

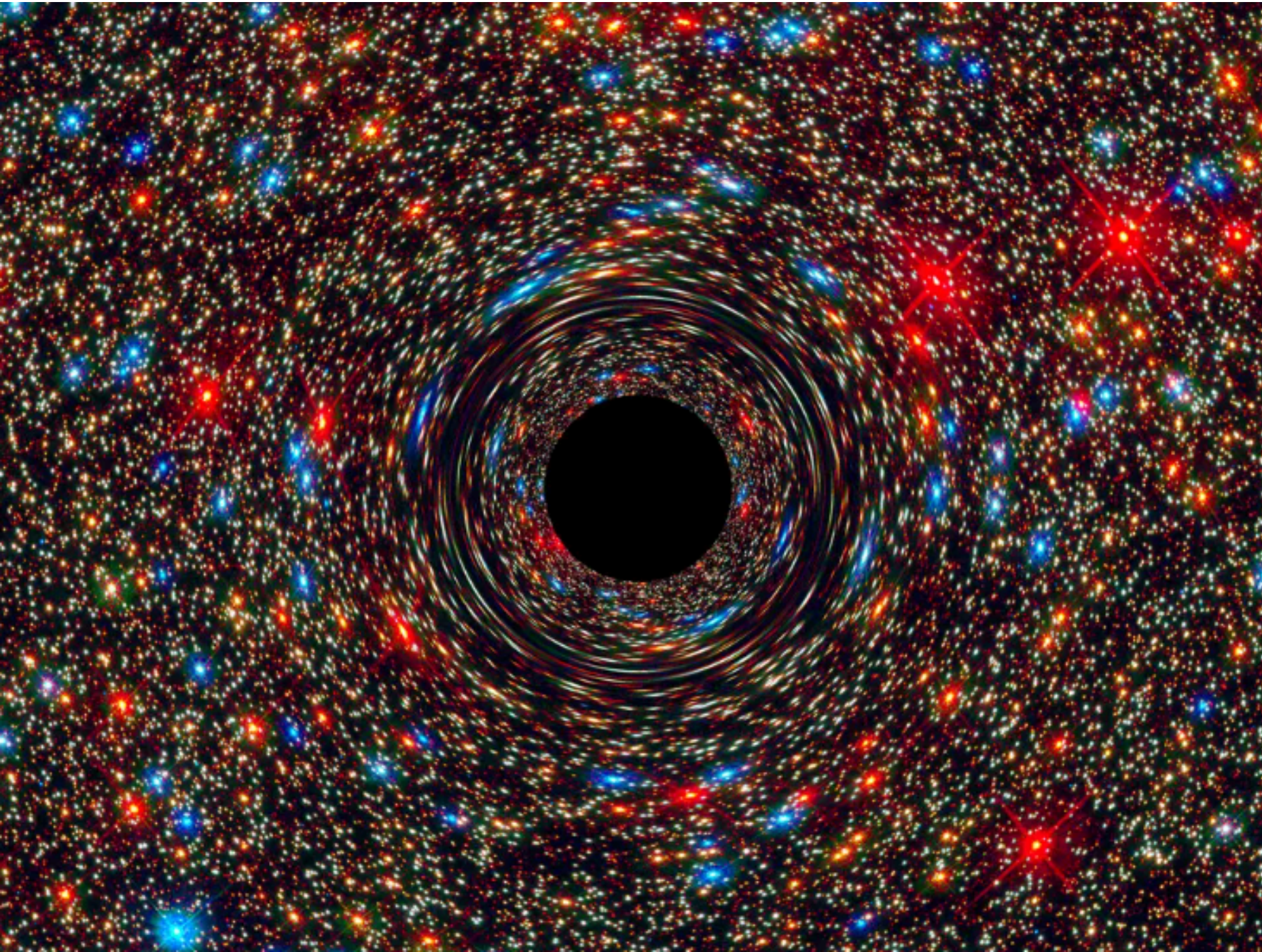
# Chaotic dynamics around black holes

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# Motivation



NASA, ESA, and D. Coe, J. Anderson, and R. van der Marel (STScI)

- Influence of **black hole** mass and spin on dynamics
- Exploring **dynamics** around black holes
- Understanding the behavior of **extended objects**
- Revealing the role of **chaos** in astrophysical systems
- Investigating the effects of **gravitational waves** on motion
- Assessing the impact of chaotic dynamics on **accretion** and **emissions**

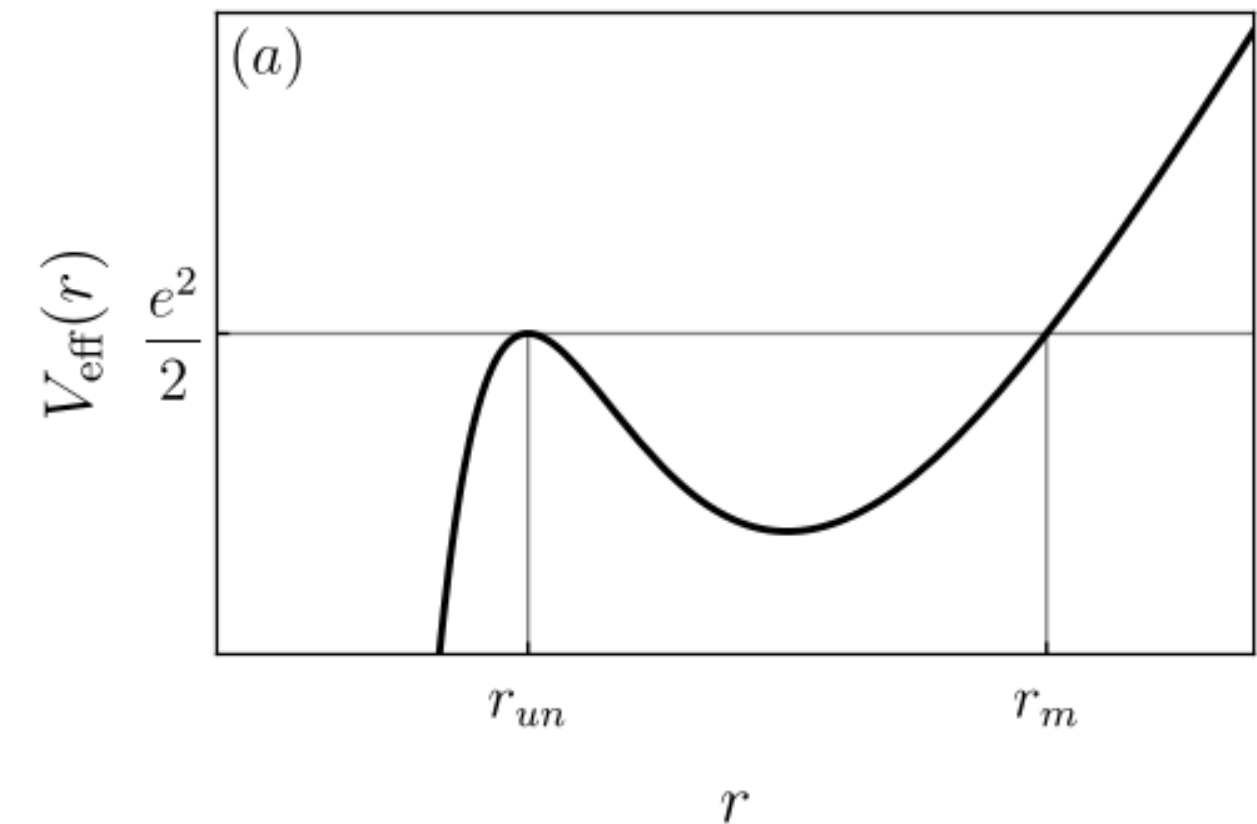
# Motion around black holes

- Point particles follow geodesics;
  - in the presence of gravitational waves the movement can become chaotic;
    - this can affect the long time behavior of the particles around the black hole.
- The dynamics of extended objects is more complicated

$$\frac{Dp^\mu}{ds} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}v^\nu S^{\alpha\beta} + F^\mu,$$
$$\frac{DS^{\mu\nu}}{ds} = 2p^{[\mu}v^{\nu]} + N^{\mu\nu},$$

$$F^\mu = -\frac{1}{6}J^{\alpha\beta\gamma\delta}\nabla^\mu R_{\alpha\beta\gamma\delta},$$

$$N^{\mu\nu} = \frac{4}{3}J^{\alpha\beta\gamma[\mu}R^{\nu]}{}_{\gamma\alpha\beta},$$

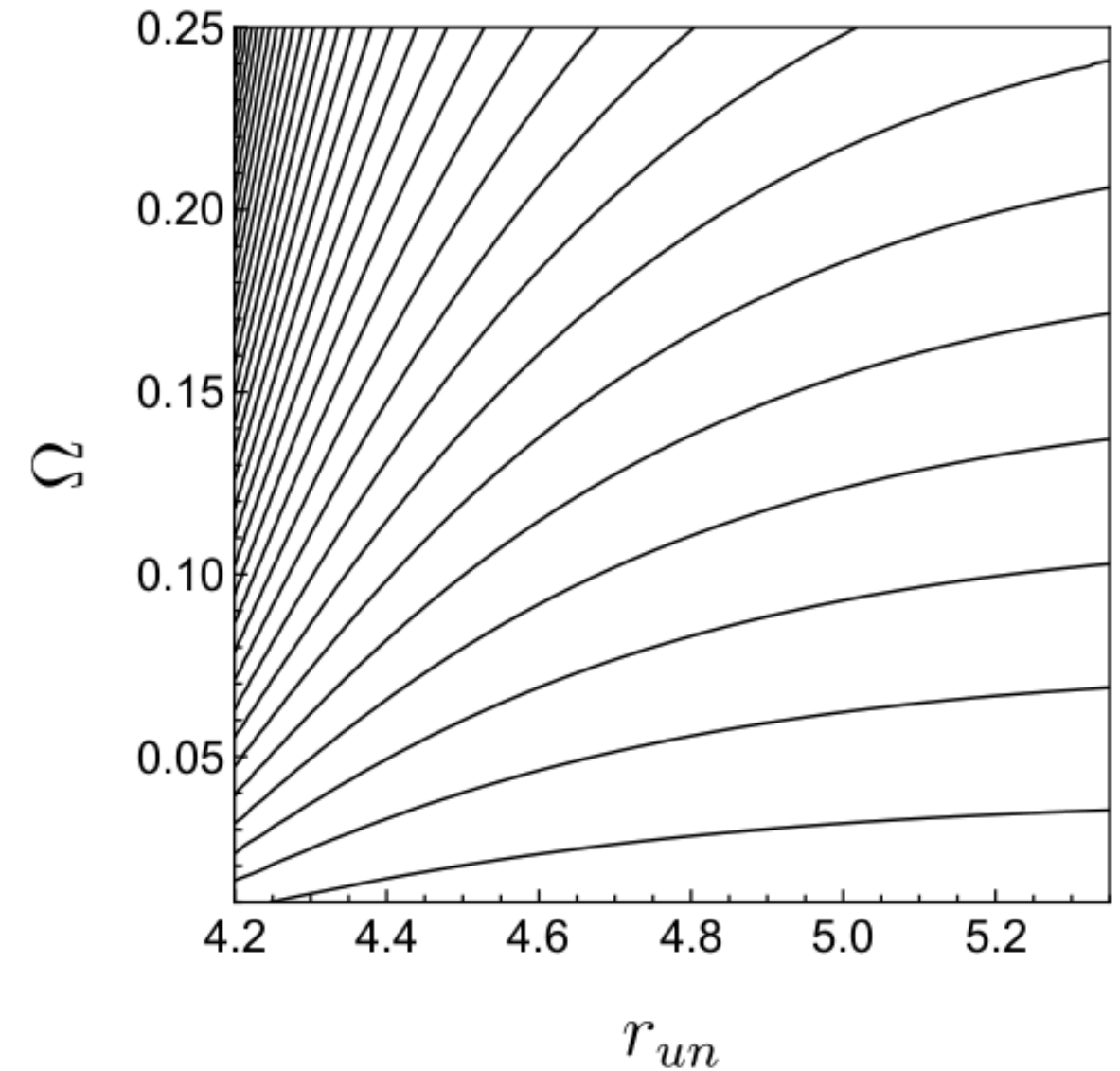


# Chaotic dynamics

- Different spacetime
- Different extended objects
- Melnikov function as a tool for chaos detection

$$M(\tau_0) = 2 \cos(\Omega \tau_0) K(\Omega),$$

$$K(\Omega) = \int_{r_{un}}^{r_m} k(r) \sin[\Omega \tau(r)] dr,$$




[1] Mosna, R.A., Rodrigues, F.F., Vieira, R.S.S.: Chaotic dynamics of a spinless axisymmetric extended body around a schwarzschild black hole. Phys. Rev. D 106, 024016 (2022).

[2] Rodrigues, F.F., Mosna, R.A. and Vieira, R.S.S.: Chaotic dynamics of pulsating spheres orbiting black holes. Gen. Rel. Grav. 56:112 (2024).

# Chaotic dynamics

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## Chaotic dynamics around black holes

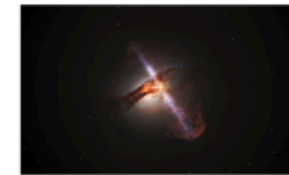
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### Motivation

In astrophysics, studying the dynamics around black holes is crucial for understanding the behavior of matter and energy in extreme gravitational fields. Black holes, with their immense gravitational pull, influence the motion of nearby stars, gas, and dust, forming luminous rotating disks. These dynamics not only reveal important aspects of how black holes interact with their surroundings but also help detect their presence through the emission of energetic particles and radiation.



These dynamics can also exhibit chaotic behavior, where small changes in initial conditions lead to vastly different outcomes. Studying this chaos is important because it could shed light into how matter behaves under extreme non-linear gravitational forces. The chaotic motion of particles and radiation near black holes can offer unique observational signatures, such as irregularities in gravitational waves or electromagnetic radiation, further enhancing our ability to study and understand these enigmatic objects.

### Motion around black holes

The following metric describes a static, spherically symmetric spacetime

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (1)$$

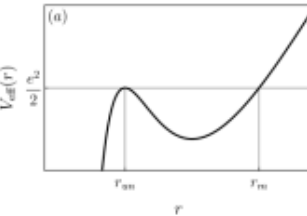
The conservation equation for the equatorial orbit of a point particle in a spacetime with metric (1) is given by

$$\frac{d}{dt} \left( 1 + \frac{c^2}{2} \right) = \frac{c^2}{2} \quad (2)$$

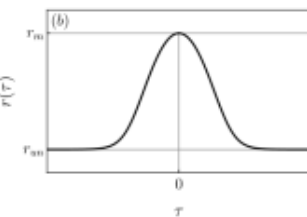
where  $l$  and  $c$  represent the specific angular momentum and energy, respectively. This equation can be expressed in terms of the effective potential

$$V_{\text{eff}} = \frac{f(r)}{2} \left( 1 + \frac{c^2}{2} \right) \quad (3)$$

The local minima and maxima of this effective potential correspond to stable and unstable circular orbits at certain  $r = r_{\pm}$ , respectively, for each fixed value of  $c$ . Consider, for example, the case where the metric (1) describes a Schwarzschild black hole. In this scenario, the effective potential  $V_{\text{eff}}$  has a local maximum corresponding to an unstable circular orbit at a radius  $r = r_{\text{un}}$ , where  $1.31 < r_{\text{un}} < 0.51$  (see the following figure). For a bounded energy levels (i.e.,  $c < 0$ ), this range is further restricted to  $1.31 < r_{\text{un}} < 0.51$  [1].



The following figure shows a typical plot of  $V_{\text{eff}}(r)$ . We note that  $V_{\text{eff}}(r)$  tends asymptotically to the unstable equilibrium point  $r_{\text{un}}$ .

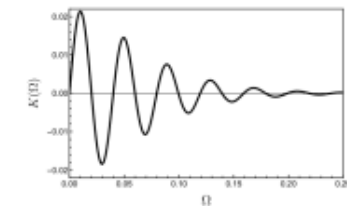


### Results

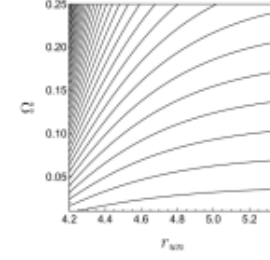
#### Schwarzschild black hole

Due to the symmetries of the spacetime, the components  $P^{(i)}$  of the quadrupole moment enter the equations of motion only via the combination  $Q = J_{22} - J_{33} - J_{33} + J_{22}$ . In order to analyze the effects of a changing shape configuration of the test body on its own translational dynamics, we consider a time-dependent  $Q(t) = \alpha \cos(\Omega t)$  with frequency  $\Omega$ , oscillating between an odd  $Q > 0$  and an even  $Q < 0$  spherical.

One way to search for chaos is by means of Melnikov's integral [4], which can be written as  $M(t) = 2\alpha \sin(\Omega t) K(t)$ . The behavior of  $K$  as a function of  $t$  can be seen in the following figure, with  $\Omega = 1$  and  $r_{\text{un}} = 4.5$  [1].



The behavior of  $K$  can also be illustrated by its  $K = 0$  contour plot in the  $(r, t)$  plane.



If the Melnikov function has isolated zeros, it implies the presence of a homoclinic intersection in the phase space. The existence of such an intersection leads to the entanglement of the stable and unstable manifolds, which is a hallmark of chaotic systems.

#### Schwarzschild (A)dS black hole

It is interesting to apply this result to the case when the spacetime is that of a spherically symmetric black hole in vacuum, possibly along with a cosmological constant, i.e., the Schwarzschild (A)dS black hole,

$$f(r) = 1 - \frac{2M}{r} + \frac{\Lambda}{3} r^2 \quad (8)$$

In this case,  $V_{\text{eff}}$  is identically zero. This shows (at quadrupole order) that the center-of-mass trajectory of a spherically symmetric test body, pulsating or not, is always a geodesic on these spacetimes [2].

#### Reissner-Nordström black hole

In the Reissner-Nordström metric, characterized by the function  $f(r)$  given by equation

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} \quad (9)$$

where  $M$  and  $Q$  are the mass and charge of the black hole, if  $Q < M$ , the system exhibits two event horizons. In this case, as demonstrated in [2], the object will show chaotic dynamics if its shape changes. This shift in the object's geometry leads to unpredictable orbital motion, further emphasizing the sensitivity of the system to perturbations.

#### Ayón-Beato-García black hole

In the Ayón-Beato-García spacetime the metric corresponds to (1) with [5]

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{2\beta Q^2}{r^3} \quad (10)$$

where  $M$  and  $Q$  are the mass and charge of the black hole, respectively. When  $Q < 0.5M$ , the system has two event horizons. As shown in [2], under these conditions, the object will experience chaotic dynamics if its shape is altered. Such a change in geometry triggers irregular orbital behavior, highlighting the system's sensitivity to even small perturbations.

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### Conclusions

We analyzed the motion of spinless spherical test bodies in a spherically symmetric spacetime with metric given by equation (1). We showed that the motion is in general not geodesic, except for the case of a cosmological vacuum. In particular for a spherical test body with oscillating radius, the force terms due to the body's quadrupole structure generally breaks the homoclinic orbit to associated with unstable fixed points of test particle motion, giving rise to homoclinic chaos. We presented examples of this phenomenon for the Schwarzschild [1], Schwarzschild (A)dS, Reissner-Nordström and Ayón-Beato-García black hole spacetimes [2].

Studying chaotic motion is crucial for understanding a variety of astrophysical phenomena. In the context of dynamics around black holes, chaos can influence the stability of orbits and the potential capture of matter or energy by the black hole. Investigating these dynamics can reveal important patterns or events, such as the formation of relativistic jets or the emission of gravitational waves, which are of great interest in astrophysics and cosmology. Furthermore, understanding chaos helps in exploring the transition between regular and chaotic orbits, offering deep insights into the limits of predictability in complex physical systems.

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### References

- [1] Mosna, R.A., Rodrigues, F.F., Vieira, R.S.S.: Chaotic dynamics of pulsating spheres orbiting black holes. Phys. Rev. D 106, 024016 (2022)
- [2] Rodrigues, F.F., Mosna, R.A., and Vieira, R.S.S.: Chaotic dynamics of pulsating spheres orbiting black holes. Gen. Rel. Grav. 56:112 (2024)
- [3] Chandrasekhar, S.: The Mathematical Theory of Black Holes. Cambridge University Press, Cambridge (1983)
- [4] Melnikov, I.V.: On the Problem of Anomalous Perturbations of the Motion of a Particle in a Resonance. Izv. Akad. Nauk SSSR Ser. Mat. 29, 576-592 (1963)
- [5] Ayón-Beato, M., García, J.: Regular Black Holes in the Exact Relativity Approach to Black Holes. Phys. Rev. D 63, 044007 (2001)



***Thank you !***

***Grazie!***