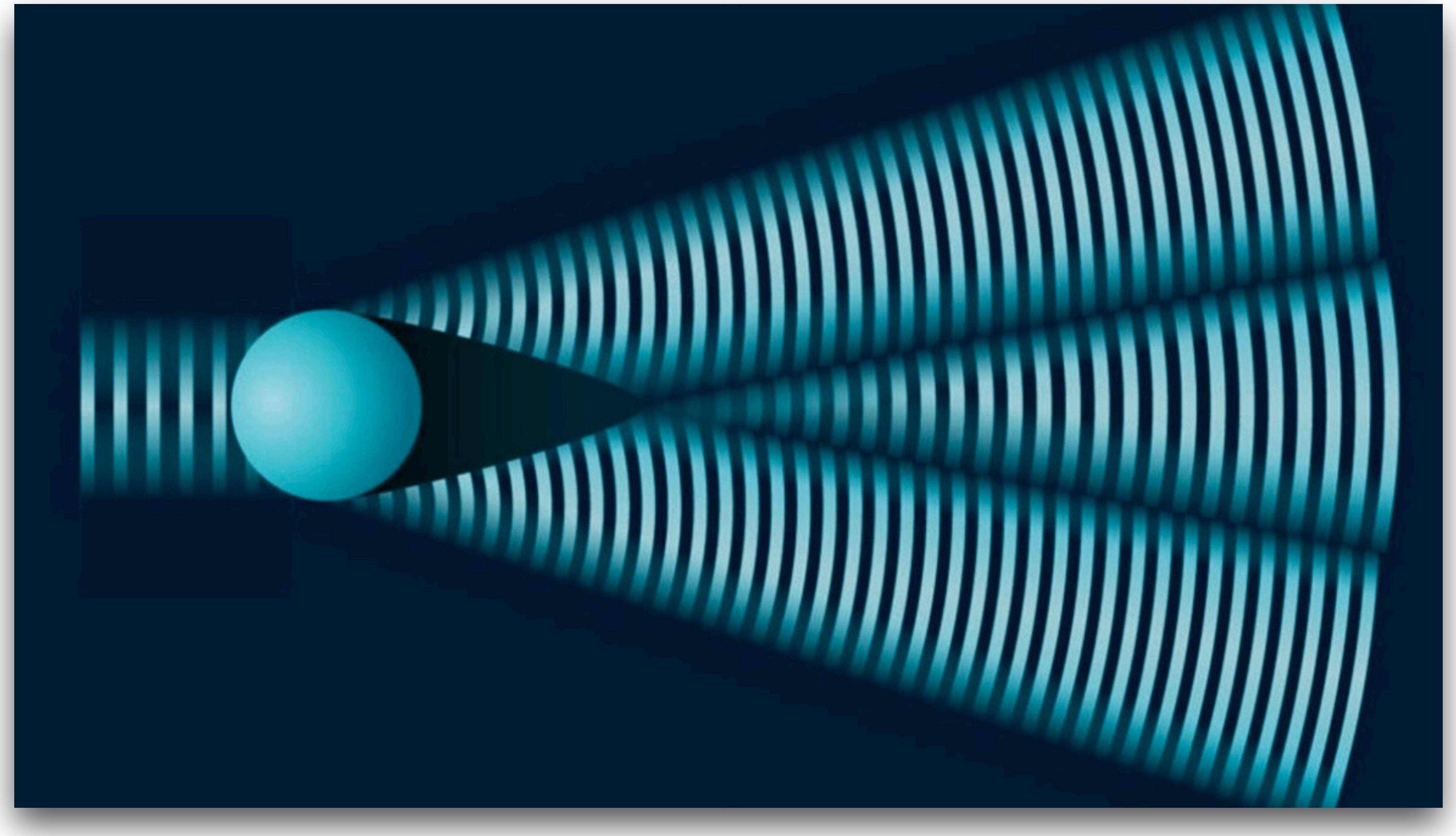
Proper time path integrals for GWs

An improved wave optics framework



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Based on: 2405.20208
GB, A.Garoffolo, A. Ricciardone, N. Bartolo, S. Matarrese

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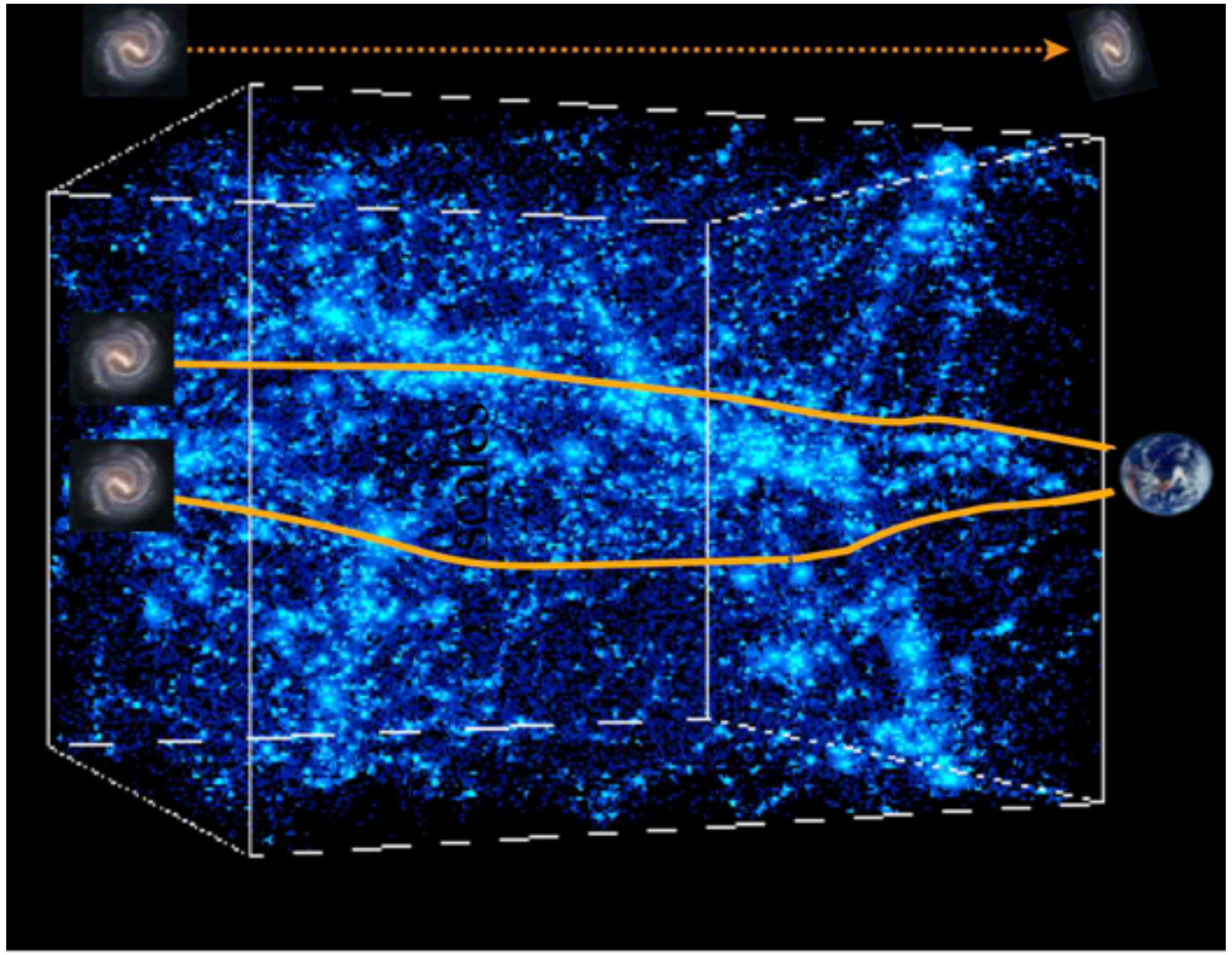
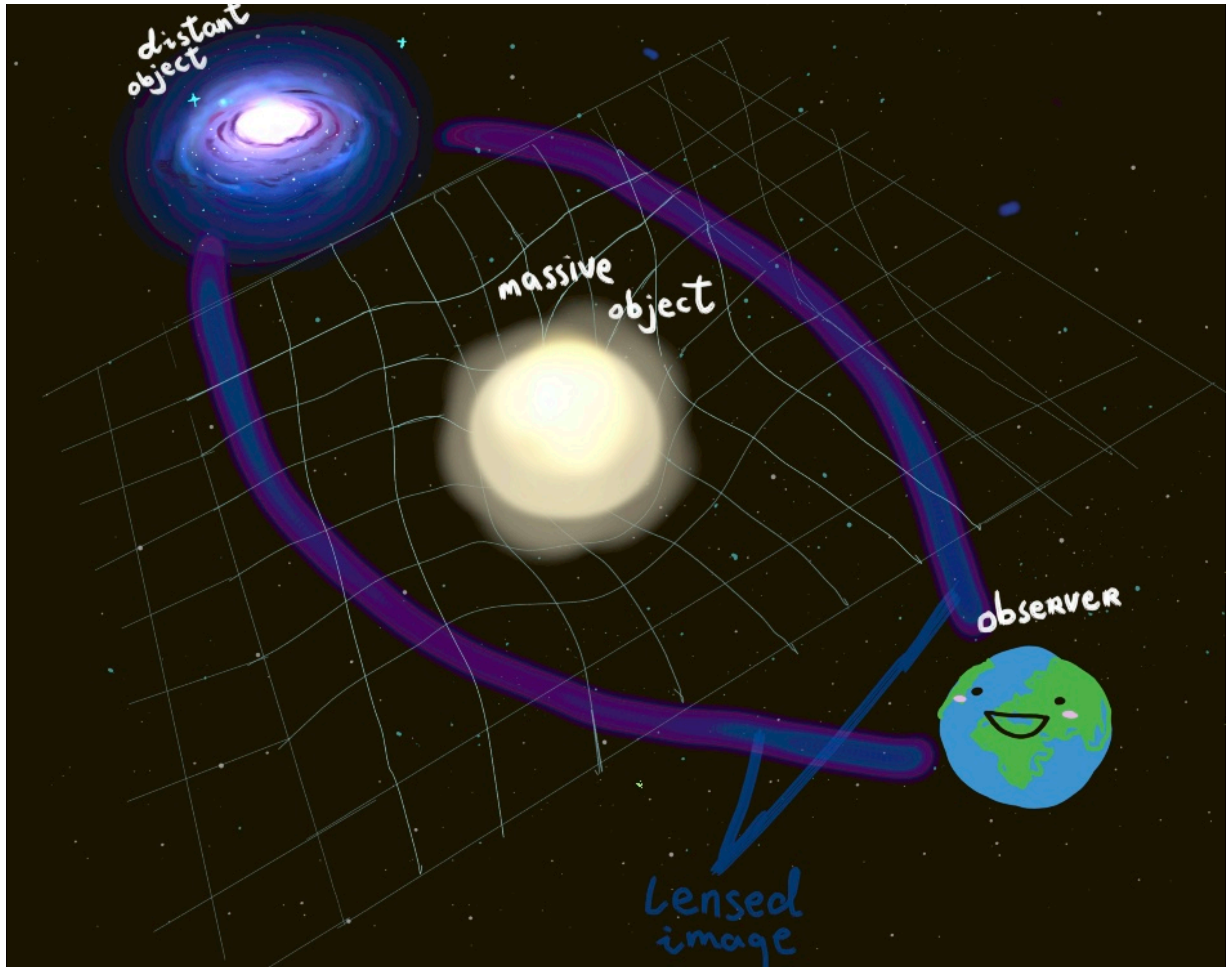
GWs through the perturbed Universe

Probe of large scale structures and compact objects

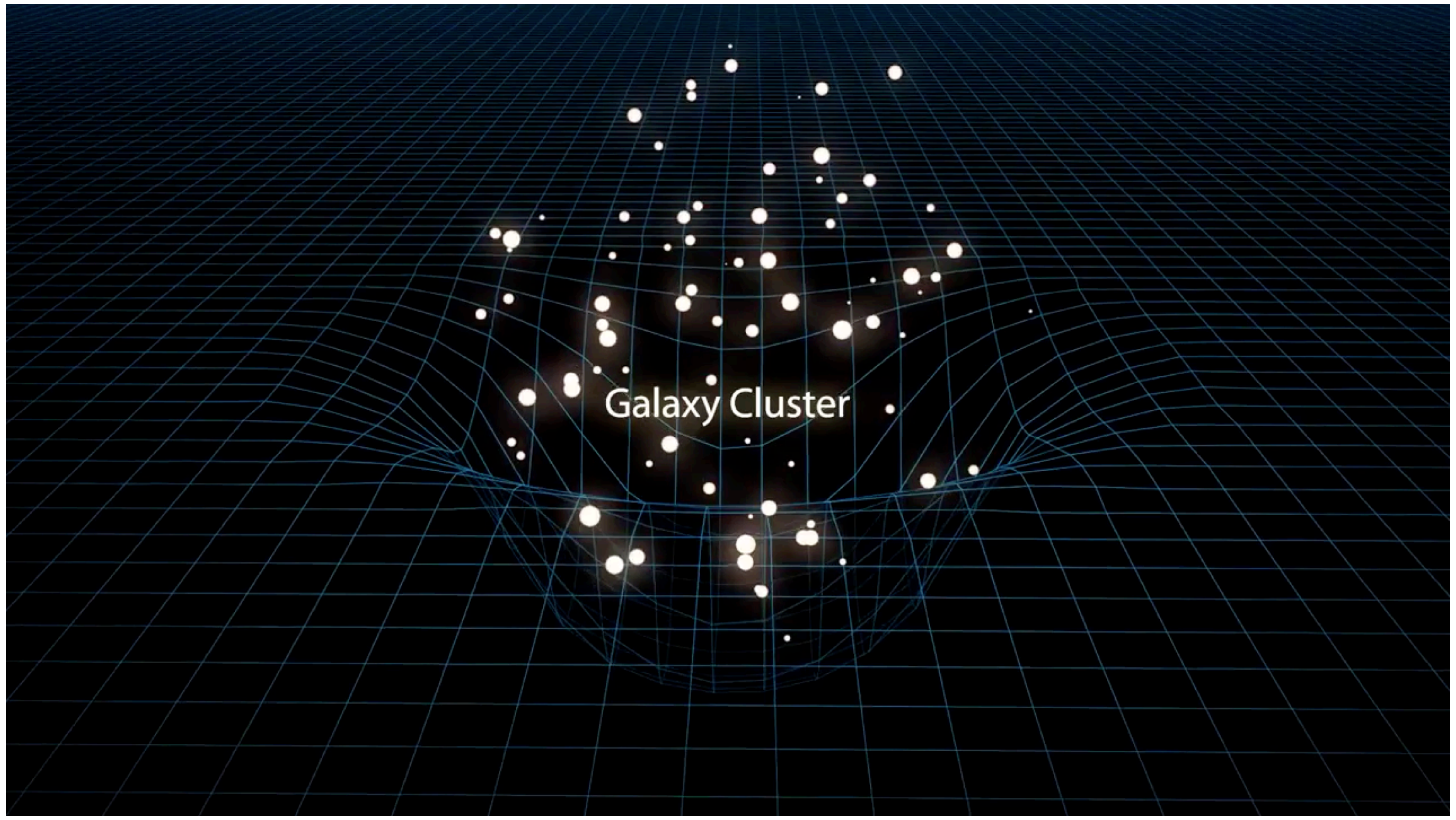




Propagation effects carry cosmological and astrophysical information

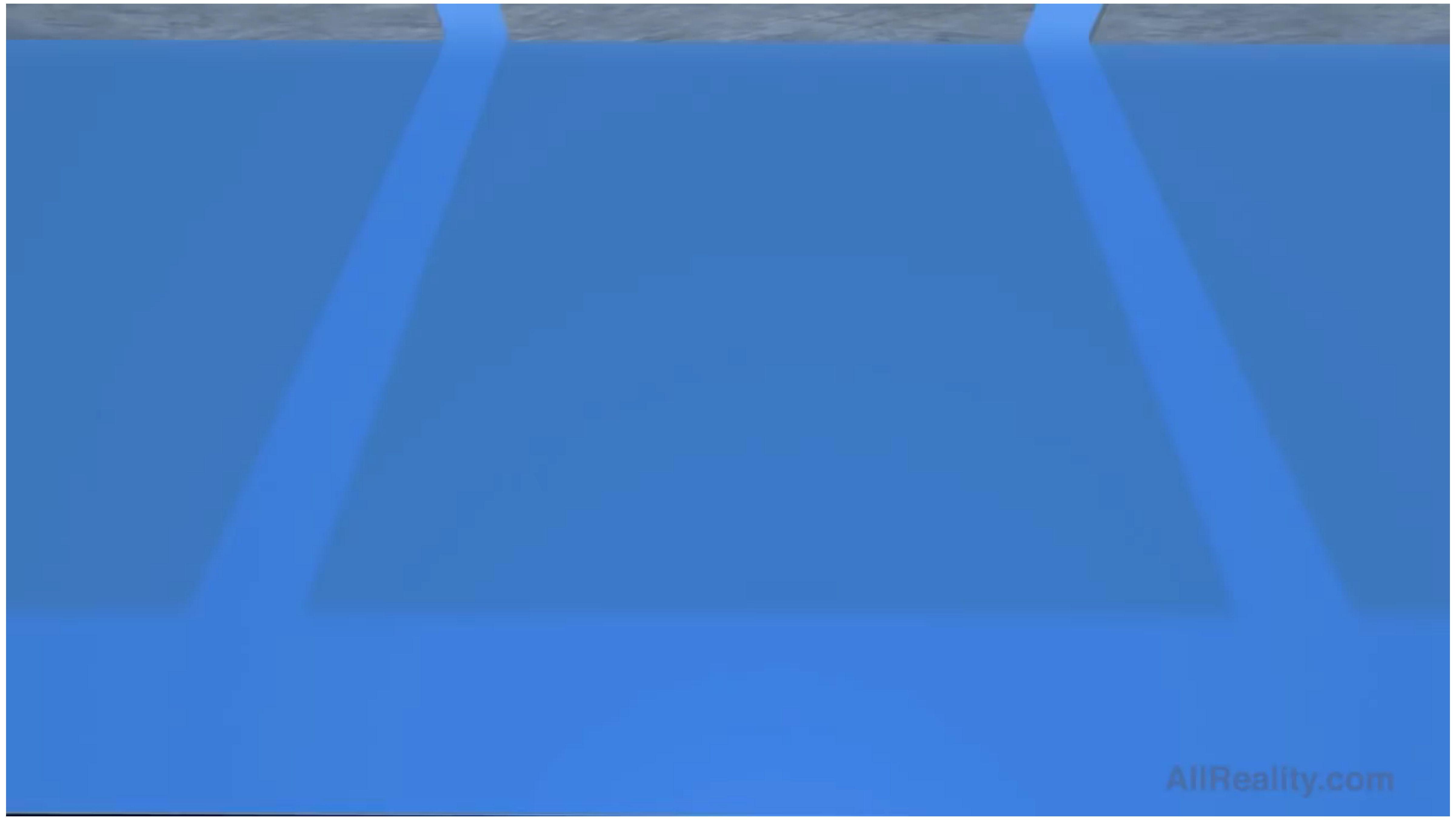
Optical regimes

High Frequency: $\omega R_S \gg 1$



Ray description

Low Frequency: $\omega R_S \lesssim 1$

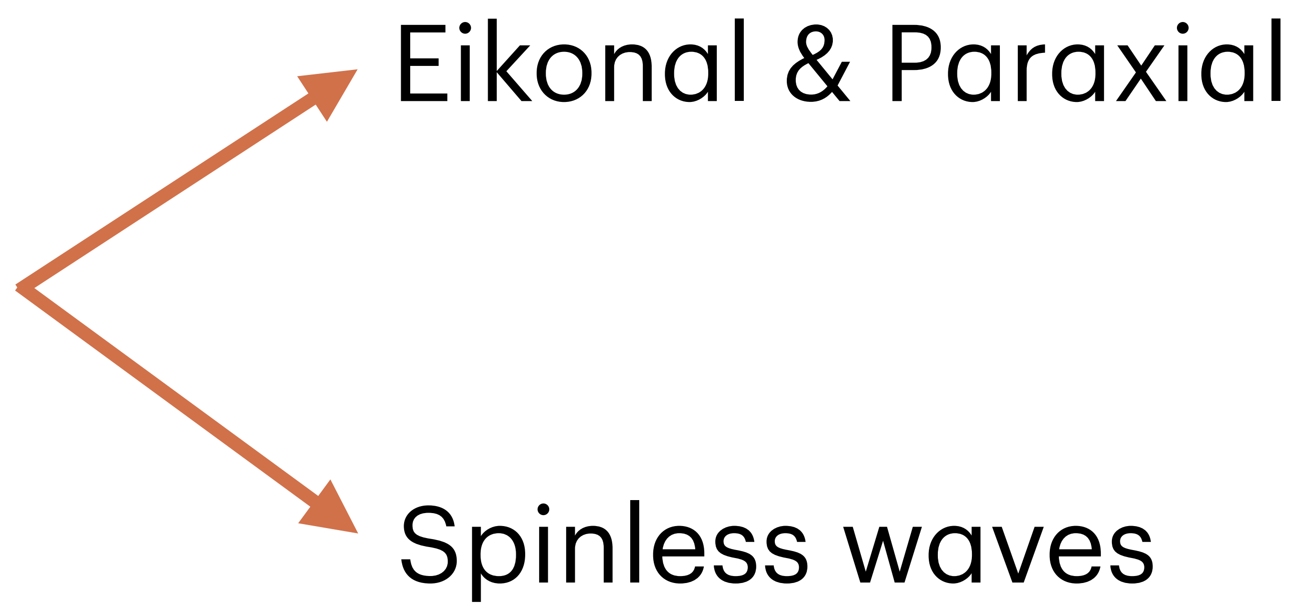


Wave effects

- Gao et al. (2102.10295): ~(0.1 - 1.6)% of MBHB with $(10^5 - 10^{6.5})M_\odot$ and $4 \leq z \leq 10$
- LISA CosGW (2204.05434): WO need to be considered for typical LISA sources

Talk’s plan

- Geometric optics (GO)
- Standard formalism of Wave optics (WO): diffraction integral
- Our work: proper time path integral

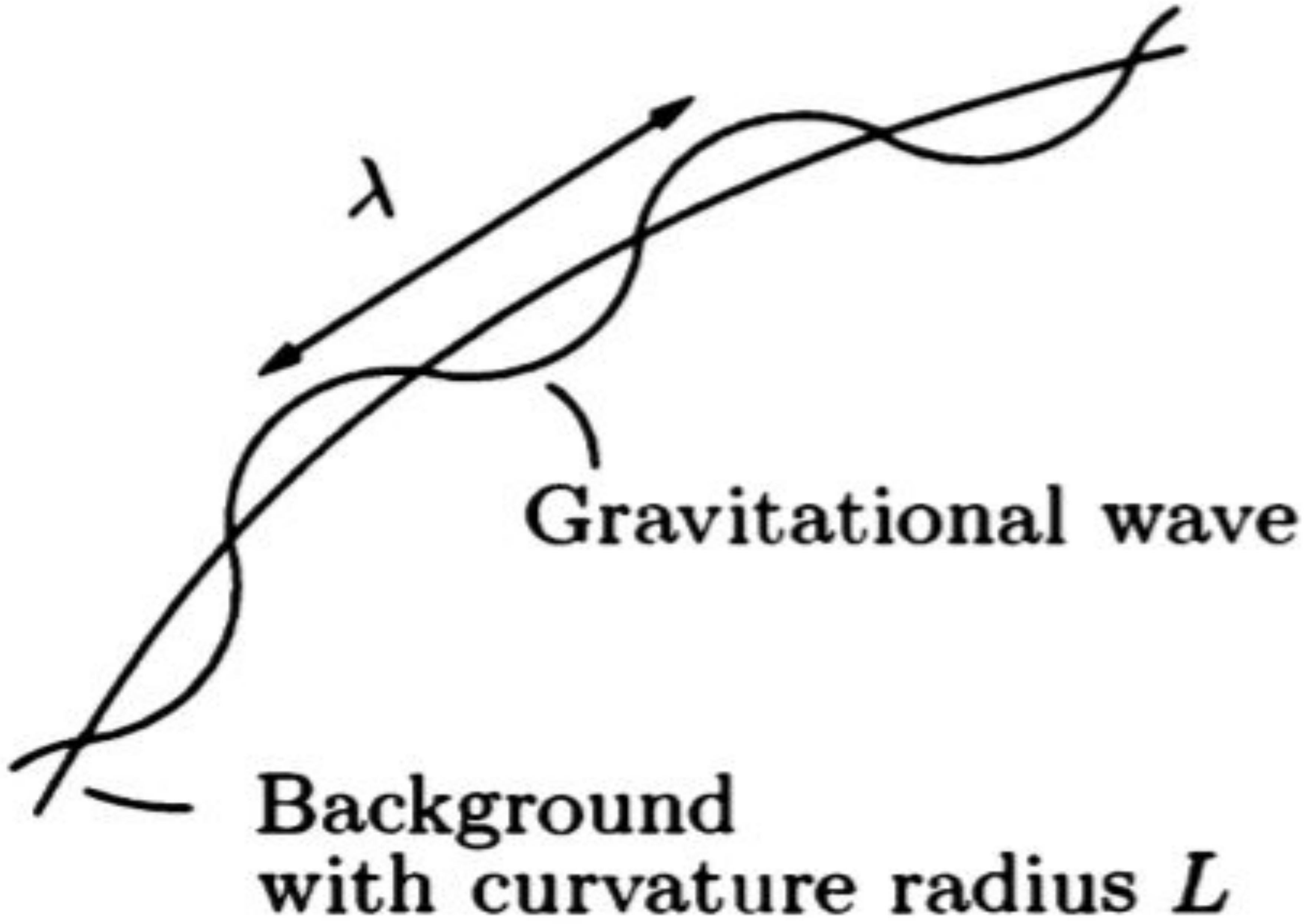


Eikonal & Paraxial

Spinless waves

Geometric optics

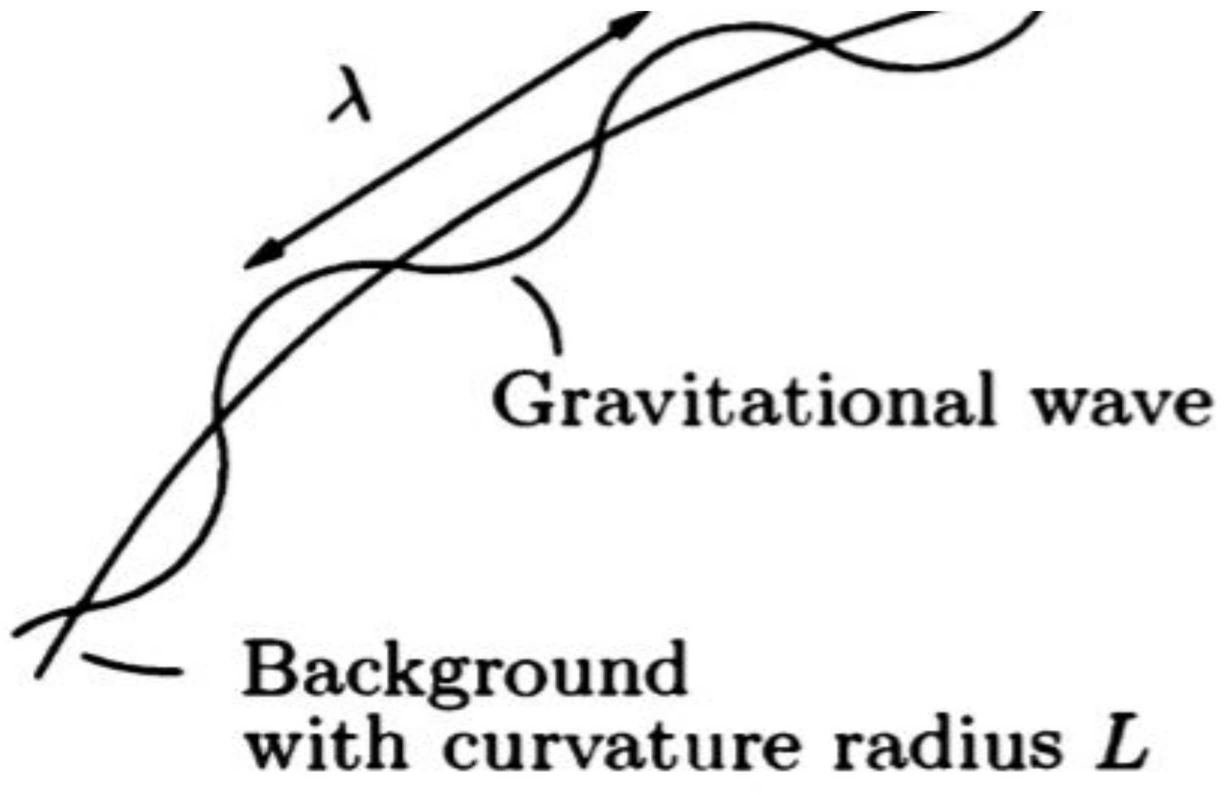
Isaacson ‘68



Geometric optics

Isaacson '68

1. GO ansatz for GW: $h_{\mu\nu} \equiv \mathscr{A}_{\mu\nu}(x)\, e^{i\omega\,\theta(x)}$ where $\omega \equiv \frac{L}{\lambda} \gg 1$

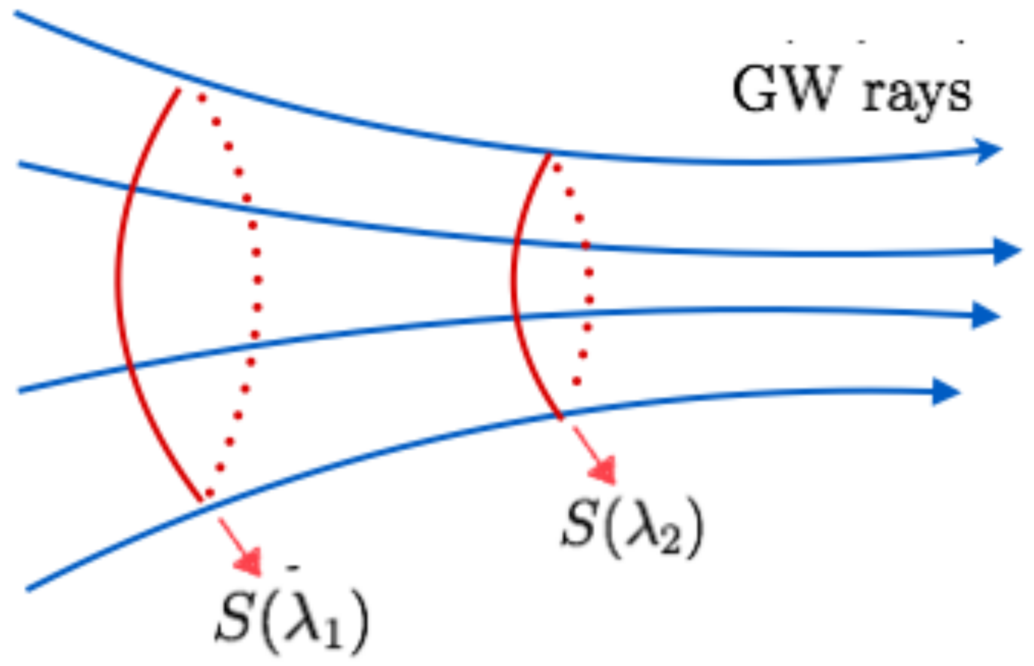


2. Organize linearized Einstein Eqs. in powers of ω:

$$\omega^2\,[\ldots] + i\omega\,[\ldots] + [\ldots] = 0$$

$$\nabla_\rho\left(\mathscr{A}^2 k^\rho\right) = 0 \longrightarrow \text{Amplitude continuity equation}$$

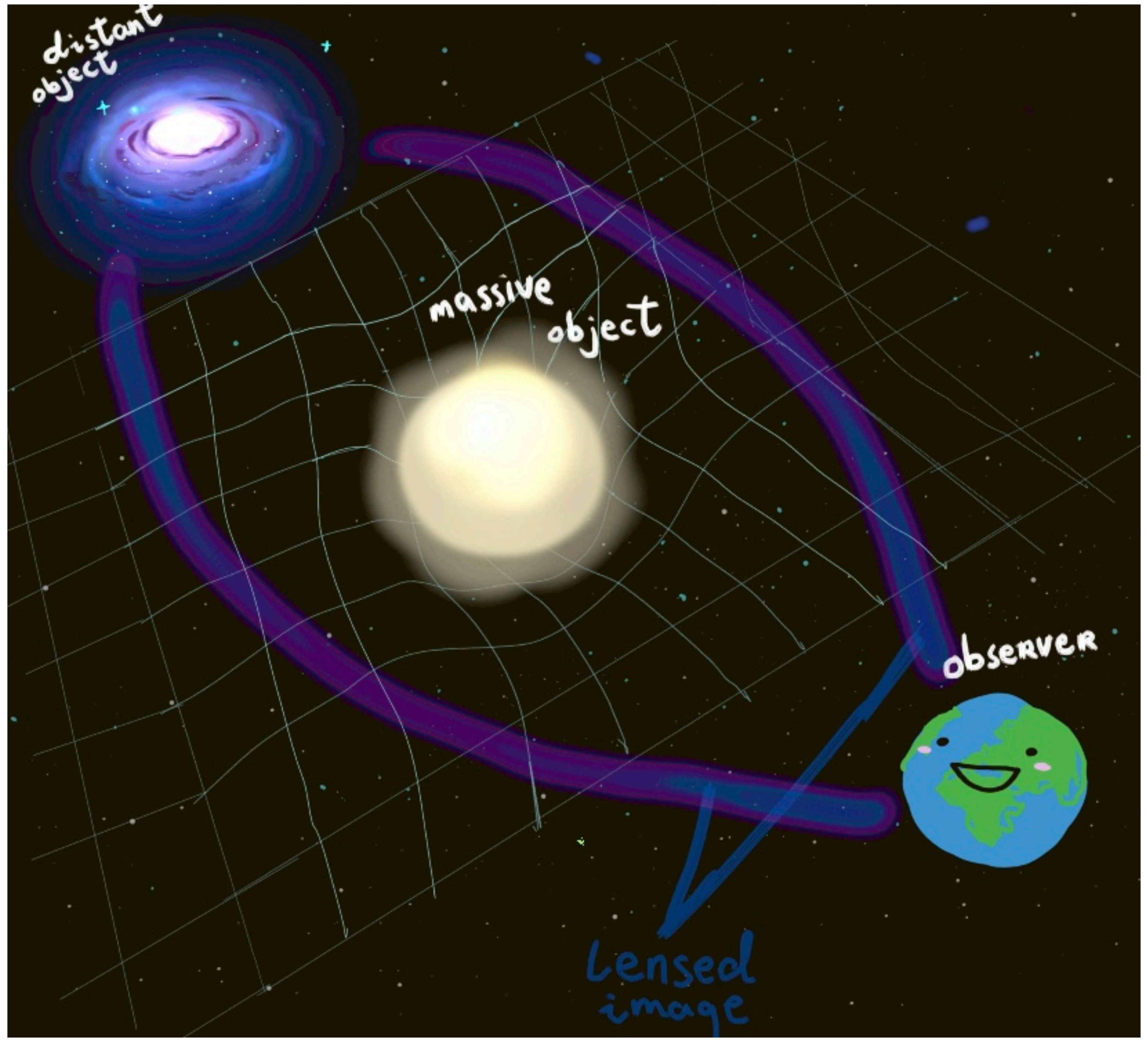
- $k_\rho \equiv \partial_\rho \theta$
- $k^\rho\,\nabla_\rho k^\mu = 0$ → There is a unique direction of propagation, and the trajectory is a null geodesic
- $g^{\mu\nu} k_\mu k_\nu = 0$



Refractive index & dispersion relation

- From GO: trajectories are null geodesics of background spacetime

$$ds^2 = -(1 + 2\,\alpha\,U(\mathbf{x}))dt^2 + (1 - 2\,\alpha\,U(\mathbf{x}))d\mathbf{x}^2$$



- Null condition = dispersion relation :

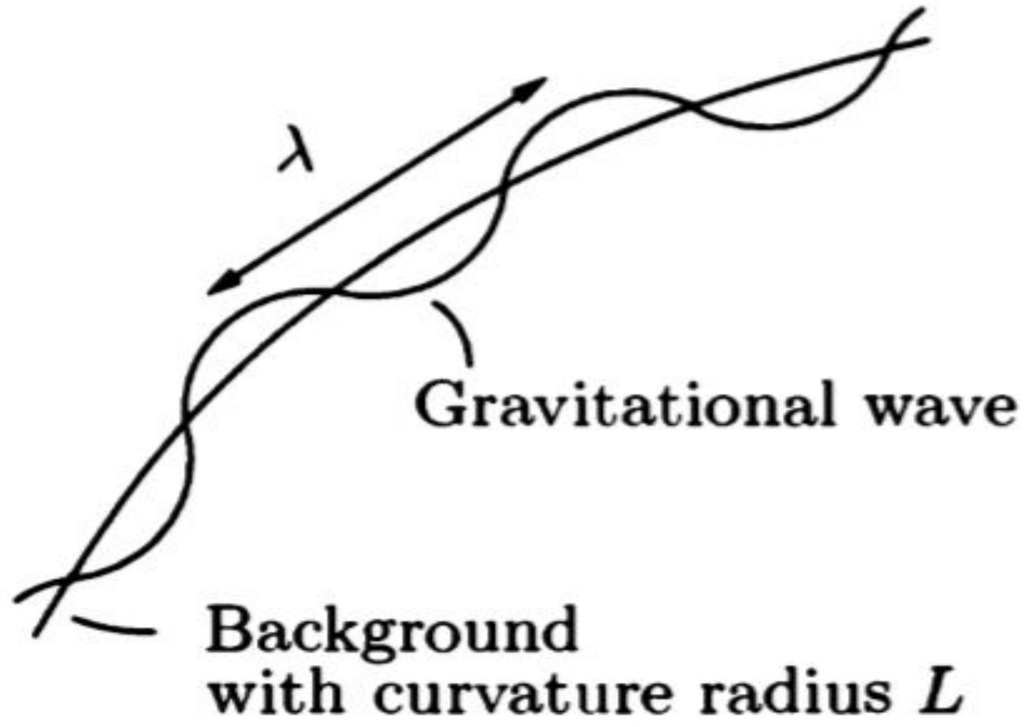
$$g^{\mu\nu}k_\mu k_\nu = 0$$

$$\mathbf{k}^2 = \omega^2(1 - 2\alpha U)^2 \equiv \omega^2 n^2$$ → Refractive index

- Geodesic equation = Fermat principle :

$$\delta\left(\int_0^{l_{cl}} dl' n(\mathbf{x}(l'))\right) = 0$$

Hamilton analogy



Isaacson ‘68

GWs

$$h_{\mu\nu} = \mathscr{A}_{\mu\nu}\, e^{i\omega\,\theta}$$

$$\omega$$

Einstein + $1/\omega \to 0$

Continuity eq. with $k_\mu = \partial_\mu \theta$

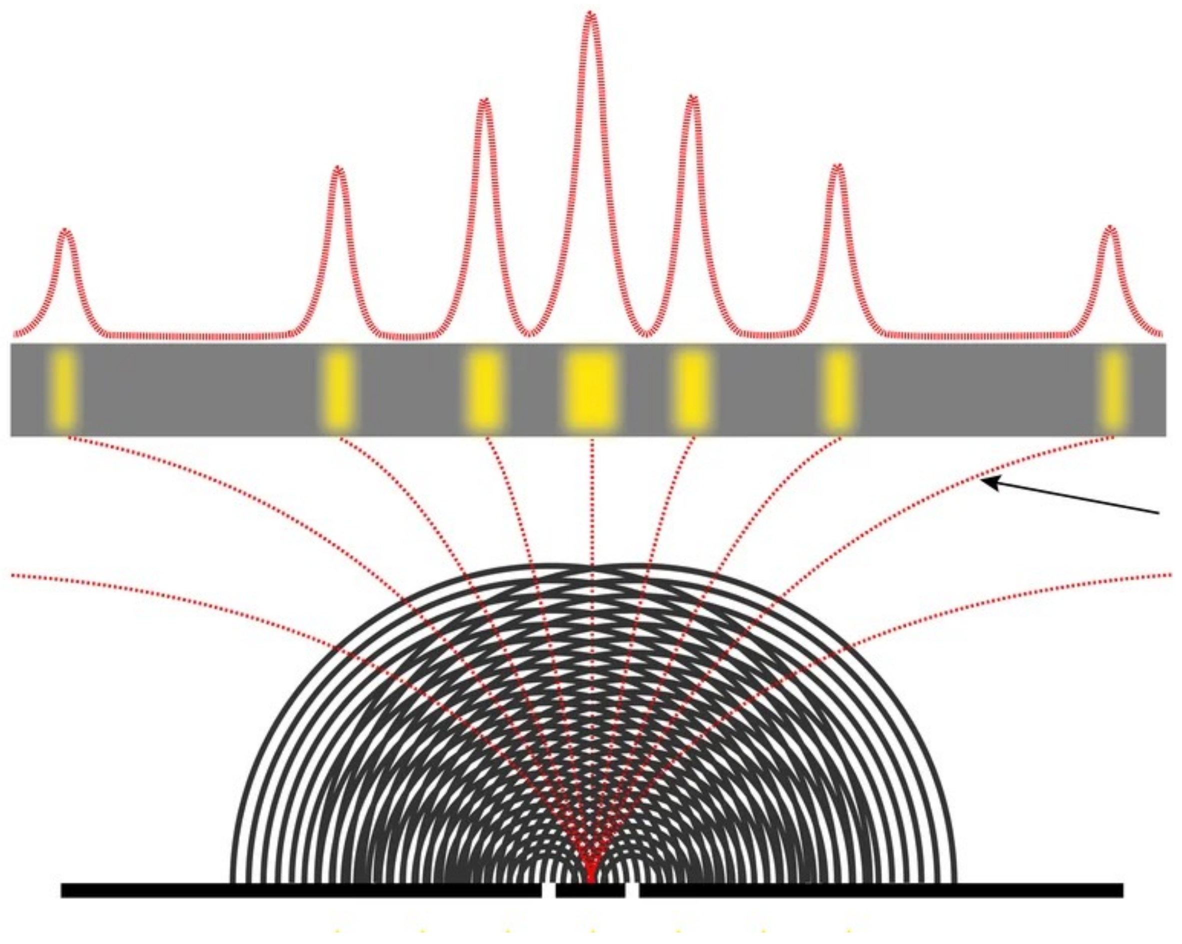
Quantum mechanics

$$\psi = \sqrt{\rho}\, e^{iS/\hbar}$$

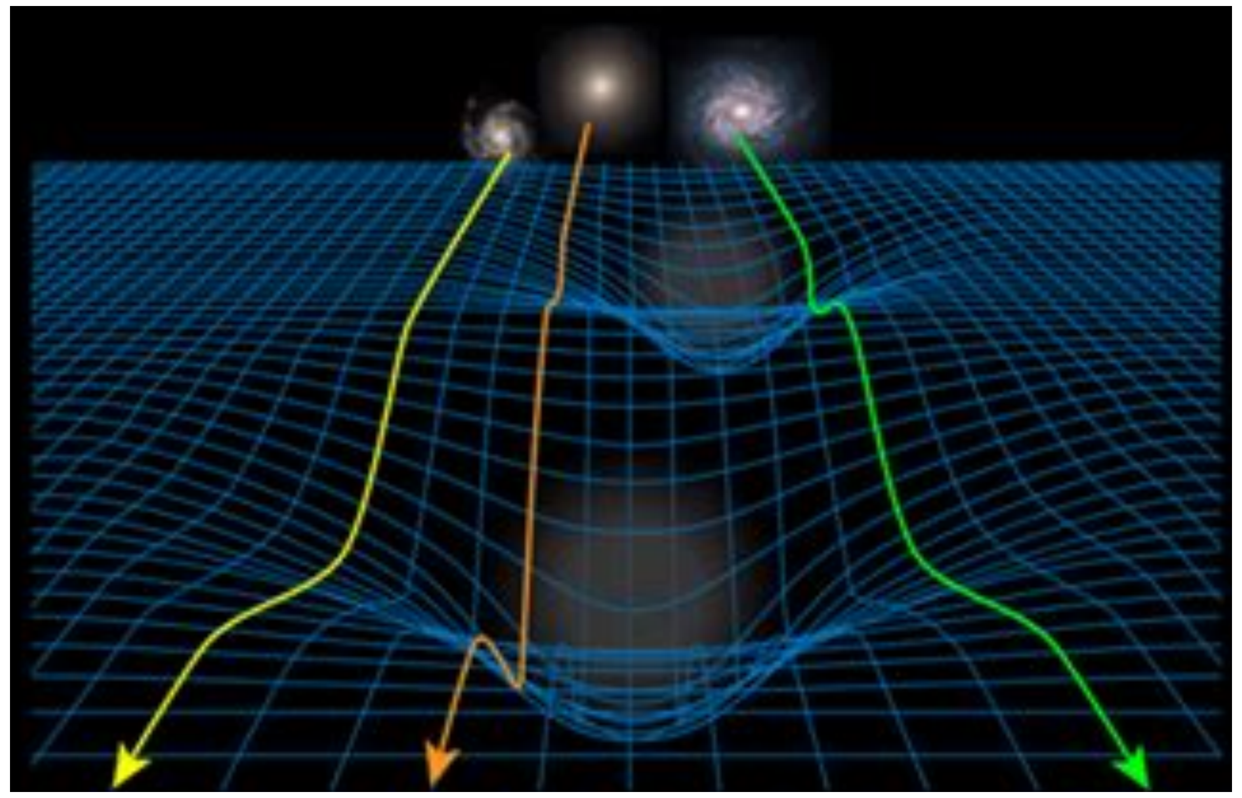
$$\hbar$$

Schrodinger + $\hbar \to 0$

Continuity eq. with $k_i = \partial_i S$



WKB

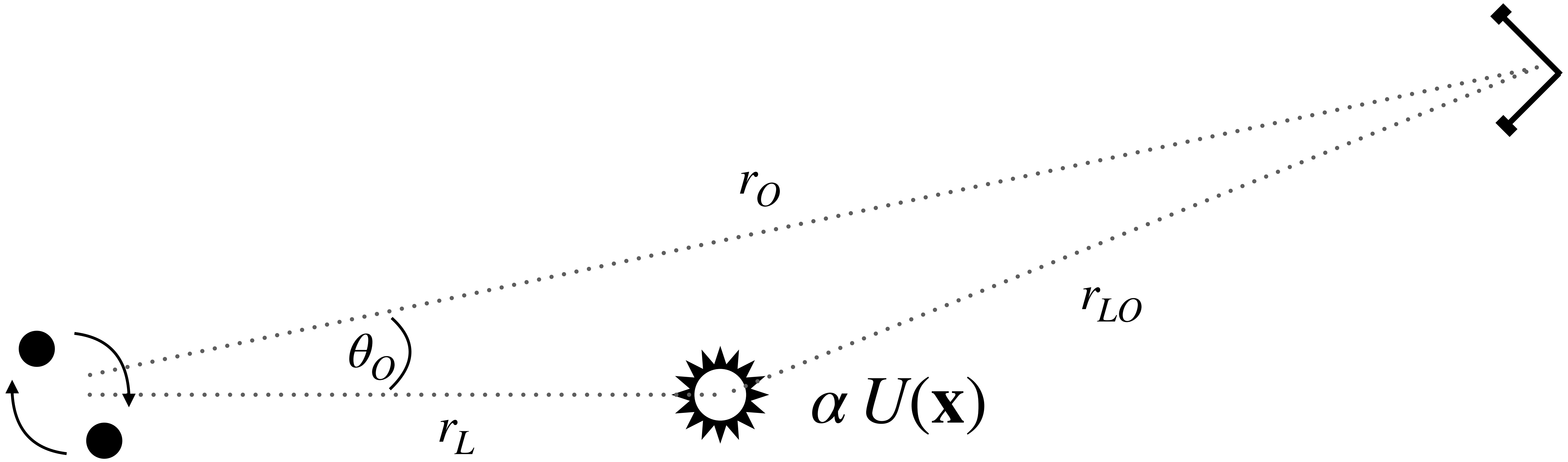


- GO is like classical limit
- Analogy between WO and QM
- Uncertainty principle: existence of trajectories

GOAL: find a Schrödinger equation

Diffraction integral for a scalar wave

Nakamura&Deguchi 1999



$$ds^2 = -\,(1 + 2\,\alpha\, U(\mathbf{x}))dt^2 + (1 - 2\,\alpha\, U(\mathbf{x}))d\mathbf{x}^2$$

Diffraction integral for a scalar wave

Nakamura&Deguchi 1999

1. Klein-Gordon Eq. For scalar wave:

$$\left[\nabla^2 + \omega^2(1 - 4\alpha U)\right] \tilde{\Psi}_\omega(\mathbf{x}) = 0$$

2. Amplification Factor:

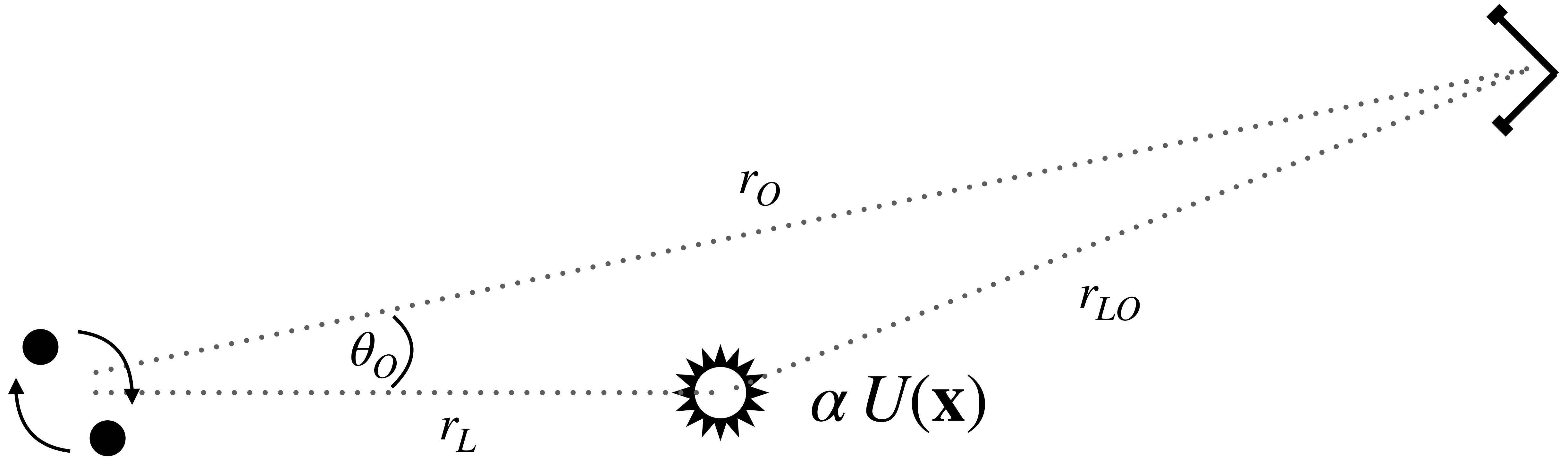
$$F(\mathbf{x}) = \tilde{\Psi}_\omega / \tilde{\Psi}_\omega^{NL}$$

3. Eikonal & Paraxial:

$$|\partial_r^2 F| \ll |2i\omega\partial_r F| \qquad \sin\theta \approx \theta$$

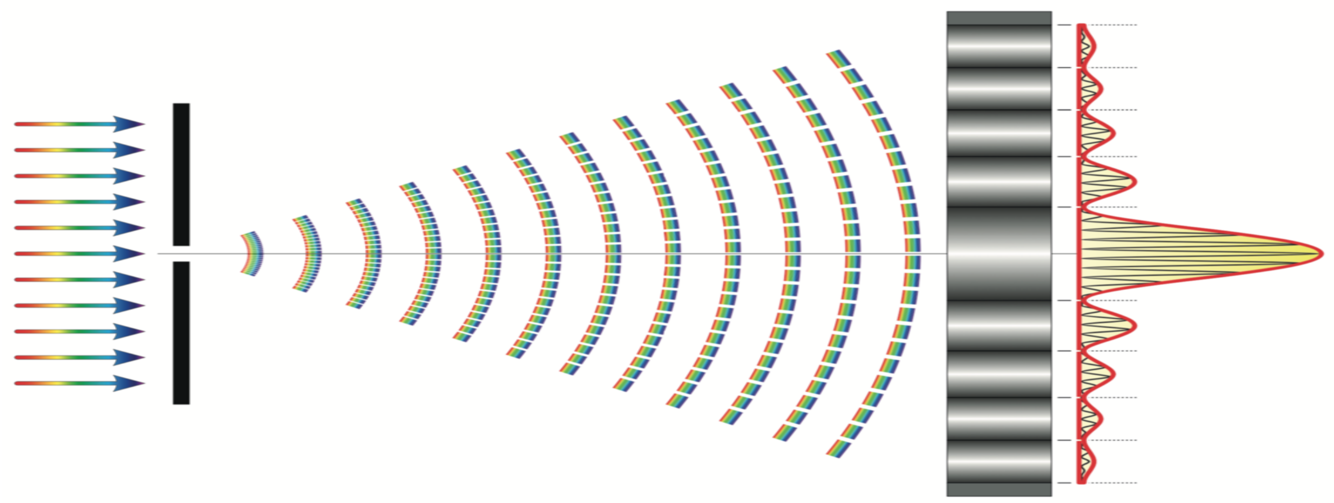
4. Schrödinger Eq.:

$$i\partial_r F = -\frac{1}{2\omega}\partial_{\boldsymbol{\theta}}^2 F + 2\alpha\omega U F \longrightarrow$$



Diffraction integral:

$$F(\vec{r}_O) = \int \mathcal{D}\boldsymbol{\theta}(r) \exp\left\{ i\omega \int_0^{r_O} dr \left[\frac{r^2}{2}|\dot{\boldsymbol{\theta}}|^2 - 2\alpha U(r, \boldsymbol{\theta}) \right] \right\}$$



Analogy between wave and quantum effects:
interference between all paths.
Geometric optics = classical limit

Diffraction integral: Pros and Cons

PROs

1. Wave optics effects are frequency dependent
2. Easy high frequency limit
3. Already used for: lens parameter estimation, constraints PBH abundance, matter PS at small scales,...

Citation per year of
Nakamura&Deguchi 1999

CONs

1. Eikonal: frequency lower bound $\omega \gg |\partial_r^2 F|/|\partial_r F|$
2. Scalar field: no polarization effects

Proper time path integrals

2405.20208

Fields vs Particles: Proper time technique

PROBLEM:

In order to have a direct transition to the GO description in terms of trajectories in the limit $\omega \to \infty$, want a path integral description QM-style, i.e. over all paths.

However, GWs are relativistic fields and QFT path integrals are over field configurations. Can we achieve this without Eikonal&Paraxial?

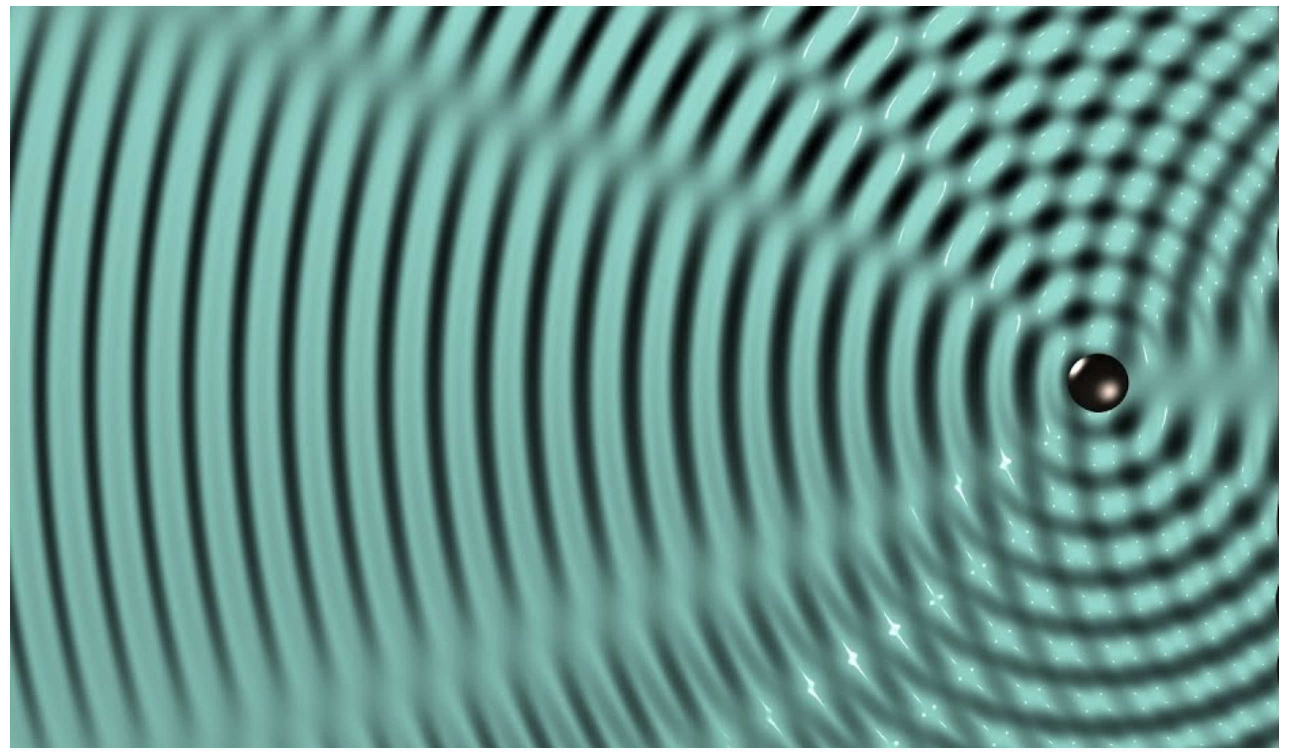
$$\begin{tikzcd}
\boxed{\int Dh_{\mu\nu}\, e^{i\,\omega\, S_f[h_{\mu\nu},\nabla_\rho h_{\mu\nu}]}} \arrow[r, shift left, "\text{Eikonal\&Paraxial}"] \arrow[r, shift right, "\text{Exactly?}"'] & \boxed{\int D\mathbf{x}\, e^{i\,\omega\, S_p[\mathbf{x},\dot{\mathbf{x}}]}}
\end{tikzcd}$$

Proper time technique

- A.K.A: worldline quantization, Schwinger proper time, energy propagator, Feynman/Fradkin...
- Already used in: QFT, QED, optics, acoustic waves, many-body, quantum cosmology...

Proper time path integral in Cosmology

Finding a Schrödinger equation without Eikonal approximation



1. Green function approach: $\left[\nabla^2 + \omega^2(1-4\alpha U)\right] G_\omega(\mathbf{x}_f, \mathbf{x}_i) = \delta^{(3)}(\mathbf{x}_f - \mathbf{x}_i)$
2. Proper time: $G_\omega(\mathbf{x}_f, \mathbf{x}_i) \equiv -\frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau}\tilde{G}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau)$
3. Schrödinger Eq.: $\frac{i}{\omega}\frac{\partial \tilde{G}_\omega}{\partial \tau} = -\frac{1}{\omega^2}\nabla^2\tilde{G}_\omega + 4\alpha U(\mathbf{x})\tilde{G}_\omega$

$\omega = 1/\hbar$

Particle action

Proper time path integral:

$$G_\omega(\mathbf{x}_f, \mathbf{x}_i) = -\frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau}\int_{\mathbf{x}(\tau'=0)=\mathbf{x}_i}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f} \underbrace{\mathcal{D}\mathbf{x}(\tau')}_{\text{Sum over paths}}\, e^{i\omega\int_0^\tau d\tau'\left[\frac{\dot{\mathbf{x}}^2}{4} - 4\alpha U\right]}$$

Exact particle-like solution **WITHOUT** Eikonal/Paraxial approximation

What you can find in 2405.20208:

1. $\omega \to \infty$ limit to recover geometric optics

   ($\delta W / \delta \tau = 0$ and $\delta W / \delta \mathbf{x} = 0$)

2. Eikonal assumption ***a posteriori*** to recover diffraction integral of Nakamura&Deguchi 1999

   ($\delta W / \delta \tau = 0$)

3. Perturbative expansion in αU and Dyson equations

4. First order solution for Coulomb-like potential

5. Massive scalar field: $\omega_m = \omega \sqrt{1 - \frac{m^2}{\omega^2}}$

6. Polarization effects

Recovering geometric optics: $\boldsymbol{\omega} \rightarrow \boldsymbol{\infty}$

$$G_\omega = -\frac{i}{\omega}\int_0^{+\infty} d\tau \int_{\mathbf{x}(\tau'=0)=\mathbf{x}_i}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f} D\mathbf{x}(\tau')\; e^{i\omega W[\mathbf{x},\dot{\mathbf{x}},\tau]}$$

Total phase: $W[\mathbf{x},\dot{\mathbf{x}},\tau] \equiv \tau + \int_0^{\tau} d\tau' L[\mathbf{x}(\tau'),\dot{\mathbf{x}}(\tau'),\tau']$

Variations w.r.t. proper time

$$\frac{\delta W}{\delta \tau} = -\hat{H} = 0$$

- Enforces particle-like dispersion relation $\hat{H} = -\mathbf{p}^2 + n^2$
- Changing variables to arc length, we have

$$W = \int_0^{l_{cl}} dl' \sqrt{1 - 4\alpha U(\mathbf{x}(l'))} = \int_0^{l_{cl}} dl' n(\mathbf{x}(l'))$$

Variations w.r.t. coordinates

$$\frac{\delta W}{\delta \mathbf{x}} = 0$$

- Fermat principle of geometric optics

$$\delta\left(\int_0^{l_{cl}} dl' n(\mathbf{x}(l'))\right) = 0$$

Removing the proper time

- Want to remove proper time integration to have **direct** representation of G_ω with only $\int Dx(\tau')$

- In **high frequency limit**, one can expand τ around its "classical" value: $\delta W/\delta\tau = 0$
(Garrod 1966)

$$W[\mathbf{x}, \dot{\mathbf{x}}, \tau] \approx W[\mathbf{x}, \dot{\mathbf{x}}, \tau_{cl}]$$

$$G_\omega = \mathcal{N} \int_{\mathbf{x}(\ell'=0)=\mathbf{x}_i}^{\mathbf{x}(\ell'=\ell_{cl})=\mathbf{x}_f} D\mathbf{x}(\ell')\, e^{i\omega\, n[\mathbf{x}(\ell)]}$$

Using Paraxial approximation ($\sin\theta \approx \theta$)
this becomes diffraction integral

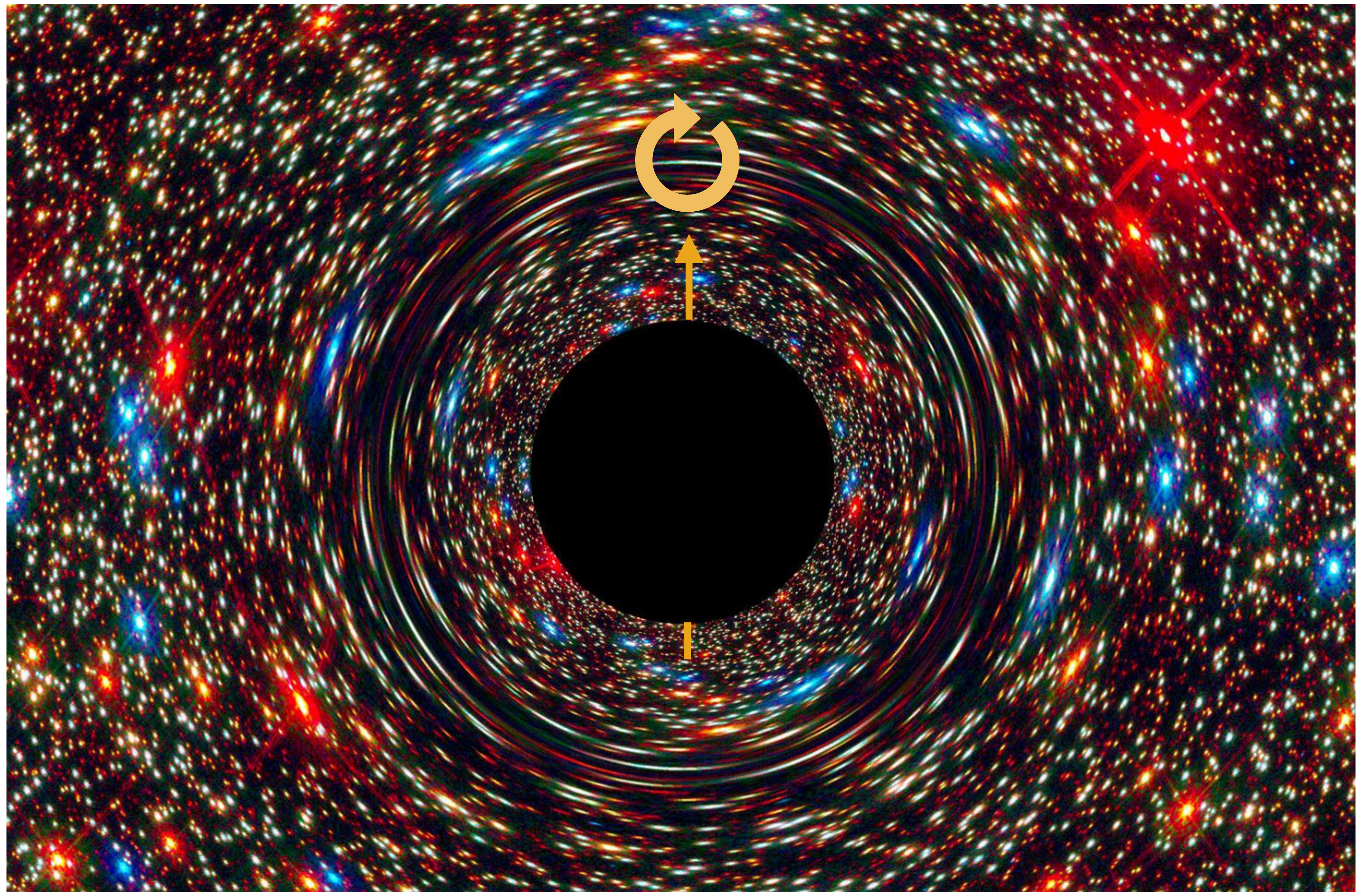
$$W[\mathbf{x}, \dot{\mathbf{x}}, \tau] \approx W[\mathbf{x}, \dot{\mathbf{x}}, \tau_{cl}] + \frac{(\tau - \tau_{cl})^2}{2} \frac{\delta^2 W}{\delta \tau^2}\Big|_{\tau_{cl}}$$

$$G_\omega = \mathcal{N} \int_{\mathbf{x}(\ell'=0)=\mathbf{x}_i}^{\mathbf{x}(\ell'=\ell_{cl})=\mathbf{x}_f} D\mathbf{x}(\ell')\, \frac{e^{i\omega\, n[\mathbf{x}(\ell)]}}{\sqrt{\delta^2 W/\delta\tau^2\big|_{\tau_{cl}}}}$$

Feynman-Garrod propagator

Polarization effects on a Kerr background

S. Teukolsky (1973)



1. Lens = Kerr BH
2. Use BH perturbation theory long-standing results
3. Perturbations of spin $s = 0, 1/2, 1, 2$ on Kerr BH satisfy Teukolsky Eq.:

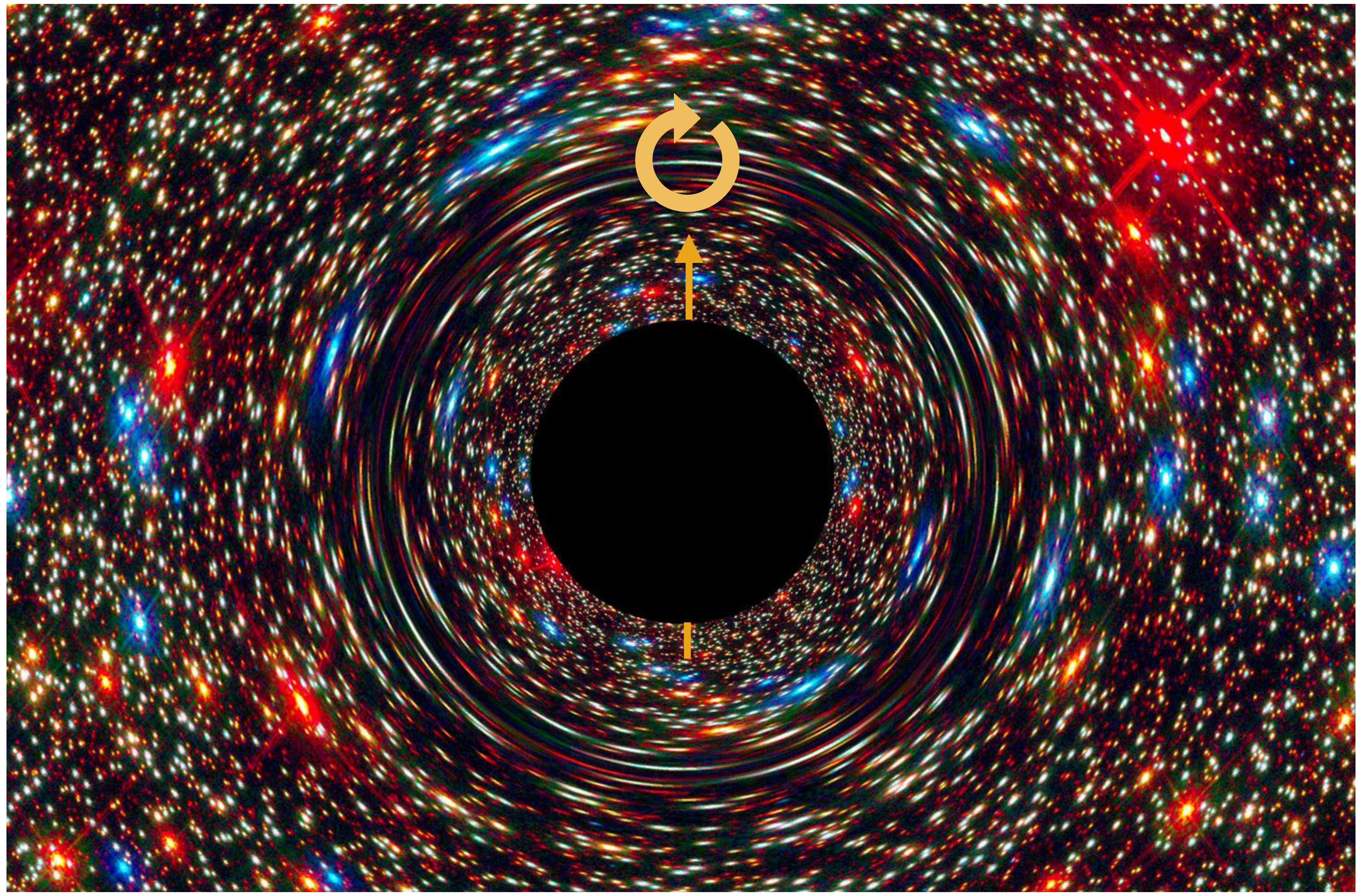
$$\hat{O}[M, a, \omega, s]\, \psi^s_\omega(r, \theta, \varphi) = 0$$

Differential operator

Newman-Penrose scalar, e.g.:
$\psi_\omega^{s=2} \supset \{\ddot{h}_+ \pm i\, \ddot{h}_\times\}$

Polarization effects on a Kerr background

Helmholtz Eq. for radial part



1. Decompose NP scalar: $\psi^s_\omega = e^{-im\varphi}\, S(\theta)\, R(r)$

2. Radial part satisfies 1D Klein-Gordon equation:

$$\frac{d^2\tilde{R}}{dr^2} + \omega^2\left[1 - 4\tilde{U}^s_{\ell m}(\omega, r)\right]\tilde{R} = 0$$

3. Same starting point, solve again with PTPI

4. In Newtonian limit:

$$4\tilde{U}^s_{\ell m}(\omega, r) \approx \underbrace{-4\frac{M}{r}}_{\text{Same as diffraction integral}} + \frac{\underbrace{\ell(\ell+1)}_{\text{Angular momentum (decomposition)}} + s(s+1)}{\omega^2 r^2} - \frac{2is}{\omega r}$$

Spin dependent terms:

Negligible in $\omega \gg 1$ limit

Summary

1. PTPI: wave-optics description without eikonal approximation (no frequency lower bound)
2. For BH lenses: include polarization effect

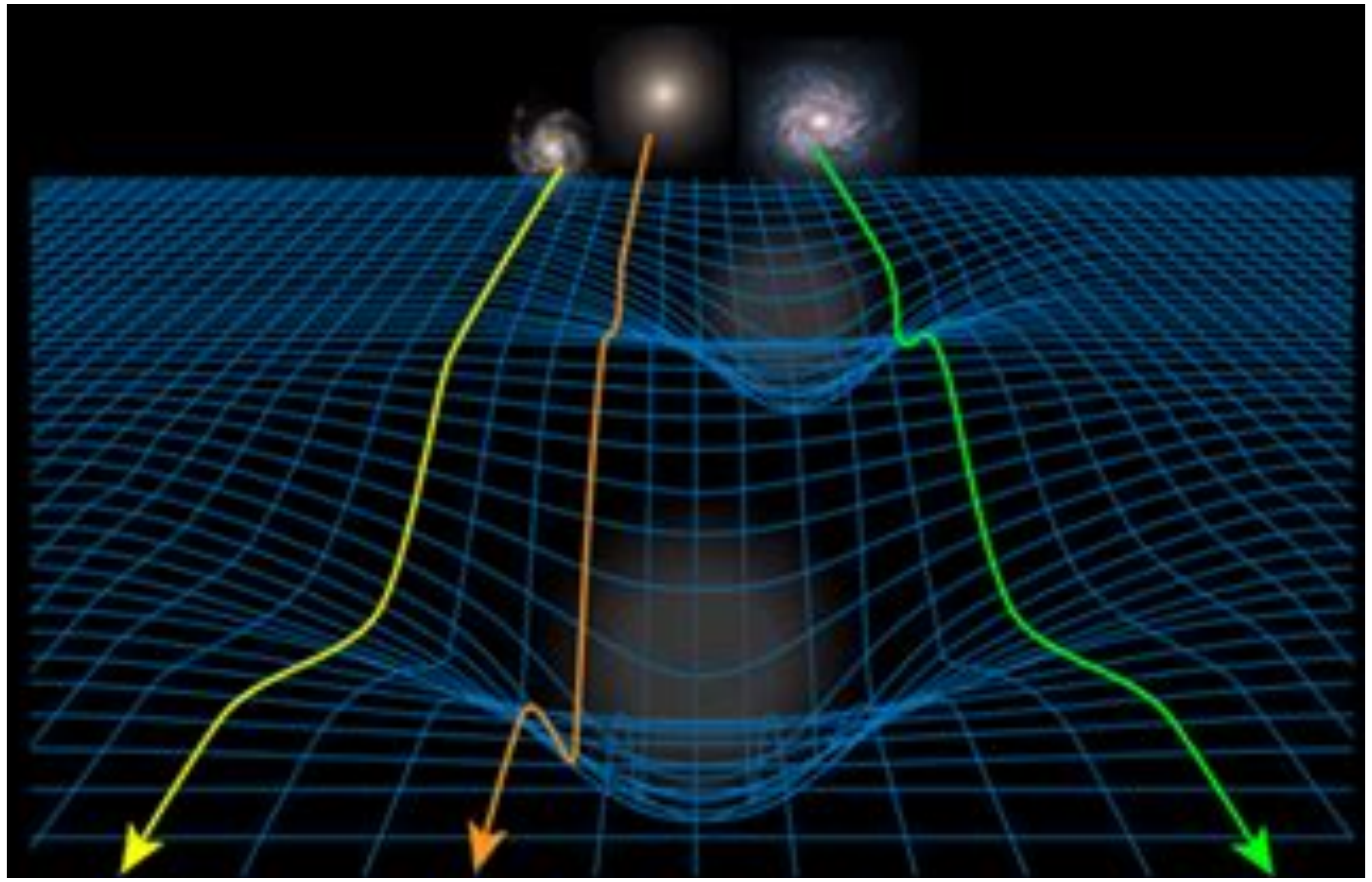
What's next

1. Numerical investigation: lens parameter estimation, constraints PBH abundance, matter PS at small scales… do they change with PTPI instead of Diffraction integral?
2. Other backgrounds
3. Many waves, many lenses

Thank you!

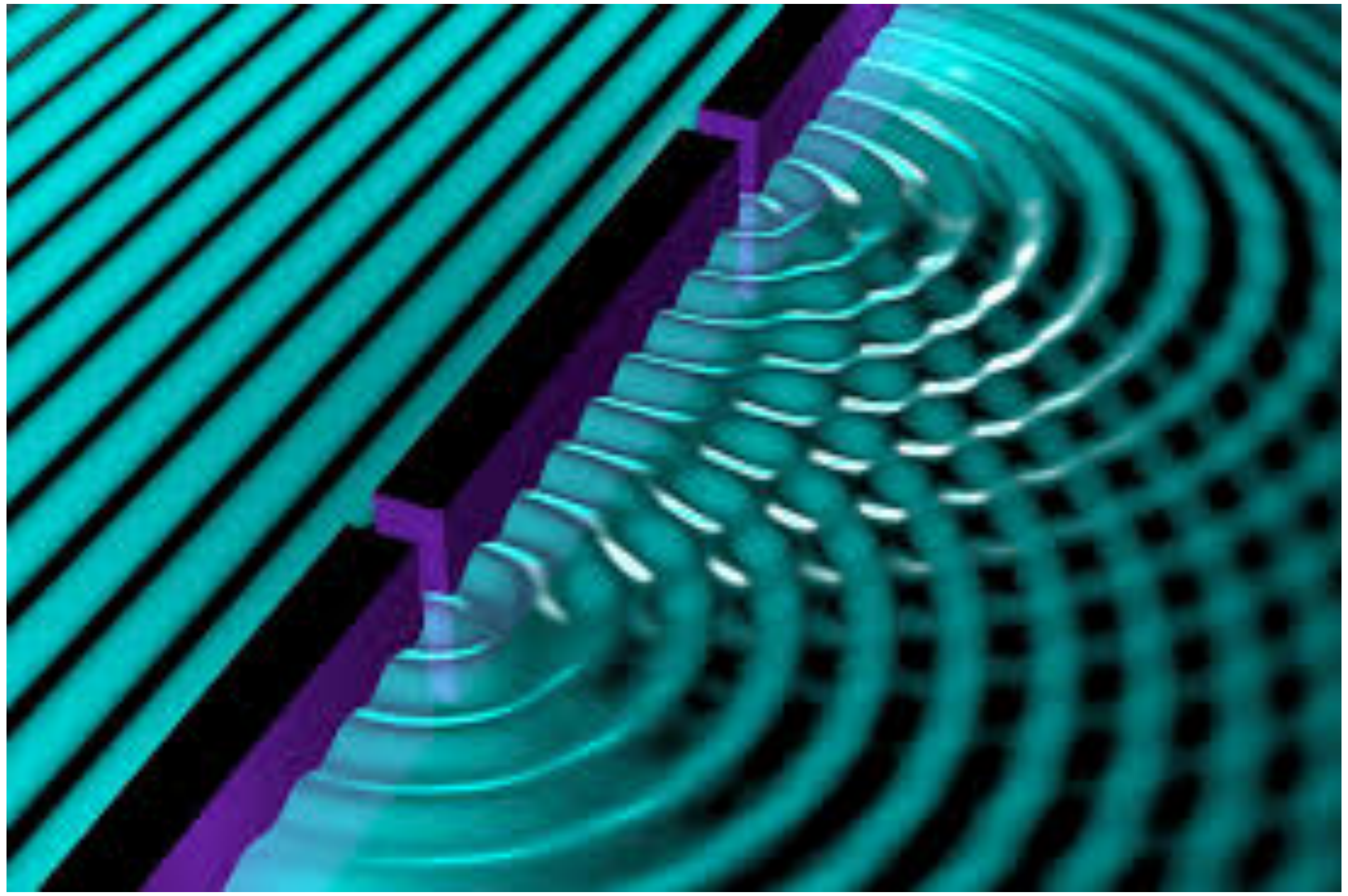
Optical regimes

High Frequency: $\omega R_S \gg 1$



Ray description

Low Frequency: $\omega R_S \lesssim 1$



Wave effects

- Gao et al. (2102.10295): ~(0.1 - 1.6)% of MBHB with $(10^5 - 10^{6.5})M_\odot$ and $4 \leq z \leq 10$
- LISA CosGW (2204.05434): WO need to be considered for typical LISA sources

3. Perturbative expansion

By assuming $\omega\alpha \ll 1$, we expand the potential term in the action and set up a perturbative expansion as in QFT/QM
(Feynman 1965)

$$G_\omega(\mathbf{x}_f, \mathbf{x}_i) = -\frac{i}{\omega}\int_0^{+\infty} d\tau\, e^{i\omega\tau}\left[\tilde{G}^{(0)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) - i\omega\, \tilde{G}^{(1)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) - \frac{\omega^2}{2}\, \tilde{G}^{(2)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) + \dots\right]$$

$$\tilde{G}^{(0)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) \equiv \int_{\mathbf{x}(\tau'=0)=\mathbf{x_i}}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f} \mathcal{D}\mathbf{x}(\tau')\, e^{i\omega S_0}\,,$$

$$\tilde{G}^{(1)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) \equiv \int_{\mathbf{x}(\tau'=0)=\mathbf{x_i}}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f} \mathcal{D}\mathbf{x}(\tau')\, e^{i\omega S_0} \int_0^\tau d\tau' V(\mathbf{x}(\tau'))\,,$$

$$\tilde{G}^{(2)}_\omega(\mathbf{x}_f, \mathbf{x}_i, \tau) \equiv \int_{\mathbf{x}(\tau'=0)=\mathbf{x_i}}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f} \mathcal{D}\mathbf{x}(\tau')\, e^{i\omega S_0} \left[\int_0^\tau d\tau_1 V(\mathbf{x}(\tau_1)) \int_0^\tau d\tau_2 V(\mathbf{x}(\tau_2))\right]$$

$$S_0 \equiv \int_0^\tau d\tau'\, \tfrac{1}{4}\left(\tfrac{d\mathbf{x}}{d\tau'}\right)^2$$

$$V(\mathbf{x}) \equiv 4\alpha U(\mathbf{x})$$

Sum over all path of the free particle action weighted by powers of the potential $V(\mathbf{x}(\tau'))$

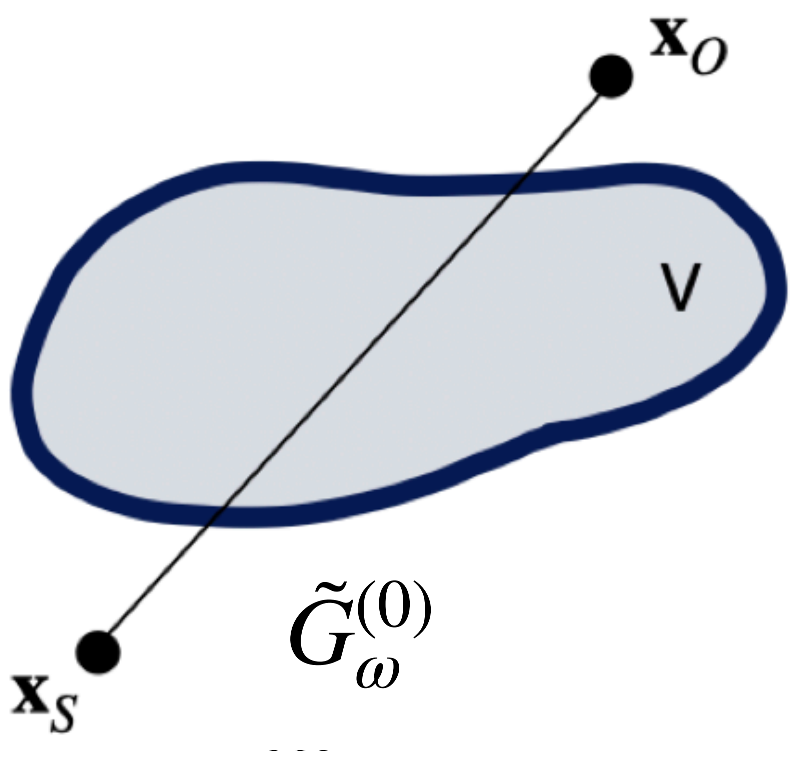
3. Born approximation and Dyson Eq.

- Potential has a role only at the point of the path at the time of scattering:

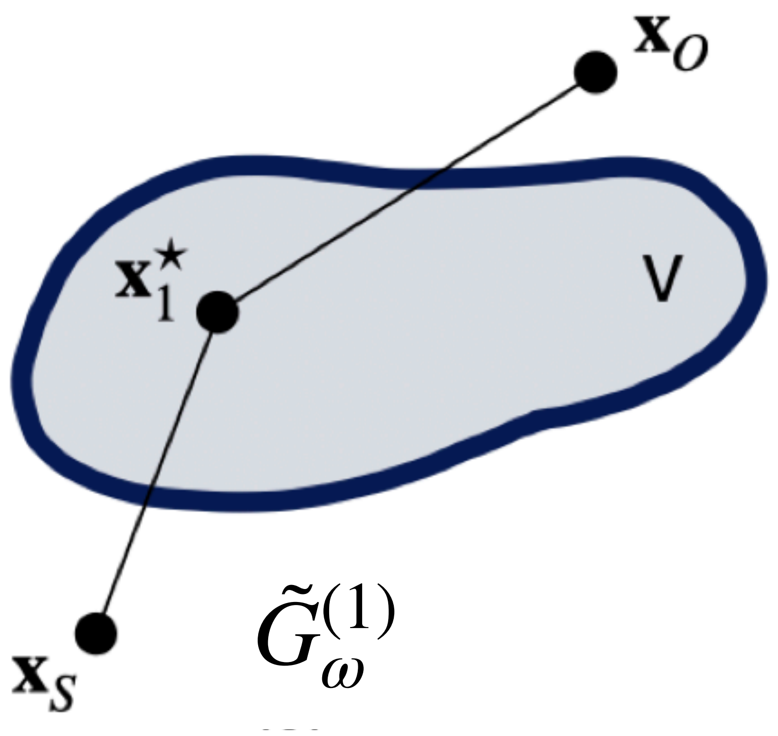
$$\tilde{G}_\omega^{(1)}(\mathbf{x}_f,\mathbf{x}_i,\tau) = \int_0^\tau d\tau_1 \int_{-\infty}^{+\infty} d\mathbf{x}_1^\star \tilde{G}_\omega^{(0)}(\mathbf{x}_f,\mathbf{x}_1^\star,\tau-\tau_1) V(\mathbf{x}_1^\star) \tilde{G}_\omega^{(0)}(\mathbf{x}_1^\star,\mathbf{x}_i,\tau_1)$$

Sum over all scattering centers

Localized scattering

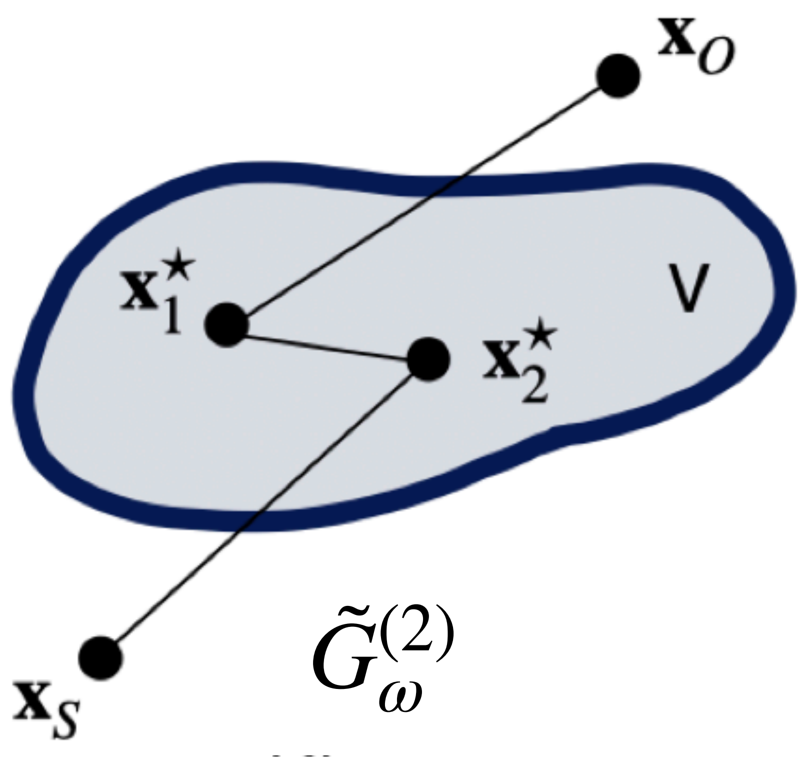


- Generalize to n^{th} order Green function: $\tilde{G}_\omega^{(n)}$ represented by free propagation in between n scattering events



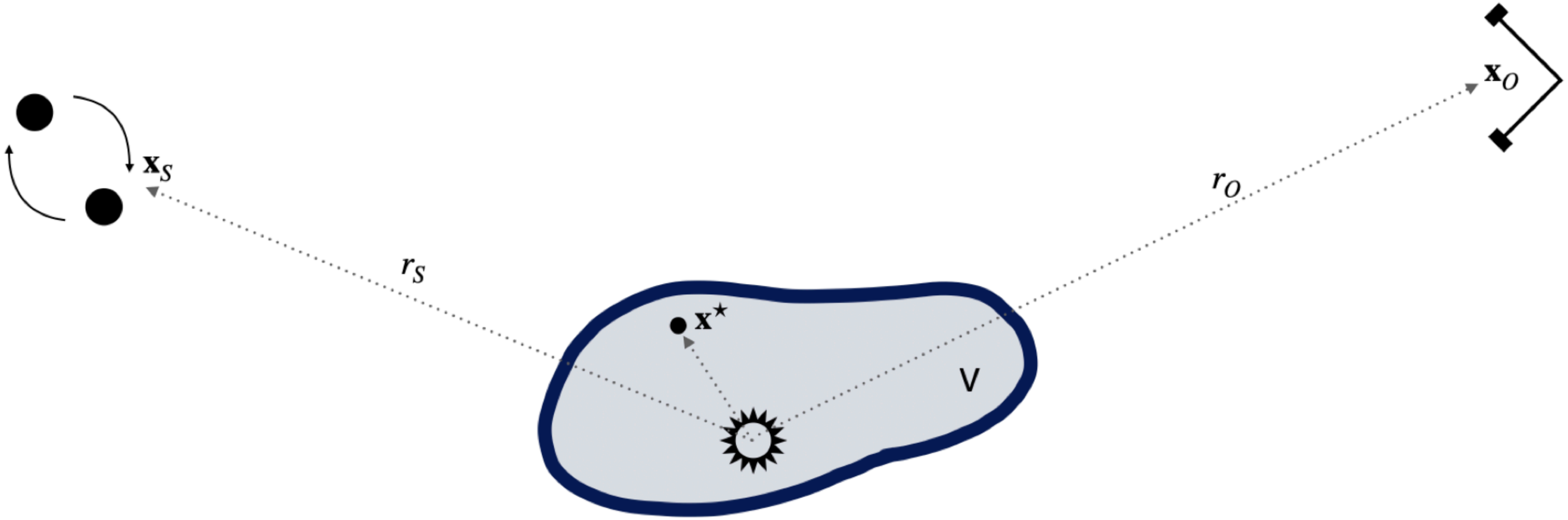
Last scattering

$$\tilde{G}_\omega(\mathbf{x}_f,\mathbf{x}_i,\tau) = \tilde{G}_\omega^{(0)}(\mathbf{x}_f,\mathbf{x}_i,\tau) - i\omega \int_0^\tau d\tau_{LS}\, \tilde{G}_\omega^{(0)}(\mathbf{x}_f,\mathbf{x}_{LS},\tau-\tau_{LS}) V(\mathbf{x}_{LS}) \tilde{G}_\omega(\mathbf{x}_{LS},\mathbf{x}_i,\tau_{LS})$$



- Dyson equation with $V(\mathbf{x})$ as self-energy

4. Coulomb-like potentials



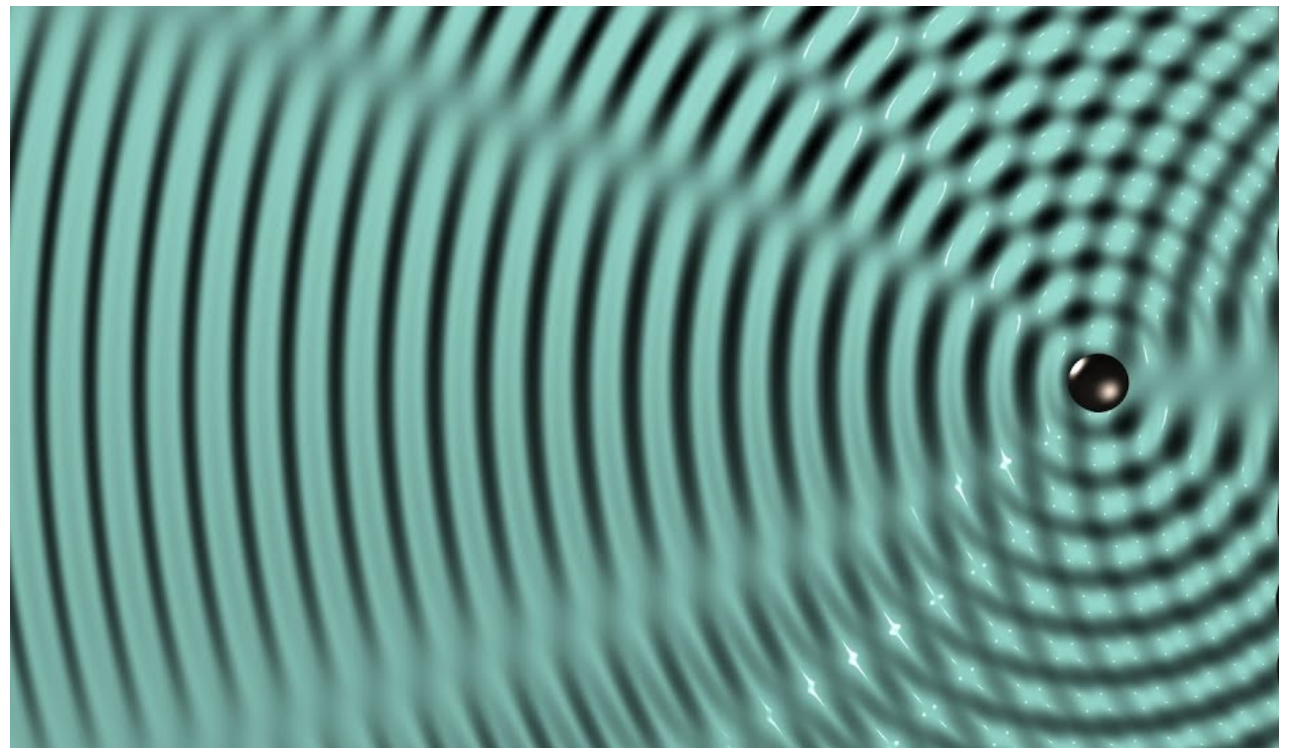
$$G_\omega(\mathbf{x}_O,\mathbf{x}_S) = G_\omega^{(0)}(\mathbf{x}_O,\mathbf{x}_S) - i\omega G_\omega^{(1)}(\mathbf{x}_O,\mathbf{x}_S) \approx$$

$$\approx -\frac{1}{4\pi}\frac{e^{i\omega(r_O+r_S)}}{r_O+r_S}\left[1+4i\alpha(\omega MG)\frac{1-i\omega(r_S+r_O)}{r_S r_O|\boldsymbol{\theta}_S-\boldsymbol{\theta}_O|^2\omega^2}\right]$$

↓

Effects goes to zero for $\lambda_{gw} \gg R_S$

Proper time path integral in Cosmology



$$G_\omega(\mathbf{x}_f,\mathbf{x}_i) = -\frac{i}{\omega}\int_0^\infty d\tau\, e^{i\omega\tau}\int_{\mathbf{x}(\tau'=0)=\mathbf{x}_i}^{\mathbf{x}(\tau'=\tau)=\mathbf{x}_f}\mathcal{D}\mathbf{x}(\tau')\, e^{i\omega\int_0^\tau\, d\tau'\left[\frac{\dot{\mathbf{x}}^2}{4}-4\alpha U\right]}$$

- Physical interpretation:

  The probability for the wave to propagate from $\mathbf{x}_i$ to $\mathbf{x}_f$ is given as the probability of the associated particle to propagate from from $\mathbf{x}_i$ to $\mathbf{x}_f$ in a fictitious time τ, integrated over all possible values of this parameter.

- Hamiltonian point of view:

$$G_\omega(\mathbf{x}_f,\mathbf{x}_i) = \langle\mathbf{x}_f|\left[\nabla^2+\omega^2(1-4\alpha U)\right]^{-1}|\mathbf{x}_i\rangle = -\frac{i}{\omega}\int_0^{+\infty} d\tau\,\langle\mathbf{x}_f|\,e^{i\omega\tau\hat{H}}|\mathbf{x}_i\rangle$$

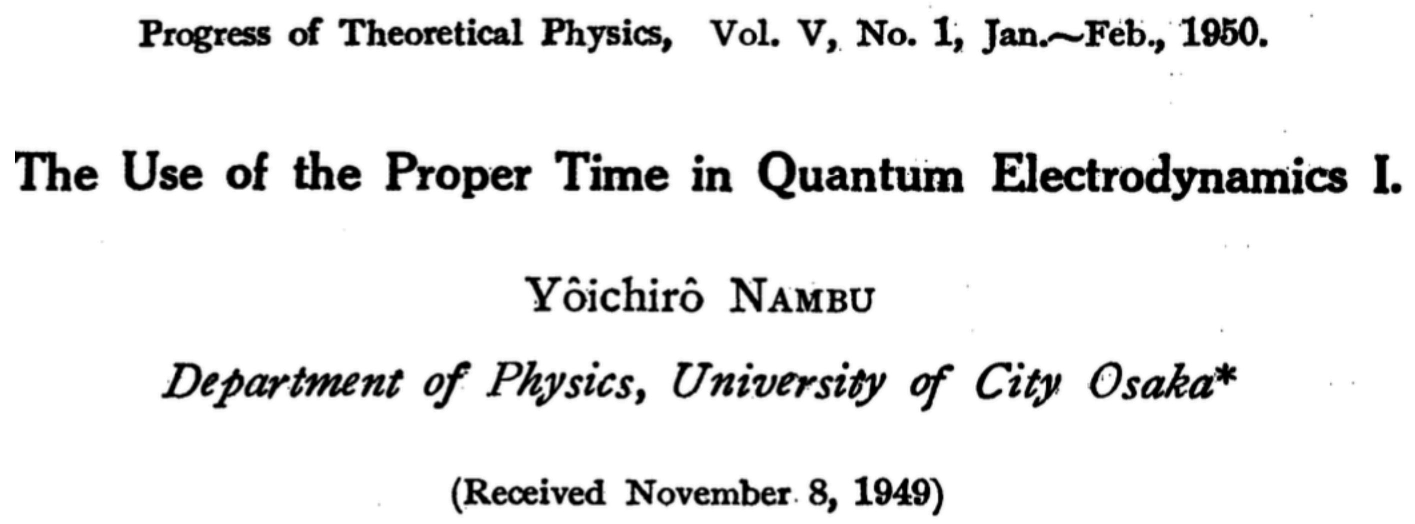
$$\hat{\mathcal{H}} \equiv -p^2 + (1-4\alpha U)$$

$$\mathbf{p} \equiv i\omega^{-1}\boldsymbol{\nabla}$$

Dispersion relation plays the role of Hamiltonian

Fields vs Particles: Proper time technique

- Also known as: worldline quantization, Schwinger proper time, energy propagator, Feynman/Fradkin…
- Already used in: QFT, QED, optics, acoustic wave propagation, many-body, quantum cosmology…



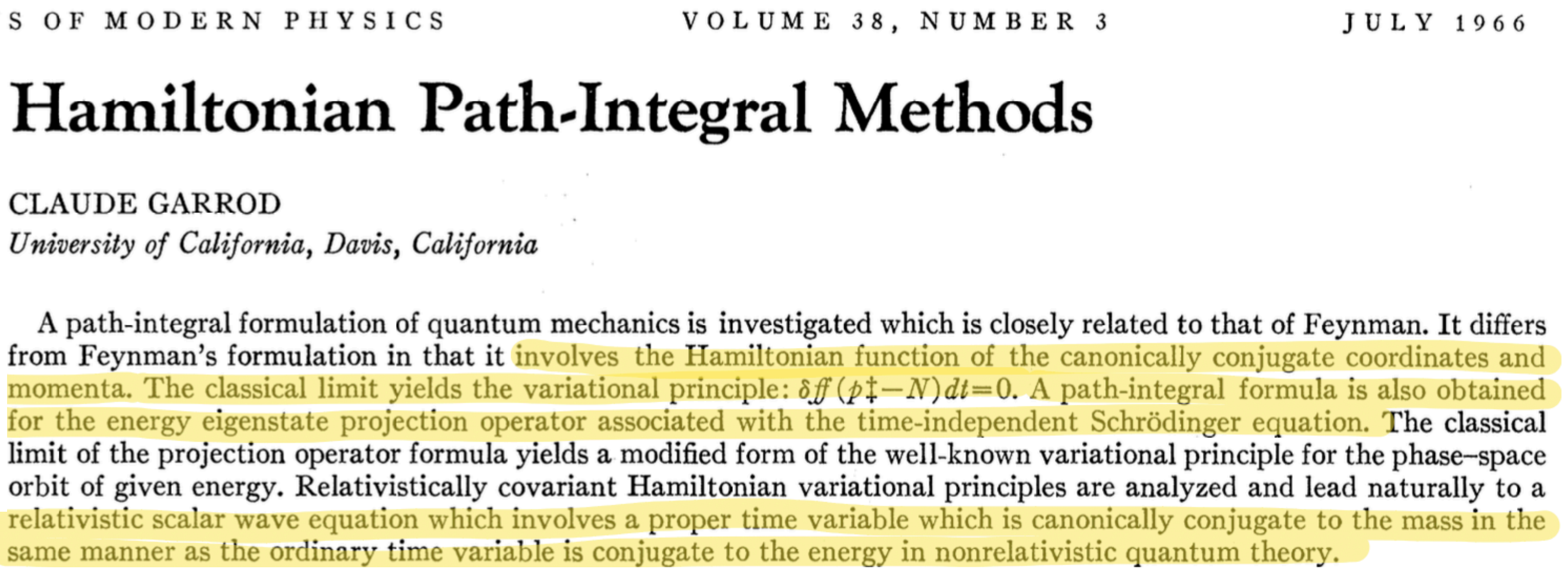
Progress of Theoretical Physics, Vol. V, No. 1, Jan.~Feb., 1950.

The Use of the Proper Time in Quantum Electrodynamics I.

Yôichirô NAMBU

*Department of Physics, University of City Osaka**

(Received November 8, 1949)



REVIEWS OF MODERN PHYSICS VOLUME 38, NUMBER 3 JULY 1966

Hamiltonian Path-Integral Methods

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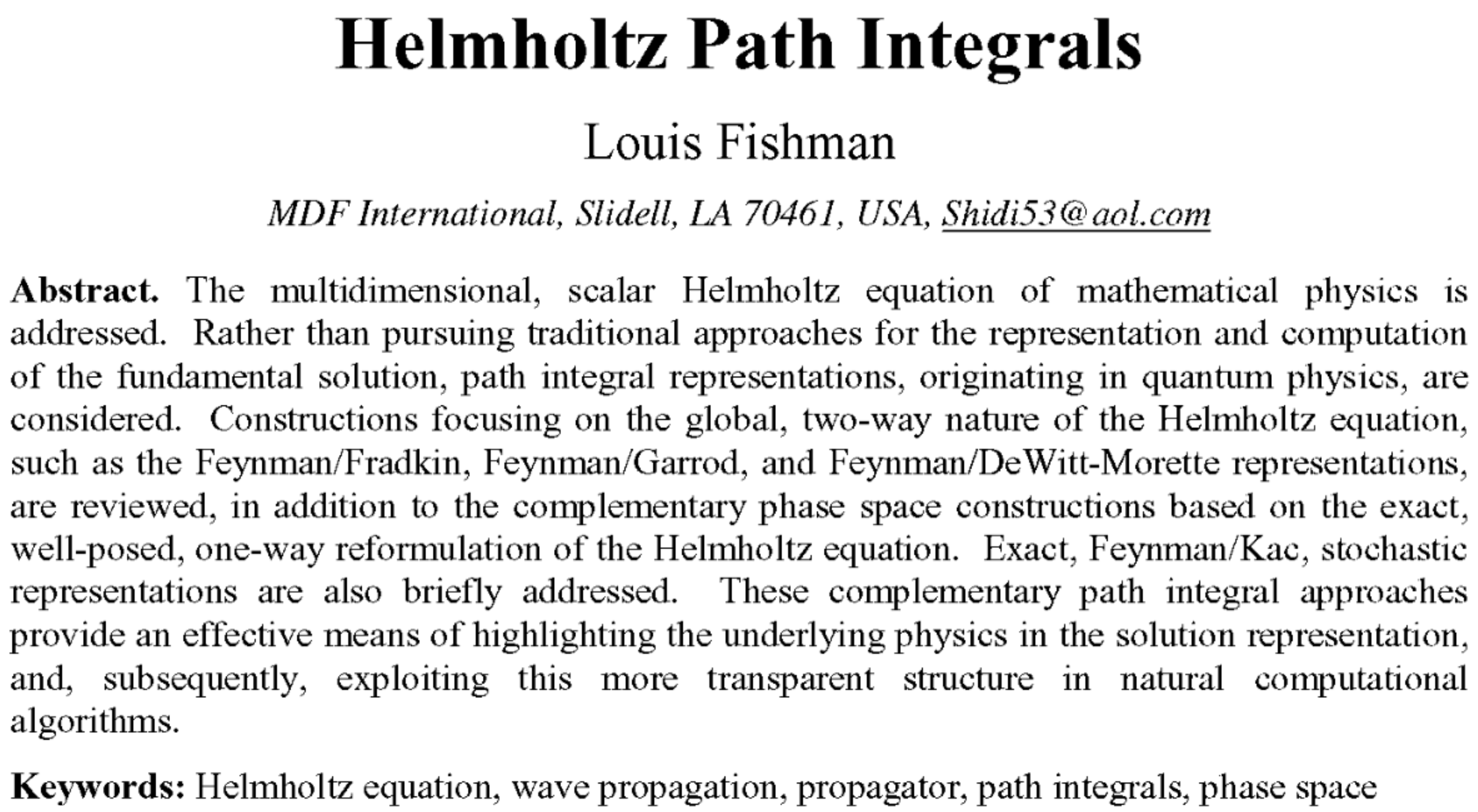
A path-integral formulation of quantum mechanics is investigated which is closely related to that of Feynman. It differs from Feynman's formulation in that it involves the Hamiltonian function of the canonically conjugate coordinates and momenta. The classical limit yields the variational principle: $\delta\iint(p\ddagger-N)dt=0$. A path-integral formula is also obtained for the energy eigenstate projection operator associated with the time-independent Schrödinger equation. The classical limit of the projection operator formula yields a modified form of the well-known variational principle for the phase–space orbit of given energy. Relativistically covariant Hamiltonian variational principles are analyzed and lead naturally to a relativistic scalar wave equation which involves a proper time variable which is canonically conjugate to the mass in the same manner as the ordinary time variable is conjugate to the energy in nonrelativistic quantum theory.



Helmholtz Path Integrals

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Abstract. The multidimensional, scalar Helmholtz equation of mathematical physics is addressed. Rather than pursuing traditional approaches for the representation and computation of the fundamental solution, path integral representations, originating in quantum physics, are considered. Constructions focusing on the global, two-way nature of the Helmholtz equation, such as the Feynman/Fradkin, Feynman/Garrod, and Feynman/DeWitt-Morette representations, are reviewed, in addition to the complementary phase space constructions based on the exact, well-posed, one-way reformulation of the Helmholtz equation. Exact, Feynman/Kac, stochastic representations are also briefly addressed. These complementary path integral approaches provide an effective means of highlighting the underlying physics in the solution representation, and, subsequently, exploiting this more transparent structure in natural computational algorithms.

Keywords: Helmholtz equation, wave propagation, propagator, path integrals, phase space

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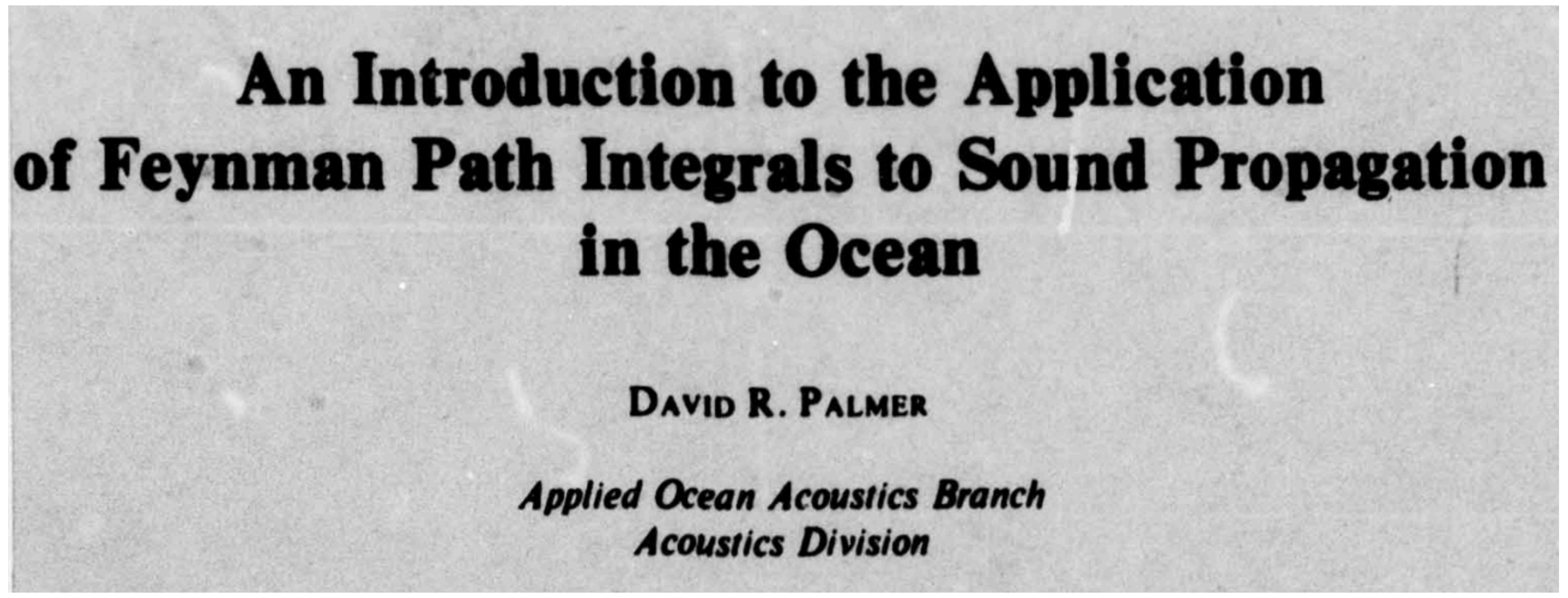

An Introduction to the Application of Feynman Path Integrals to Sound Propagation in the Ocean

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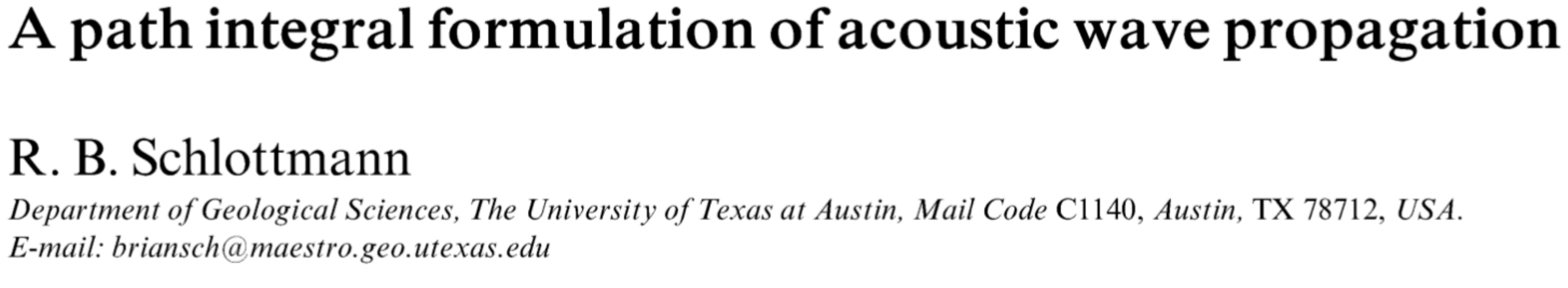

A path integral formulation of acoustic wave propagation

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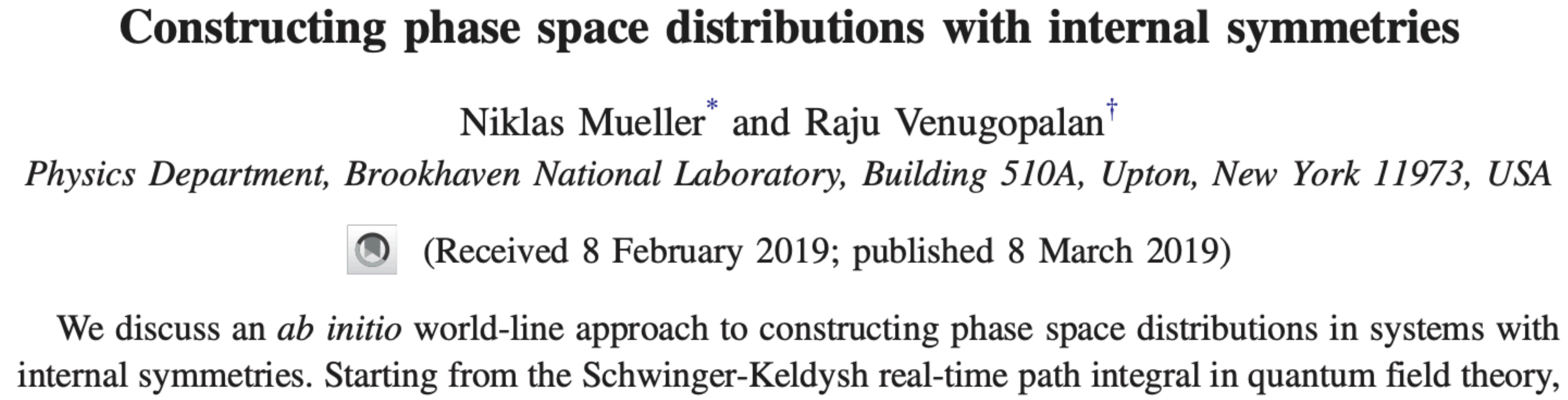

Constructing phase space distributions with internal symmetries

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We discuss an *ab initio* world-line approach to constructing phase space distributions in systems with internal symmetries. Starting from the Schwinger-Keldysh real-time path integral in quantum field theory,