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Observing GRB 221009A at very-high energy

Collaborators: G. Bonnoli, L. Nava, M. Roncadelli, F. Tavecchio

The Fifth Gravi-Gamma-Nu Workshop

9-11 October 2024

Outline

- GRB 221009A
    - Axion-like particles
    - Results with ALP model
    - Lorentz Invariance Violation
    - Results with LIV model
    - Discussion
- Other hints at ALP existence
- Conclusions

GRB 221009A

GRB 221009A

- Extremely luminous Gamma Ray Burst (GRB) at redshift $z = 0.151$
- Observed by:
  - Fermi-GBM, Fermi-LAT (Fermi, 2023), Swift (Williams+2023)
  - **LHAASO** up to $E \simeq$ (13-18) TeV within 2000 s after the initial burst (LHAASO Collaboration, 2022, 2023a,b)

BUT strong absorption for $E \gtrsim O(10)$ TeV due to the **extragalactic background light (EBL)** at $z = 0.151$ in Conventional Physics (CP)

EBL	15 TeV		18 TeV		100 TeV		251 TeV	
	τ_{CP}	P_{CP}	τ_{CP}	P_{CP}	τ_{CP}	P_{CP}	τ_{CP}	P_{CP}
D	12.7	3×10^{-6}	19.4	4×10^{-9}	350	2×10^{-152}	9654	~0
G	9.4	8×10^{-5}	13.1	2×10^{-6}	246	2×10^{-107}	9502	~0
FR	10.1	4×10^{-5}	14.1	7×10^{-7}	333	2×10^{-145}	15411	~0
SL	12.8	3×10^{-6}	18.3	10^{-8}	220	3×10^{-96}	>9251	~0

τ_{CP} -> optical depth; P_{CP} -> photon survival probability
D -> EBL model by Domínguez et al., 2011
G -> EBL model by Gilmore et al. 2012
FR -> EBL model by Franceschini & Rodighiero 2017
SL -> EBL model by Saldana-Lopez et al. 2021

QUESTION:

How can we have detected this GRB up to $E \simeq$ (13-18) TeV?

ANSWER:

with **axion-like particles (ALPs)** !!!

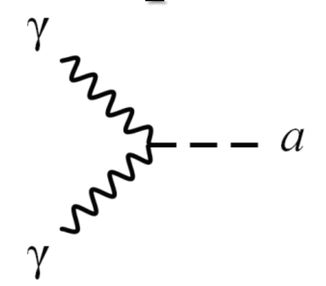
G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

Axion-like particles

Axion-like particles (ALPs)

- Predicted by **String Theory**
- Very light particles ($m_a < 10^{-8}$ eV)
- Spin 0
- **Interaction with** two **photons** (coupling $g_{a\gamma\gamma}$)
- Subdominant interactions with other particles
- Possible candidate for dark matter
- Produce **spectral** and **polarization effects**

Two photons



$$\mathcal{L}_{a\gamma} = g_{a\gamma\gamma}\, \mathbf{E}\cdot\mathbf{B}\, a$$

In an external B field

Photon-ALP oscillations

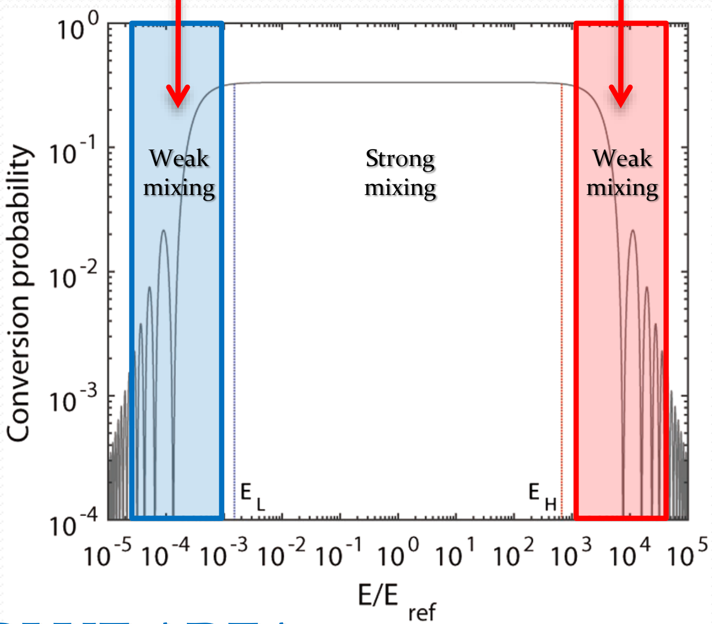
ALPs in astrophysical contest

- ALPs very **elusive** in laboratory experiments (low coupling) → **astrophysical environment** is the **best opportunity** to study ALPs and ALP effects (*for free*)
- Photon/ALP beam with energy $E >> m_a$
- ALPs induce **modifications to astrophysical spectra**
  - For $E < 10$ GeV → negligible photon absorption due to EBL
    - **Photon-ALP interaction** produces effective **photon absorption**
  - For $E > 10$ GeV → photons absorbed by EBL ($\gamma\gamma \rightarrow e^+e^-$), **ALPs** are **not absorbed**
    - **Photon-ALP oscillations increase medium transparency**
- ALPs induce **modifications to photon polarization** (*birefringence*, *dichroism*)
- **HINTS** at ALP existence:
  - Explain how flat spectrum radio quasars (FSRQs) can emit up to 400 GeV
    F. Tavecchio, M. Roncadelli, G. Galanti and G. Bonnoli, Phys. Rev. D, 86, 085036 (2012).
  - Solve the anomalous redshift dependence of blazar spectra
    G. Galanti, M. Roncadelli, A. De Angelis and G. F. Bignami, MNRAS 493, 1553 (2020).
  - GRB 221009A
    G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio and G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

ALP-induced irregularities

ALP mass & plasma effects

QED vacuum polarization & CMB effects



- Photon-ALP conversion probability $P_{\gamma \to a}(E, m_a, g_{a\gamma\gamma}, B, y)$, y -> distance
- Highlighted zones predict **spectral irregularities** and **polarization features** in observational data
- Constraints on $g_{a\gamma\gamma}$ and m_a but the firmest is $g_{a\gamma\gamma} < 6.6 \times 10^{-11}$ GeV^{-1} for $m_a < 0.02$ eV (CAST collaboration, 2017)

BLUE AREA:

- Spectral effects investigated in:

  D. Wouters, P. Brun, Phys. Rev. D 86, 043005 (2012).

  Fermi-LAT Collaboration, Phys. Rev. Lett. 116, 161101 (2016).

  CTA Consortium, JCAP 02, 048 (2021).

- Polarization effects studied in:

  G. Galanti, Phys. Rev. D 107, 043006 (2023).

  G. Galanti, M. Roncadelli, F. Tavecchio, E. Costa, Phys. Rev. D 107, 103007 (2023).

RED AREA:

- Spectral effects investigated in:

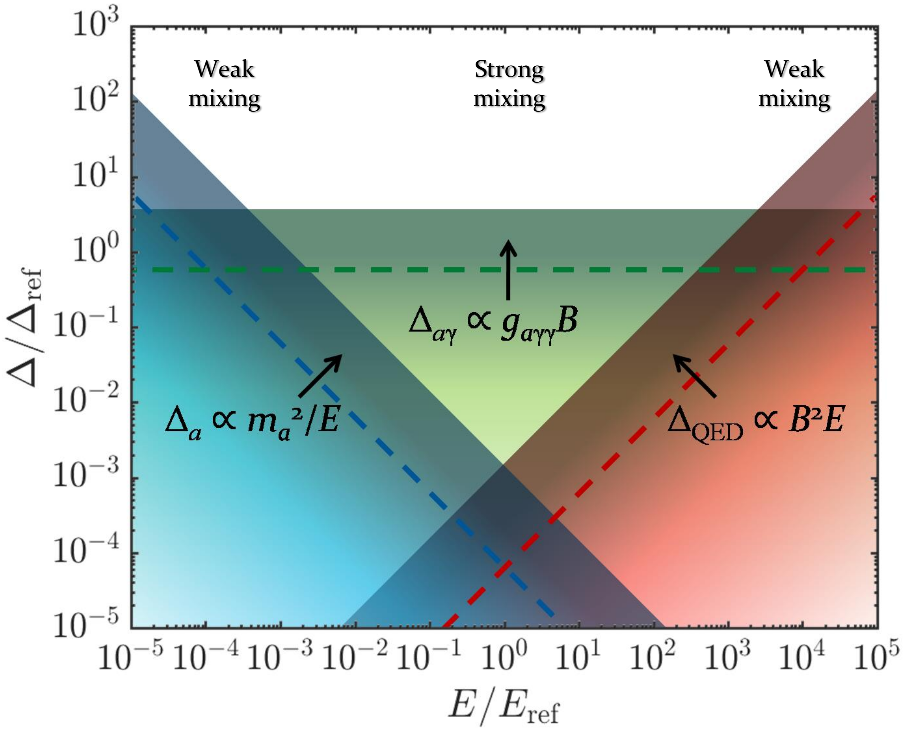
  G. Galanti, F. Tavecchio, M. Roncadelli, C. Evoli, MNRAS 487, 123 (2019).

  G. Galanti, F. Tavecchio, M. Landoni, MNRAS 491, 5268 (2020).

- Polarization effects studied in:

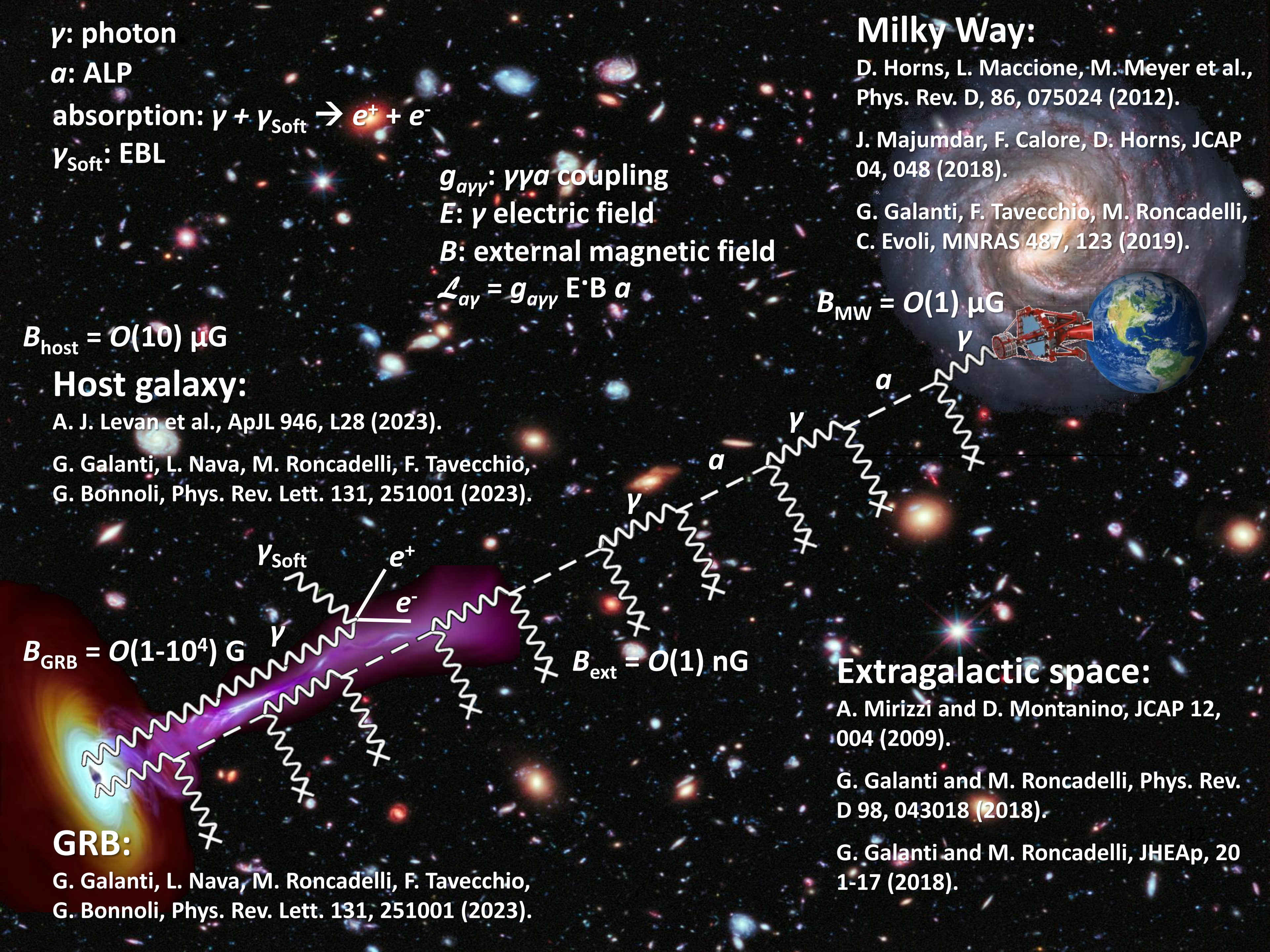
  G. Galanti, Phys. Rev. D 107, 043006 (2023).

Photon-ALP interaction



- Photon-ALP conversion probability $P_{\gamma \to a}(E, m_a, g_{a\gamma\gamma}, B, y)$, y -> distance
- $P_{\gamma \to a} = f(\Delta_{a\gamma}, \Delta_a, \Delta_{QED}, \ldots)$
- Δ terms -> in the γ-ALP **mixing matrix**:
  - γ-ALP **mixing strength**
  - medium **absorption**, **dispersion**, **magnetization**
- $\Delta_{a\gamma}(g_{a\gamma\gamma}, B)$ -> γ-ALP **mixing** term
- $\Delta_a(m_a, E)$ -> **ALP mass** term; $\Delta_{pl}(\omega_{pl}, E)$ -> **plasma** term (similar behavior)
- $\Delta_{QED}(B, E)$ -> **QED one-loop**; $\Delta_{CMB}(E)$ -> **CMB** dispersion (similar behavior)
- If $\Delta_{a\gamma} >> \Delta_a, \Delta_{QED}$, other Δ terms -> **strong mixing**
  - $P_{\gamma \to a} = \sin^2(\Delta_{a\gamma} y) \simeq g_{a\gamma\gamma}^2 B^2 y^2/4$ if $\Delta_{a\gamma} y << 1$ -> $P_{\gamma \to a}$ **maximal**, ***E* independent**
- If $\Delta_{a\gamma} < \Delta_a$ and/or Δ_{QED} and/or other Δ terms -> **weak mixing**
  - $P_{\gamma \to a}$ progressively **vanishes**

Results with ALP model


γ: photon
a: ALP
absorption: $\gamma + \gamma_{\rm Soft} \rightarrow e^+ + e^-$
$\gamma_{\rm Soft}$: EBL
$g_{a\gamma\gamma}$: $\gamma\gamma a$ coupling
E: γ electric field
B: external magnetic field
$\mathcal{L}_{a\gamma} = g_{a\gamma\gamma}\, E \cdot B\, a$
Milky Way:
D. Horns, L. Maccione, M. Meyer et al., Phys. Rev. D, 86, 075024 (2012).
J. Majumdar, F. Calore, D. Horns, JCAP 04, 048 (2018).
G. Galanti, F. Tavecchio, M. Roncadelli, C. Evoli, MNRAS 487, 123 (2019).
$B_{\rm MW} = O(1)$ μG
$B_{\rm host} = O(10)$ μG
Host galaxy:
A. J. Levan et al., ApJL 946, L28 (2023).
G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).
γ
a
γ
a
γ
$\gamma_{\rm Soft}$
e^+
e^-
γ
$B_{\rm GRB} = O(1\text{-}10^4)$ G
$B_{\rm ext} = O(1)$ nG
Extragalactic space:
A. Mirizzi and D. Montanino, JCAP 12, 004 (2009).
G. Galanti and M. Roncadelli, Phys. Rev. D 98, 043018 (2018).
G. Galanti and M. Roncadelli, JHEAp, 20 1-17 (2018).
GRB:
G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

γ: photon
a: ALP
absorption: $\gamma + \gamma_{Soft} \rightarrow e^+ + e^-$
γ_{Soft}: EBL

$g_{a\gamma\gamma}$: $\gamma\gamma a$ coupling
E: γ electric field
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$\mathcal{L}_{a\gamma} = g_{a\gamma\gamma}\, E \cdot B\, a$

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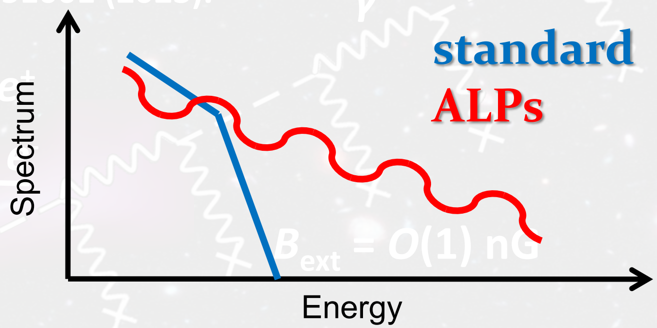
SPECTRUM

- **Standard physics**
  - photon absorption
- **With ALPs**
  - lower photon absorption
  - spectral irregularities



$B_{GRB} = O(1\text{-}10^4)$ G

$B_{ext} = O(1)$ nG

Extragalactic space:

A. Mirizzi and D. Montanino, JCAP 12, 004 (2009).

G. Galanti and M. Roncadelli, Phys. Rev. D 98, 043018 (2018).

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GRB:

G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

γ-ALP conversion for GRB 221009A

GRB:

- **Negligible** γ-ALP conversion even with parameters maximizing $P_{\gamma \to a}$
  - $B'_{\rm GRB} = 2$ G, distance $R = 2 \times 10^{17}$ cm, $\Gamma = 45$ at $t \sim 2000$ s
  - → γ-ALP beam propagation length $\Delta R' \sim R/\Gamma \sim 5 \times 10^{15}$ cm

Host Galaxy:

- **Disk-like galaxy** observed **edge-on** with **GRB** in the **center** (Levan+2023). We take:
  - i) **Starburst** galaxy -> M82 as a model (Lopez-Rodriguez+2021) -> $B_{\rm host} = O(20\text{-}50)$ μG
  - ii) **Spiral** galaxy (Beck 2016) -> $B_{\rm host} = O(5\text{-}10)$ μG

Extragalactic space:

- **EBL model** -> Saldana-Lopez et al. 2021: recent, satellite data, used by LHAASO
- **Domain-like model** for $B_{\rm ext}$ (Galanti & Roncadelli, 2018) -> limits: 10^{-7} nG < $B_{\rm ext}$ < 1.7 nG on $L_{\rm dom} = O(1)$ Mpc (Pshirkov+2016). We take:
  - i) **$B_{\rm ext}$ = 1 nG** with 0.2 Mpc < $L_{\rm dom}$ < 10 Mpc: **favored** by (Rees & Setti, 1968; Kronberg+1999)
  - ii) **$B_{\rm ext} < 10^{-15}$ G** -> very **conservative** scenario

Milky Way:

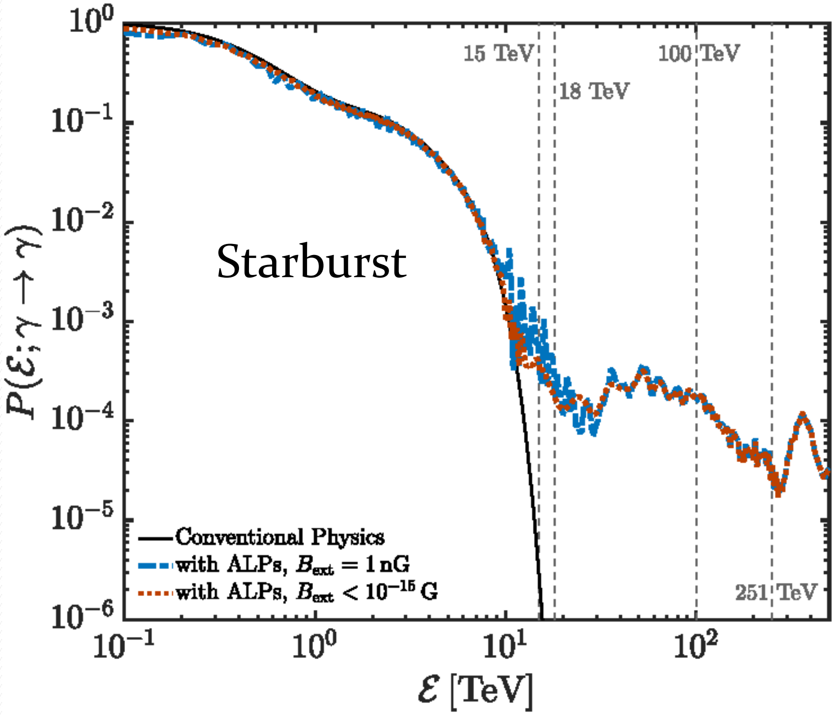
- $B_{\rm MW}$ **map by Jansson & Farrar** (Jansson & Farrar, 2012a,b)

Total Effect:

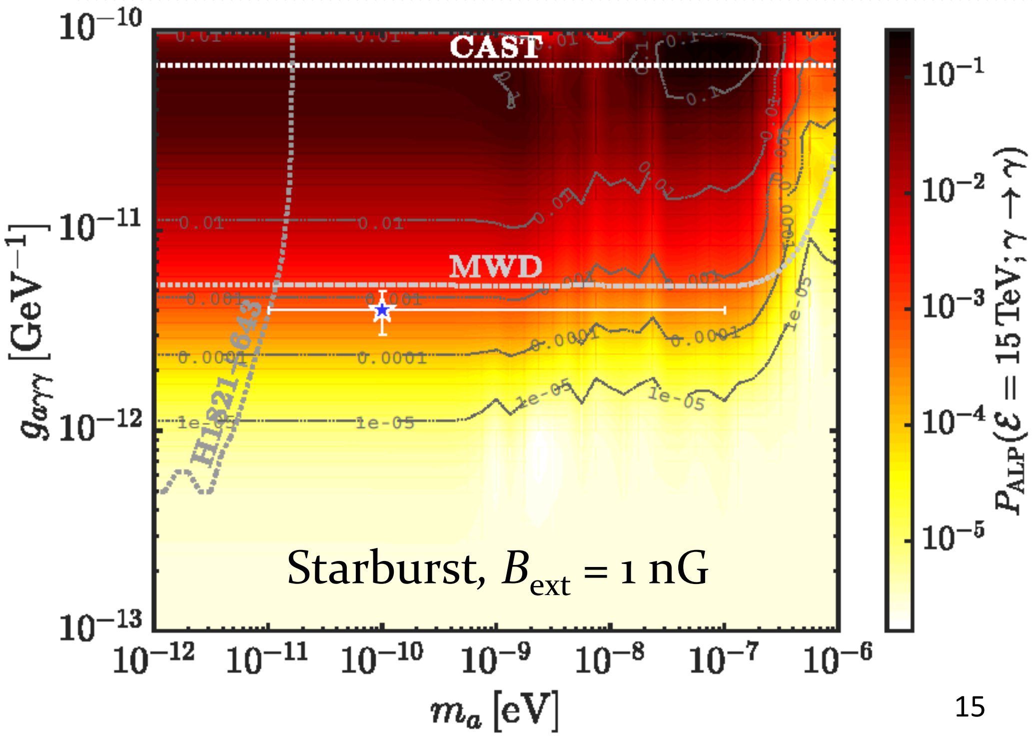
- γ–ALP interaction in **all media** → **final photon survival probability** $P_{\rm ALP}$

G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

Photon survival probability



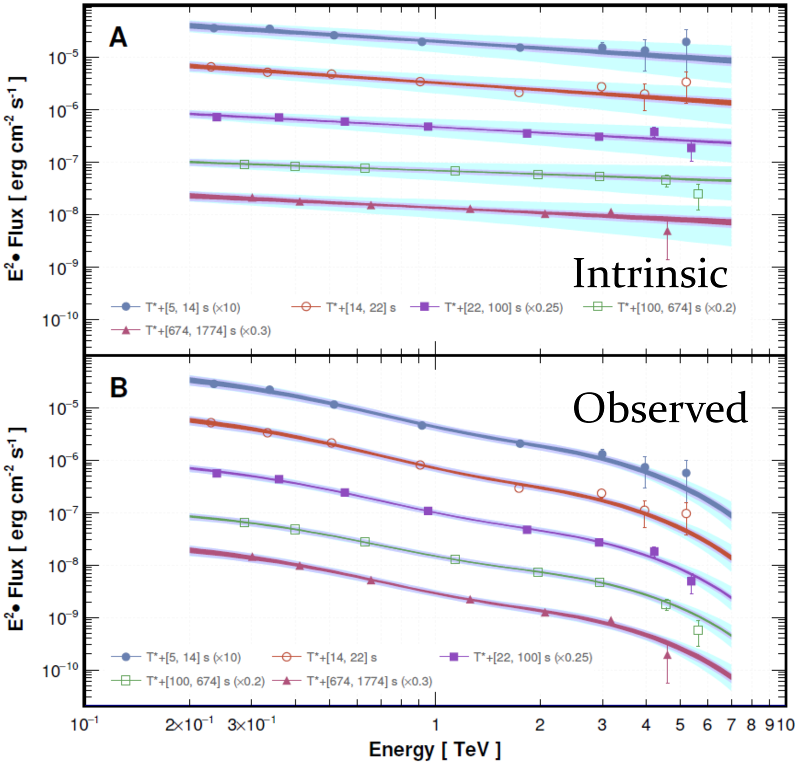
- $P_{CP}(E, \gamma \to \gamma)$:
  - @ 15 TeV → ~3 × 10^-6^
  - @ 18 TeV → ~1 × 10^-8^
  - @ 100 TeV → ~3 × 10^-96^
  - @ 251 TeV → ~0
- $P_{ALP}(E, \gamma \to \gamma)$:
  - @ 15 TeV → **~6 × 10^-4^**
  - @ 18 TeV → **~3 × 10^-4^**
  - @ 100 TeV → ~2 × 10^-4^
  - @ 251 TeV → ~3 × 10^-5^



- We take $g_{a\gamma\gamma} = (3 - 5) \times 10^{-12}$ GeV^{-1}; $m_a = (10^{-11} - 10^{-7})$ eV
- Within all most stringent ALP bounds (Sisk-Reynés+2022; Dessert+2022)
- **Explain LHAASO** event
- **Compatible with other ALP hints**

G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

GRB 221009A spectrum



Adapted from LHAASO Collaboration, Science 380, 1390 (2023).

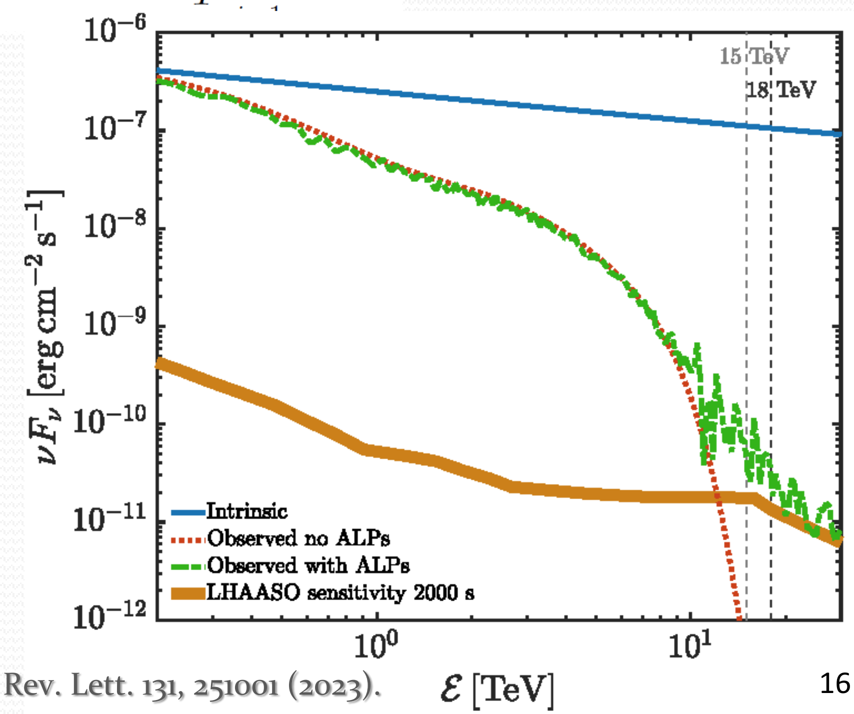
- **LHAASO spectrum** (only **WCDA**):
  - E = (0.2 – 7) TeV
  - In 5 time intervals Δt_i
- **Intrinsic** spectrum -> power law with **no cutoff**:

$$\mathcal{F}_i(\mathcal{E}) \equiv \frac{\mathrm{d}N}{\mathrm{d}\mathcal{E}} = A\left(\frac{\mathcal{E}}{\mathrm{TeV}}\right)^{-\gamma}$$

- **Average** intrinsic spectrum

$$\mathcal{F}_{\rm int} \equiv \langle \mathcal{F}_i \rangle = \frac{1}{T}\sum_{i=1}^{5} \Delta t_i \mathcal{F}_i$$

T -> total duration



- **Observed** spectrum -> extended to ~20 TeV:

$\mathcal{F}_{\rm obs}(\mathcal{E}) = P(\mathcal{E}; \gamma \to \gamma)\mathcal{F}_{\rm int}(\mathcal{E})$ with $P = P_{\rm CP}$ or $P_{\rm ALP}$

$\nu F_\nu = \mathcal{E}^2 \mathcal{F}_{\rm obs}$

- **CP does NOT work** for E above ~10 TeV
- **ALPs solve** the **problem** -> **hint at ALP**

G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

Lorentz Invariance Violation

Lorentz Invariance Violation (LIV)

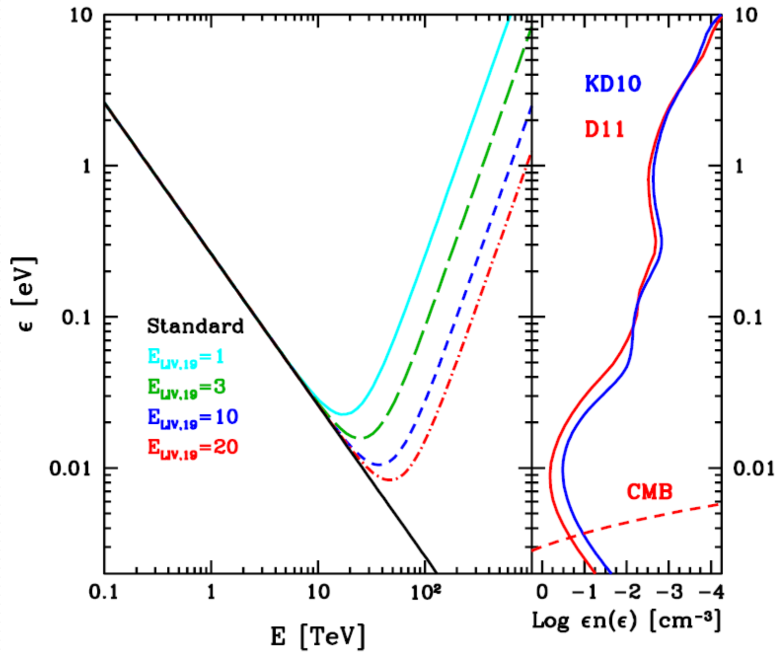
- Predicted by quantum gravity models for $E > 10^{19}$ GeV (Mattingly 2005)

- Effects on standard physics processes (Coleman&Glashow 1999; Jacobson+2003; Liberati 2013):
  - Photon decay
  - Photon splitting
  - **Modification of dispersion relations**

- **Modified photon dispersion relation**

$$E^2 - p^2 = -\frac{E^{n+2}}{E_{\rm LIV}^n}$$

E -> energy
p -> momentum
$E_{\rm LIV}$ -> LIV parameter

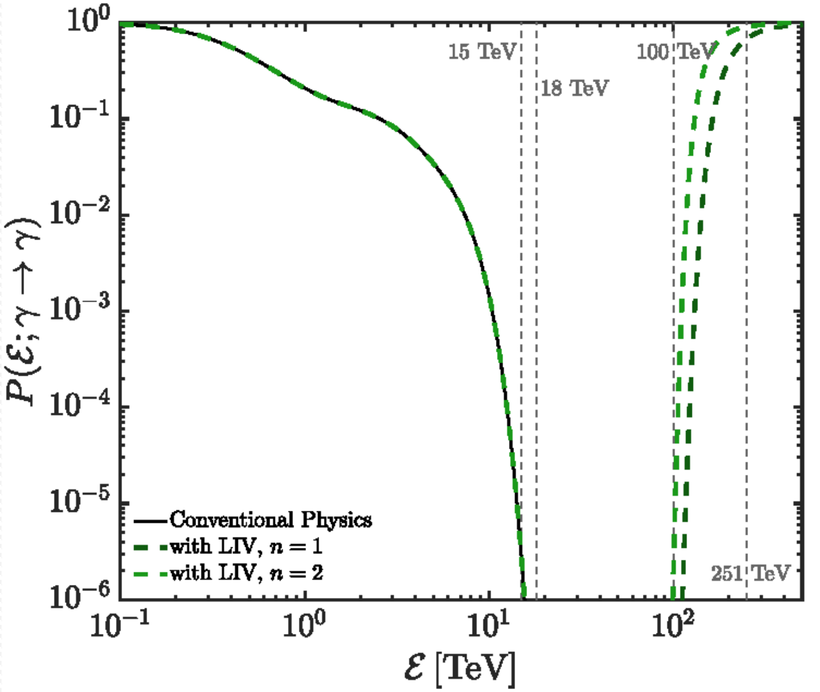


F. Tavecchio and G. Bonnoli, A&A 585, A25 (2016)

- → **Modification** of the **threshold** of the $\gamma\gamma \to e^+e^-$ process
  - Hundreds-TeV photons interact with optical/UV photons
  - **Smaller** photon absorption

Results with LIV model

LIV effect



- $P_{CP}(E, \gamma \rightarrow \gamma)$:
  - @ 15 TeV → ~3×10^{-6}
  - @ 18 TeV → ~1×10^{-8}
  - @ 100 TeV → ~3×10^{-96}
  - @ 251 TeV → ~0
- $P_{LIV,\, n=1}(E, \gamma \rightarrow \gamma)$:
  - @ 15 TeV → ~as in CP
  - @ 18 TeV → ~as in CP
  - @ 100 TeV → ~5×10^{-10}
  - @ 251 TeV → **~0.68**
- $P_{LIV,\, n=2}(E, \gamma \rightarrow \gamma)$:
  - @ 15 TeV → ~as in CP
  - @ 18 TeV → ~as in CP
  - @ 100 TeV → ~2×10^{-6}
  - @ 251 TeV → **~0.91**

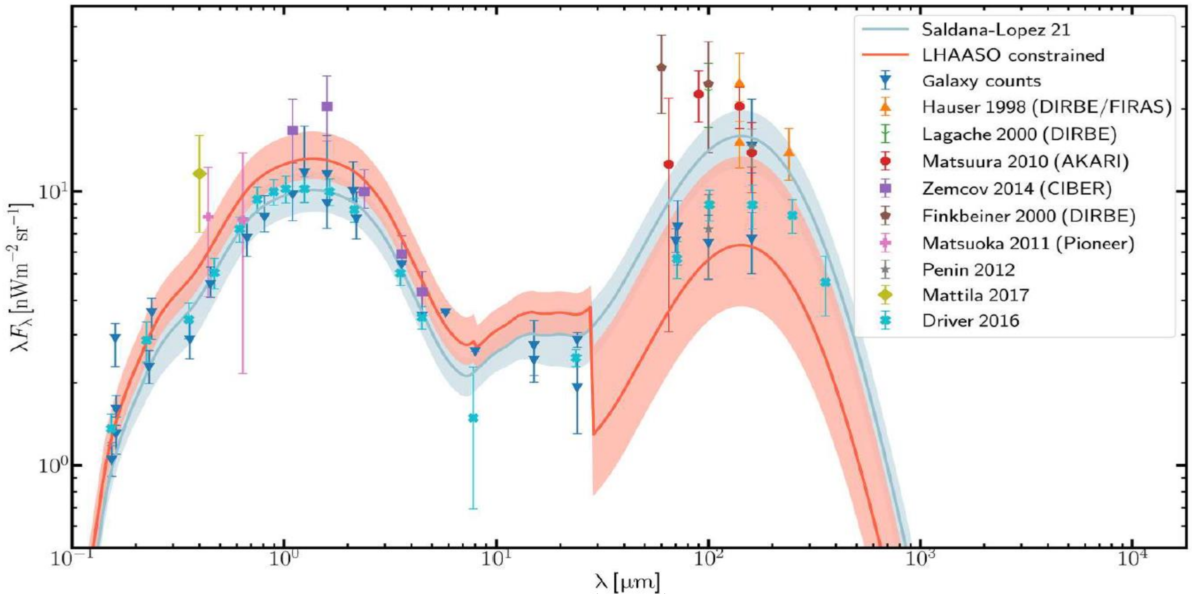
- LIV bounds (Lang+2019)
  - $E_{LIV,\, n=1} > 10^{20}$ GeV
  - $E_{LIV,\, n=2} > 2 \times 10^{12}$ GeV
- We take
  - $E_{LIV,\, n=1} = 3 \times 10^{20}$ GeV
  - $E_{LIV,\, n=2} = 5 \times 10^{12}$ GeV

- LIV-induced modifications sizable above ~40 TeV
  - **cannot explain LHAASO** event

G. Galanti, L. Nava, M. Roncadelli, F. Tavecchio, G. Bonnoli, Phys. Rev. Lett. 131, 251001 (2023).

Discussion

LHAASO constraints on EBL



LHAASO Collaboration, Sci. Adv. 9, 46 (2023).

- GRB 221009A used to **constrain EBL** (**no ALPs**)

BUT

- Required EBL level *appears* too close to **galaxy counts**
- "Physical" EBL level -> **higher** than galaxy counts

- **BUT how much higher?**
  - We cannot count all galaxies -> measure at $z = 2$ more difficult than at $z = 1$
  - Maximal contribution to far-IR (50-80%) from galaxies at $z > 1$ (Devlin+2009; Saldana-Lopez+2021)
  - Star formation rate peaks at $z = 2$-3 (Madau & Dickinson, 2014)
  - JWST inferred stellar mass density much higher than expected (Labbé+2023)

- → Indication for a "physical" **EBL level** sufficiently **higher than galaxy counts ->** Saldana-Lopez et al. EBL more physical and realistic (see also Desai+2019)

GRB 221009A (2)

- **GRB 221009A challenges conventional physics** (CP) -> **EBL absorption**
  - Event @ (13-18) TeV (**LHAASO**) → **extremely problematic** within CP
  - **LHAASO spectrum** → **extremely problematic** beyond ~10 TeV

- **ALPs** can **explain** the **detections** and the **LHAASO spectrum**
  - Within all current bounds about $g_{a\gamma\gamma}$ and m_a
  - LHAASO spectrum up to (13–18) TeV confirms and strengthens our results
  - **Hint** at **ALP existence**
  - Same ALP parameters used in **previous hints at ALPs**

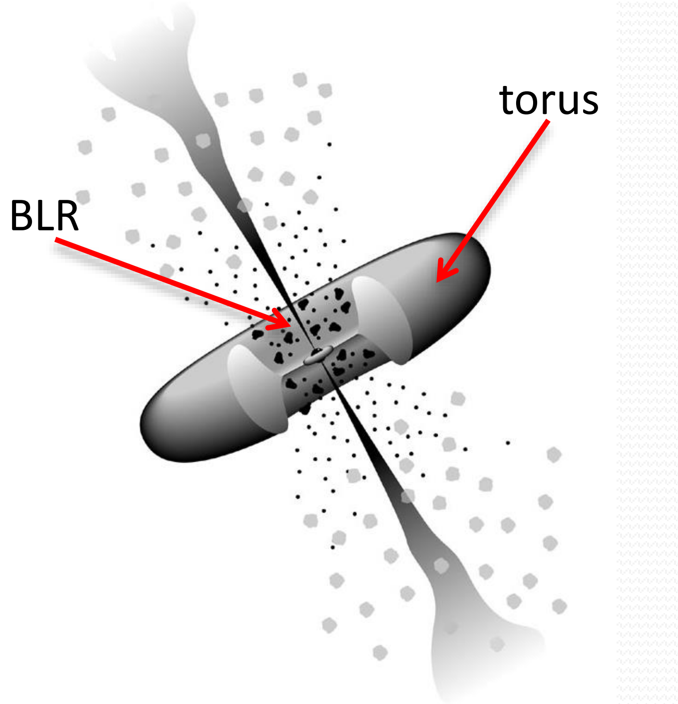
- **LIV not satisfactory**
  - **Cannot explain LHAASO** detection and spectrum

- **FINAL QUESTION:** Possible first **ALP indirect detection** (?)

Other hints at ALP existence

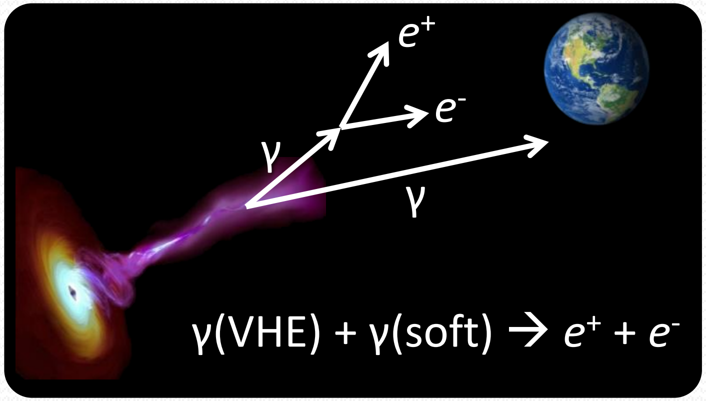
Active Galactic Nuclei (AGN)

- Super massive black holes (10^6 – 10^9 $M_\odot$)
- Accretion disk
- Collimated jets (if towards us -> **blazars**)
- Photons produced at the jet base



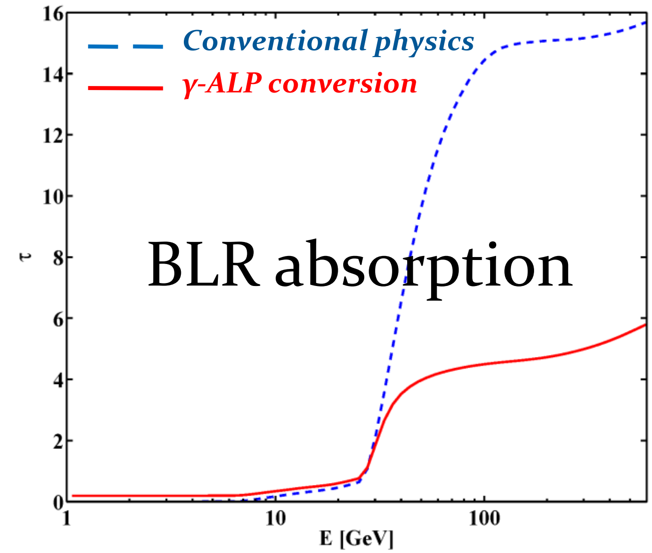
BL Lacs:
- No broad line region (BLR)
- No dusty torus
- Absorption due to the extragalactic background light (EBL) for $E > 100$ GeV

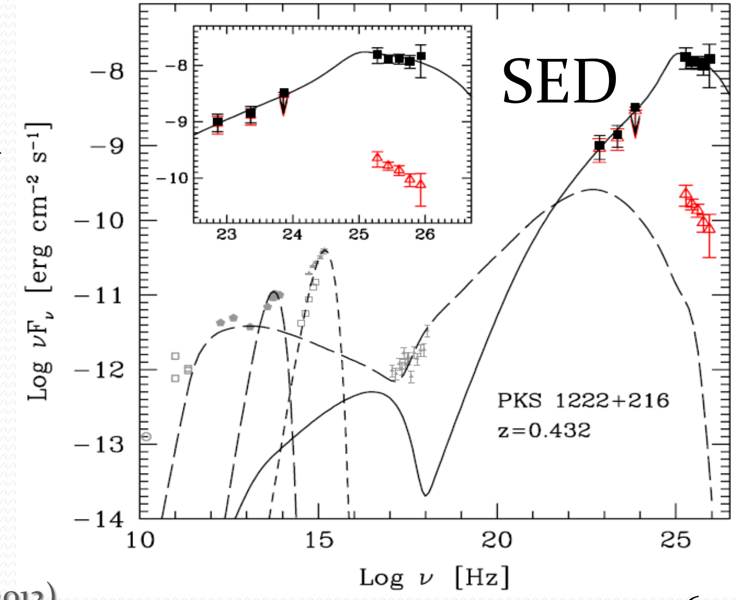
Flat spectrum radio quasars (FSRQs):
- Absorption due to the BLR for $E > 20$ GeV
- Absorption due to the dusty torus for $E > 300$ GeV
- Absorption due to the EBL for $E > 100$ GeV



ALPs in FSRQs

- **High BLR absorption** → no photons with $E > 20$ GeV predicted **BUT**
- **Photons observed up to 400 GeV**
- **Why?** Photon/ALP conversions?
  - $B_{jet} = 0.2$ G and scales as $1/y$
  - $g_{a\gamma\gamma} = O(10^{-11})$ GeV^{-1}, $m_a < O(10^{-10}$ eV$)$
  - BLR $n_{e,BLR} = 10^{10}$ cm^{-3}
- Photon-ALP **conversion** before the BLR – **reconversion** outside the BLR
- -> BLR absorption **REDUCED**
- Physically motivated flux (SED)
- **Hint** at **ALP existence**





F. Tavecchio, M. Roncadelli, G. Galanti, G. Bonnoli, Phys. Rev. D, 86, 085036 (2012).

Anomalous z dependence of Blazars

- We consider all BL Lacs (HBL and IBL) with strong VHE spectrum:
  - In flare
  - $E > 100$ GeV
  - redshift up to $z = 0.6$

- Emitted spectra → **power law**

$$\Phi_{\rm em}(E) = \hat{K}_{\rm em}\, E^{-\Gamma_{\rm em}}$$

- Observed spectrum → **power law**

$$\Phi_{\rm obs}(E_0, z) = \hat{K}_{\rm obs}(z)\, E_0^{-\Gamma_{\rm obs}(z)}$$

- Emitted – observed spectrum relation

$$\Phi_{\rm obs}(E_0, z) = P_{\gamma \to \gamma}(E_0, z)\, \Phi_{\rm em}\big(E_0(1+z)\big)$$

- We **deabsorb** the **observed spectrum**:
  - if no ALPs → EBL absorption only
  - **with ALPs** → EBL absorption and photon-ALP oscillations

G. Galanti, M. Roncadelli, A. De Angelis, G. F. Bignami, MNRAS 493, 1553 (2020).

Anomalous z dependence of Blazars (2)

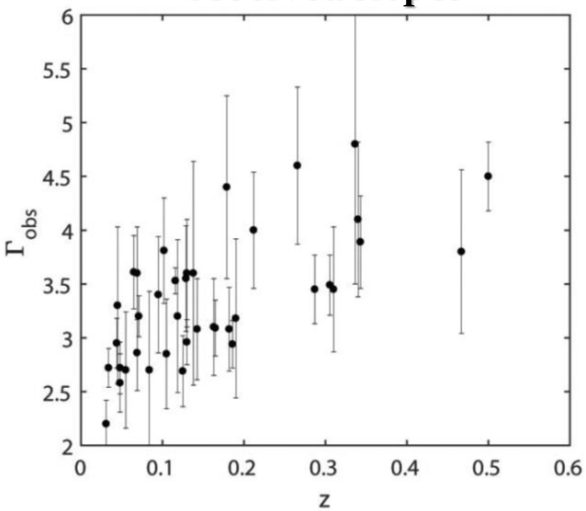
Conventional Physics (CP):

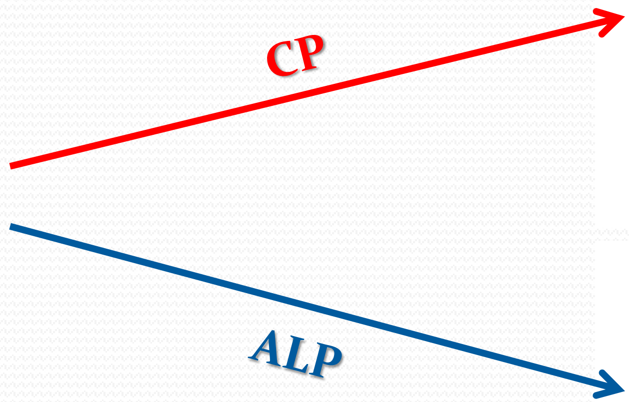
- **Anomalous redshift dependence** of blazar spectra

$$\Phi_{\rm em}^{\rm CP}(E_0(1+z)) = e^{\tau_\gamma^{\rm FR}(E_0,z)}\, K_{\rm obs}(z) \left(\frac{E_0}{E_{0,*}}\right)^{-\Gamma_{\rm obs}(z)}$$

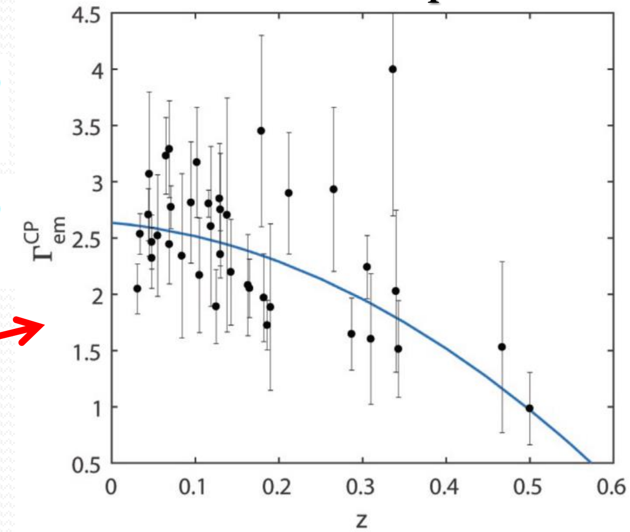
$$\Phi_{\rm em}^{\rm CP,BF}(E_0(1+z)) = K_{\rm em}^{\rm CP}(z) \left(\frac{E_0(1+z)}{E_{0,*}}\right)^{-\Gamma_{\rm em}^{\rm CP}(z)}$$

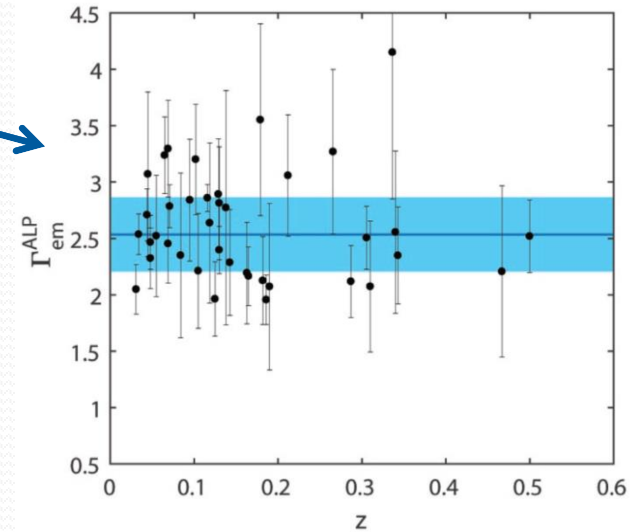
observed slopes





emitted slopes





With ALPs:

$$\Phi_{\rm em}^{\rm ALP}(E_0(1+z)) = \left(P_{\gamma\to\gamma}^{\rm ALP}(E_0,z)\right)^{-1} K_{\rm obs}(z) \left(\frac{E_0}{E_{0,*}}\right)^{-\Gamma_{\rm obs}(z)}$$

- Anomaly **SOLVED**

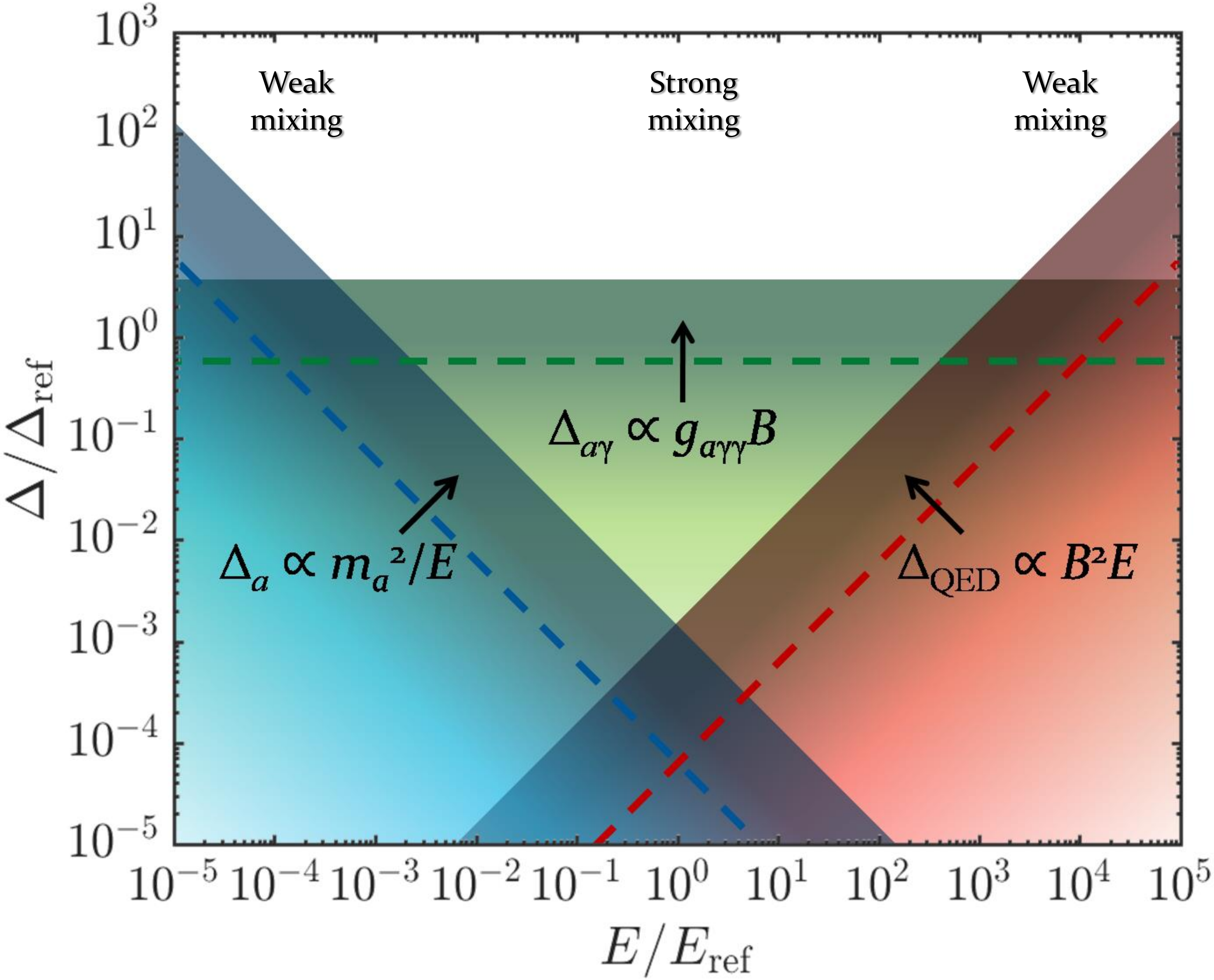
$$\Phi_{\rm em}^{\rm ALP,BF}(E_0(1+z)) = K_{\rm em}^{\rm ALP}(z) \left(\frac{E_0(1+z)}{E_{0,*}}\right)^{-\Gamma_{\rm em}^{\rm ALP}(z)}$$

Hint at **ALP existence** (a new study gets similar results, see Dong+2023)

G. Galanti, M. Roncadelli, A. De Angelis, G. F. Bignami, MNRAS 493, 1553 (2020).

Conclusions

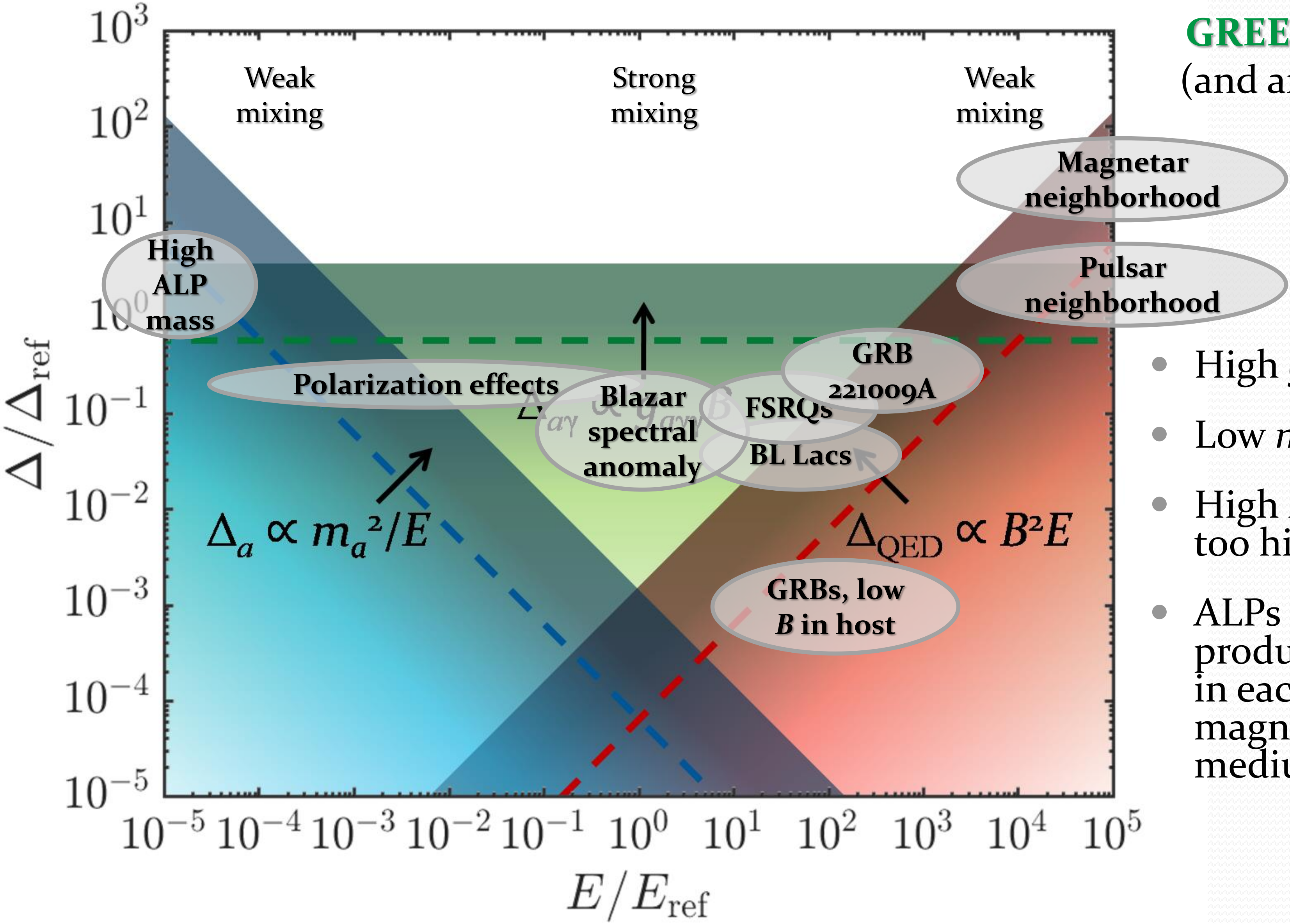
So… Where to look for ALPs??



GREEN AREA
(and around it)

- High $g_{a\gamma\gamma}$
- Low m_a and ω_{pl}
- High B but not too high

So… Where to look for ALPs??



GREEN AREA
(and around it)

- High $g_{a\gamma\gamma}$
- Low m_a and ω_{pl}
- High B but not too high
- ALPs **do not** produce effects in each magnetized medium

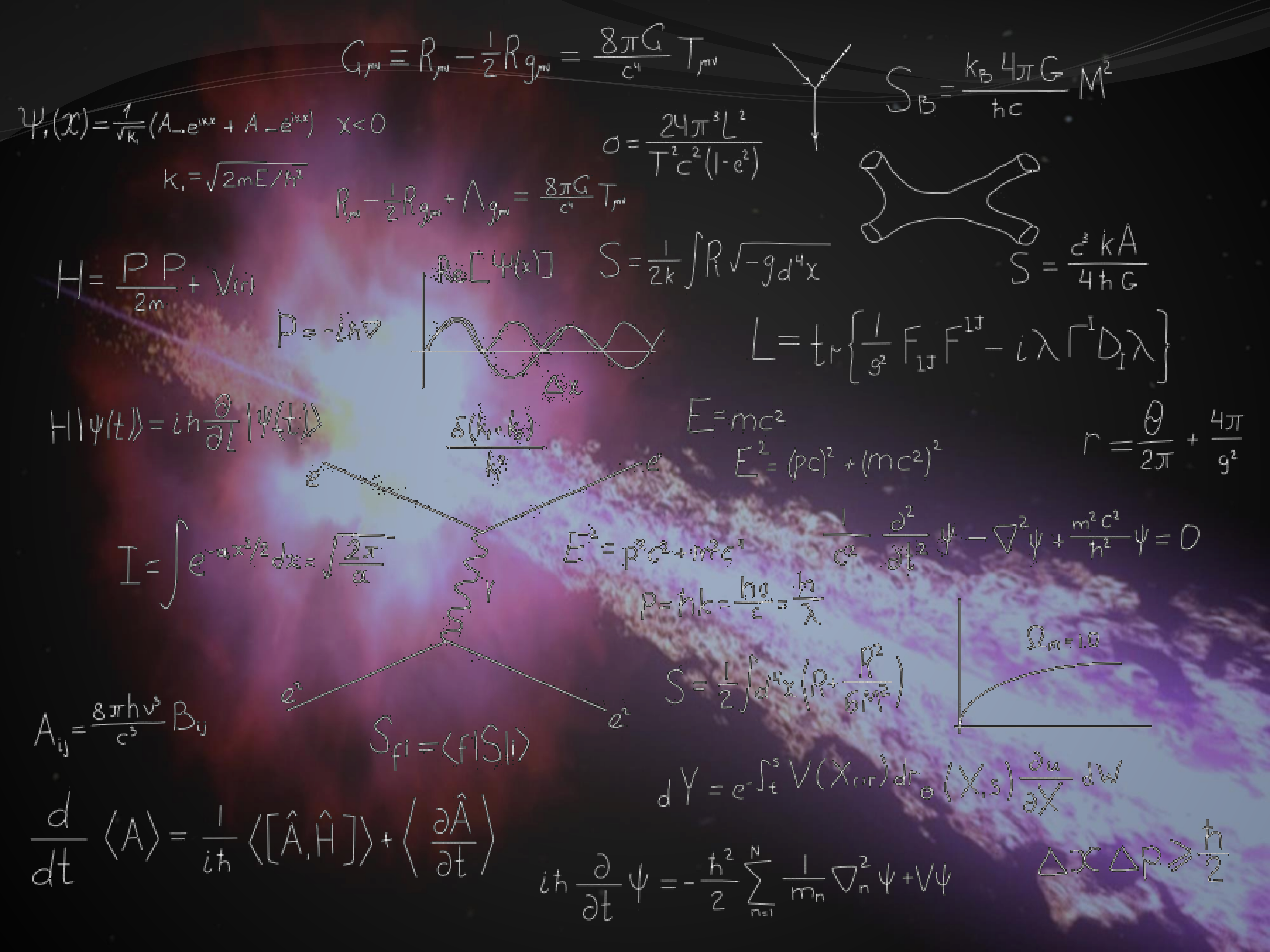
DO ALPs EXIST?

- **Three hints** from astrophysical spectra with **same ALP parameters**
  - **Two indications** from **blazars**
  - The most recent and **strongest one** from **GRB 221009A**
- Additional hints from photon polarization expected

FINAL ANSWER:

- Within few years
- **Confirmed** or **disproved**:
  - From new spectral data by ASTRI, CTA, LHAASO
  - Possible polarization data from IXPE, NGXP, COSI, AMEGO
  - From laboratory experiments such as ALPSII, IAXO

Thank you



$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
$S_B = \frac{k_B 4\pi G}{\hbar c} M^2$
$\Psi_1(x) = \frac{1}{\sqrt{k_1}} (A_{-} e^{ikx} + A_{-} e^{ikx})$ $x < 0$
$\sigma = \frac{24\pi^3 L^2}{T^2 c^2 (1 - e^2)}$
$k_1 = \sqrt{2mE/\hbar^2}$
$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$
$H = \frac{P\,P}{2m} + V(r)$
$\mathrm{Re}[\Psi(x)]$
$S = \frac{1}{2k} \int R \sqrt{-g}\, d^4x$
$S = \frac{c^3 kA}{4\hbar G}$
$p = -i\hbar\nabla$
$L = \mathrm{tr}\{\frac{1}{g^2} F_{IJ} F^{IJ} - i\lambda \Gamma^I D_I \lambda\}$
Δx
$H|\psi(t)\rangle = i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle$
$E = mc^2$
$E^2 = (pc)^2 + (mc^2)^2$
$r = \frac{\theta}{2\pi} + \frac{4\pi}{g^2}$
e^-
$\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \psi - \nabla^2 \psi + \frac{m^2 c^2}{\hbar^2} \psi = 0$
$I = \int e^{-ax^2/2} dx = \sqrt{\frac{2\pi}{a}}$
$E^2 = p^2 c^2 + m^2 c^4$
γ
$p = \hbar k = \frac{h\nu}{c} = \frac{h}{\lambda}$
$S = \frac{1}{2} \int d^4x \, (R + \dots)$
$A_{ij} = \frac{8\pi h \nu^3}{c^3} B_{ij}$
$S_{fi} = \langle f | S | i \rangle$
$\frac{d}{dt} \langle A \rangle = \frac{1}{i\hbar} \langle [\hat{A}, \hat{H}] \rangle + \langle \frac{\partial \hat{A}}{\partial t} \rangle$
$i\hbar \frac{\partial}{\partial t} \psi = -\frac{\hbar^2}{2} \sum_{n=1}^{N} \frac{1}{m_n} \nabla_n^2 \psi + V\psi$
$\Delta x \Delta p \geq \frac{\hbar}{2}$