frac{(x+\xi)(1-2x+\xi)}{\xi(1+\xi)(1-\xi^2)}$$

$$p^U_{g/q}\left(x,\frac{\xi}{x}\right) = C_F\frac{(x+\xi)(2-x+\xi)}{\xi x(1+\xi)}$$

$$p^U_{g/g}\left(x,\frac{\xi}{x}\right) = -C_A\frac{x^2-\xi^2}{\xi x(1-\xi^2)}\left[1-\frac{2\xi}{1-x}-\frac{2(1+x^2)}{(x-\xi)(1+\xi)}\right]$$

Evolution kernels at one loop

- We have computed these functions also in the **longitudinally polarised** case:

$$p^L_{q/q}\left(x,\frac{\xi}{x}\right) = -C_F\frac{(x+\xi)(x-1-2\xi)}{(1+\xi)\xi(1-x)}$$

$$p^L_{q/g}\left(x,\frac{\xi}{x}\right) = -2n_f T_R\frac{x+\xi}{\xi(1+\xi)^2}$$

$$p^L_{g/q}\left(x,\frac{\xi}{x}\right) = C_F\frac{(x+\xi)^2}{x\xi(1+\xi)}$$

$$p^L_{g/g}\left(x,\frac{\xi}{x}\right) = \frac{C_A(\xi+x)\left(-\xi^2(2\xi+1)+\xi+(\xi-3)x^2+\left(\xi^2+3\right)x\right)}{(1-\xi^2)\xi(1+\xi)(1-x)x}$$

- and in the **transversely polarised** case:

$$p^T_{q/q}\left(x,\frac{\xi}{x}\right) = 2C_F\frac{x+\xi}{(1+\xi)(1-x)}$$

$$p^T_{q/g}\left(x,\frac{\xi}{x}\right) = p^T_{g/q}\left(x,\frac{\xi}{x}\right) = 0$$

$$p^T_{g/g}\left(x,\frac{\xi}{x}\right) = 2C_A\frac{(x+\xi)^2}{(1+\xi)^2(1-x)x}$$

Evolution equations

- Defining the **anti-quark** distributions as:

$$F_{\bar{q}/H}^{[U,T]}(x,\xi,\Delta^2;\mu) = -F_{q/H}^{[U,T]}(-x,\xi,\Delta^2;\mu)$$

$$F_{\bar{q}/H}^{[L]}(x,\xi,\Delta^2;\mu) = +F_{q/H}^{[L]}(-x,\xi,\Delta^2;\mu)$$

- one can construct **non-singlet** and **singlet** combinations:

$$F^{[\Gamma],-} = F_{q/H}^{[\Gamma]} - F_{\bar{q}/H}^{[\Gamma]} \qquad F^{[\Gamma],+} = \begin{pmatrix} \sum_{q=1}^{n_f} F_{q/H}^{[\Gamma]} + F_{\bar{q}/H}^{[\Gamma]} \\ F_{g/H}^{[\Gamma]} \end{pmatrix}$$

- The evolution equations **decouple** and can be written in a **DGLAP-like** fashion:

$$\frac{dF^{[\Gamma],\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty \frac{dy}{y}\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)F^{[\Gamma],\pm}\left(\frac{x}{y},\xi,\mu\right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)$$

DGLAP region ERBL contribution

- $\mathscr{P}_{1,2}^{[\Gamma]\pm,[0]}$ are appropriate combinations of the functions $p_{i/j}^{\Gamma}$ presented above.

Numerical setup

- The one-loop evolution kernels for *all polarisations* are now implemented in **APFEL++** and are available through **PARTONS** allowing for LO GPD evolution in momentum space.

- We achieved a stable numerical implementation over the full range $\mathbf{0 \leq \xi \leq 1}$:

  - numerical checks that both the **DGLAP** and **ERBL** limits are recovered,

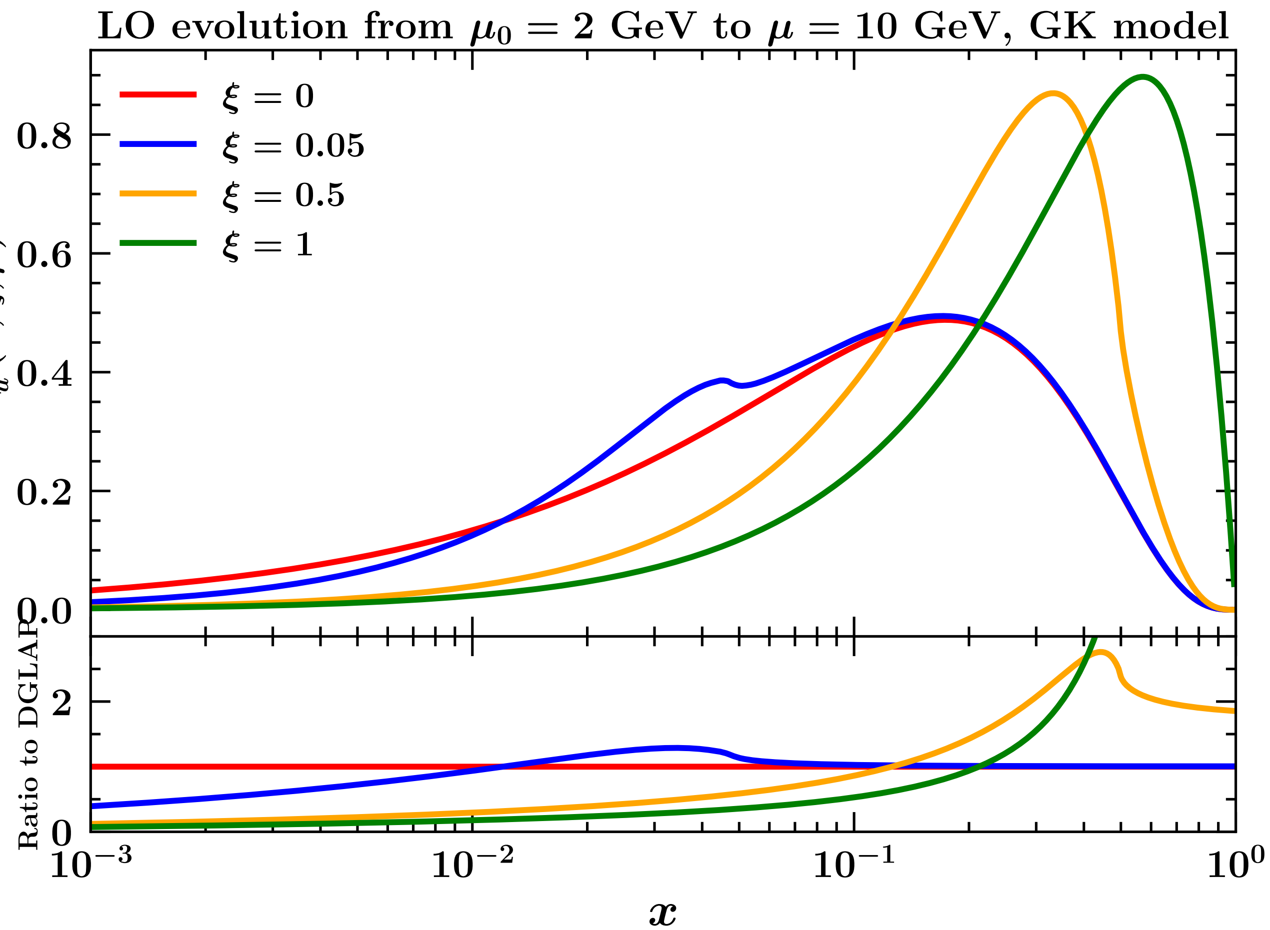
  - numerical check of **polynomiality** conservation.

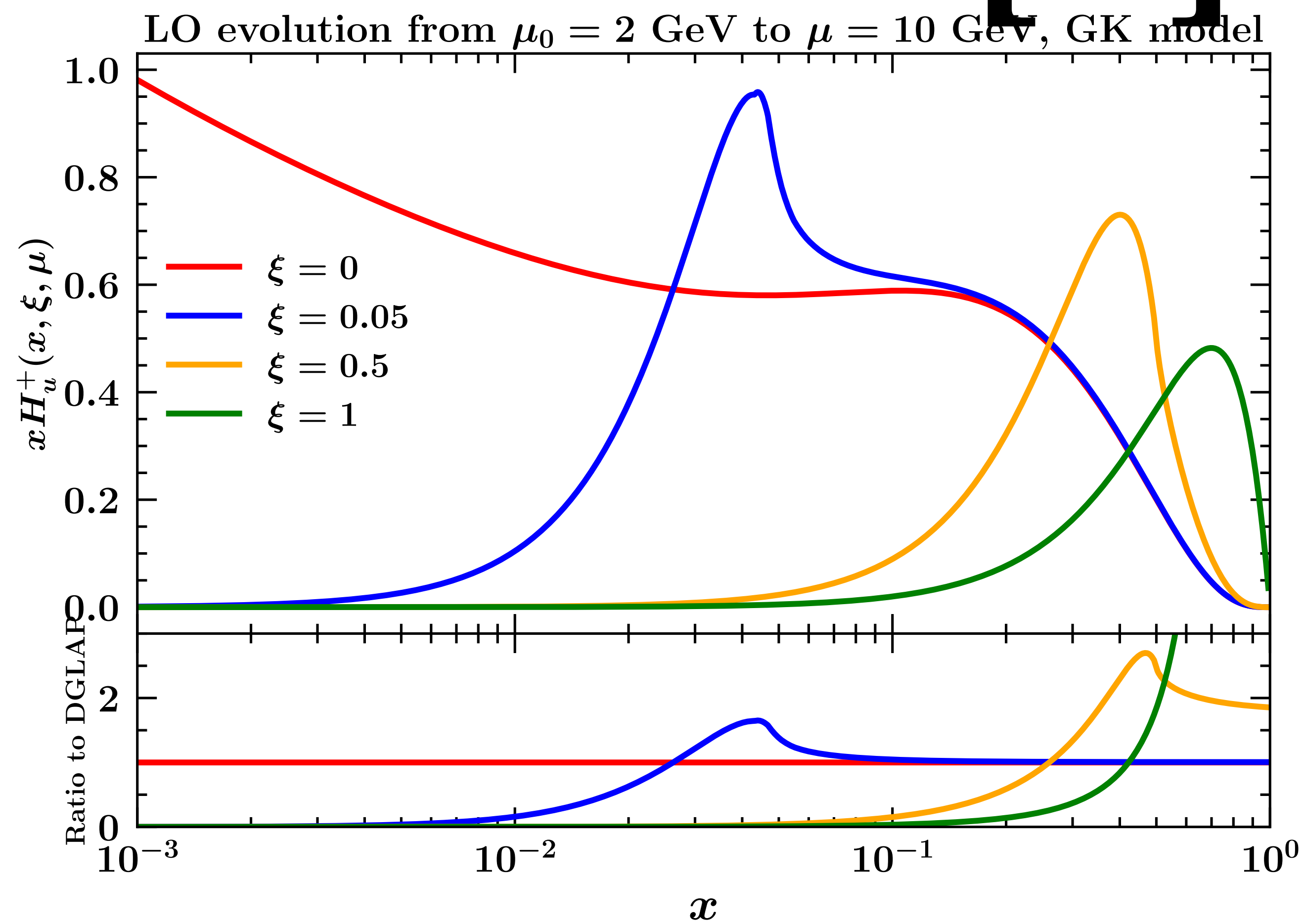
- Numerical tests use the *realistic* Goloskokov-Kroll (GK) model for proton GPDs [*Eur. Phys. J. C* 53 (2008) 367-384] as implemented in **PARTONS** as an initial-scale set of distributions:

  - we consistently used $H_{i/H}$ for unpolarised, $\widetilde{H}_{i/H}$ for longitudinally polarised, and $H^T_{i/H}$ for transversely polarised evolution.

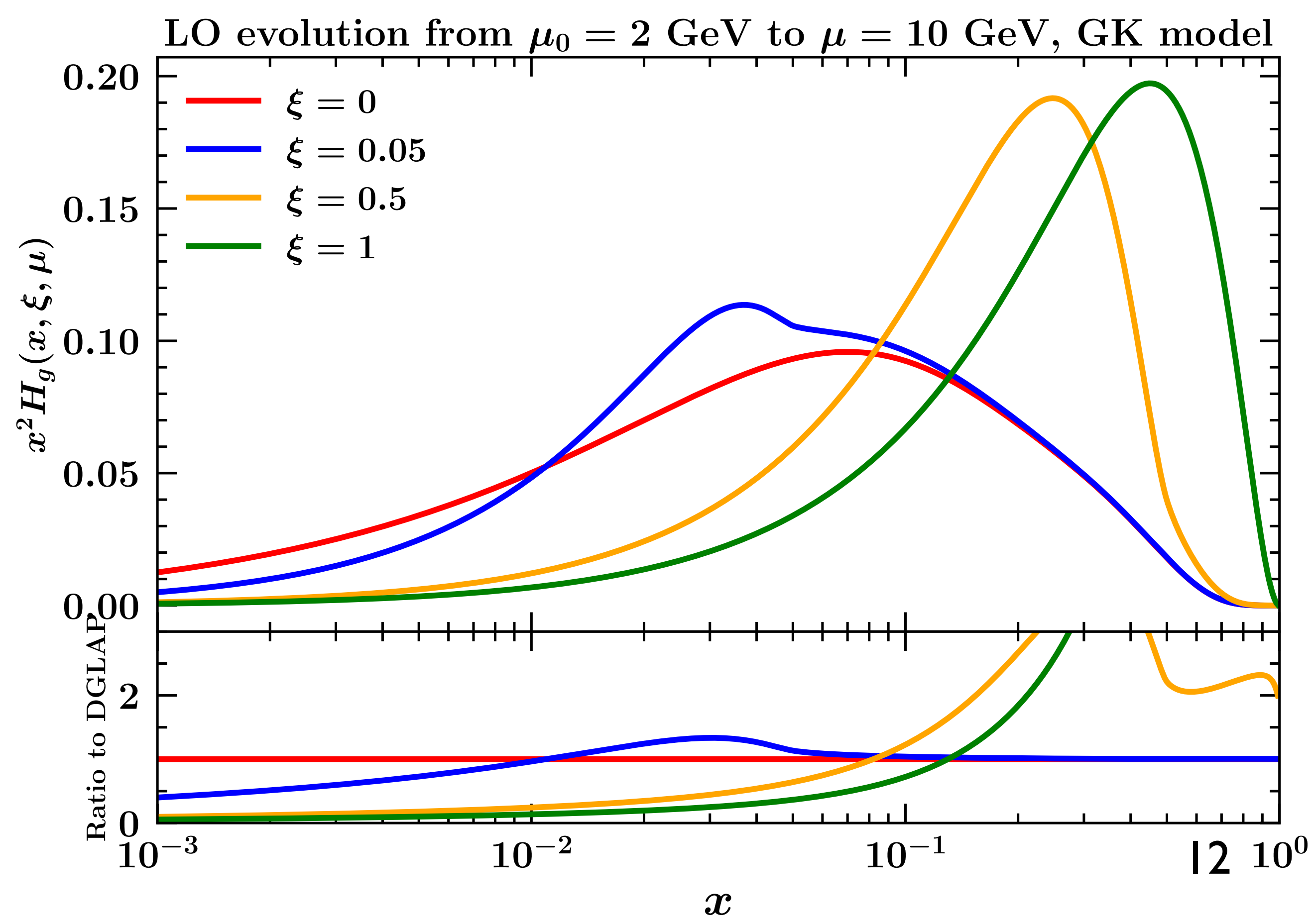
  - GPDs are evolved from 2 to 10 GeV in the **variable-flavour-number scheme**, *i.e.* accounting for heavy-quark-threshold crossing, at $\Delta^2 = -0.1$ GeV^2.

Evolution and DGLAP limit [U]

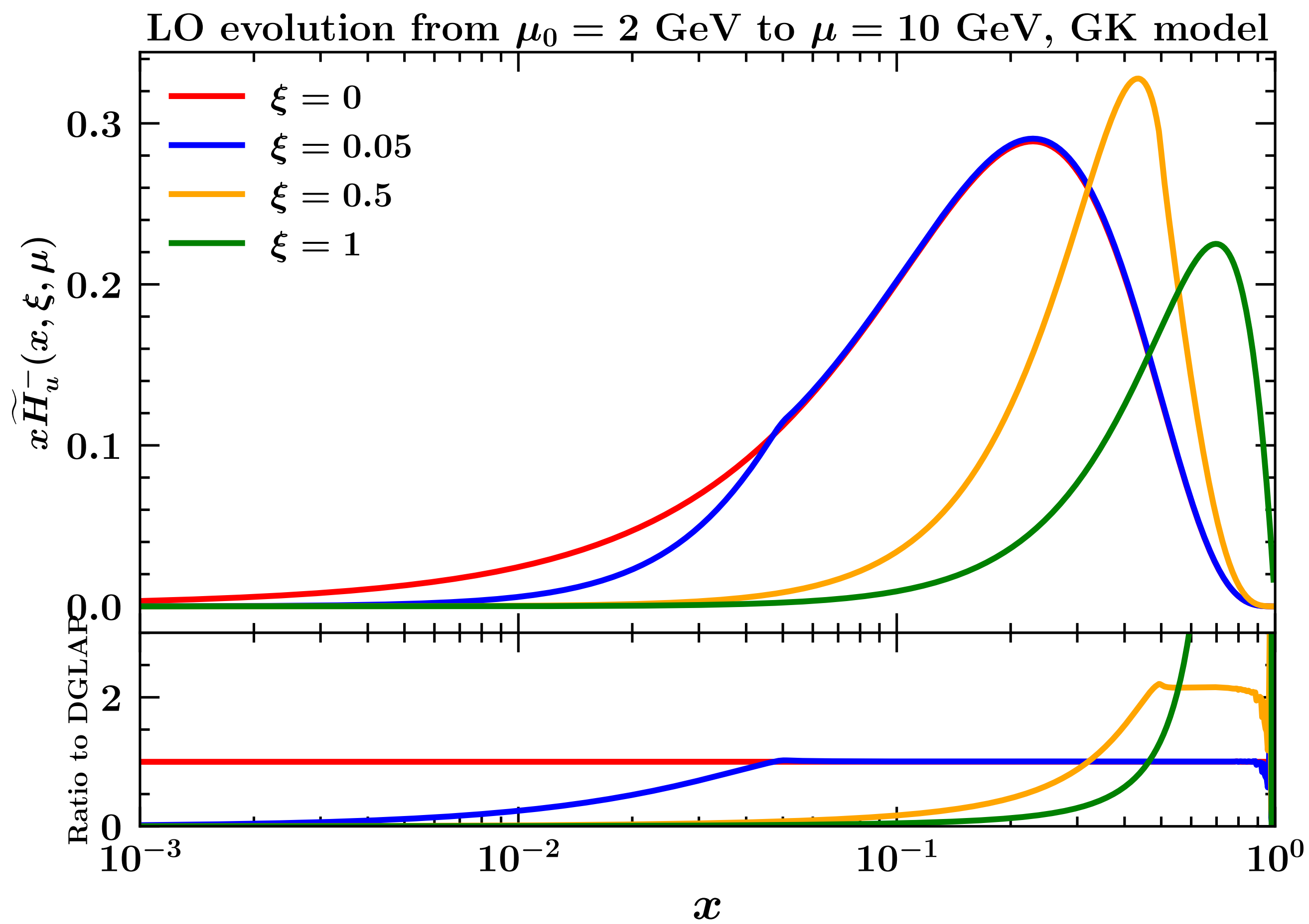


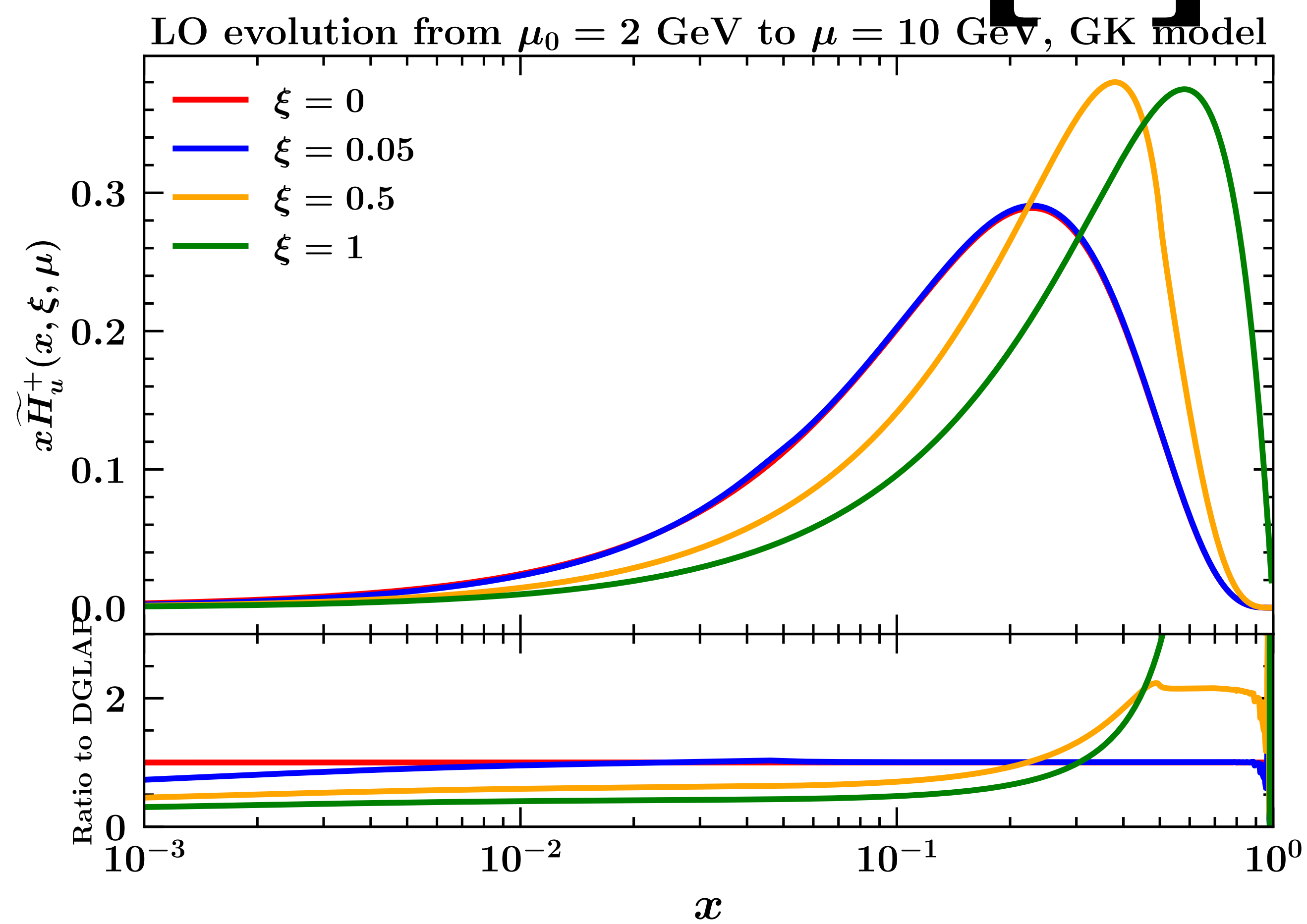


- **DGLAP limit** reproduced within 10^{-5} relative accuracy.
- GPD evolution may significantly deviate from DGLAP evolution.
- The evolution generates a cusp at $x = \xi$ but the distribution remains **continuous** at this point.



Evolution and DGLAP limit [L]





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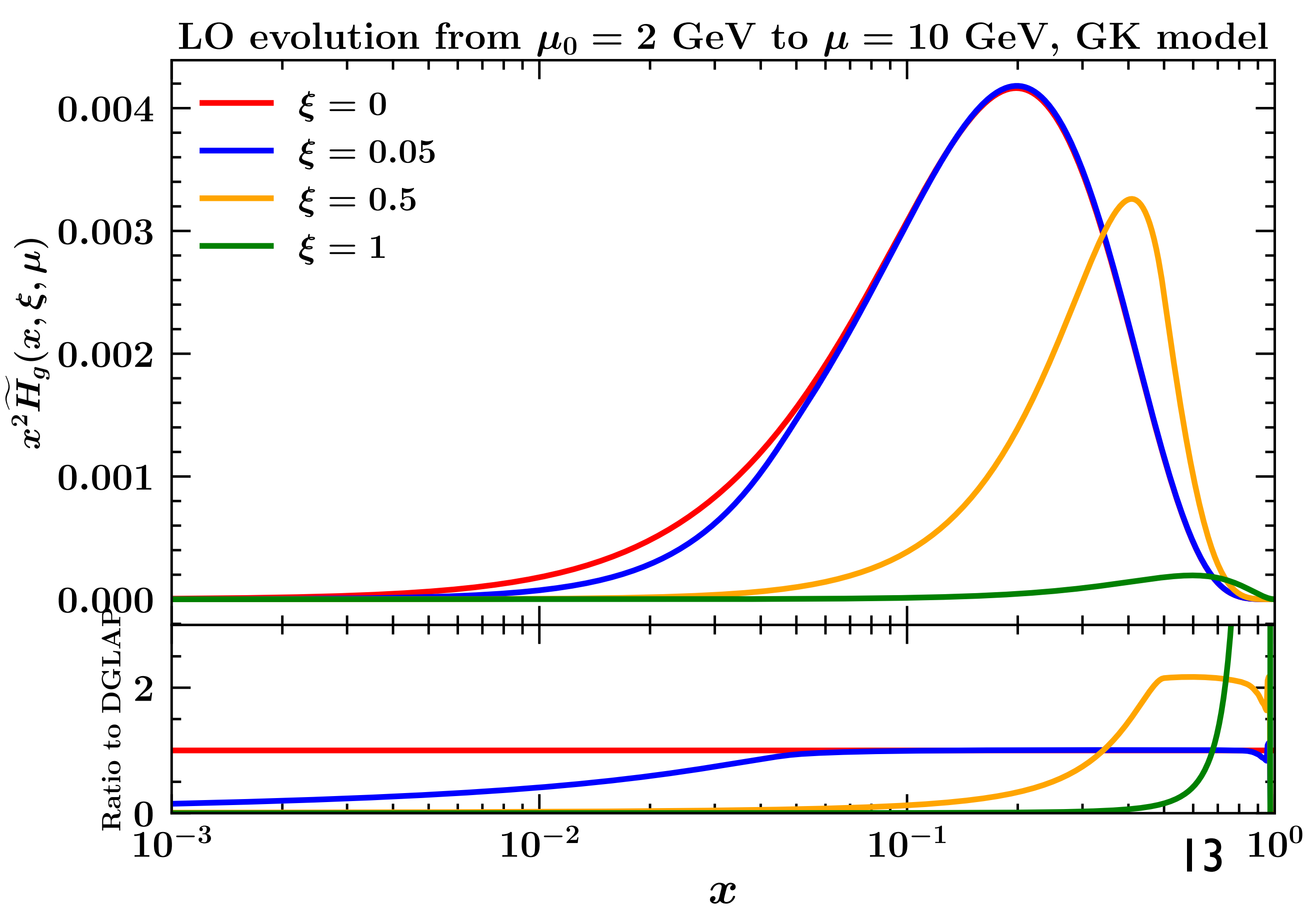
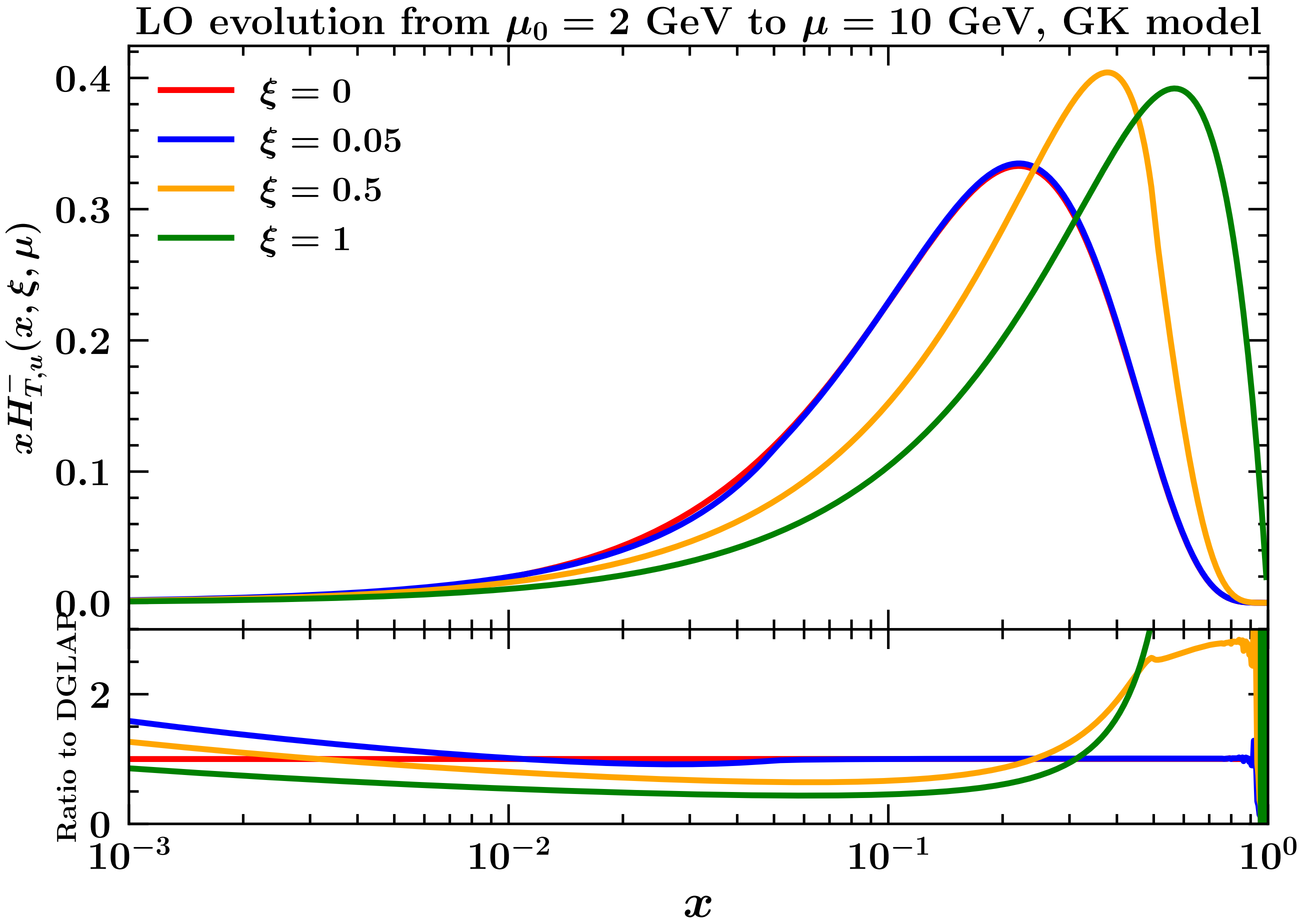
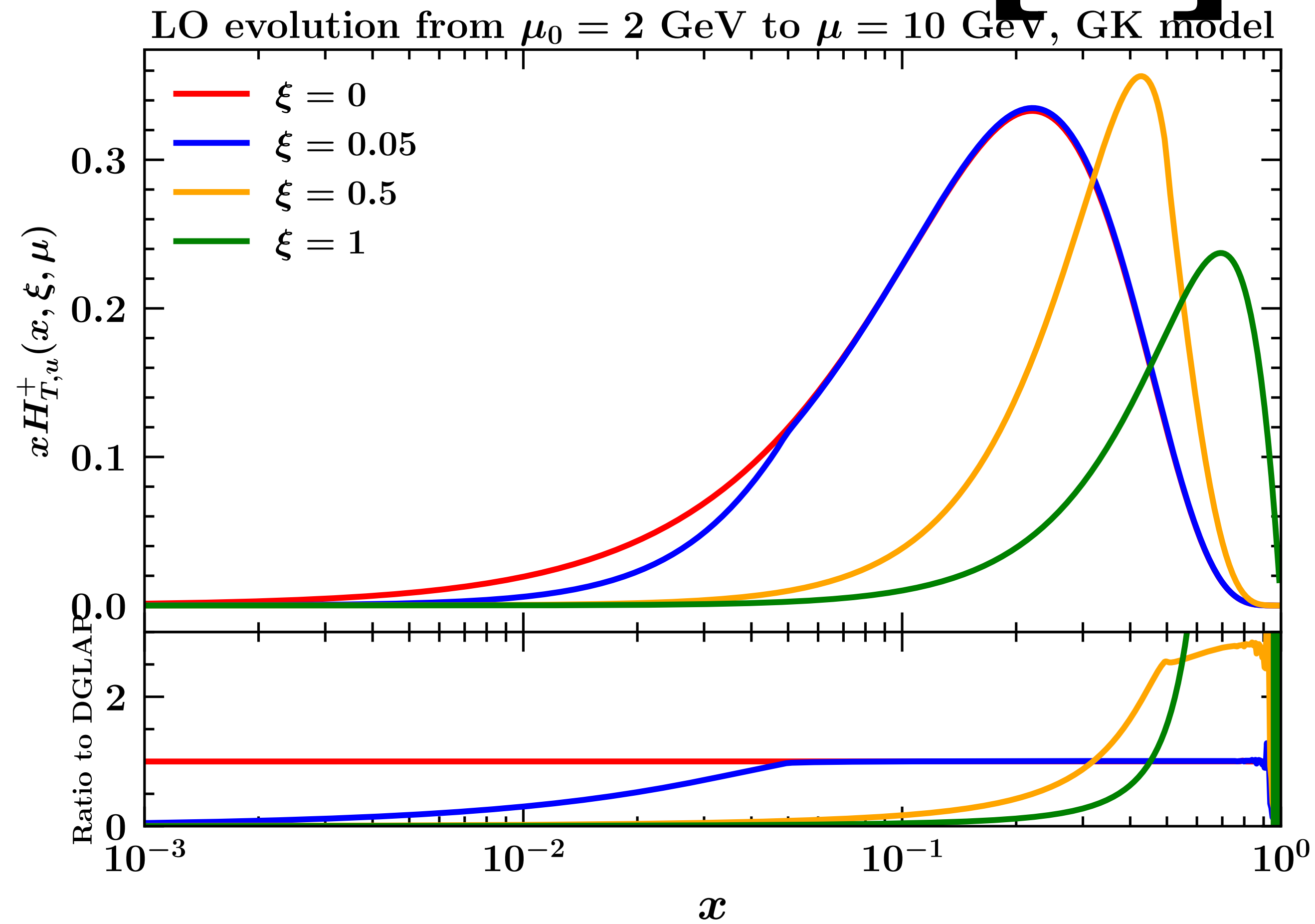


Evolution and DGLAP limit [T]





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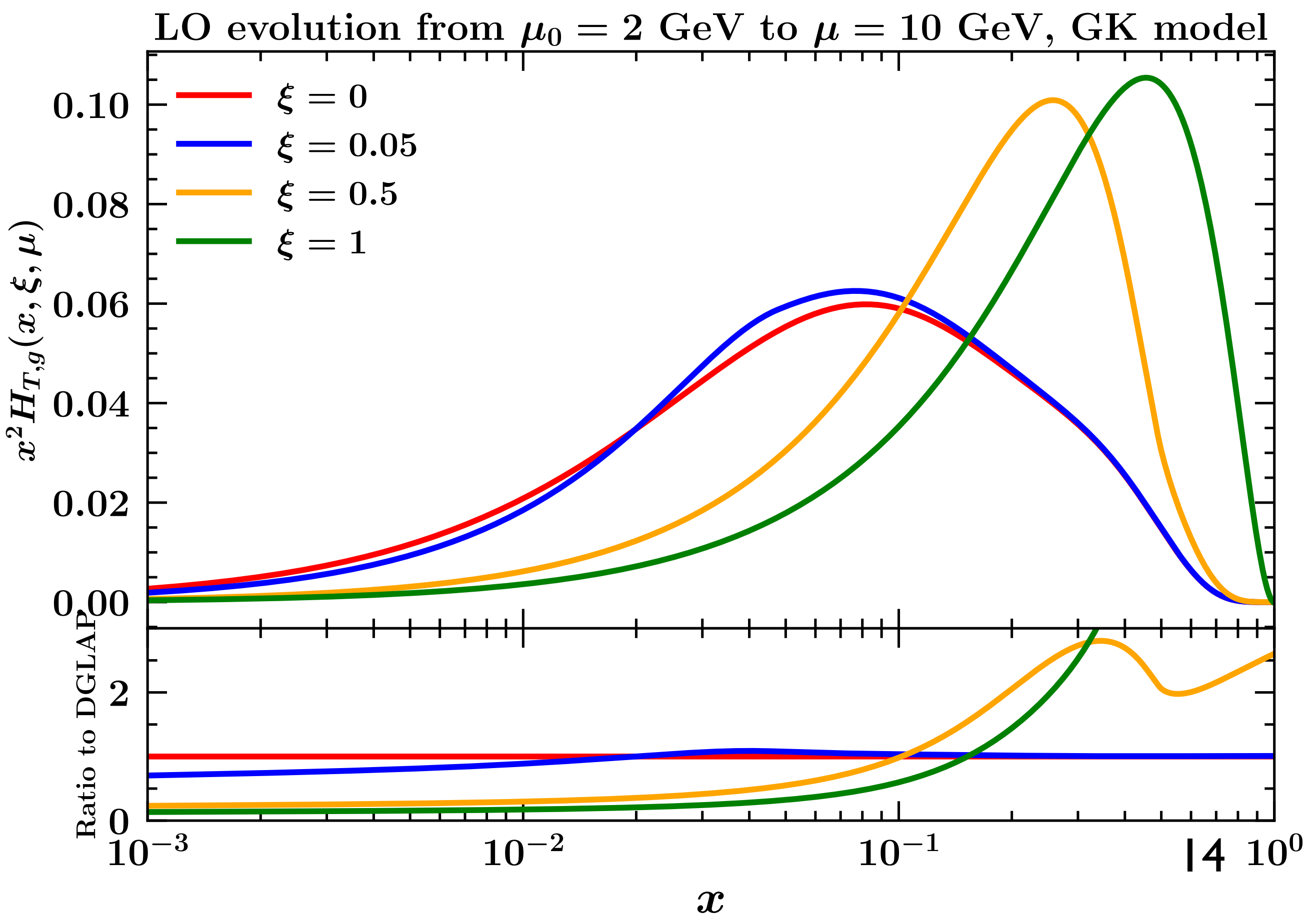


Polynomiality

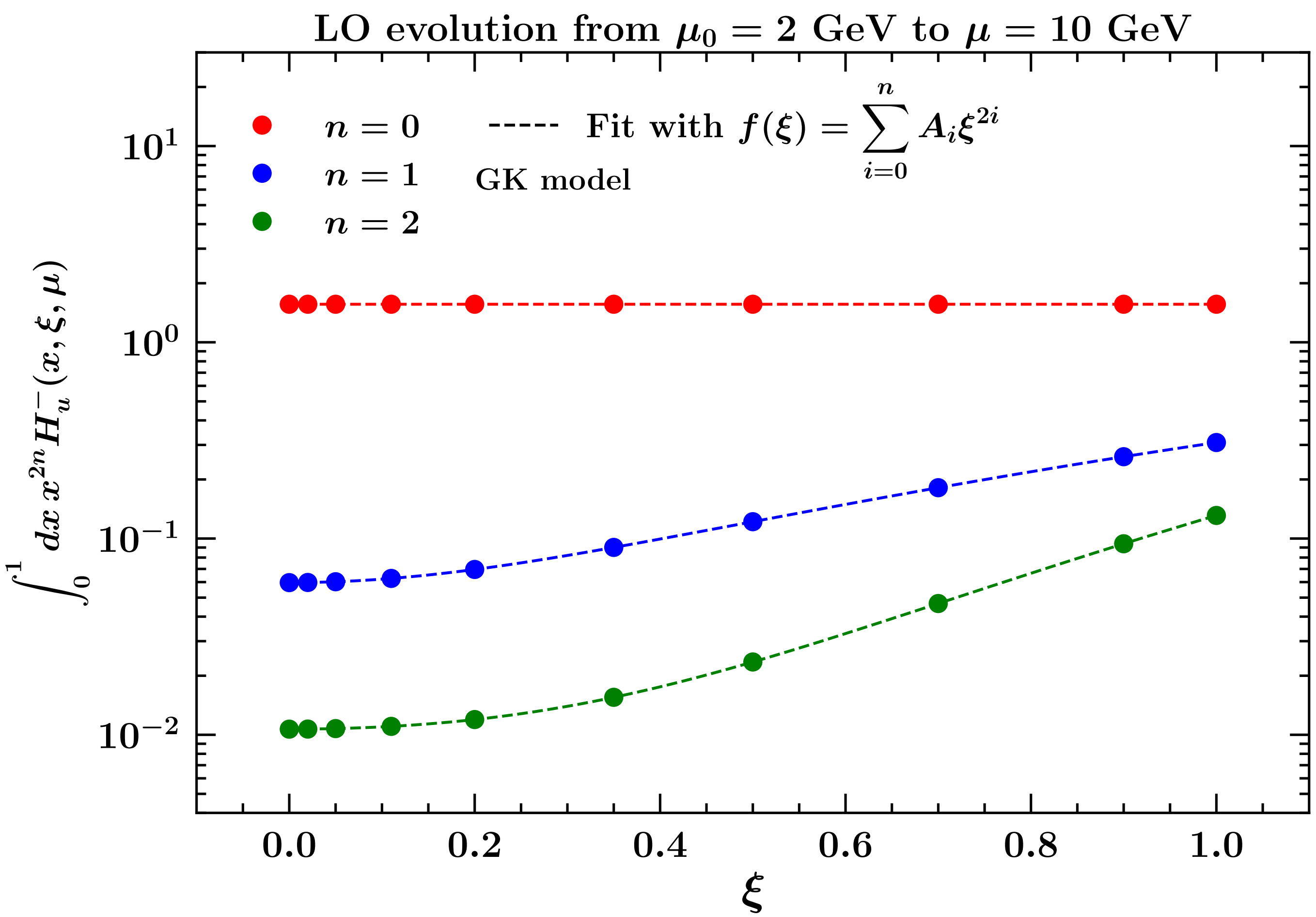
- GPD evolution should preserve **polynomiality**.
  [Xiang-Dong Ji, *J.Phys.G* 24 (1998) 1181-1205] [A.V. Radyushkin, *Phys.Lett.B* 449 (1999) 81-88]
- The following relations for Mellin moments of GPDs must hold at **all scales**:

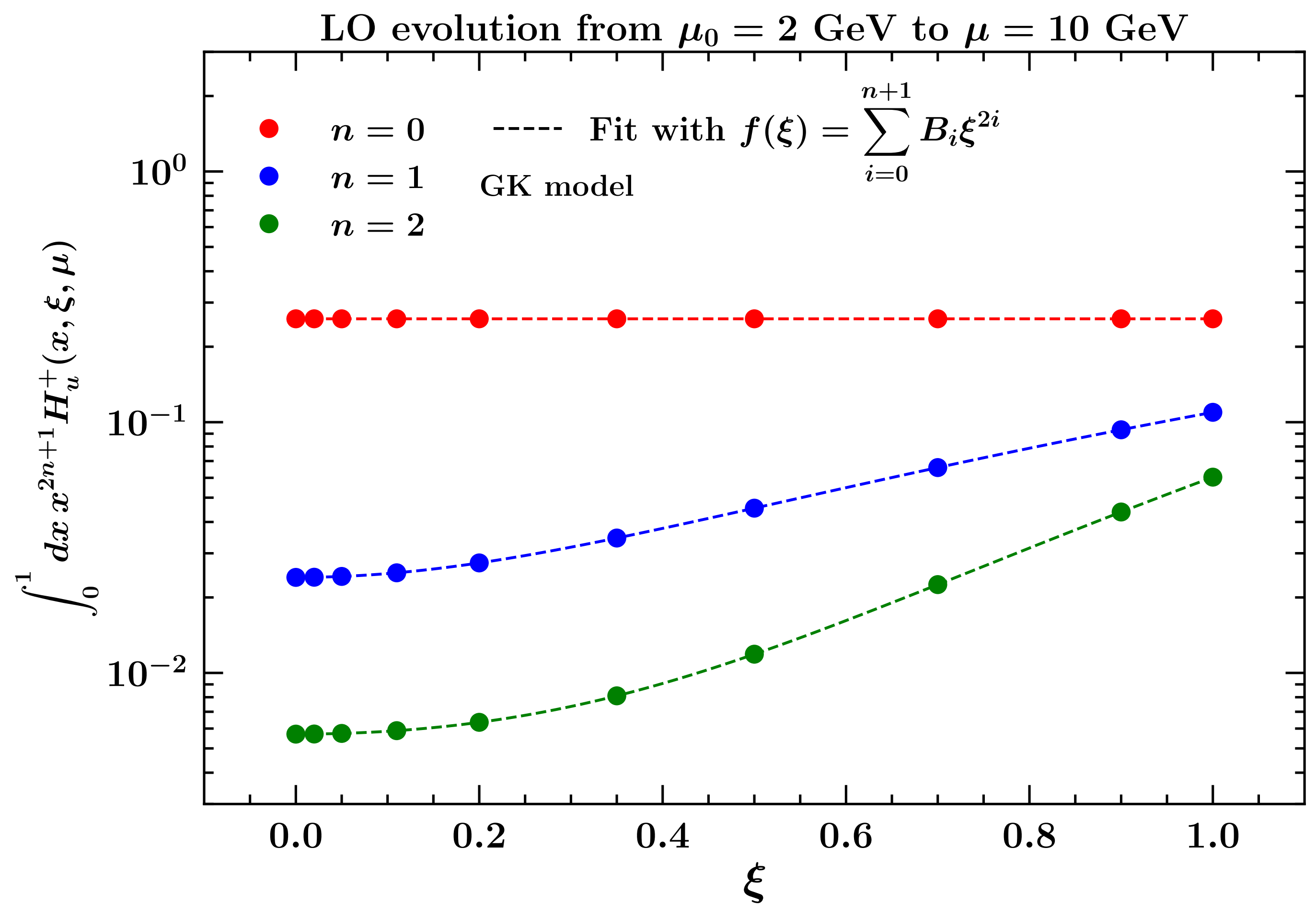
$$\int_0^1 dx\, x^{2n} F_q^{[\Gamma]-}(x,\xi,\mu) = \sum_{k=0}^{n} A_k^{[\Gamma]}(\mu)\xi^{2k}$$

$$\int_0^1 dx\, x^{2n+1} F_q^{[\Gamma]+}(x,\xi,\mu) = \sum_{k=0}^{n(+1)} B_k^{[\Gamma]}(\mu)\xi^{2k}$$

- Polynomiality predicts that the first moment ($n = 0$) of the *non-singlet* distribution is **constant** in ξ.
- The coefficient of the ξ^{2n+2} term of the *singlet* (D-term), only present in the *unpolarised* case, is absent in the GK model and is *not* generated by evolution:
  - hence also the first moment of the singlet is expected to be **constant** in ξ.
- For larger values of n, one can just check that the behaviour in ξ follows the **expected power law in** ξ.

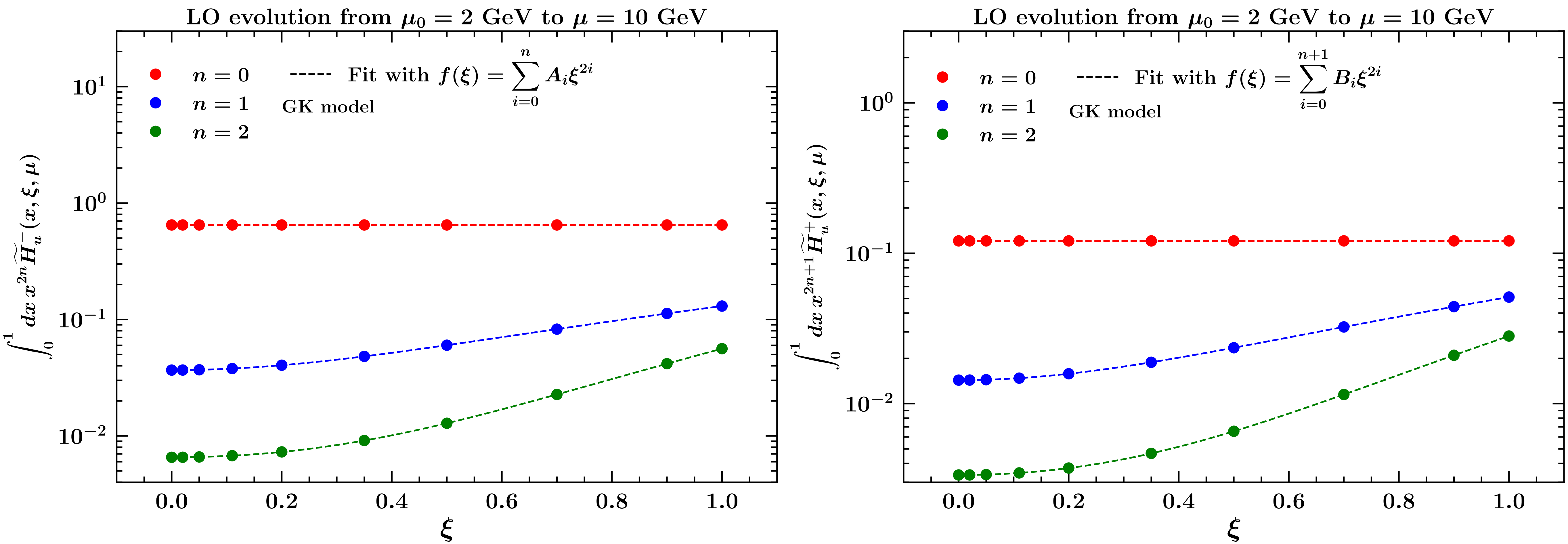
Polynomiality [U]





- **First moment** for both singlet and non-singlet is indeed **constant** in ξ:
  - this was expected and the expectation is very nicely fulfilled.
- **Second and third moments** follow the expected law:
  - including odd-power terms in the fit gives coefficients very close to zero.
  - B_{n+1} in the singlet is consistently found to be compatible with zero (no D-term).

Polynomiality [L]



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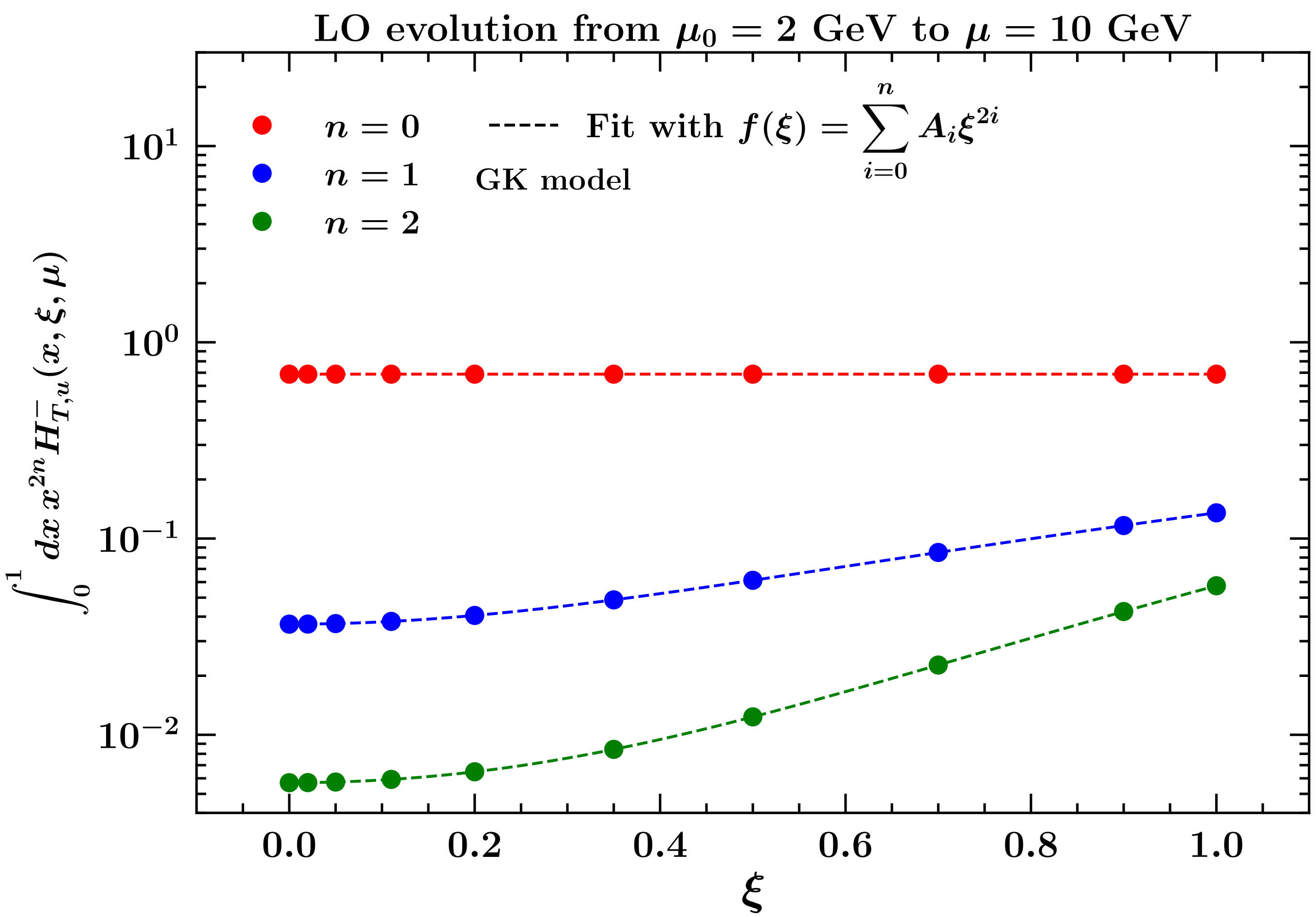

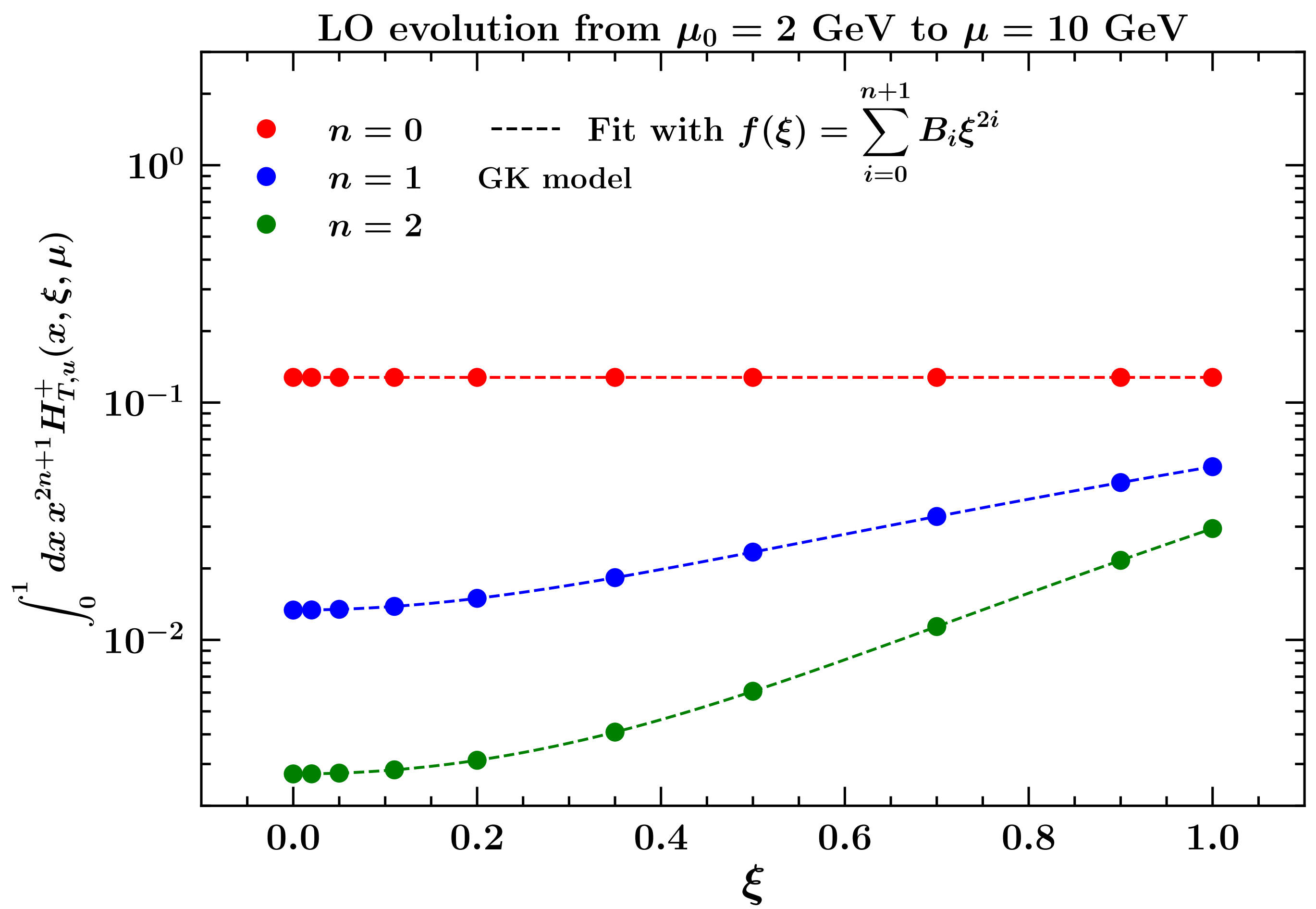

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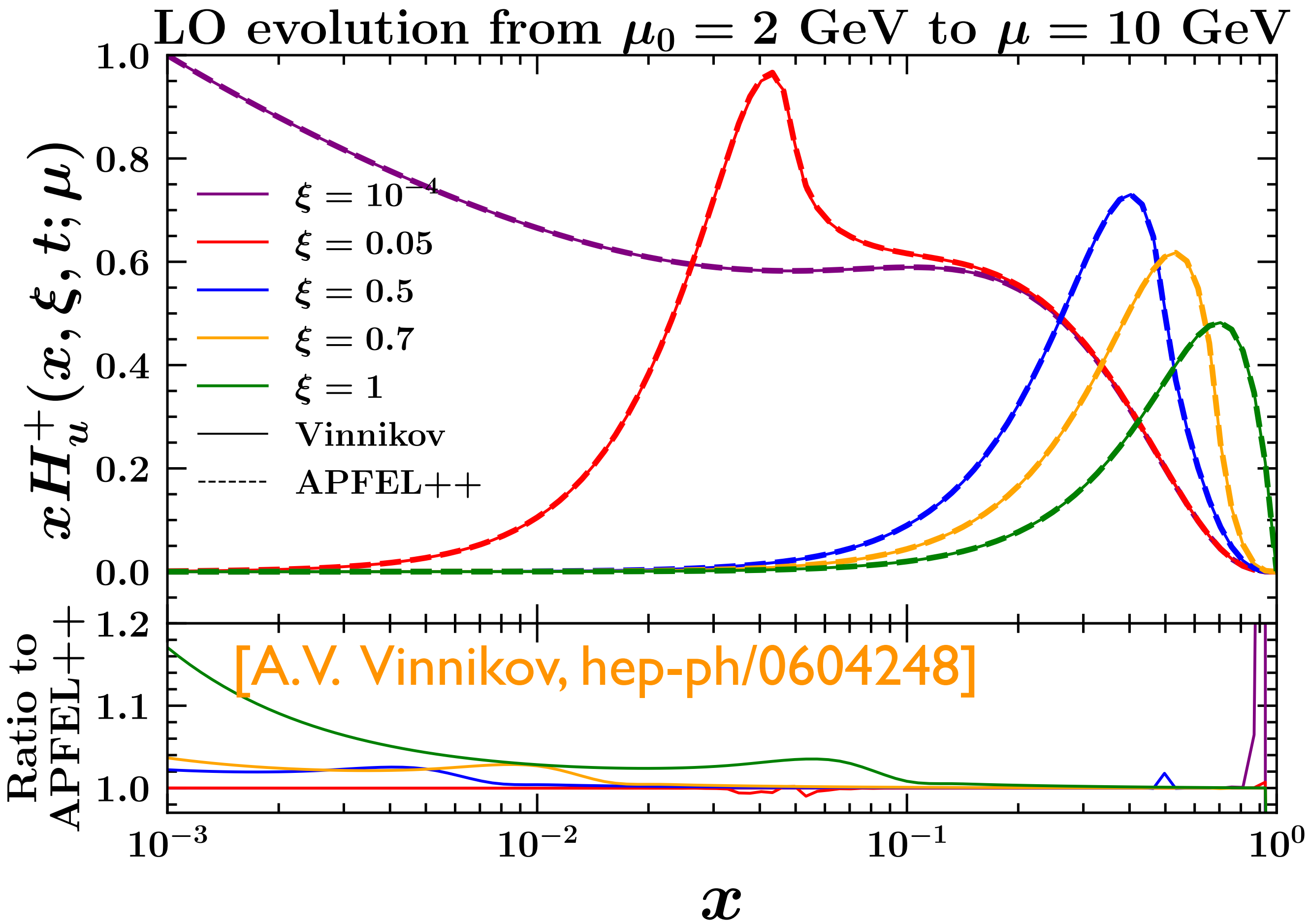
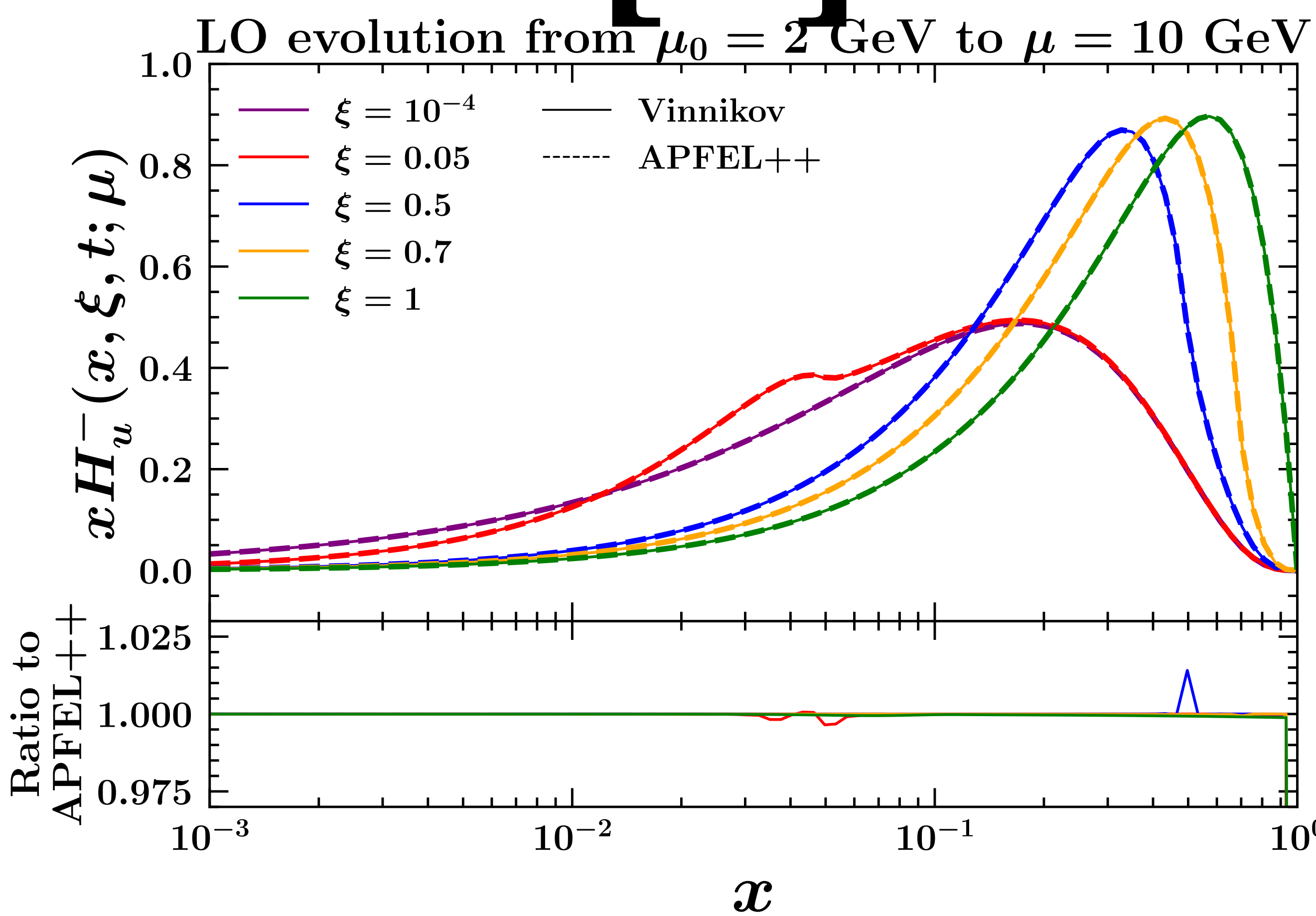
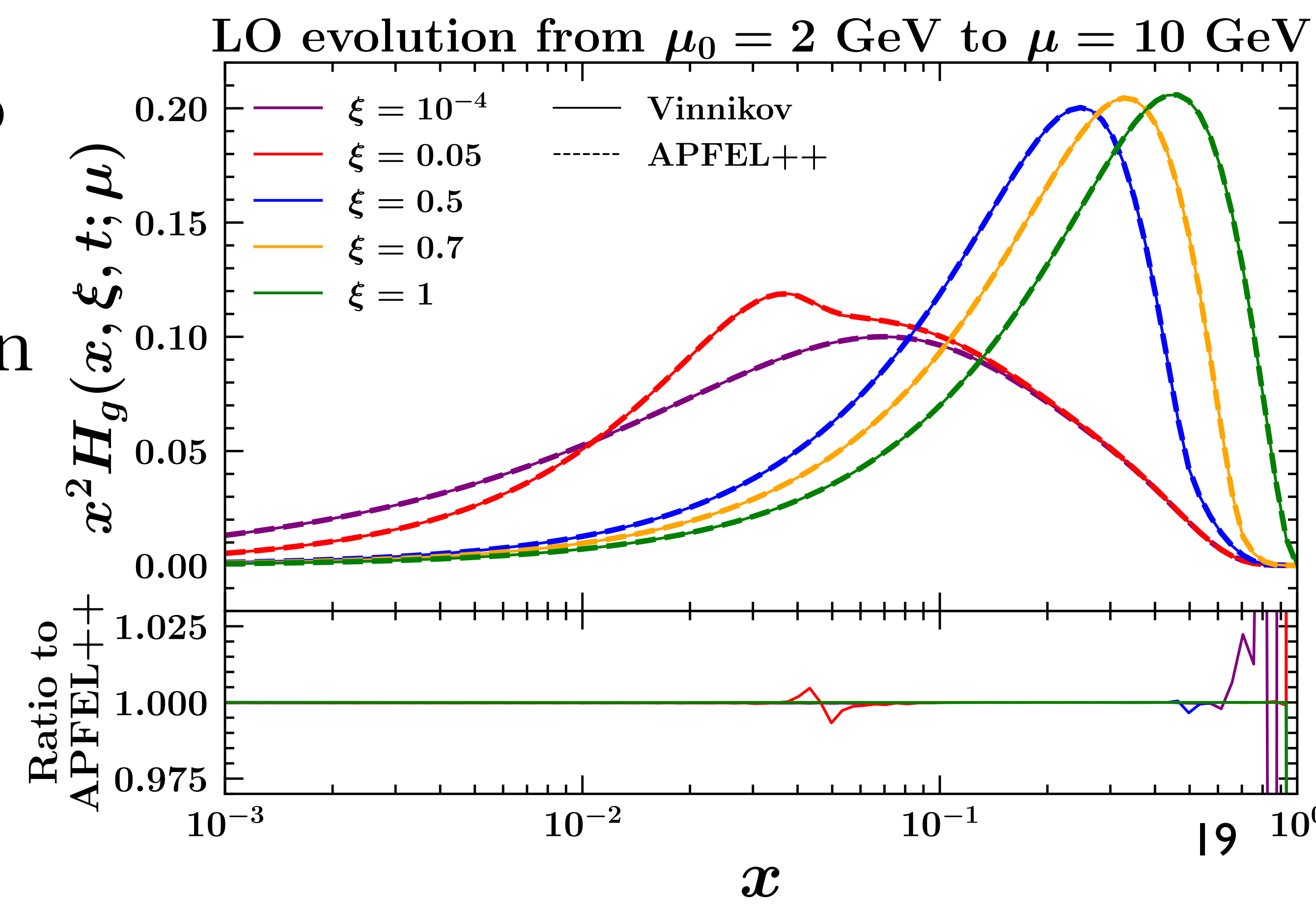
APFEL++ vs. Vinnikov [U]





- **Good agreement** between the two codes.
- Better agreement after fixing a bug in Vinnikov's code:
  - affecting the region $\xi > 2/3$,
  - due to an incorrect construction of the interpolation grid in this region.



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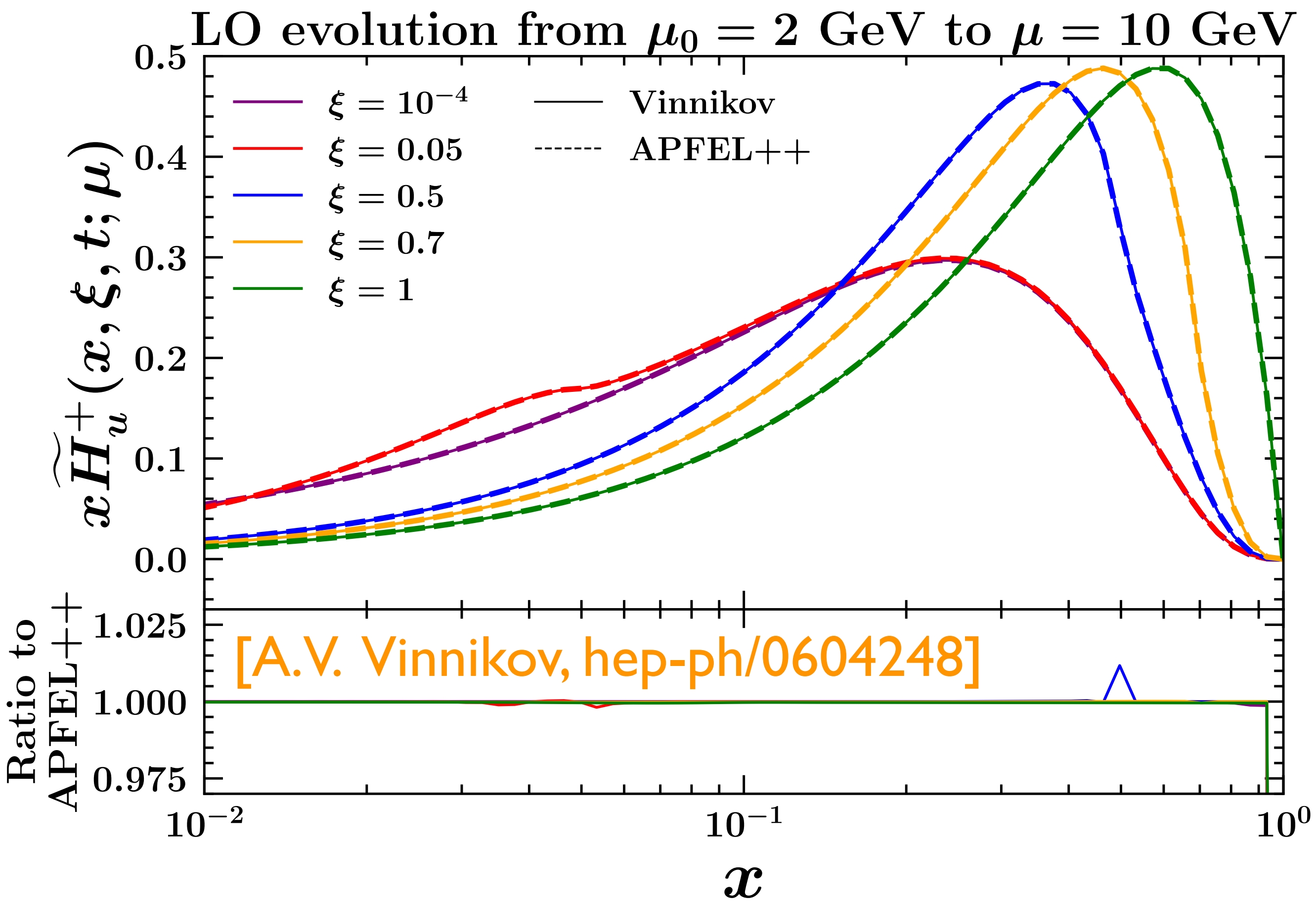

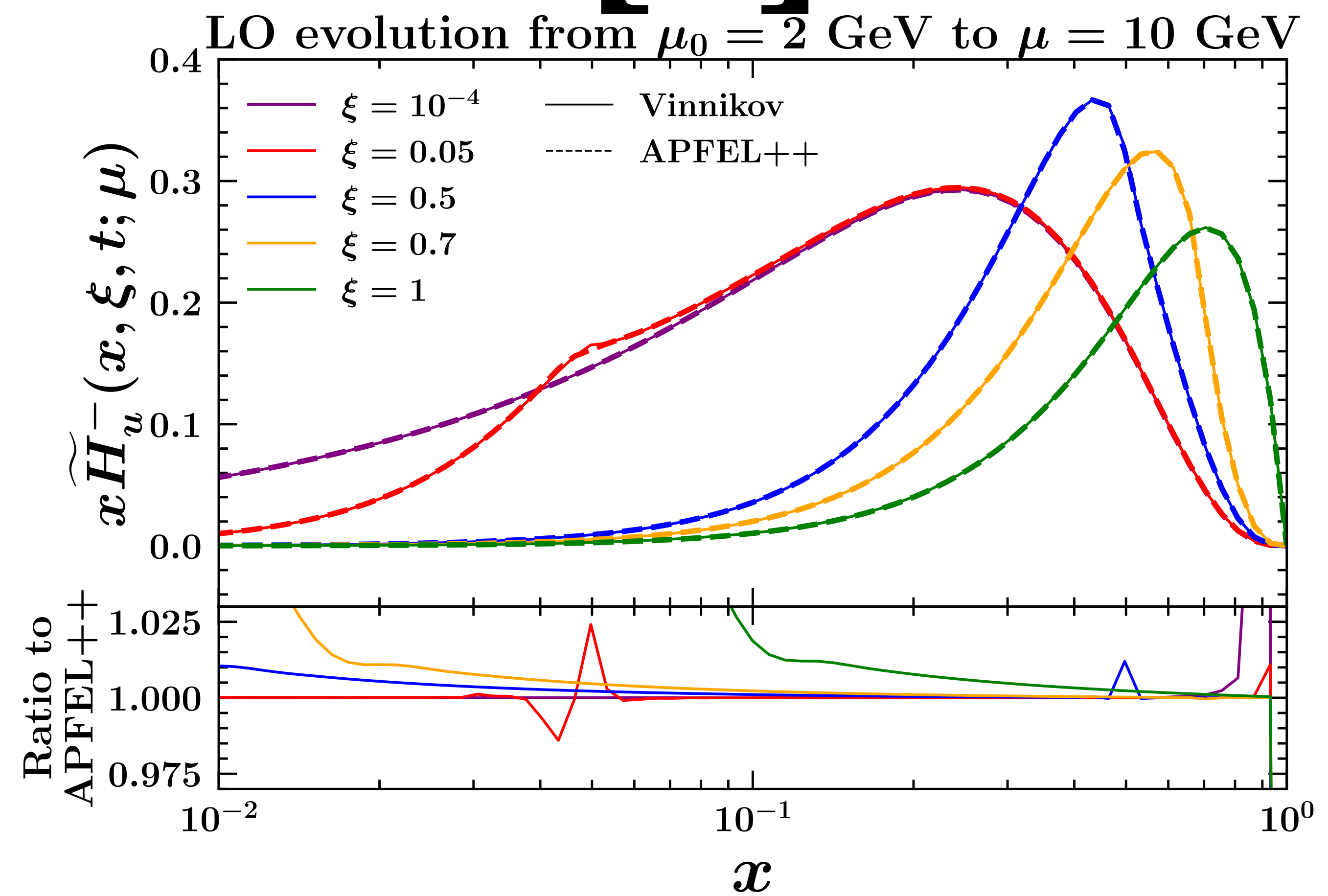

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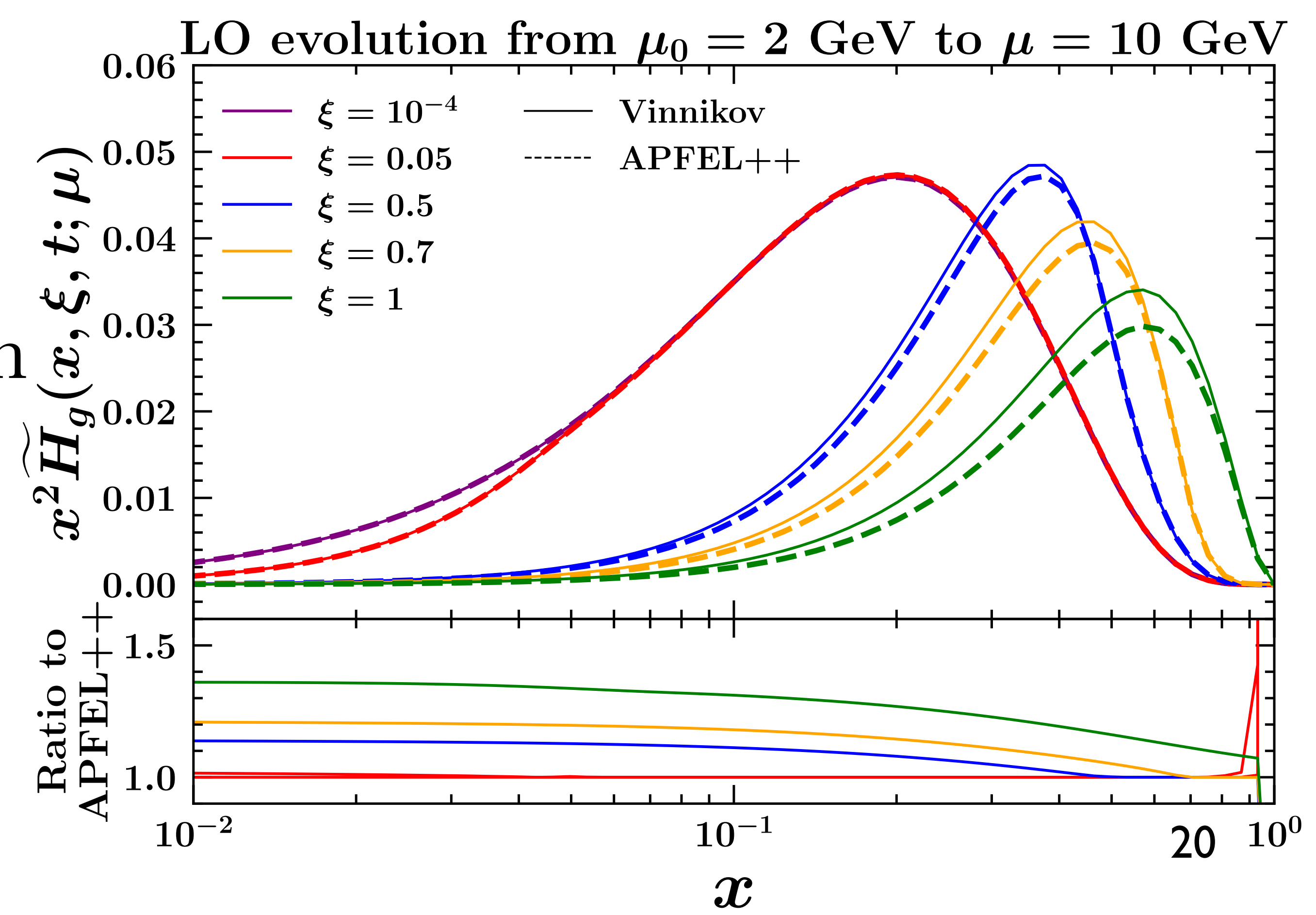


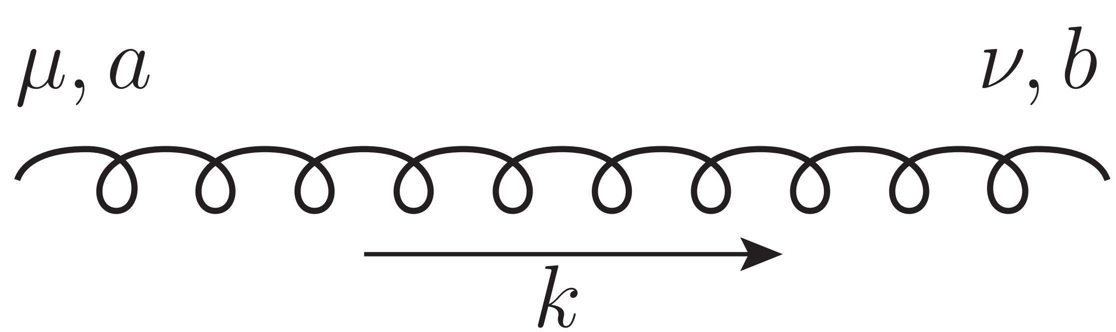
Conclusions and outlook

- We have **revisited LO GPD evolution** in momentum space:
  - *Ab-initio* calculation of the LO unpolarised splitting kernels based on Feynman diagrams in light-cone gauge for **all twist-2 operators**.
  - GPD evolution equations recasted in a DGLAP-like form convenient for implementation.
  - Various analytical properties of the kernels highlighted and numerically checked.
  - DGLAP (and ERBL) limits correctly recovered within excellent accuracy.
  - Evolution conserves polynomiality (and agrees with conformal-space evolution).
  - the code (APFEL++) is public and available within PARTONS
    https://github.com/vbertone/apfelxx
    http://partons.cea.fr/partons/doc/html/index.html
- **Next steps**:
  - **middle term:** benchmark of public evolution codes (discussion already started),
  - **longer term:** (re)calculation and implementation of NLO corrections (already on the way).

Back up

GPD definition

- The use of light-cone gauge implies:
  - the **absence of the Wilson line**,
  - a simpler gluon GPD written in terms of the **gluon field** and not the field strength,
  - the **absence of ghosts** in perturbative calculations,
  - **more complicated** gluon propagator:

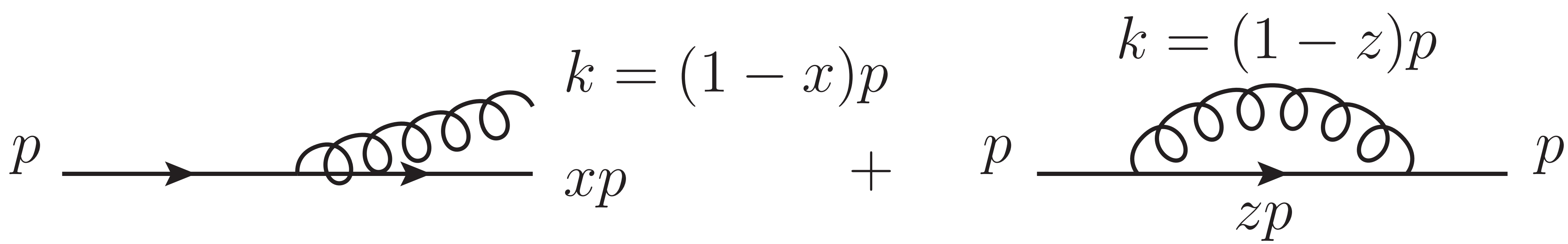


$$\mathcal{D}^{\mu\nu}_{ab}(k) = \frac{i\delta_{ab} d^{\mu\nu}(k)}{k^2 + i0}, \quad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\text{Reg.}}}$$

GPD definition

$$\mathcal{D}^{\mu\nu}(k) = \frac{id^{\mu\nu}(k)}{k^2 + i0}\,, \qquad d^{\mu\nu}(k) = -g^{\mu\nu} + \frac{k^\mu n^\nu + k^\nu n^\mu}{(nk)_{\text{Reg.}}}$$

- The linear (eikonal) propagator $(nk)^{-1}$ needs to be **regularised**:
  - it separately gives rise to non-integrable end-point singularities in real-emission graphs and to plain divergences in virtual graphs,
  - the two **cancel** giving an **integrable** result.



$$\frac{1}{(nk)} \sim \frac{1}{1-x} \qquad + \qquad \delta(1-x)\int \frac{dk}{(nk)} \sim \delta(1-x)\int \frac{dz}{1-z}$$

$$\sim \left(\frac{1}{1-x}\right)_+$$

GPD evolution

- Using dimensional regularisation in $4-2\varepsilon$ dimensions, the **UV** renormalisation of GPDs can be implemented in a multiplicative fashion:

$$F_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu) = \lim_{\varepsilon\to 0}\sum_{j=q,g}\int_{-1}^{1}\frac{dy}{|y|}Z_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s(\mu),\varepsilon\right)\hat{F}_{j/H}^{[\Gamma]}(y,\xi,\Delta^2;\varepsilon,\mu^{-\varepsilon})$$

- In the $\overline{\mathrm{MS}}$ scheme, renormalisation constants have the following structure:

$$Z_{ij}^{[\Gamma]}(z,\kappa,\alpha_s,\varepsilon) = \delta_{ij}\delta(1-z) + \sum_{n=1}^{\infty}\left(\frac{\alpha_s}{4\pi}\right)^n\sum_{p=1}^{n}\frac{1}{\overline{\varepsilon}^p}Z_{ij}^{[\Gamma],[n,p]}(z,\kappa)$$

- with:

$$\frac{1}{\overline{\varepsilon}} = \frac{S_\varepsilon}{\varepsilon} = \frac{1}{\varepsilon} + \ln 4\pi - \gamma_{\mathrm{E}} + \mathcal{O}(\varepsilon)$$

- Exploiting the independence of the bare GPDs on μ (for $\varepsilon\to 0$), one can derive a **RGE** governing the evolution of renormalised GPDs w.r.t. μ:

$$\frac{dF_{i/H}^{[\Gamma]}(x,\xi,\Delta^2;\mu)}{d\ln\mu^2} = \sum_{k=q,g}\int_{-1}^{1}\frac{dz}{|z|}\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s(\mu)\right)F_{k/H}^{[\Gamma]}(z,\xi,\Delta^2;\mu)$$

GPD evolution

- The evolution kernels $\mathscr{P}$ are related to the normalisation constants Z as follows:

$$\mathcal{P}_{ik}^{[\Gamma]}\left(\frac{x}{z},\frac{\xi}{x},\alpha_s\right)=\lim_{\varepsilon\to 0}\sum_j\int_{-1}^{1}\frac{dy}{|y|}\frac{dZ_{ij}^{[\Gamma]}\left(\frac{x}{y},\frac{\xi}{x},\alpha_s,\varepsilon\right)}{d\ln\mu^2}Z_{jk}^{[\Gamma]-1}\left(\frac{y}{z},\frac{\xi}{y},\alpha_s,\varepsilon\right)$$

- where the inverse of the renormalisation constant Z^{-1} is defined as:

$$\sum_j\int_{-1}^{1}\frac{dw}{|w|}Z_{ij}^{[\Gamma]}\left(\frac{w}{x},\frac{\xi}{w},\alpha_s,\varepsilon\right)Z_{jk}^{[\Gamma]-1}\left(\frac{z}{w},\frac{\xi}{z},\alpha_s,\varepsilon\right)=\delta_{ik}\delta\left(1-\frac{z}{x}\right)$$

- If factorisation holds, the evolution kernels $\mathscr{P}$ **must be finite**:
  - consider the factorisation of a Compton form factor: $\mathcal{F}(Q)=C(\mu/Q,\alpha_s(\mu))\otimes F(\mu)$
  - Being $\mathscr{F}$ a physical observable, it has to be independent of μ order by order in α_s:

$$C^{-1}\otimes\frac{d\mathcal{F}}{d\ln\mu^2}=0=\left[\frac{d\ln C(\mu,\alpha_s(\mu))}{d\ln\mu^2}+\mathcal{P}(\alpha_s(\mu))\right]\otimes F(\mu)$$

  - Since the coefficient function C is finite, so must be $\mathscr{P}$.

GPD evolution

- The finiteness of the evolution kernels $\mathscr{P}$ has important consequences on the structure of the renormalisation constants Z:

$$\mathcal{P} = \frac{d\ln Z}{d\ln\mu^2} = \overline{\beta}(\alpha_s,\varepsilon)\frac{\partial\ln Z}{\partial\alpha_s}$$

- but:

$$Z = 1 + \sum_{n=1}^{\infty}\alpha_s^n\sum_{p=1}^{n}\frac{1}{\varepsilon^p}Z^{[n,p]} = 1 + \sum_{p=1}^{\infty}\frac{1}{\varepsilon^p}\sum_{n=p}^{\infty}\alpha_s^n Z^{[n,p]} = 1 + \sum_{p=1}^{\infty}\frac{1}{\varepsilon^p}Z^{[p]}(\alpha_s)$$

- so that:

$$\frac{\partial\ln Z}{\partial\alpha_s} = Z^{-1}\frac{\partial Z}{\partial\alpha_s} = \frac{1}{\varepsilon}\frac{\partial Z^{[1]}}{\partial\alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon^2}\right)$$

- Since $\overline{\beta}(\alpha_s,\varepsilon) = -\varepsilon\alpha_s + \beta(\alpha_s)$, it follows that:

$$\mathcal{P} = -\alpha_s\frac{\partial Z^{[1]}}{\partial\alpha_s} + \mathcal{O}\left(\frac{1}{\varepsilon}\right)$$

- The evolution kernels are extracted from the **single pole** of the renormalisation constants **to all orders** in α_s.

- The finiteness of $\mathscr{P}$ implies that the residual $\mathscr{O}(1/\varepsilon)$ has to be **identically zero**:

  - higher-order-pole coefficients $Z^{[n]}$, $n > 1$, are related to $Z^{[1]}$ and β.

GPD evolution

- The kernels $\mathscr{P}$ admit the **perturbative expansion**:

$$\mathcal{P}(\alpha_s) = \alpha_s \sum_{n=0}^{\infty} \alpha_s^n \mathcal{P}^{[n]}$$

- At one loop, *i.e.* the leading order, one simply finds:

$$\mathcal{P}^{[0]} = -Z^{[1,1]}$$

- At two loops:

$$\mathcal{P}^{[1]} = -2Z^{[2,1]}$$

- But with the additional constraints that:

$$Z^{[2,2]} = \frac{1}{2}\beta_0 Z^{[1,1]} + \frac{1}{2} Z^{[1,1]} \otimes Z^{[1,1]}$$

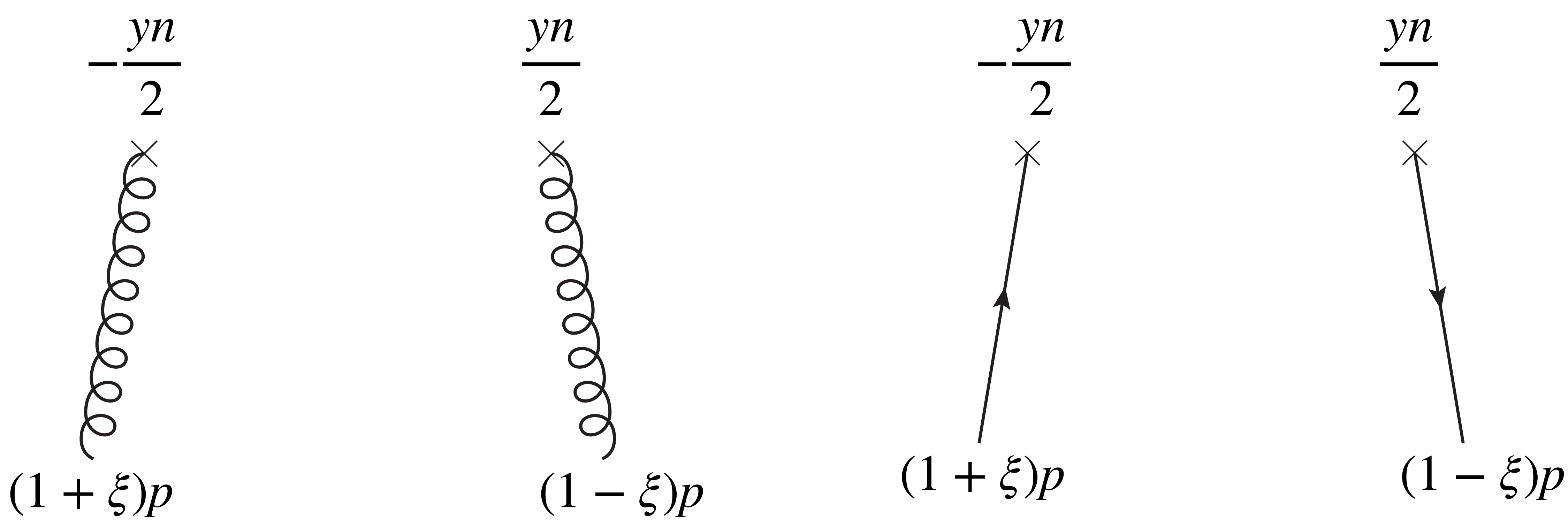
- An explicit two-loop calculation must fulfil this identity, thus providing a **strong check of the calculation** itself.

Parton-in-parton GPDs at LO

- At $\mathcal{O}(1)$:

$$\psi_q(x) = \psi_q^{(0)}(x) \qquad A_a^j(x) = A_a^{(0),j}(x)$$

- One immediately finds that the only non-zero GPDs are g/g and q/q:



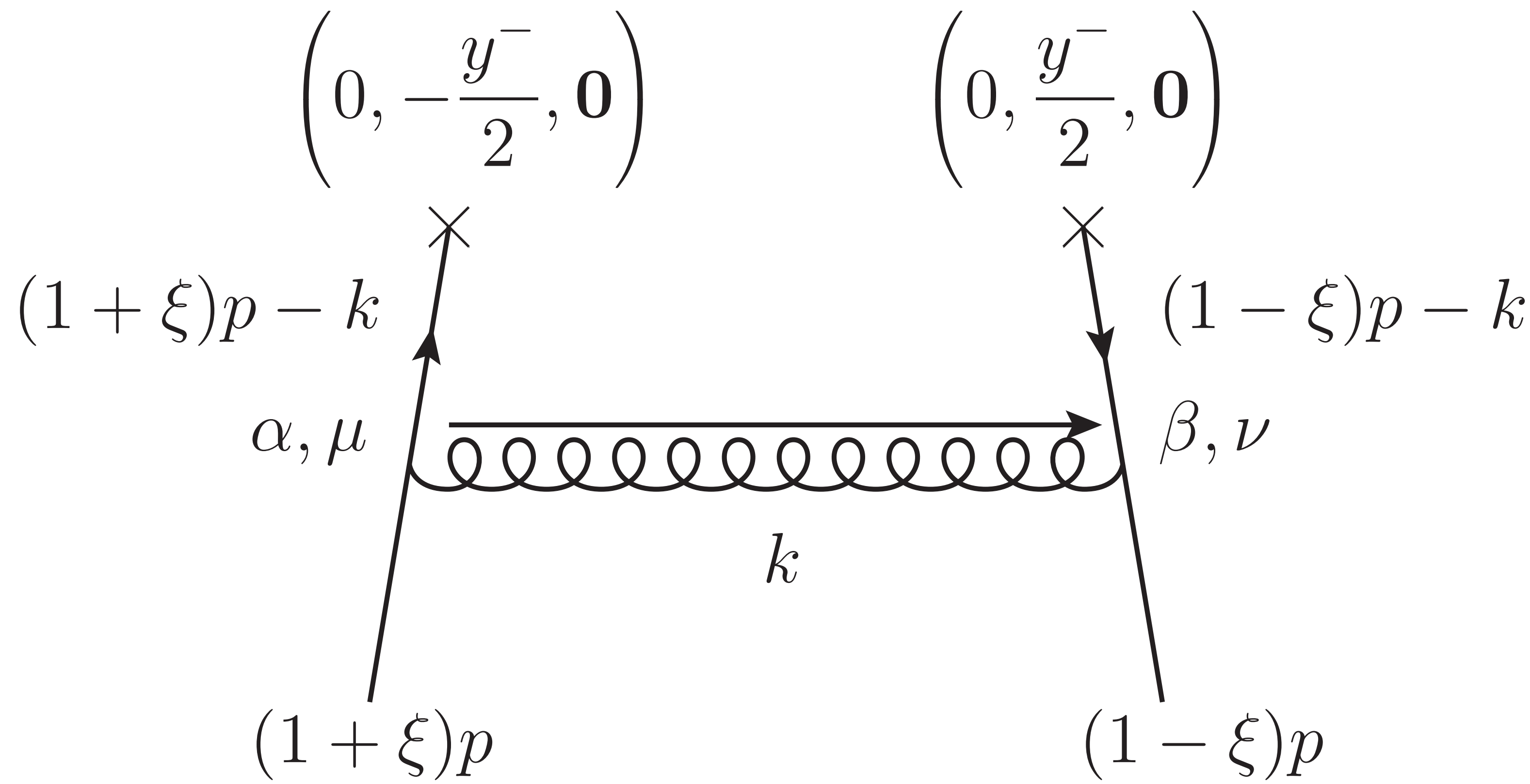
$$F_{g/g}^{[U][U],[0]}(x,\xi) = F_{g/g}^{[L][L],[0]}(x,\xi) = F_{g/g}^{[T][T],[0]}(x,\xi) = (1-\xi^2)\delta(1-x)$$

$$F_{q/q}^{[U][U],[0]}(x,\xi) = F_{q/q}^{[L][L],[0]}(x,\xi) = F_{q/q}^{[T][T],[0]}(x,\xi) = \sqrt{1-\xi^2}\,\delta(1-x)$$

- No divergences at this order and thus **no need for renormalisation**.
- This calculation sets the **normalisation** of GPDs.

Parton-in-parton GPDs at NLO

- At $\mathcal{O}(\alpha_s)$ for the q/q channel one has to compute *one single* "real" diagram:



- This produces:

$$\frac{\alpha_s}{4\pi}\hat{F}_{q/q}^{[\Gamma],[1],\text{real}}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\int_{-\infty}^{\infty}\frac{dy_-}{2\pi}e^{i(1-x)p_+y_-}\,\text{Tr}\left[R_{qq}^{[\Gamma]}(y_-,\xi,\varepsilon)\Lambda_q^{[\Gamma]}\right]$$

- with:

$$R_{qq}^{[\Gamma]}(y_-,\xi,\varepsilon) = \frac{\alpha_s}{4\pi}iC_F\int\frac{d^{4-2\varepsilon}k}{(2\pi)^{2-2\varepsilon}}e^{-ik_+y_-}\frac{\gamma^\mu[(1+\xi)\not{p}-\not{k}]\,\Gamma_q\,[(1-\xi)\not{p}-\not{k}]\gamma^\nu d_{\mu\nu}(k)}{[((1+\xi)p-k)^2+i0][((1-\xi)p-k)^2+i0]}$$

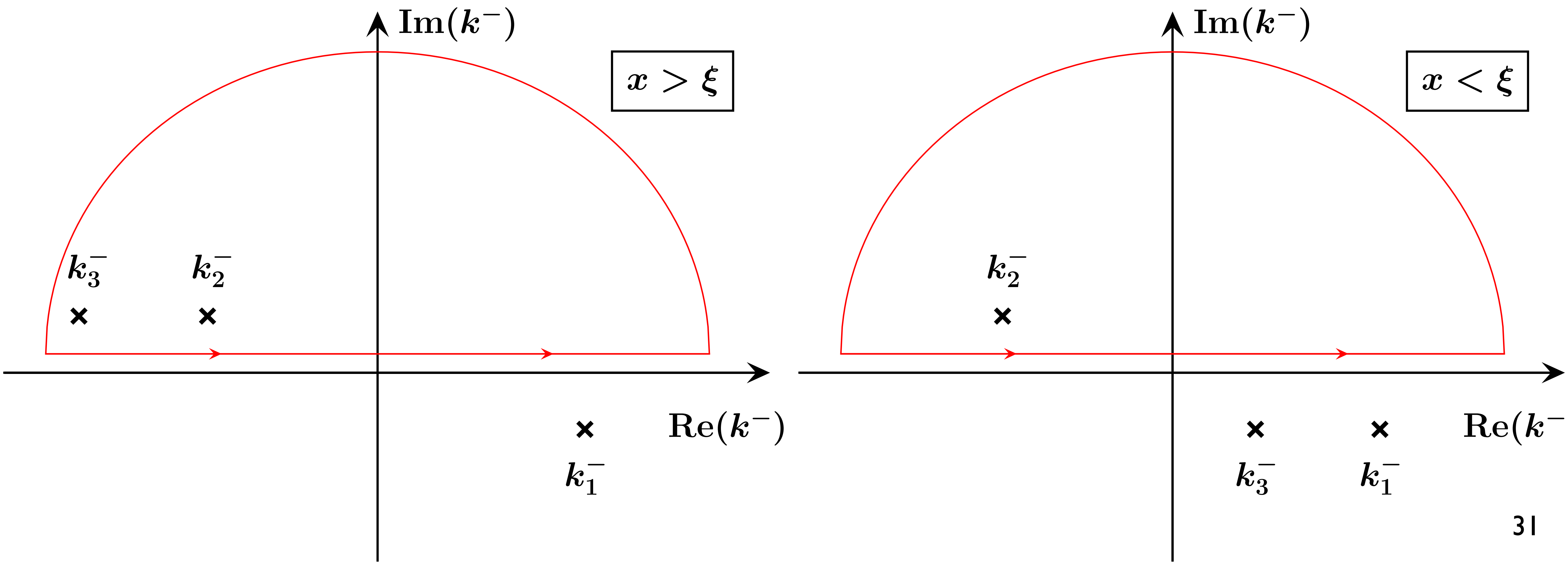
Parton-in-parton GPDs at NLO

- After the trivial integration over k^+ and the evaluation of contractions and traces, one finds:

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \int \frac{d^{2-2\varepsilon}\mathbf{k}_T}{(2\pi)^{2-2\varepsilon}}\mathbf{k}_T^2 \int_{-\infty}^{+\infty} dk^- \frac{A(x,\xi)+B(x,\xi)p^+k^-/\mathbf{k}_T^2}{(k^- - k_1^-)(k^- - k_2^-)(k^- - k_3^-)}$$

$$k_1^- = \frac{\mathbf{k}_T^2}{2(1-x)p^+} - i(1-x)\eta \quad k_2^- = -\frac{\mathbf{k}_T^2}{2(x+\xi)p^+} + i(x+\xi)\eta \quad k_3^- = -\frac{\mathbf{k}_T^2}{2(x-\xi)p^+} + i(x-\xi)\eta$$

- Assuming $x,\xi>0$, the pole structure depends on the sign of $x-\xi$:



Parton-in-parton GPDs at NLO

- The final result looks like this:

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\theta(1-x)\left[\theta(x+\xi)p_{q/q}^{\Gamma}\left(x,\frac{\xi}{x}\right)+\theta(x-\xi)p_{q/q}^{\Gamma}\left(x,-\frac{\xi}{x}\right)\right]\mu^{2\varepsilon}S_\varepsilon\int_0^\infty \frac{dk_T^2}{k_T^{2+2\epsilon}}$$

- Strictly speaking:

$$\int_0^\infty \frac{dk_T^2}{k_T^{2+2\epsilon}} = \mu_0^{-2\epsilon}\left(\frac{1}{\varepsilon_{\mathrm{UV}}}-\frac{1}{\varepsilon_{\mathrm{IR}}}\right)=0 \quad \Rightarrow \quad \hat{F}_{q/q}^{[\Gamma][\Lambda],[1],\mathrm{real}}(x,\xi,\varepsilon)=0$$

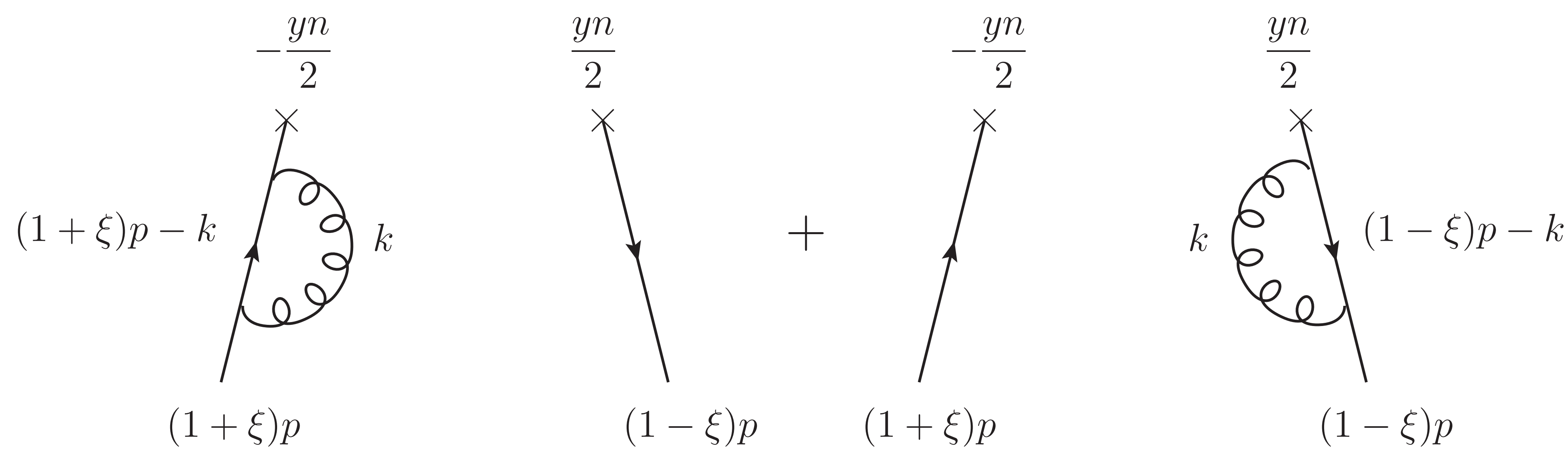
- We are only concerned with the **UV part**: the IR one has to cancel against the partonic cross section when computing a physical observable (**IR safety**).

$$\hat{F}_{q/q}^{[\Gamma],[1],\mathrm{real}}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\theta(1-x)\left[\theta(x+\xi)p_{q/q}^{\Gamma}\left(x,\frac{\xi}{x}\right)+\theta(x-\xi)p_{q/q}^{\Gamma}\left(x,-\frac{\xi}{x}\right)\right]\frac{(\mu^2/\mu_0^2)^\varepsilon}{\overline{\varepsilon}}+\mathrm{IR}$$

Evolution kernels at one loop

- The **virtual** contribution (common to all polarisations) is computed as:

$$\frac{1}{2}\Bigg[$$



$$\Bigg]$$

- The final result is:

$$\hat{F}^{[\Gamma],[1]}_{q/q}(x,\xi,\varepsilon) = \sqrt{1-\xi^2}\Bigg\{\theta(1-x)\left[\theta(x+\xi)p^{\Gamma}_{q/q}\left(x,\frac{\xi}{x}\right)+\theta(x-\xi)p^{\Gamma}_{q/q}\left(x,-\frac{\xi}{x}\right)\right]$$

$$+\ \delta(1-x)C_F\left[\frac{3}{2}-\ln\left(\left|1-\frac{\xi^2}{x^2}\right|\right)-2\int_0^1\frac{dz}{1-z}\right]\Bigg\}\frac{\mu^{2\varepsilon}}{\overline{\varepsilon}}+\ \mathrm{IR}$$

- The resulting evolution kernel is:

$$\mathcal{P}^{[\Gamma],[0]}_{qq}(y,\kappa) = \theta(1-y)\left[\theta(1+\kappa)p^{\Gamma}_{q/q}(y,\kappa)+\theta(1-\kappa)p^{\Gamma}_{q/q}(y,-\kappa)\right]$$

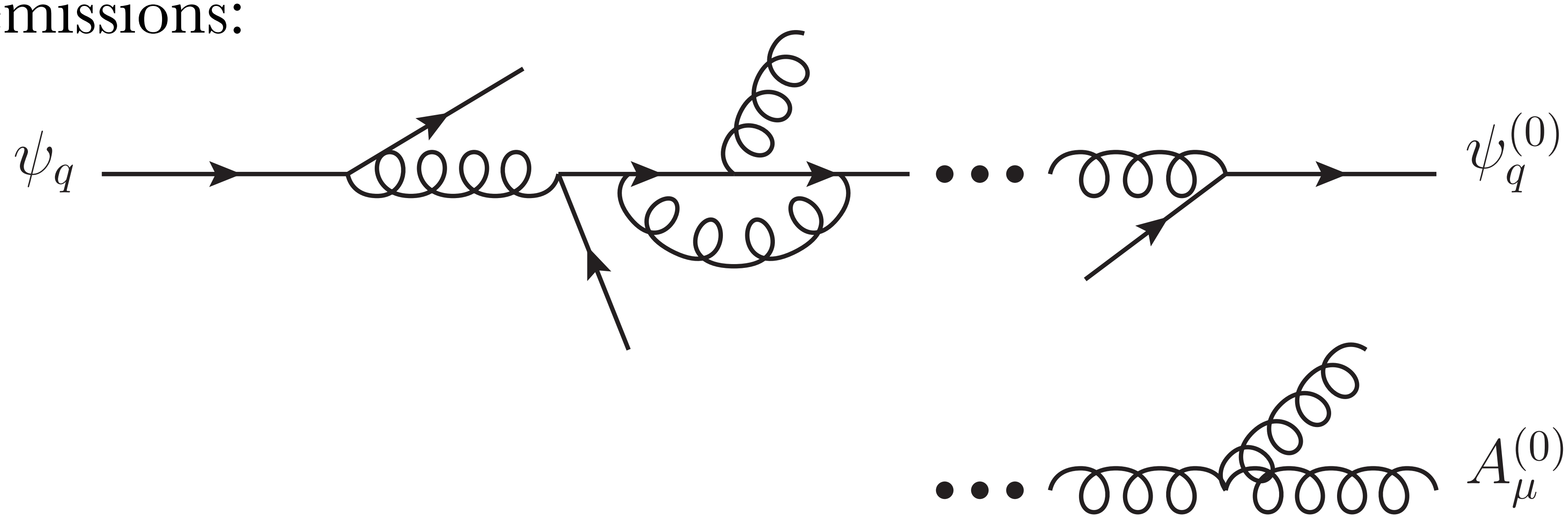
$$+\ \delta(1-y)C_F\left[\frac{3}{2}-\ln\left(\left|1-\kappa^2\right|\right)-2\int_0^1\frac{dz}{1-z}\right]$$

$$\kappa=\frac{\xi}{x}$$

Parton-in-parton GPDs

- The partonic fields that appear in the operator definition of the GPD correlators are **interacting fields**.

- Interacting fields reduce to **free fields** after an arbitrary number of *real* and *virtual* emissions:



- Additional radiation gives rise to **perturbative corrections** and the **need for renormalisation**.

- Free partonic fields eventually **annihilate** the appropriate partonic states:

$$\psi_q^{(0)}(x)|k,s\rangle_q = e^{-ik\cdot x}u_{q,s}(k)|0\rangle$$

$$\psi_q^{(0)}(x)|k,s\rangle_{\overline{q}} = e^{ik\cdot x}v_{q,s}(k)|0\rangle$$

$$A_a^{(0),j}(x)|k,s\rangle_g = e^{-ik\cdot x}e^j_{a,s}(k)|0\rangle$$

- All other combinations give zero.

Parton-in-parton GPDs

- In light-cone gauge:

$$\hat{F}^{[\Gamma][\Lambda]}_{g/g,q}(x,\xi) = \frac{(n\cdot p)(x^2-\xi^2)}{2(N_c^2-1)x}\int\frac{dy}{2\pi}e^{-ix(n\cdot p)y}\left\langle(1-\xi)p,s'\left|A^\mu_a\left(\frac{yn}{2}\right)\Gamma_{g,\mu\nu}A^\nu_a\left(-\frac{yn}{2}\right)\right|(1+\xi)p,s\right\rangle_{g,q}\Lambda_{s's}$$

$$\hat{F}^{[\Gamma][\Lambda]}_{q/g,q,\overline{q},q',\overline{q}'}(x,\xi) = \frac{1}{2N_c}\int\frac{dy}{2\pi}e^{-ix(n\cdot p)y}\left\langle(1-\xi)p,s'\left|\overline{\psi}^i_q\left(\frac{yn}{2}\right)\Gamma^{ij}_q\psi^j_q\left(-\frac{yn}{2}\right)\right|(1+\xi)p,s\right\rangle_{g,q,\overline{q},q',\overline{q}'}\Lambda_{s's}$$

- The projectors $\Lambda_{s's}$ are introduced for *convenience* to project out the physical partonic spin/helicity states that contribute to the amplitude:

$$\left.\begin{array}{l}\Lambda_{s's}\overline{u}_{q,s'}((1-\xi)p)u_{q,s}((1+\xi)p) = \Lambda_q = \sqrt{1-\xi^2}\,\{\not{p},\not{p}\gamma^5, i\sigma^{\mu\nu}P_\nu\gamma^5\} \\ \Lambda_{s's}e^{\mu*}_{s'}((1-\xi)p)e^\nu_s((1+\xi)p) = \Lambda^{\mu\nu}_g = \{-g^{\mu\nu}_T, -i\varepsilon^{\mu\nu}_T, -R^\mu R^\nu - L^\mu L^\nu\}\end{array}\right. \in\{U,L,T\}$$

- These quark-in-quark combinations:

$$\hat{F}^{\mathrm{NS},\pm}_{q/q} = (\hat{F}_{q/q}-\hat{F}_{q/q'})\pm(\hat{F}_{q/\overline{q}}-\hat{F}_{q/\overline{q}'})$$

$$\hat{F}^{\mathrm{NS,V}}_{q/q} = \hat{F}^{\mathrm{NS},-}_{q/q} + n_f(\hat{F}_{q/q'}-\hat{F}_{q/\overline{q}'})$$

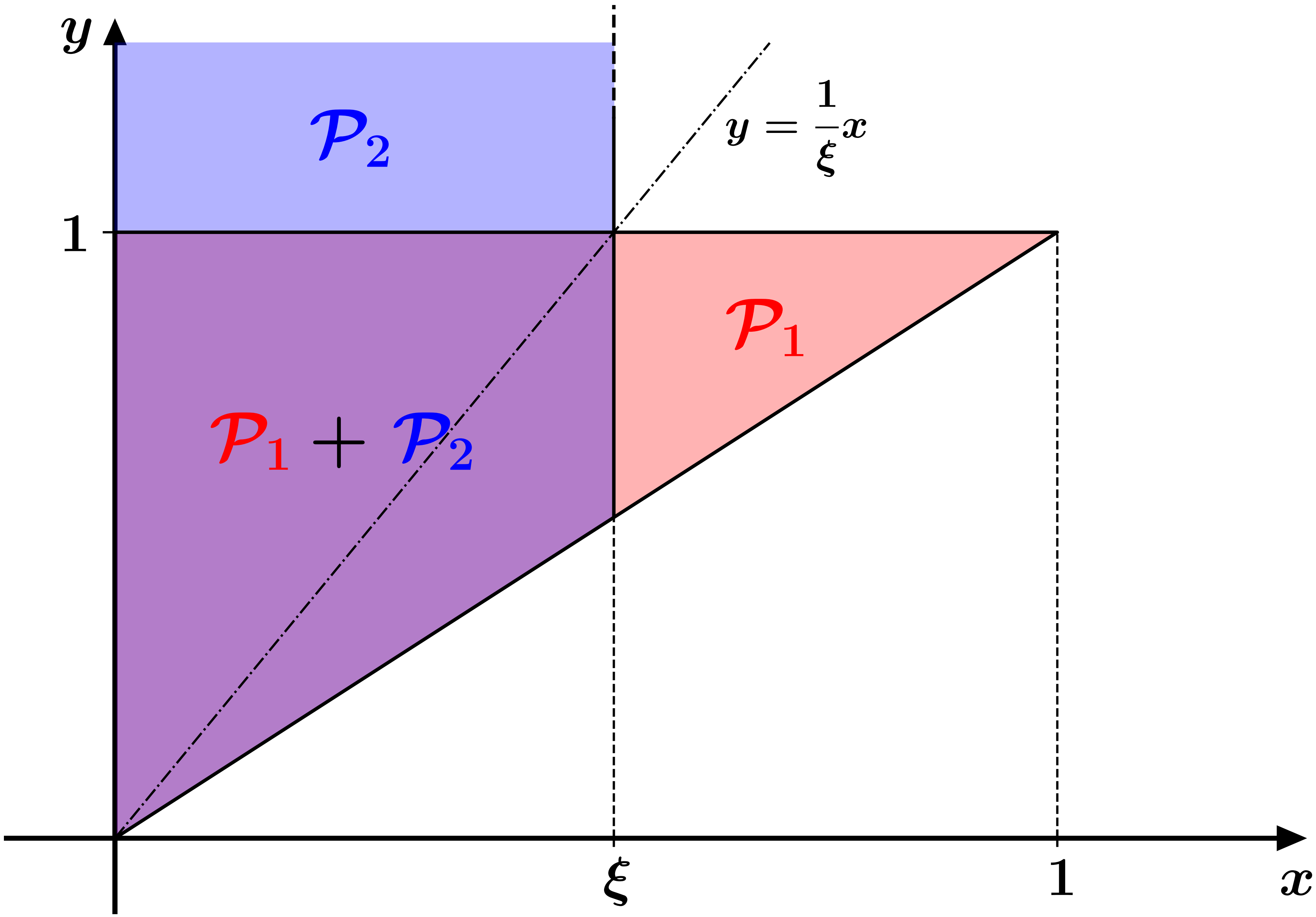
$$\hat{F}^{\mathrm{SG}}_{q/q} = \hat{F}^{\mathrm{NS},+}_{q/q} + n_f(\hat{F}_{q/q'}+\hat{F}_{q/\overline{q}'})$$

- are particularly convenient when implementing the evolution.

Properties of the kernels

$$\frac{dF^{[\Gamma],\pm}(x,\xi,\mu)}{d\ln\mu^2} = \frac{\alpha_s(\mu)}{4\pi}\int_x^\infty \frac{dy}{y}\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)F^{[\Gamma],\pm}\left(\frac{x}{y},\xi,\mu\right)$$

$$\mathcal{P}^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right) + \theta(\xi-x)\mathcal{P}_2^{[\Gamma]\pm,[0]}\left(y,\frac{\xi}{x}\right)$$



Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- In the limit $\kappa \to 0$ the **DGLAP** splitting functions are recovered:

$$\lim_{\kappa\to 0} \mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)P^{[\Gamma]\pm,[0]}(y)$$

- In the limit $\kappa \to 1/x$ the **ERBL** non-singlet kernel in the unpolarised case is recovered:
  e.g. [Mikhailov, Radyushkin, *Nucl.Phys.B* 254 (1985) 89-126]
  or [Blümlein, Geyer, Robaschik, *Phys.Lett.B* 406 (1997) 161-170]

$$\frac{1}{2u-1}\mathcal{P}^{[U]-,[0]}\left(\frac{2t-1}{2u-1},\frac{1}{2t-1}\right) = C_F\left[\theta(u-t)\left(\frac{t-1}{u}+\frac{1}{u-t}\right) - \theta(t-u)\left(\frac{t}{1-u}+\frac{1}{u-t}\right)\right]_+$$

with $[f(t,u)]_+ \equiv f(t,u) - \delta(u-t)\int_0^1 du'\, f(t,u')$

We have also derived singlet and non-singlet ERBL kernels for the other polarisations.

- **Continuity** of GPDs at the crossover point $x = \xi$ ($\kappa = 1$) guaranteed:

$$\lim_{\kappa\to 1} \mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) = \text{finite} \qquad \mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \propto (1-\kappa)$$

Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- Valence **sum rule** (polynomiality of the first moment of the **unpolarised non-singlet**):

$$\int_0^1 dx\, F^{[U],-}(x,\xi,\Delta^2;\mu) = \mathrm{FF}(\Delta^2) \quad \Rightarrow \quad \int_0^1 dz \left[\mathcal{P}_1^{[U]-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_2^{[U]-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = 0$$

- As consequence of the **Ji's sum rule** one also finds: [Ji, *Phys. Rev. Lett.* 78 (1997) 610-613]

$$\int_0^1 dx\, x \left[F_q^{[U],+}(x,\xi,\Delta^2;\mu) + F_g^{[U],+}(x,\xi,\Delta^2;\mu)\right] = \text{constant in } \xi \text{ and } \mu$$

- that leads to:

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,qq}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gq}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\left(\mathcal{P}_{2,qq}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right) + \mathcal{P}_{2,gq}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right)\right] = 0$$

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,qg}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \mathcal{P}_{1,gg}^{[U]+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\left(\mathcal{P}_{2,qg}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right) + \mathcal{P}_{2,gg}^{[U]+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right)\right] = 0$$

- These identities were analytically verified in [*Eur. Phys. J. C* 82 (2022) 10, 888].

Properties of the kernels

$$\mathcal{P}^{[\Gamma]\pm,[0]}(y,\kappa) = \theta(1-y)\mathcal{P}_1^{[\Gamma]\pm,[0]}(y,\kappa) + \theta(\kappa-1)\mathcal{P}_2^{[\Gamma]\pm,[0]}(y,\kappa) \qquad \kappa = \frac{\xi}{x}$$

- The ξ-independence of the **1st moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz \left[\mathcal{P}_{1,ij}^{L,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_{2,ij}^{L,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

- This is true and we also find that the q/q and and q/g channels are identically zero, *i.e.* the first moment of $F_{q/H}^{[L],+}$ is **scale independent**:
  - physical observable connected with the anti-symmetric part of the EMT.
- The ξ-independence of the **2nd moment of longitudinally polarised** GPDs implies:

$$\int_0^1 dz\, z \left[\mathcal{P}_1^{L,-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_2^{L,-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

- Similar arguments apply to **transversely pol.** GPDs and lead to the verified constraints:

$$\int_0^1 dz \left[\mathcal{P}_1^{T,-,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi}{y}\mathcal{P}_2^{T,-,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,qq}^{T,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_{2,qq}^{T,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

$$\int_0^1 dz\, z \left[\mathcal{P}_{1,gg}^{T,+,[0]}\left(z,\frac{\xi}{yz}\right) + \frac{\xi^2}{y^2}\mathcal{P}_{2,gg}^{T,+,[0]}\left(\frac{z\xi}{y},\frac{1}{z}\right)\right] = \text{constant in } \xi$$

The ERBL limit

- The limit $\xi \to \mathbf{1}$ ($\kappa \to 1/x$) we should reproduce the **ERBL equation**.

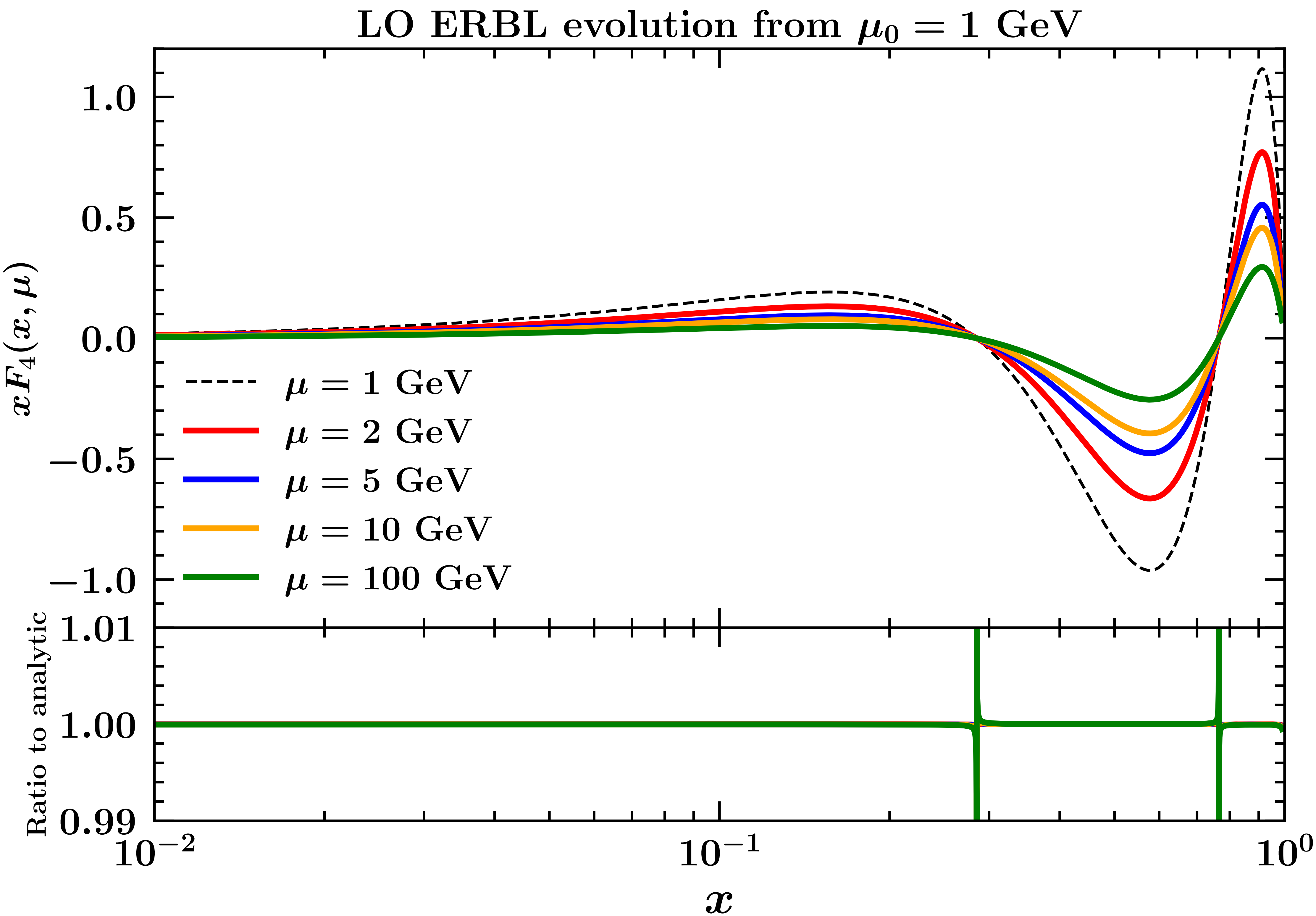
- It is well known that in this limit **Gegenbauer polynomials** decouple upon LO evolution, such that:

$$F_{2n}(x,\mu_0) = (1-x^2)C_{2n}^{(3/2)}(x) \quad \Rightarrow \quad F_{2n}(x,\mu) = \exp\left[\frac{V_{2n}^{[0]}}{4\pi}\int_{\mu_0}^{\mu} d\ln\mu^2\alpha_s(\mu)\right] F_{2n}(x,\mu_0)$$

- where the kernels $V_{2n}^{[0]}$ can be read off, for example, from [Brodsky, Lepage, *Phys.Rev.D* 22 (1980) 2157] or [Efremov, Radyushkin, *Phys.Lett.B* 94 (1980) 245-250].

- We have compared this expectation with the numerical results for GPD evolution by setting $\kappa = 1/x$ and using a Gegenbauer polynomial as an initial-scale GPD.

The ERBL limit



- **ERBL limit** reproduced within less than 10^{-5} relative accuracy,
- Same accuracy for **higher-degree** Gegenbauer polynomials.

Conformal-space evolution

- In order to check that LO GPD evolution ($\xi \neq 0$) in conformal space is diagonal in a **realistic** case, we have considered the RDDA:

$$H_q(x, \xi, \mu_0) = \int_\Omega \mathrm{d}\beta \mathrm{d}\alpha \delta\left(x - \beta - \xi\alpha\right) q(|\beta|)\pi(\beta, \alpha)$$

with:

$$q(x) = \frac{35}{32} x^{-1/2}(1-x)^3, \quad \pi(\beta, \alpha) = \frac{3}{4}\frac{((1-|\beta|)^2 - \alpha^2)}{(1-|\beta|)^3}$$

We have evolved the 4th moment:

$$\mathcal{C}_4^-(\xi, \mu) = \xi^4 \int_{-1}^{1} dx\, C_4^{(3/2)}\left(\frac{x}{\xi}\right) H_q(x, \xi, \mu)$$

from $\mu_0 = 1$ GeV using the (analytic) conformal-space evolution and the (numerical) momentum-space evolution.

we found excellent agreement.