Statistical Physics and Ecology are they a good match?

~~(Probably Not)~~

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PhD Seminars: Season 8, Episode 2

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SAPIENZA
Università di Roma

Introduction

Why theoretical physics and ecology?

An ecosystem is a system made of large number of interacting species

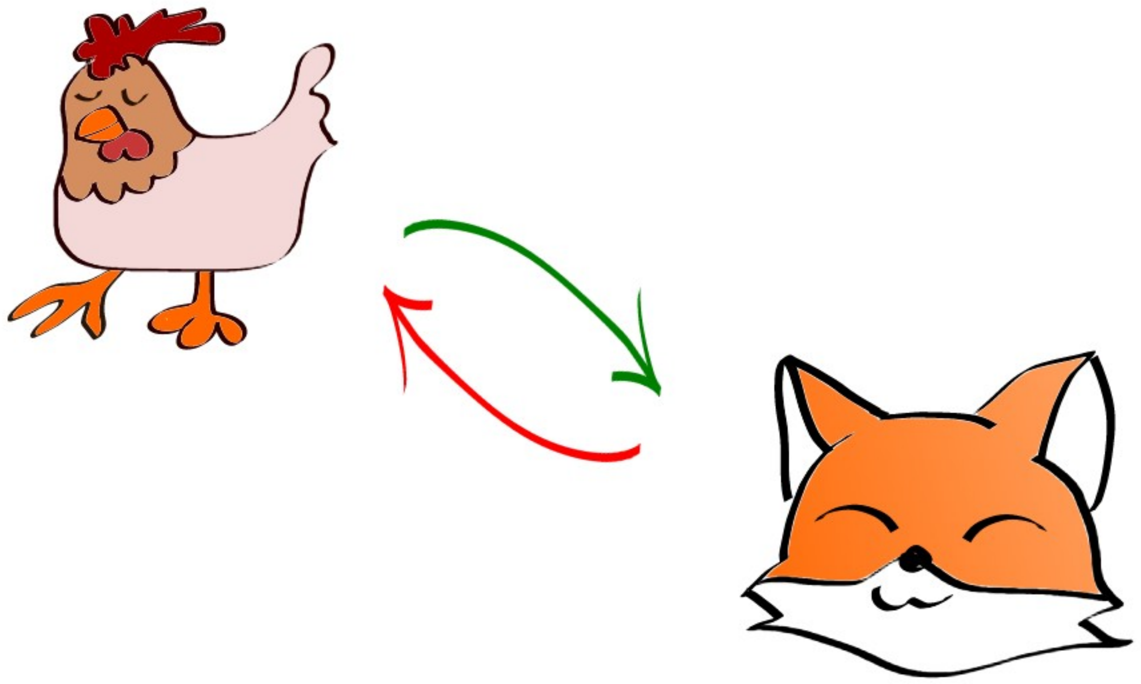
Many open challenges:

- Effect of new species addition, mutation or alien
  (*i.e.*, blue crab, grey vs red squirrel, Nile perch in Lake Victoria, plant invasion)
- Effect of change in ecological parameters
  (due to human intervention or climate change)
- Role of biodiversity and effect of its reduction

Example of ecological model: Lotka-Volterra

Lotka-Volterra Model

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = Ax - Bxy & \text{Prey} \\ \frac{\mathrm{d}y}{\mathrm{d}t} = -Cy + Dxy & \text{Predator} \end{cases}$$



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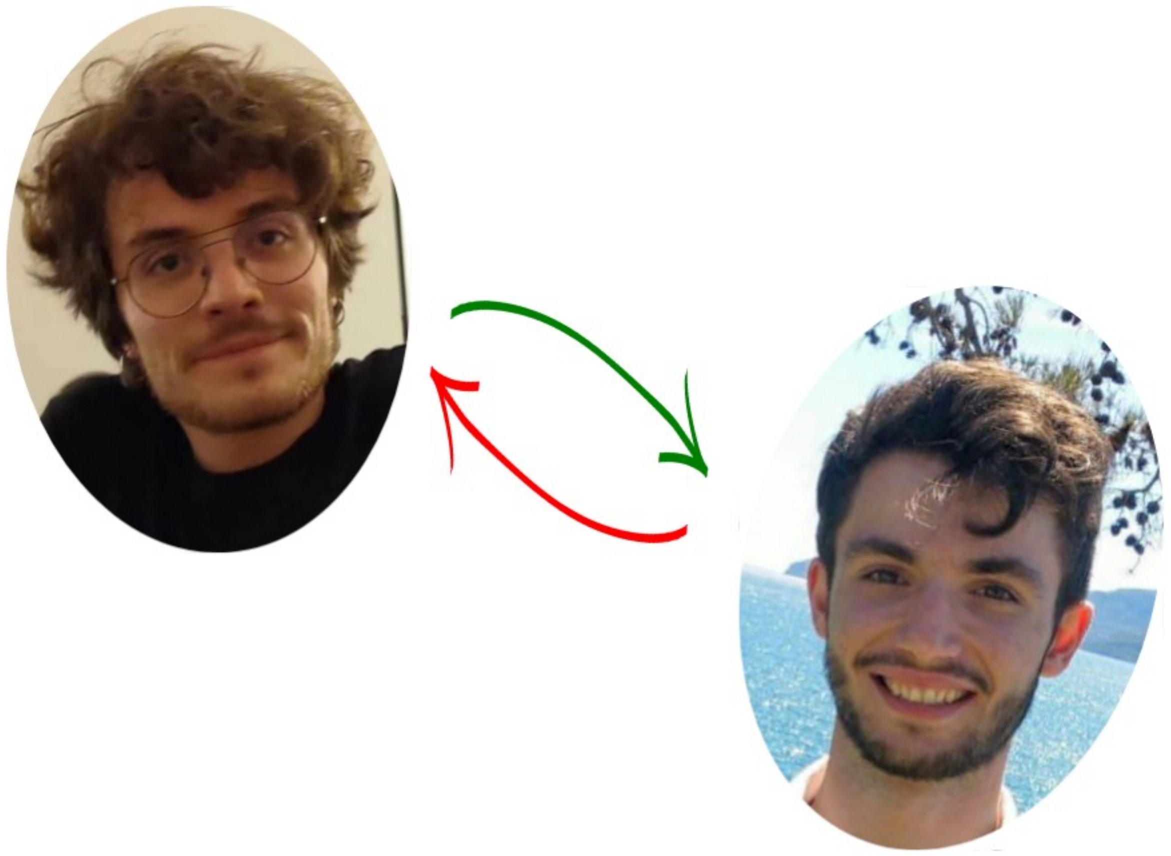
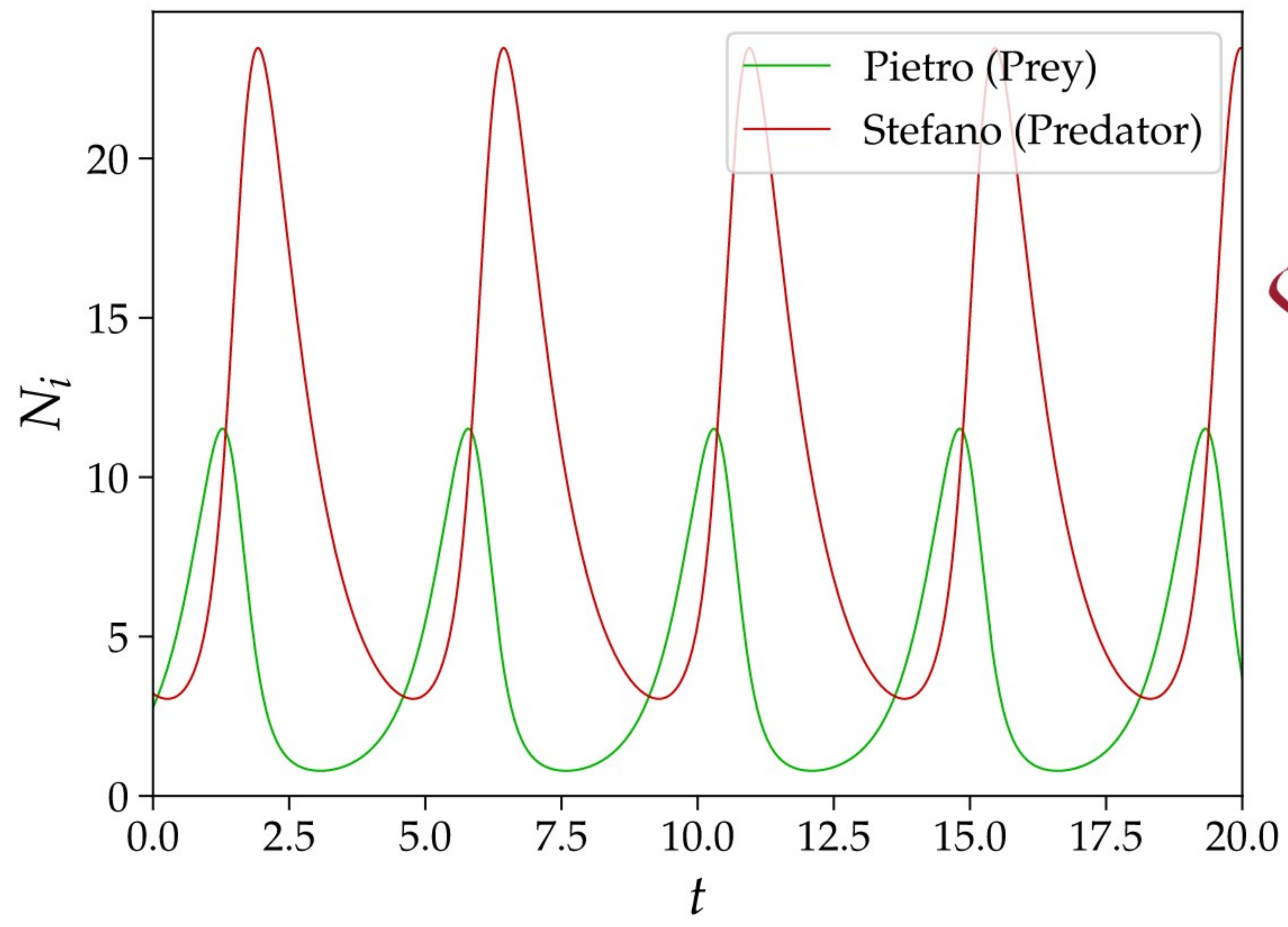
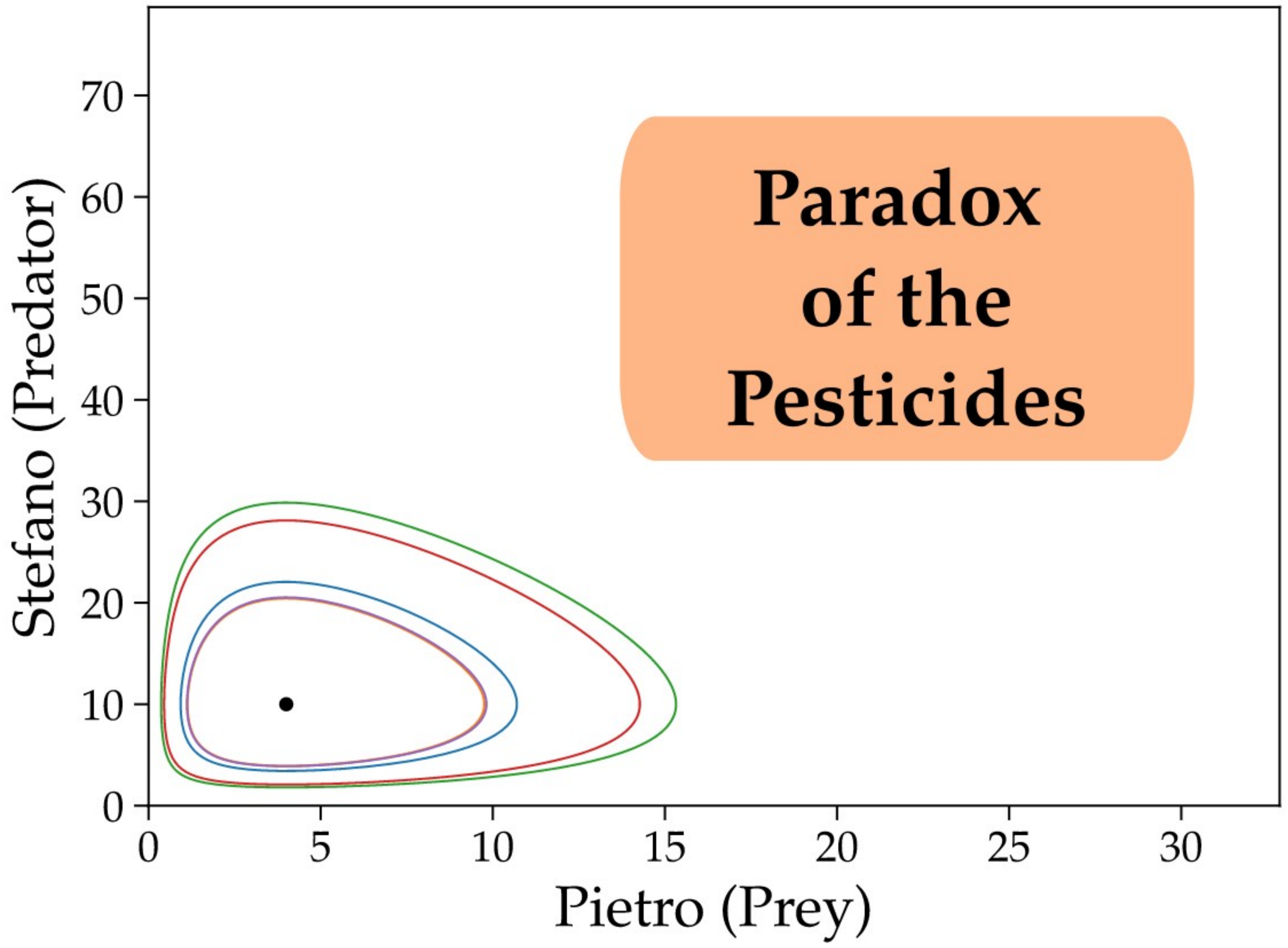


- A: prey growth rate
- C: predator mortality rate
- B and D: effect of interaction on prey and predator

↳ Many strong assumptions and approximations!

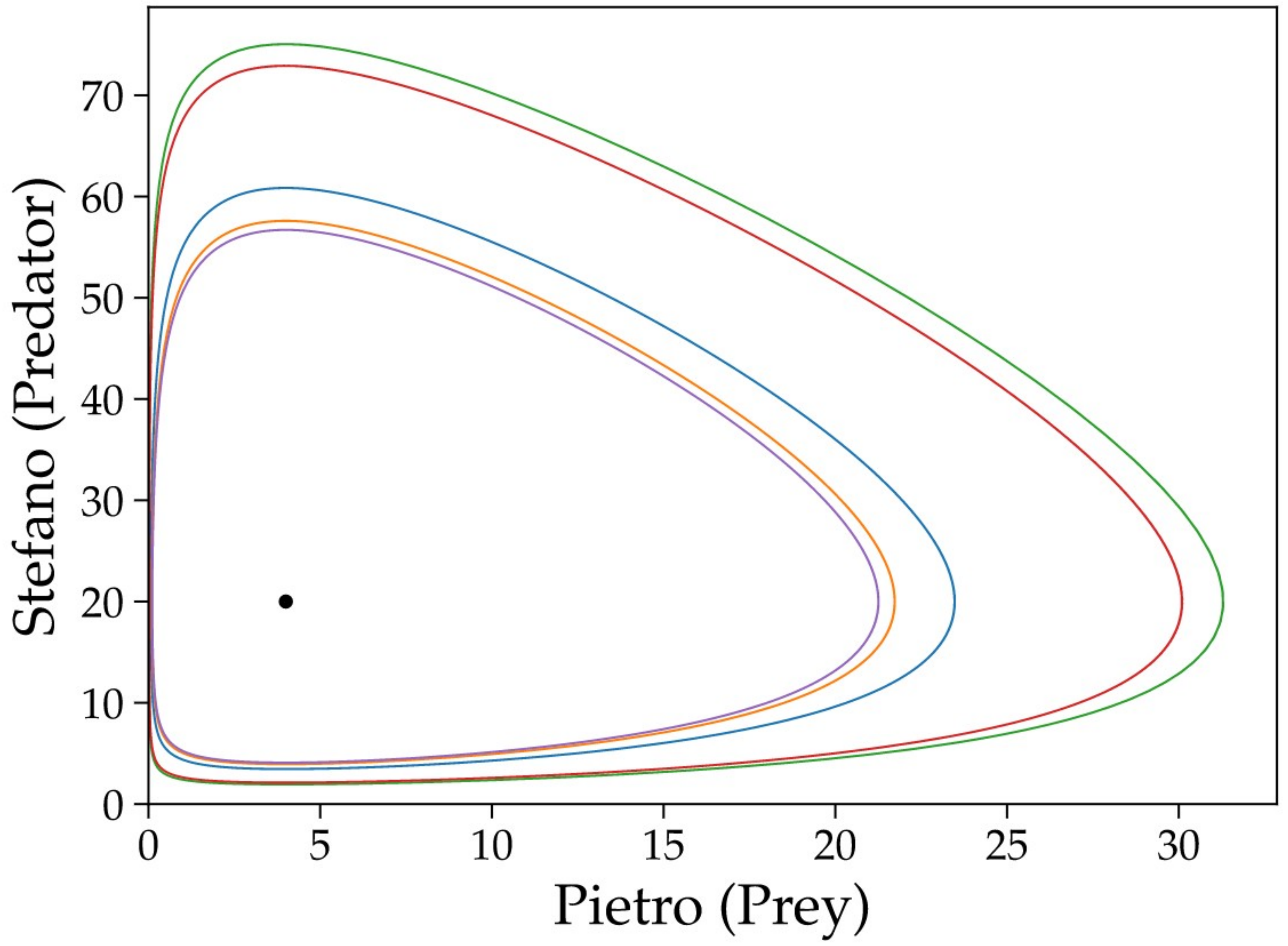


Oscillations in the number of preys and predators



Increasing prey growth rate A

→



Generalized Lotka-Volterra Model

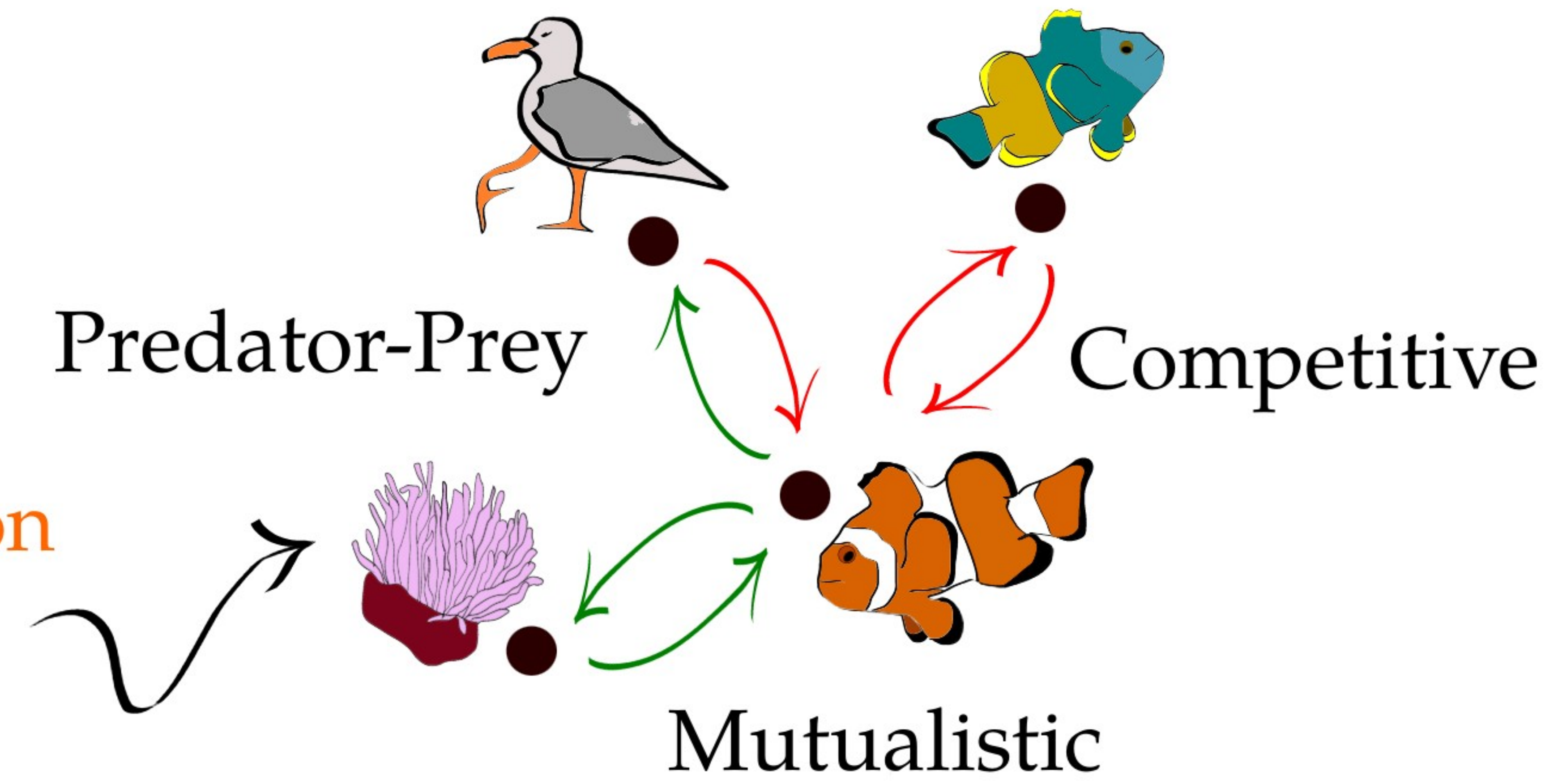
$$\frac{dx_i}{dt} = x_i \left[\left(r_i - \frac{x_i}{K_i} \right) - \sum_{j=1}^{N} A_{ij} x_j \right]$$

Species' Abundance

Growth Rate

Carrying Capacity

Interaction Matrix

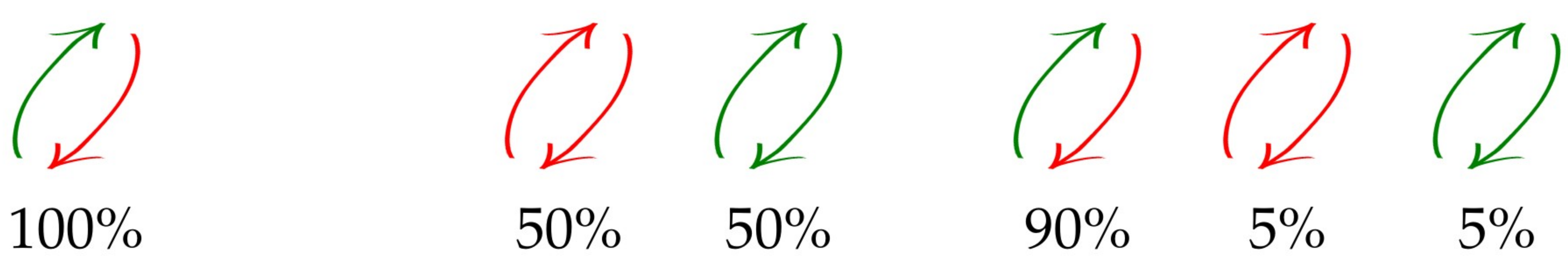


⟹

	+	−	−	+
−		−	−	−
−	+		+	+
+	−	+		−
+	+	−	+	

Sign Pattern

Antagonistic Model — **Competitive-Mutualistic Model** — **Mixture Model**



Feasibility: existence of ecologically meaningful equilibrium populations

Surviving Equilibrium Population $\quad \vec{x}^{\,\star} = \mathbf{B}^{-1}\,\vec{r} \quad \longrightarrow \quad B_{ij} = \dfrac{\delta_{ij}}{K_i} + A^{\star}_{ij}$

Structural stability: sensitivity of equilibrium populations to changes in ecological parameters

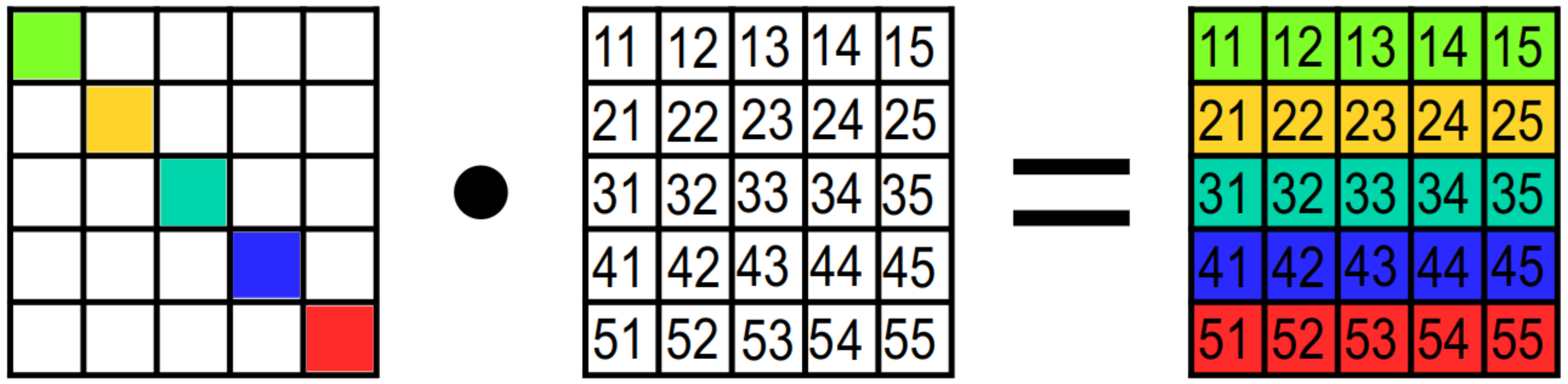
$$r_i \longrightarrow r_i + \xi_i \quad \Longrightarrow \quad \frac{\partial x^{\star}_i}{\partial \xi_j} = (\mathbf{B}^{-1})_{ij} \quad \Longrightarrow \quad \lambda_i(\mathbf{B}) \neq 0 \quad \forall\, i = 1, \ldots, N$$

Linear stability: stability with respect to small perturbations around fixed point population

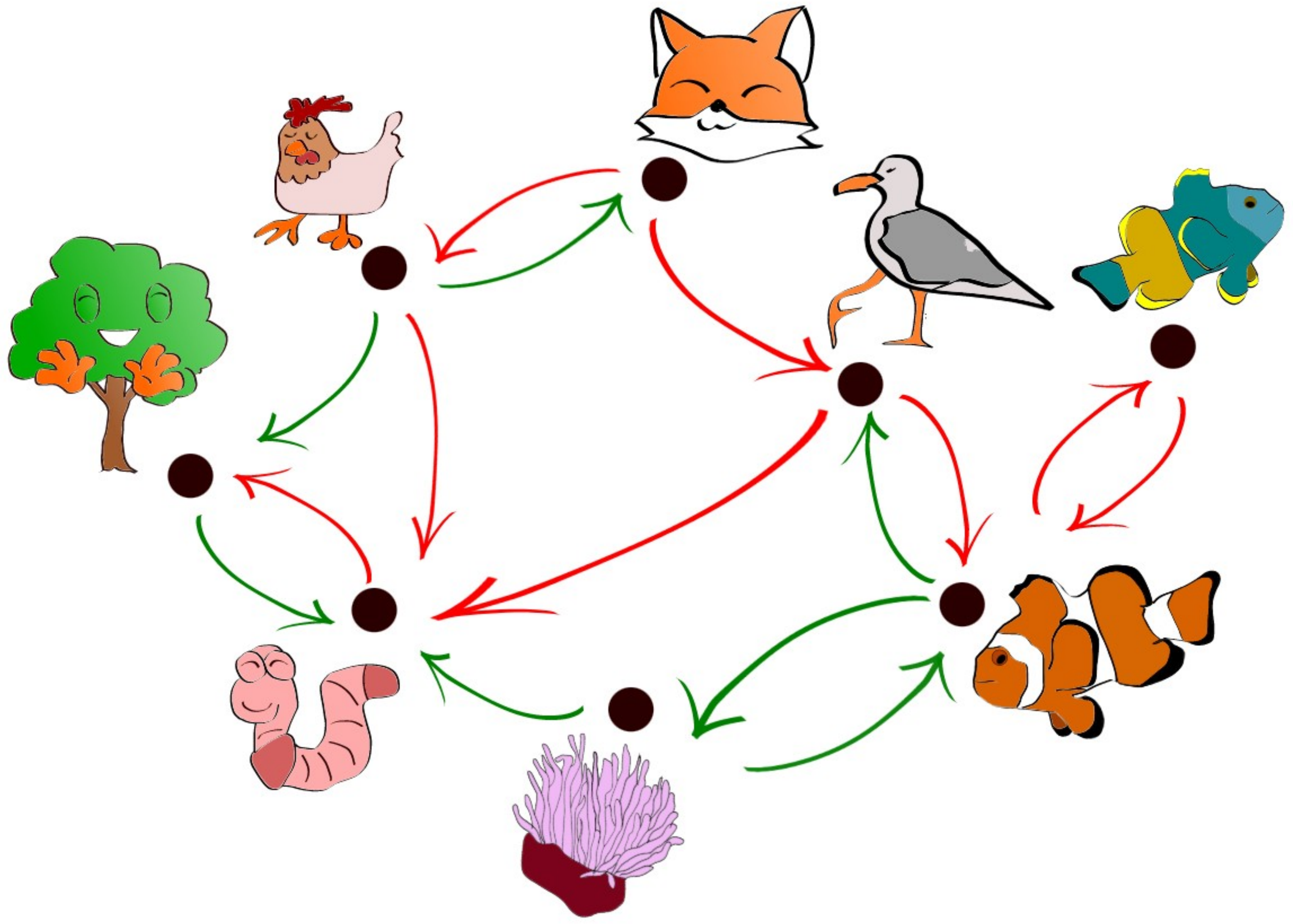
$$\Longrightarrow \quad \Re(\lambda_i(\mathbf{J})) < 0 \quad \forall\, i = 1, \ldots, N$$

Jacobian

$$J^{\star}_{ij} = x^{\star}_i\, B_{ij}$$



Stripy Structure



$$\frac{dx_i}{dt} = x_i \left[\left(r_i - \frac{x_i}{K_i} \right) - \sum_{j=1}^{N} A_{ij} x_j \right]$$

Different types of stability related to spectral properties of B and J

$$B_{ij} = \frac{\delta_{ij}}{K_i} + A^\star_{ij} \quad , \quad J_{ij} = x^\star_i \, B_{ij}$$

Easy to identify network and kind of interaction
Hard to quantify strenght of interactions

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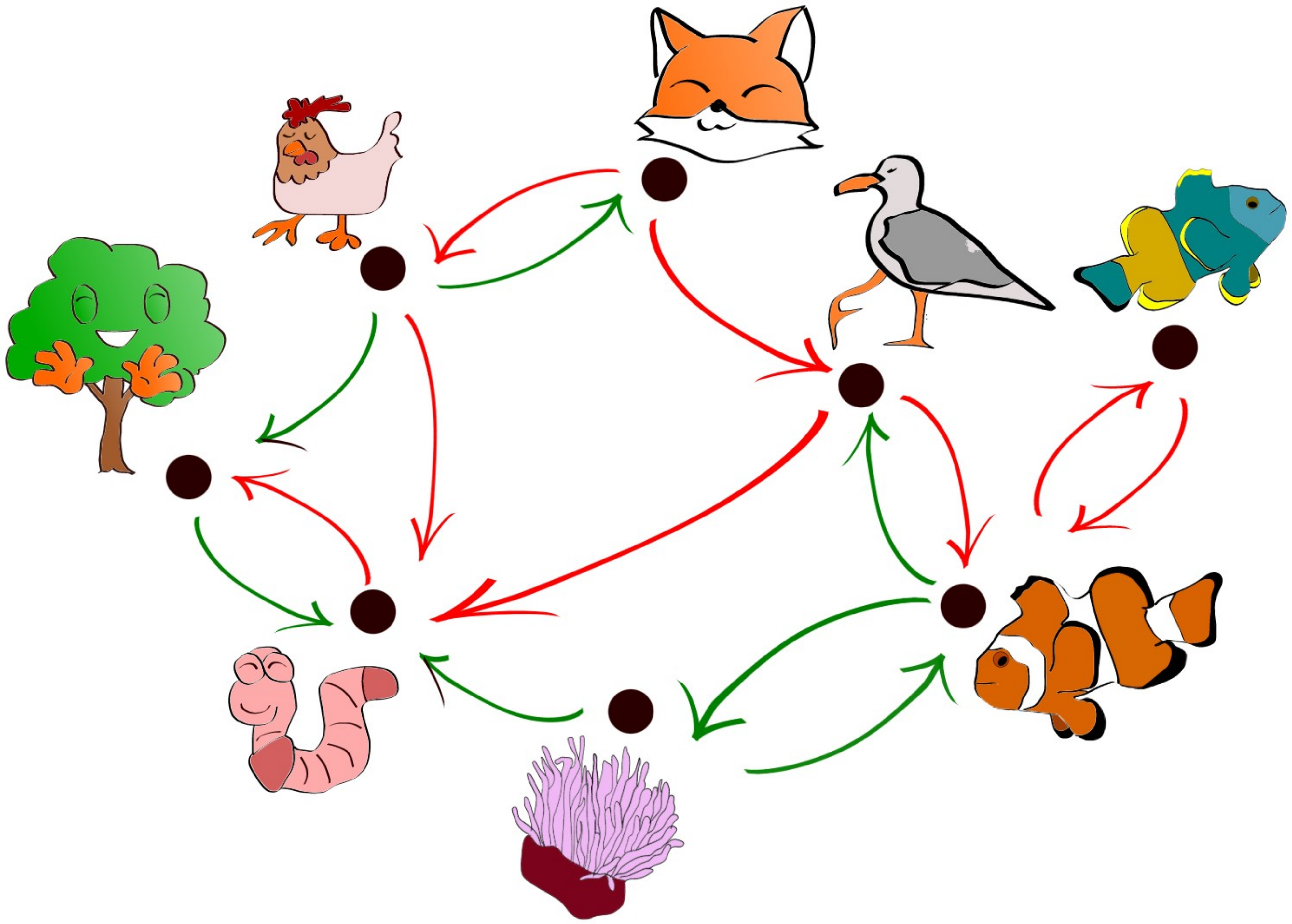
We focus on the macroscopic behaviour
we do not expect all the details to matter

$\Longrightarrow$ **Random Matrix Approach**



Ecology


RANDOM
MATRIX
THEORY

Classical results on stability

Ecological parameters $\longrightarrow$ Random variables [R. M. May, *Nature* (1972)]

$\longrightarrow$ A_{ij} i.i.d. with $\langle A_{ij} \rangle = 0$, $\langle A_{ij}^2 \rangle = \sigma^2$ and $A_{ij} = A_{ji}$ while $K_i = 1$

$\Longrightarrow$ Wigner Semicircle Law

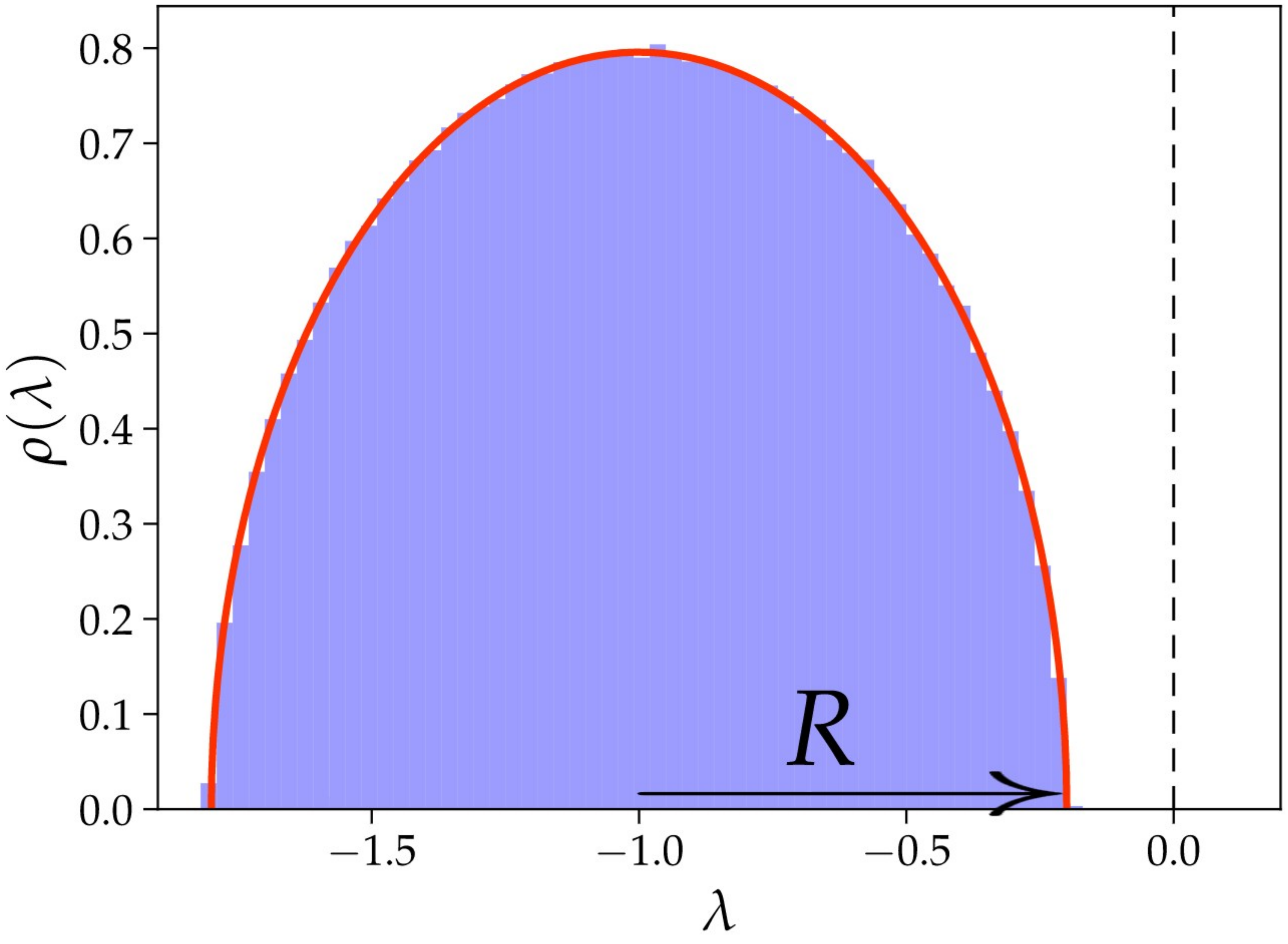
$$R = 2\sigma\sqrt{N}$$

Complexity-Stability trade-off

(**May's Paradox:** Size-dependent Stability)

No complexity–stability relationship in empirical ecosystems

[C. Jacquet et al., *Nature Communications* (2016)]



Going beyond symmetry ($A_{ij} \neq A_{ji}$)

+

Specifying nature of the interactions

$$\frac{< A_{ij} A_{ji} >}{\sigma^2} = \tau$$

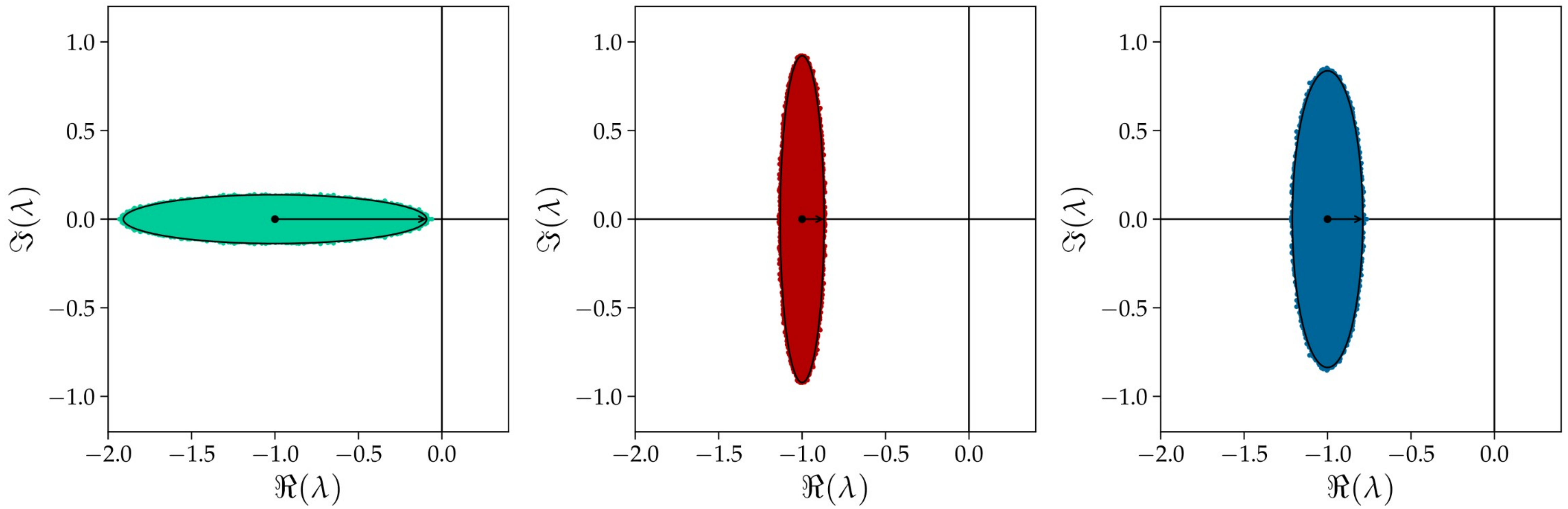
$\Longrightarrow$

Elliptic Law

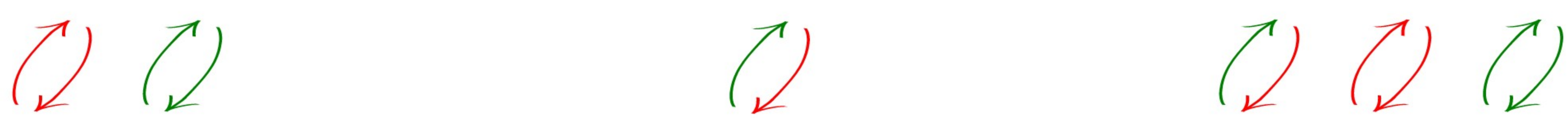
$$R = \sigma(1+\tau)\sqrt{N}$$

Interactions have quantitative effect
Same qualitative behaviour

[S. Allesina, S. Tang, *Nature* (2012)]



Competitive-Mutualistic | Antagonistic | Mixture



50% 50% | 100% | 90% 5% 5%

Sign Stability

A matrix **M** is **sign stable** if any matrix **M'** with the same topology and sign pattern is stable, independently from the absolute value of their non-zero elements:

$$\mathbf{M} \text{ is sign stable} \iff \Re(\lambda_i(\mathbf{M}')) < 0 \;\; \forall\, \mathbf{M}' \, : \, M'_{ij} = y_{ij} M_{ij}$$

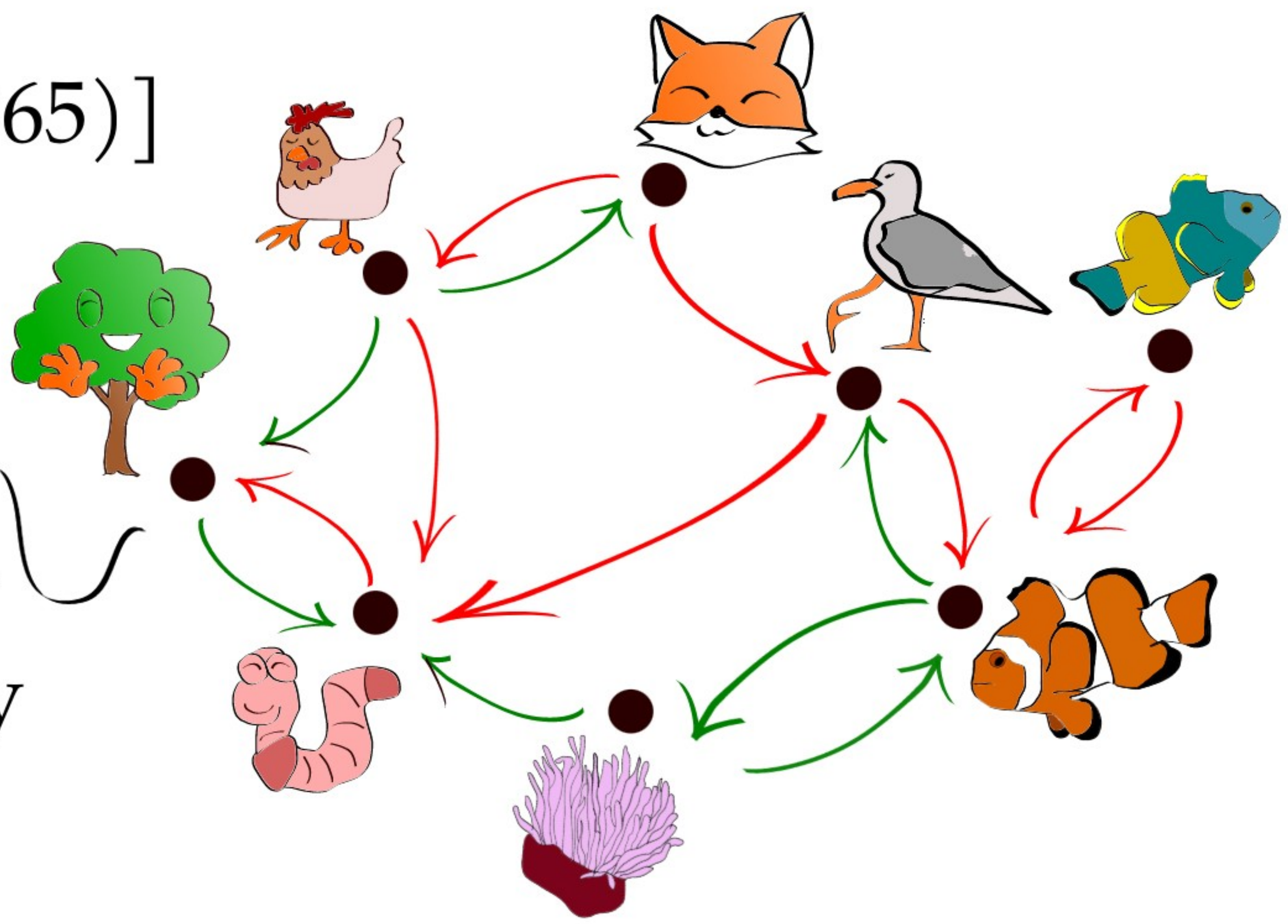
Apply to all matrices in equivalence class:

$$\mathcal{M}(\mathbf{M}) := \left\{ \mathbf{M}' \in \mathbb{R}^{N\times N} : \mathrm{sign}(M'_{ij}) = \mathrm{sign}(M_{ij}) \right\}$$

[J. Quirk, R. Ruppert, *The Review of Economic Studies* (1965)]

$\Longrightarrow$ Interesting applications in ecology:

Sign stability can apply
to this cartoon!



Sign stability can appear a too strong condition to be useful.

Conditions (necessary and sufficient) in the case $M_{ii} < 0$:

1. $M_{ij}M_{ji} \leq 0, \quad \forall\, i \neq j$ $\longrightarrow$ Sign pattern
2. $M_{i_m i_1} = 0, \;\; \forall\, i_1 \neq i_2 \neq \cdots \neq i_m,$ such that $\quad M_{i_1 i_2} M_{i_2 i_3} \ldots M_{i_{m-1} i_m} \neq 0$ $\longrightarrow$ Network Topology

$\Longrightarrow$ Antagonistic trees are sign stable

Antagonistic trees can be mapped into antisymmetric trees:

- Imaginary spectrum (no diagonal)
- With negative (disordered) diagonal are stable irrespective of system size

$\Longrightarrow$ **Absolute Stability**

What about locally tree-like graphs?

Local Sign Stability

Let's extend this idea to matrices that are only locally sign-stable: a matrix M is locally sign stable if its finite neighbourhoods are, almost certainly, sign stable.

Example: antagonistic graph which is locally like a tree (loops are typically long).

Criterion:
$\mathbf{M}_N$ is (strongly) locally sign stable $\Longrightarrow$ $\lim_{N\to\infty}\langle \Re[\lambda_1(\mathbf{M}_N)]\rangle < \infty$ (**Absolute stability**)

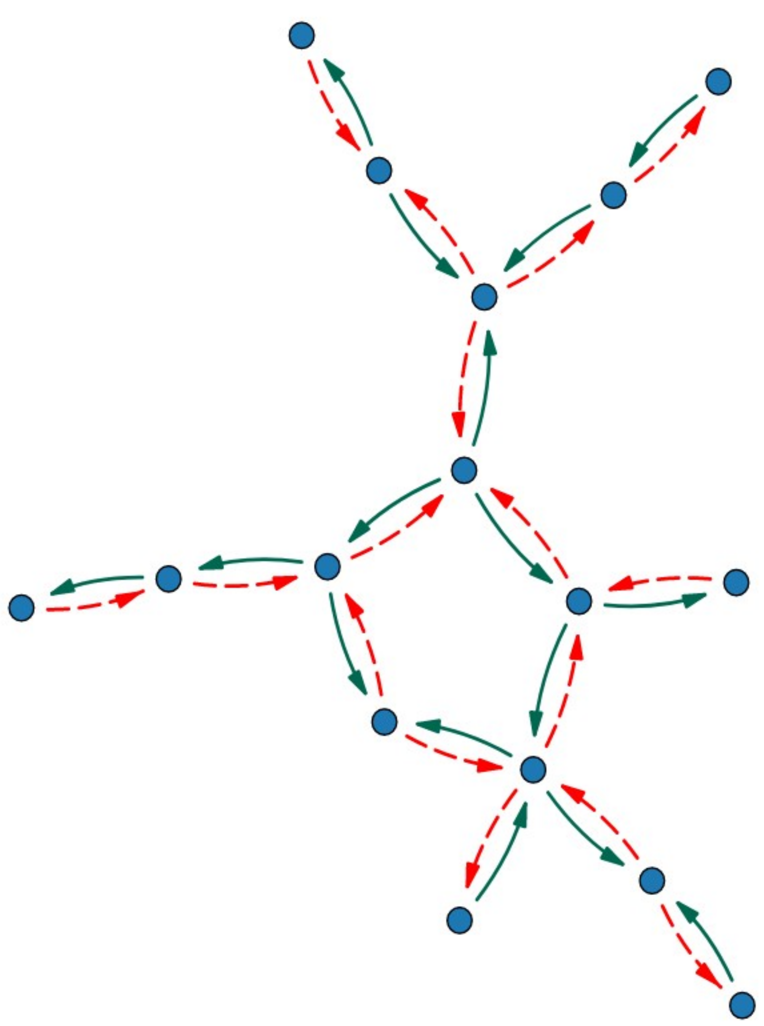
[P. Valigi, C. Cammarota, I. Neri, arXiv:2303.09897 (2023)]

Numerical tests

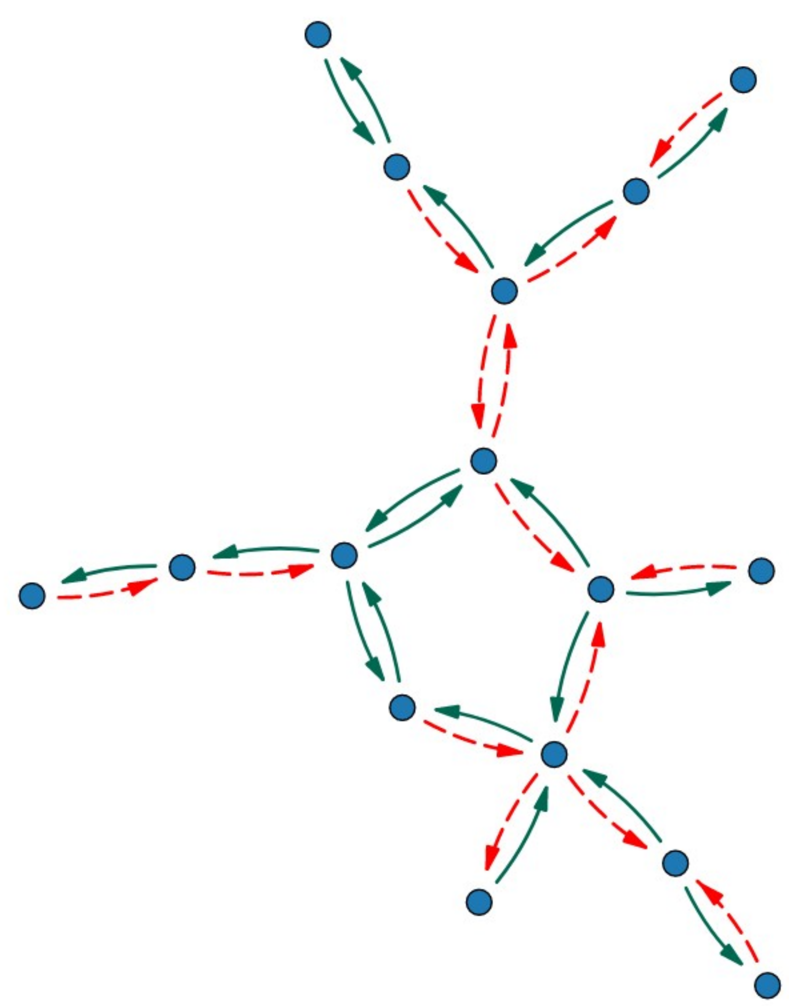
(Local) sign stability → predator-prey interactions

(Local) sign stability → no directed cycles (no feedback loops)

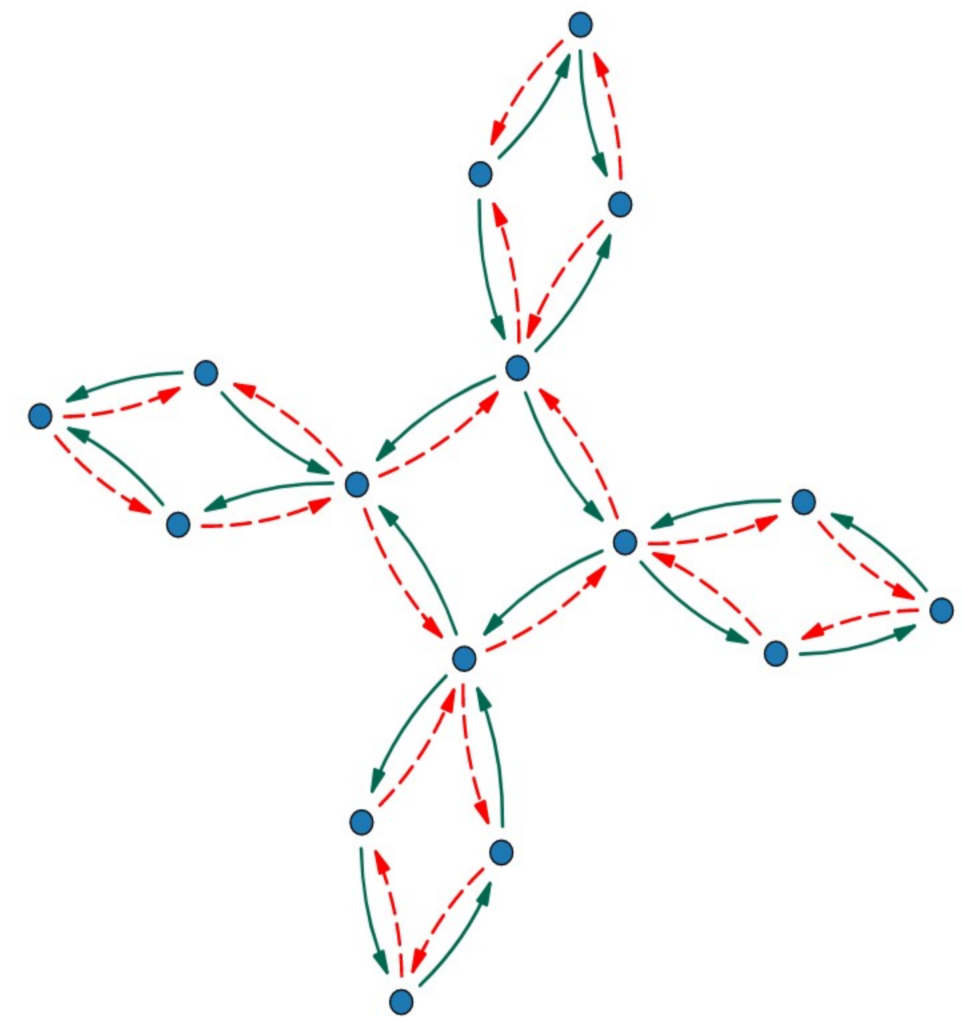
Antagonistic
Erdős-Renyi
(**SLSS**)



Mixture
Erdős-Renyi
(Not LSS)

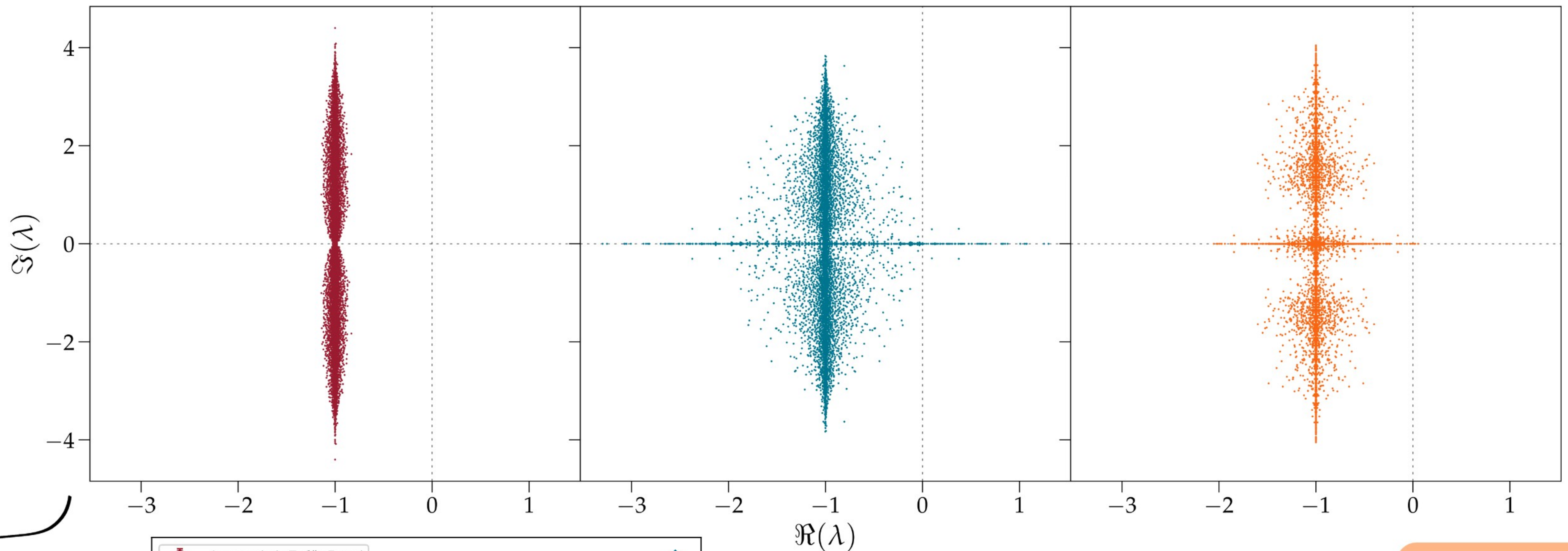


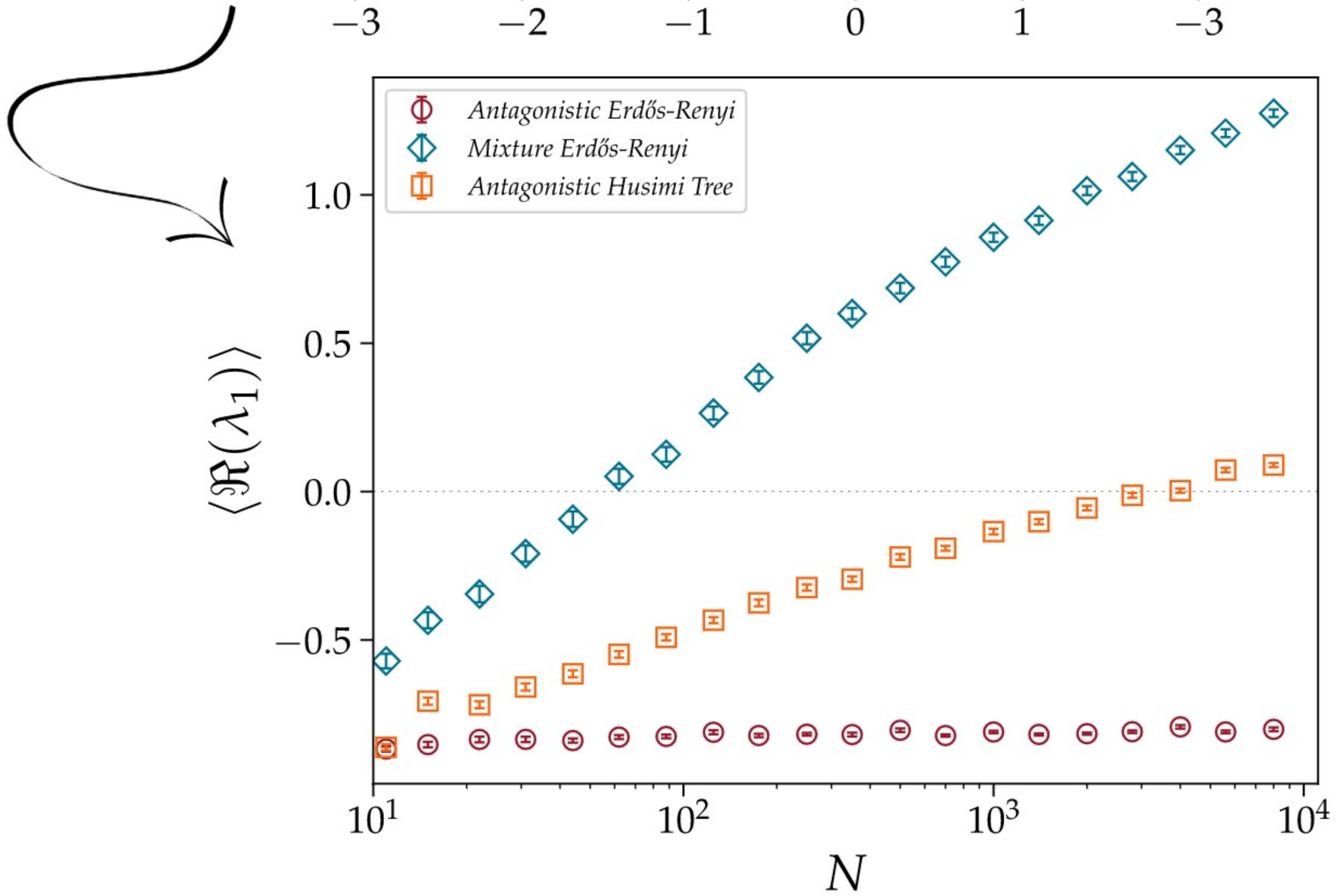
Antagonistic
Husimi Tree
(Not LSS)



We consider 3 matrix structures inspired by ecological stability

$\mathbf{B}_{\mathrm{Id}} = \mathbf{A} - \mathbf{1}$ **Shifted Interaction-like**





SLSS $\implies \lim_{N\to\infty} \langle \Re[\lambda_1] \rangle < \infty$

absolute stability

No Complexity Stability trade-off

Not LSS $\implies \lim_{N\to\infty} \langle \Re[\lambda_1] \rangle = \infty$

size-dependent stability

Conclusions

- Including nontrivial structure can bring out qualitatively new features in spectra of sparse random matrices

- Strong local sign stability enable the Complexity-Stability trade-off to be overcome and can be employed in ecosystems models

Take-Home Message

- Physicists can (try to) make a contribution to improve our understanding of ecosystems behaviour

Thank you!