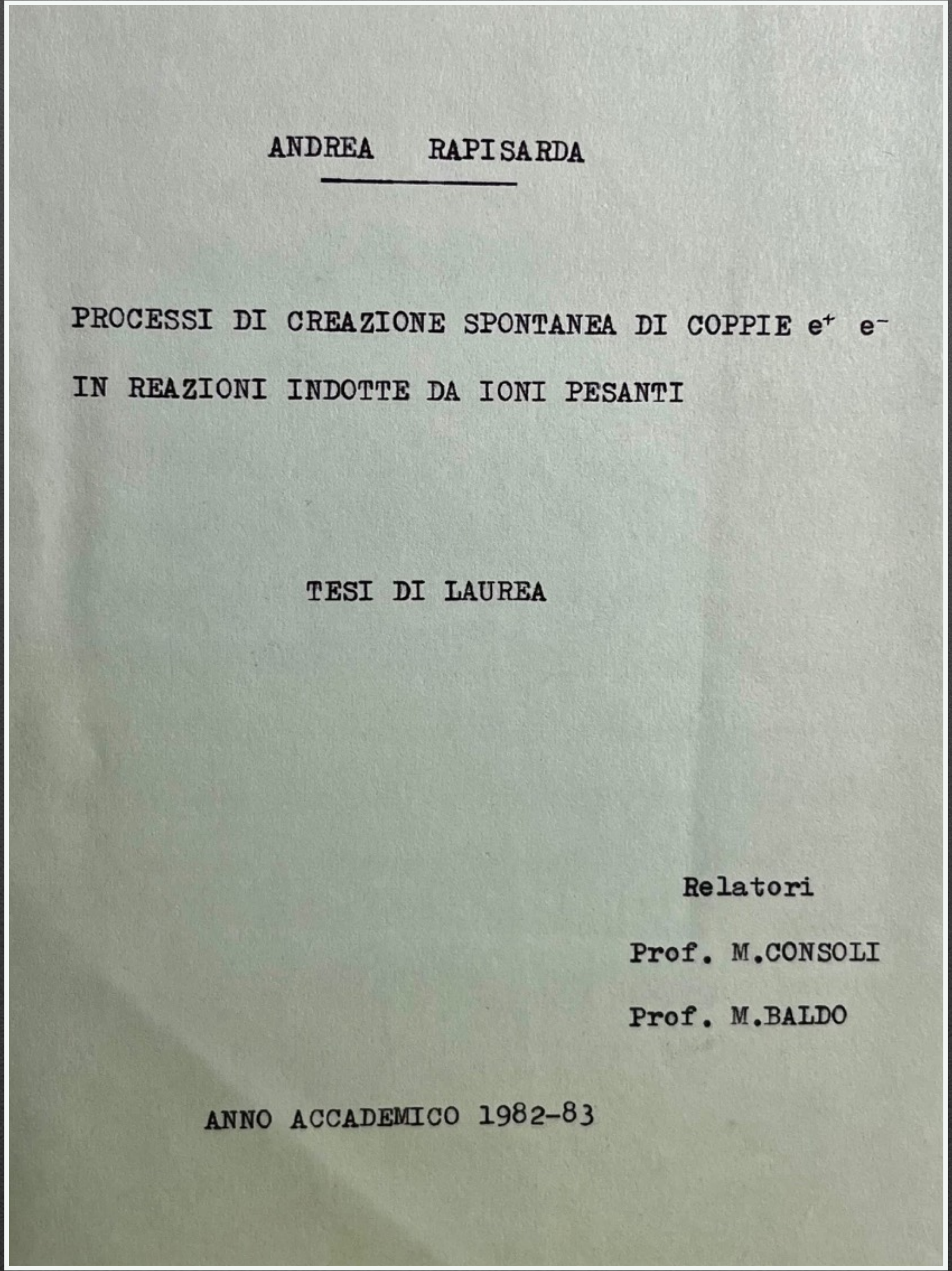
From Chaos to Complexity

Andrea Rapisarda

Dipartimento di Fisica e Astronomia "Ettore Majorana" and INFN
Università di Catania, Italy
Complexity Science Hub Vienna, Austria

Early days

My collaboration with Marcello started after my degree in physics in 1983 with frequent visits at the Niels Bohr Institute in Copenhagen



ANDREA RAPISARDA

PROCESSI DI CREAZIONE SPONTANEA DI COPPIE e^+ e^-
IN REAZIONI INDOTTE DA IONI PESANTI

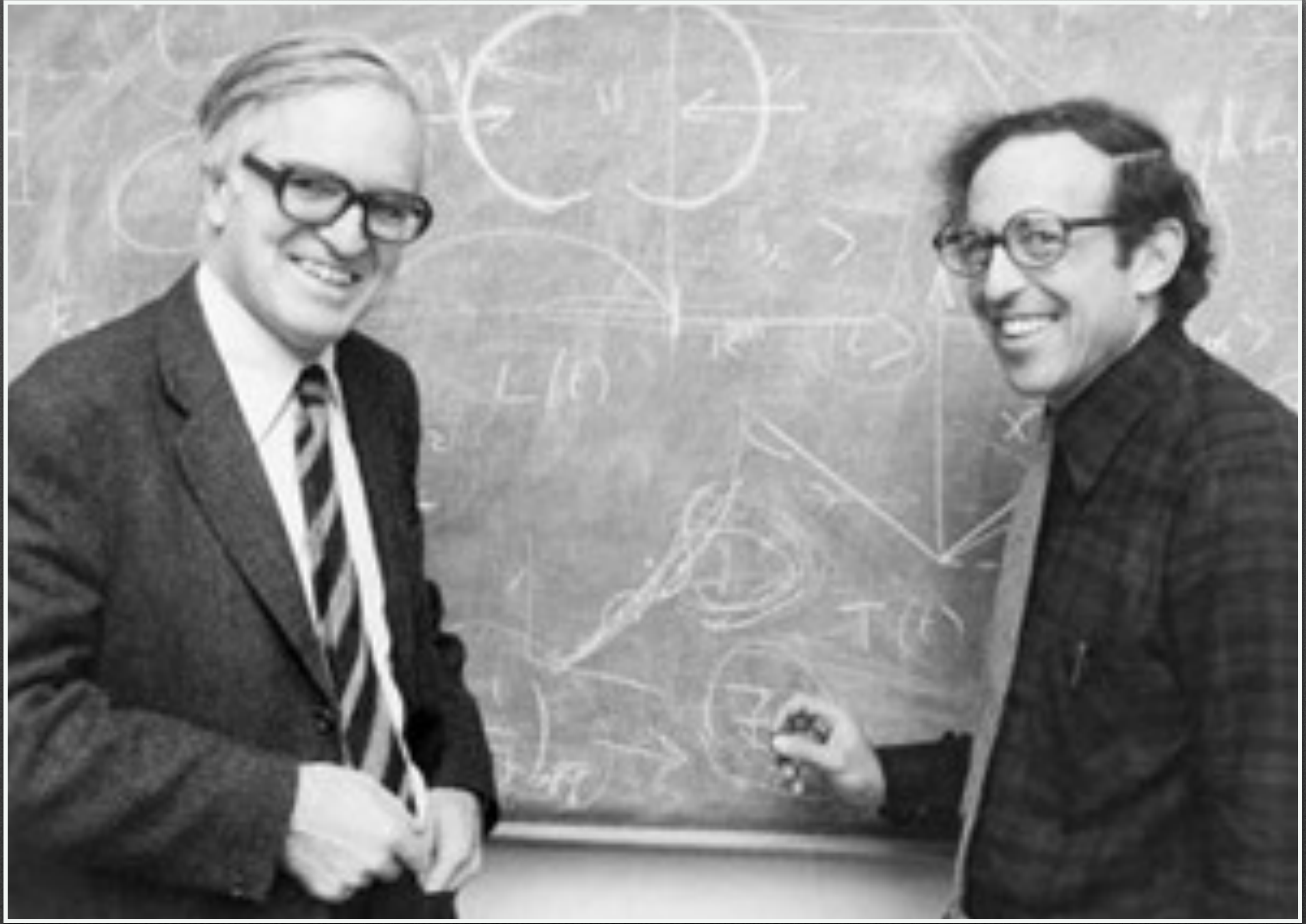
TESI DI LAUREA

Relatori
Prof. M.CONSOLI
Prof. M.BALDO

ANNO ACCADEMICO 1982-83

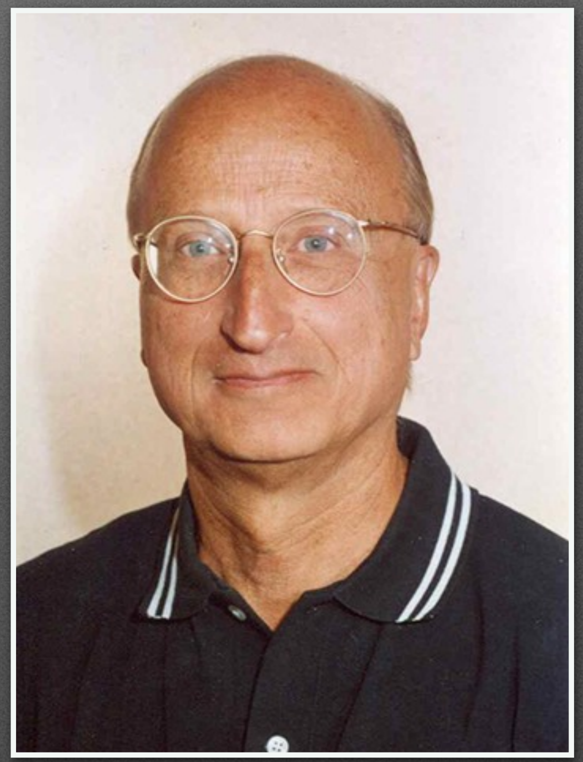
Early days

Early days



Aage Bohr and Ben Mottelson

Nobel prize winners in 1975
for their studies on the nuclear structure



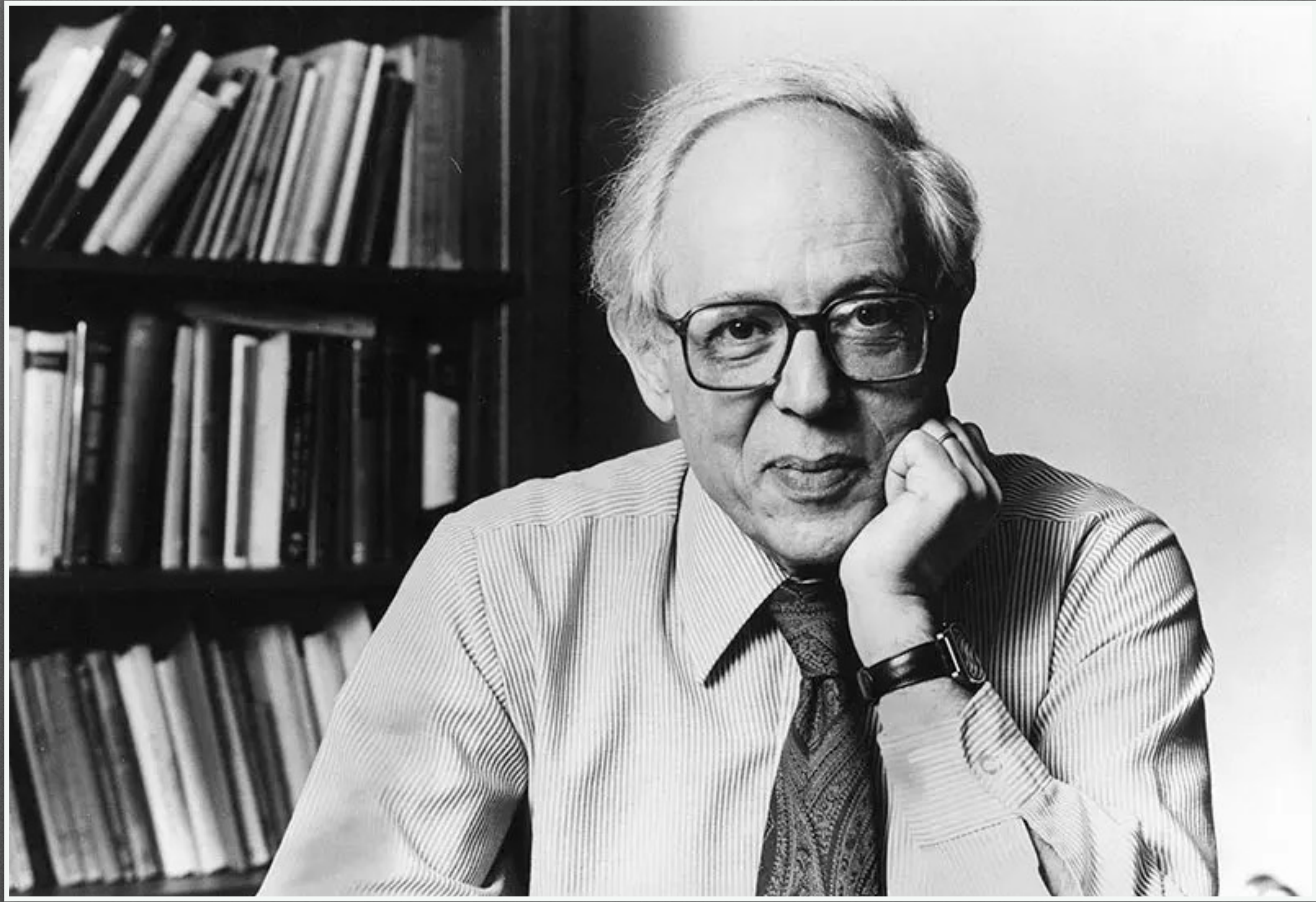
Ricardo Broglia and Aage Winther

working on heavy ion collisions

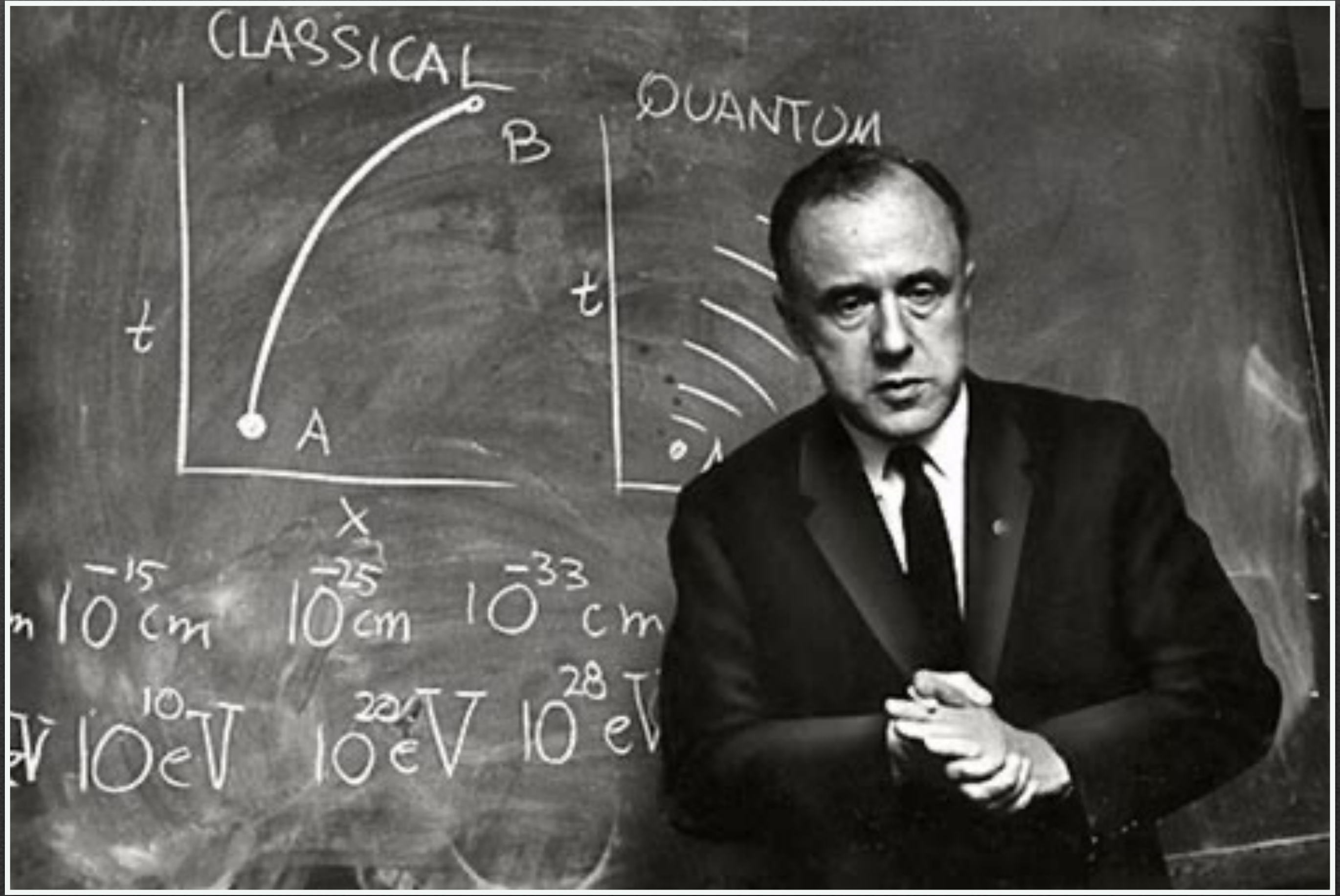
Early days

At that time NBI was a fantastic place, a sea port where you could meet very famous physicists who spent there several periods, like Abraham Pais, Sir Archibald Wheeler and many others.

It was also the only place where physicists coming from Soviet Union or East Germany could come very easily.



A. Pais



J.A. Wheeler

Early days

Life in Copenhagen - March 1987

Early days

Erice School on Nuclear Physics - October 1986

Early days

Erice School on Nuclear Physics - October 1986

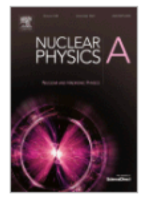
Italy *vs* Rest-of-the-world

First papers with Marcello on transfer reactions

ELSEVIER

Nuclear Physics A

Volume 472, Issue 2, 28 September 1987, Pages 333-357



Multiparticle transfer and frictional forces in heavy ion collisions

M. Baldo, A. Rapisarda, R.A. Broglia [a b], A. Winther

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Abstract

We study the frictional forces in a low-energy heavy ion collision by solving the independent particle transfer between two potentials moving on prescribed trajectories. We conclude that, although one may extract both a tangential and a radial frictional force for the initial stage of the collision, they will be different in the final stage of the reaction and they depend on the shell structure. The strong coherence of the transfer process shows up in the probability of remaining in the initial ground state, which is strongly enhanced over the result obtained from incoherent transfer. This indicates that a measurement of the absorption in the entrance channel is a sensitive measure of the effect of two-body collisions and other relaxation mechanisms.

1987

Nuclear Physics **A490** (1988) 471–484
North-Holland, Amsterdam

MICROSCOPIC THEORY OF MULTIPARTICLE TRANSFER AND OF FUSION IN THE REACTION $^{40}Ca+^{40}Ca$

M. BALDO and A. RAPISARDA

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R.A. BROGLIA

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and
INFN sez. Milano, Via Celoria 16, 20133 Milano, Italy
and
Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

A. WINTHER

Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark

Received 20 January 1988
(Revised 12 July 1988)

Abstract: The role of transfer in fusion reactions is studied within the framework of a microscopic model. In the case of the $^{40}Ca+^{40}Ca$ reaction, the coupling to transfer channels is essentially adiabati[c] and explains about one third of the observed enhancement over unrenormalized potential barrie[r] estimates. Taking into account the excitation of low-lying vibrations in the adiabatic approximatio[n] brings theory in overall agreement with the data. Predictions for transfer processes are also presente[d] which can be used to further discriminate between the potentials.

1988

PHYSICAL REVIEW C VOLUME 41, NUMBER 3 MARCH 1990

Theory of transfer reactions in peripheral heavy-ion collisions

A. Rapisarda

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and Istituto Nazionale di Fisica Nucleare, Sezione di Catania, Corso Italia 57, I-95129 Catania, Italy

M. Baldo

Istituto Nazionale di Fisica Nucleare, Sezione di Catania, Corso Italia 57, I-95129 Catania, Italy

R. A. Broglia

Dipartimento di Fisica, Università di Milano, Istituto Nazionale di Fisica Nucleare, Sezione di Milano, Via Celoria 16, I-20133 Milano, Italy
and The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

A. Winther

The Niels Bohr Institute, University of Copenhagen, DK-2100 Copenhagen Ø, Denmark

(Received 26 April 1989; revised manuscript received 5 October 1989)

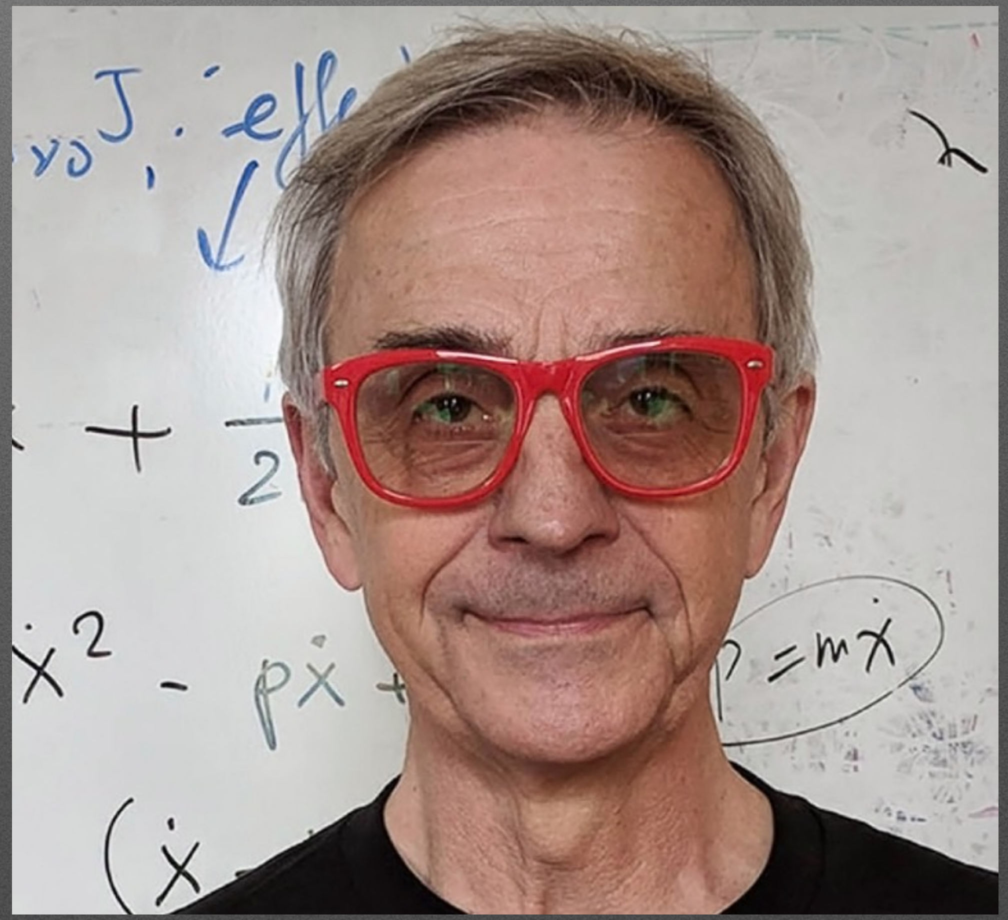
The total absorption from the elastic channel due to transfer and inelastic processes in peripheral heavy-ion collisions at low bombarding energies is calculated in a microscopic coupled-channel approach. It is demonstrated for the first time that considering the depopulation of the entrance channel as an incoherent depopulation due to transfer processes is a good approximation. Using the corresponding absorptive potential within the framework of the Born approximation to calculate the transfer to individual channels, the results of full coupled-channels calculations are accurately reproduced.

1990

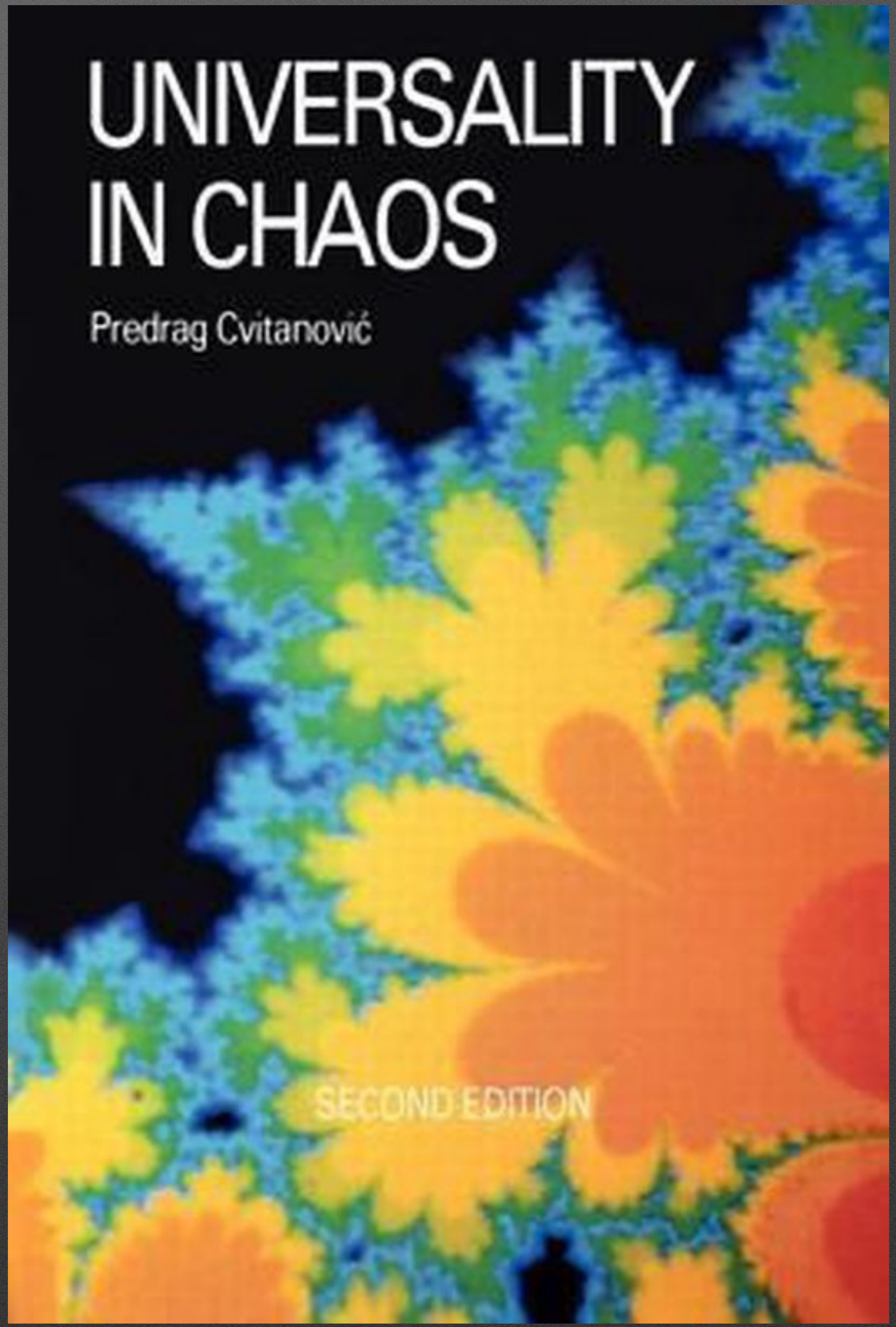
Early days

At those times, at NBI the chaos group of *Predrag Cvitanović* was very active, hosting several international guests…

and thus me and Marcello started to get interested also in *chaos theory*



P. Cvitanović



In 1991 first paper with Marcello on *Chaotic Scattering*

VOLUME 66, NUMBER 20 PHYSICAL REVIEW LETTERS 20 MAY 1991

Coexistence of Regular and Chaotic Scattering in Heavy-Ion Collisions

Andrea Rapisarda[(1),(2)] and Marcello Baldo[(2)]

[(1)]*Centro Siciliano di Fisica Nucleare e Struttura della Materia, Corso Italia 57, 95129 Catania, Italy*

[(2)]*Istituto Nazionale di Fisica Nucleare, Sezione di Catania, Corso Italia 57, 95129 Catania, Italy*

(Received 26 November 1990)

Classical dynamics of heavy-ion scattering is investigated in the case of a collision between a supposed spherical nucleus, ^{28}Si, and a deformed one, ^{24}Mg, at energies above the Coulomb barrier. Evidence of regular and irregular motion is found. The chaotic behavior justifies the presence of Ericson's fluctuations observed for this reaction, while the presence of regular motion embedded in the chaotic region could be the crucial point to explain the nature of the observed isolated resonances, once the semiclassical theory is applied.

Chaotic scattering around the Coulomb barrier

VOLUME 60, NUMBER 6 PHYSICAL REVIEW LETTERS 8 FEBRUARY 1988

Classical Irregular Scattering and Its Quantum-Mechanical Implications

R. Blümel and U. Smilansky
Max Planck Institute for Quantum Optics, 8046 Garching, Federal Republic of Germany, and Department of Nuclear Physics, The Weizmann Institute of Science, 76100 Rehovot, Israel
(Received 28 September 1987)

We analyze the effect of irregular classical scattering on the corresponding quantum-mechanical scattering matrix. Using semiclassical arguments, we show that the fluctuations in the S matrix and the cross sections are consistent with a random-matrix description (Ericson fluctuations). The results are illustrated by a numerical solution of a simple quantum problem, whose classical counterpart displays irregular scattering.

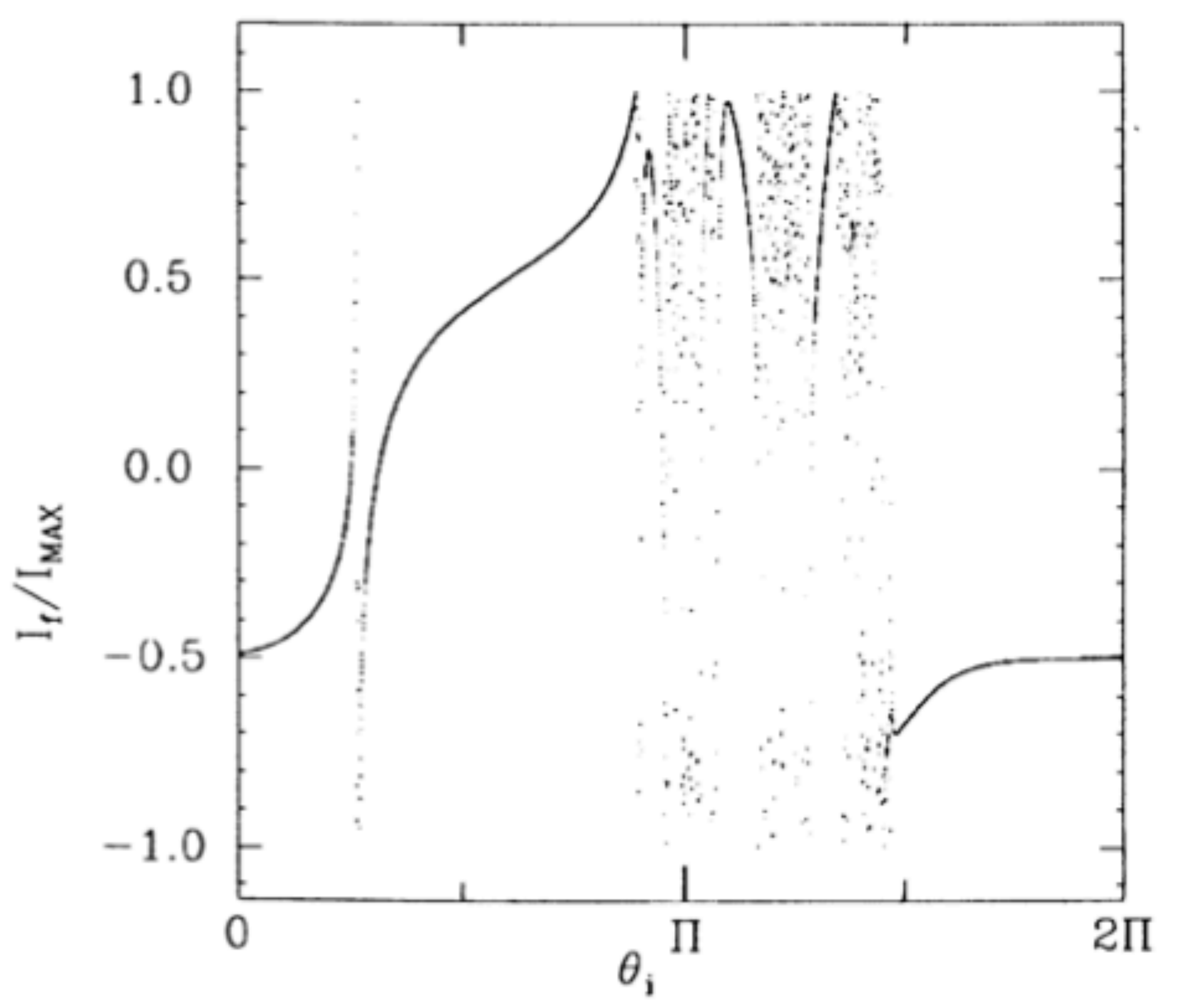
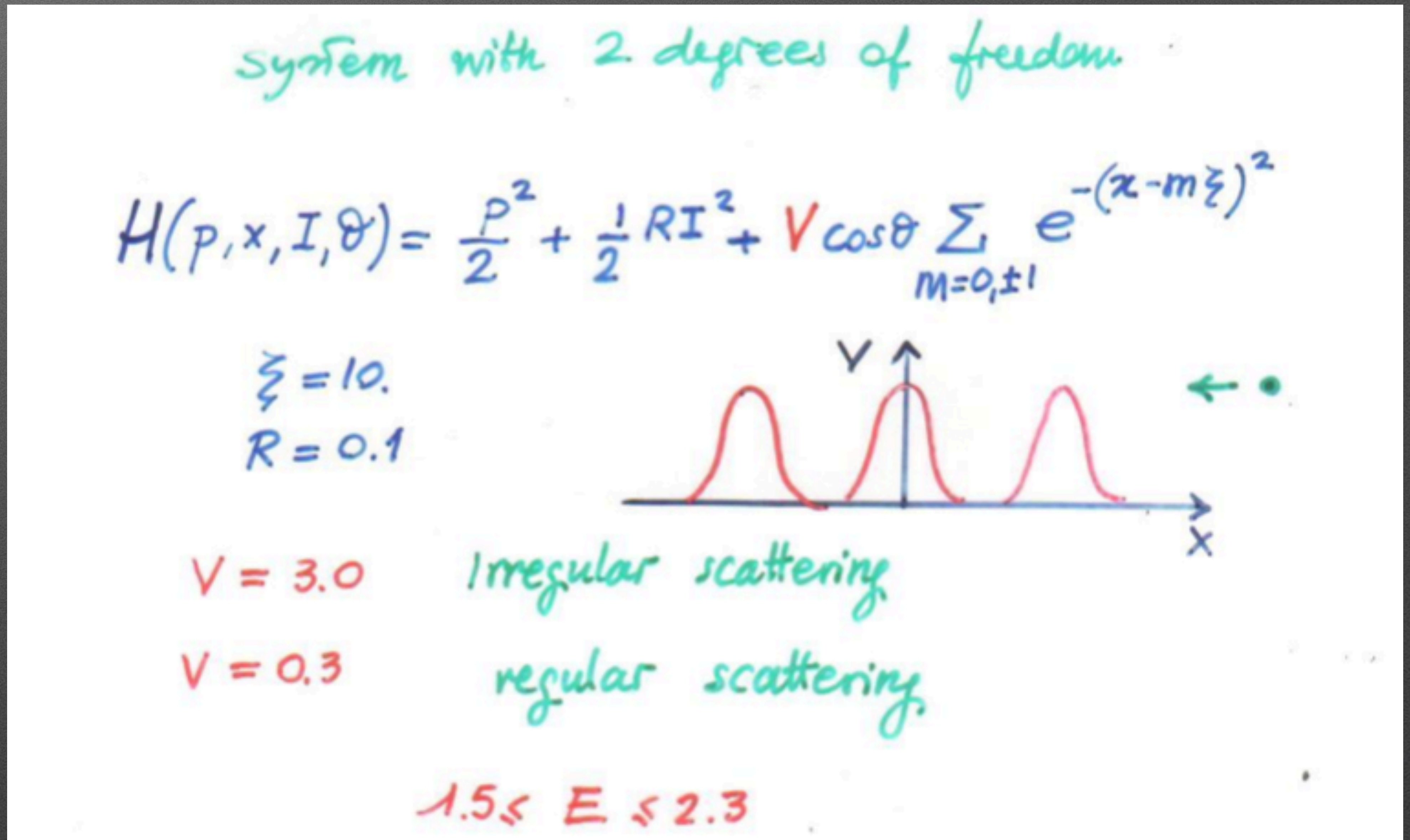
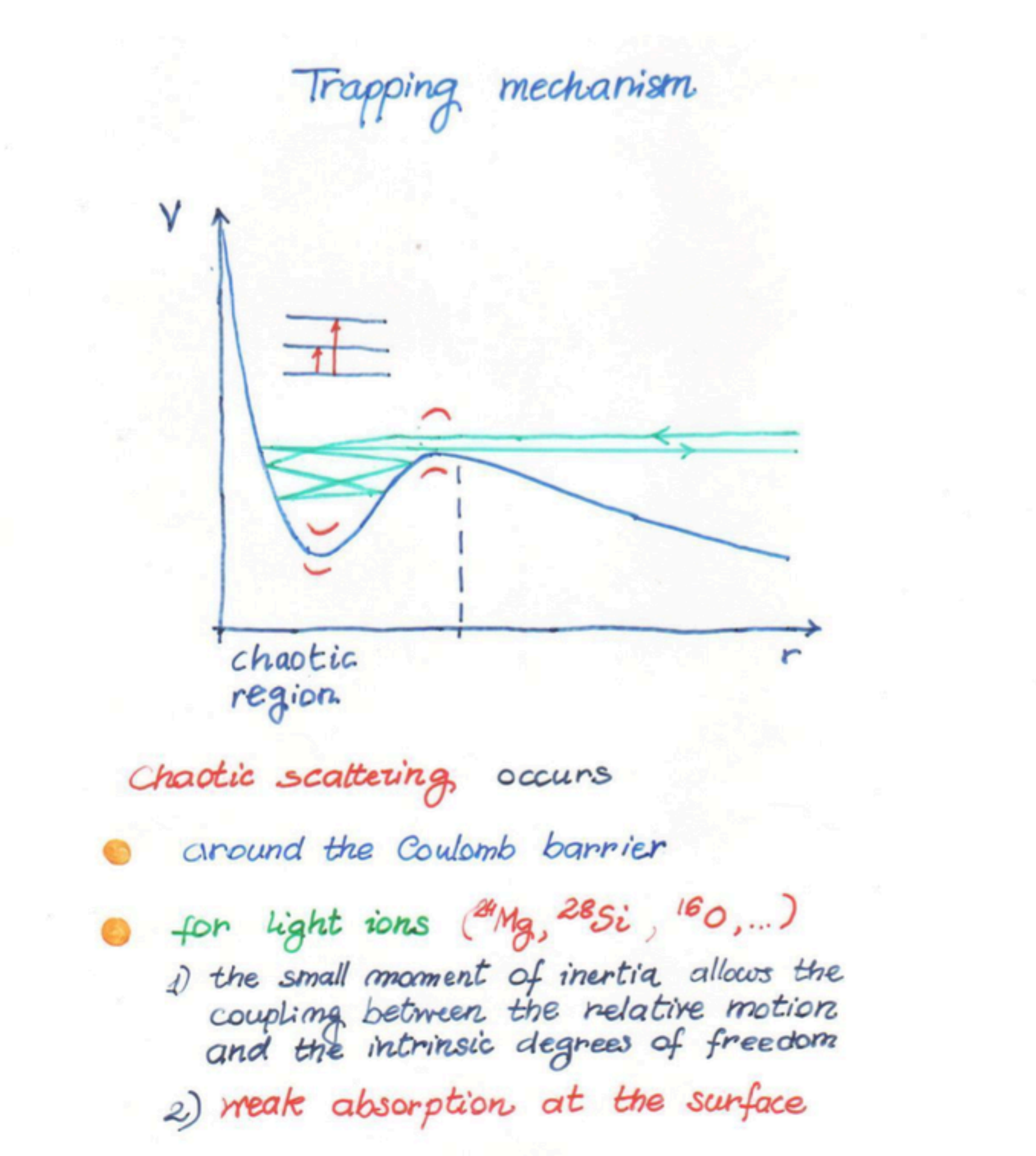
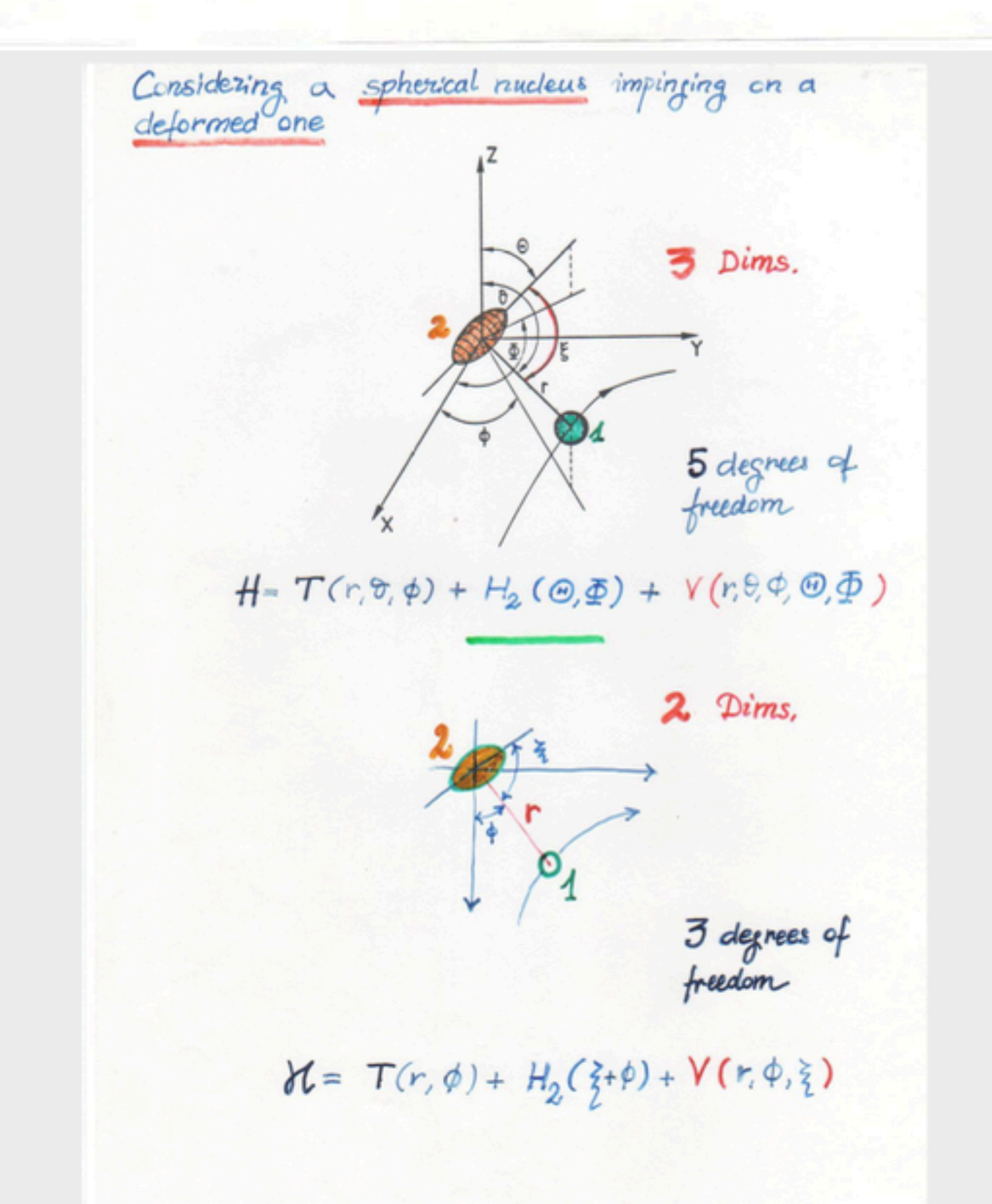


FIG. 1. The function $I_f(\theta_i, I_i = 7\hbar)$ showing IS for $E = 1.7$, $R = 0.1$, and $V = 3.0$.



Chaotic scattering around the Coulomb barrier





Realistic Nuclear effective potential considered

α_{20} is the important deformation parameter which controls chaotic behavior

Proximity potential (*)

$$V_{NUCL} = 4\pi b \gamma \mathcal{R} \, \psi(s(\xi))$$

$$\psi(s(\xi)) = \begin{cases} -\frac{1}{2}(s-2.54)^2 - 0.0852\,(s-2.54)^3 & s \leq 1.2511 \\ -3.437 \exp(-s/0.75) & s > 1.2511 \end{cases}$$

$$s(\xi) = \frac{r - R_1^0 - R_2(\xi)}{b}$$

$b = 1$ fm diffuseness parameter

$$R_i^0 = \left(1.28 A_i^{1/3} - 0.76 + 0.8 A_i^{-1/3}\right) \text{ fm} \quad i = 1, 2$$

$$R_2(\xi) = R_2^0 \left[1 + \alpha_{20} Y_{20}(\xi)\right]$$

α_{20} = deformation parameter

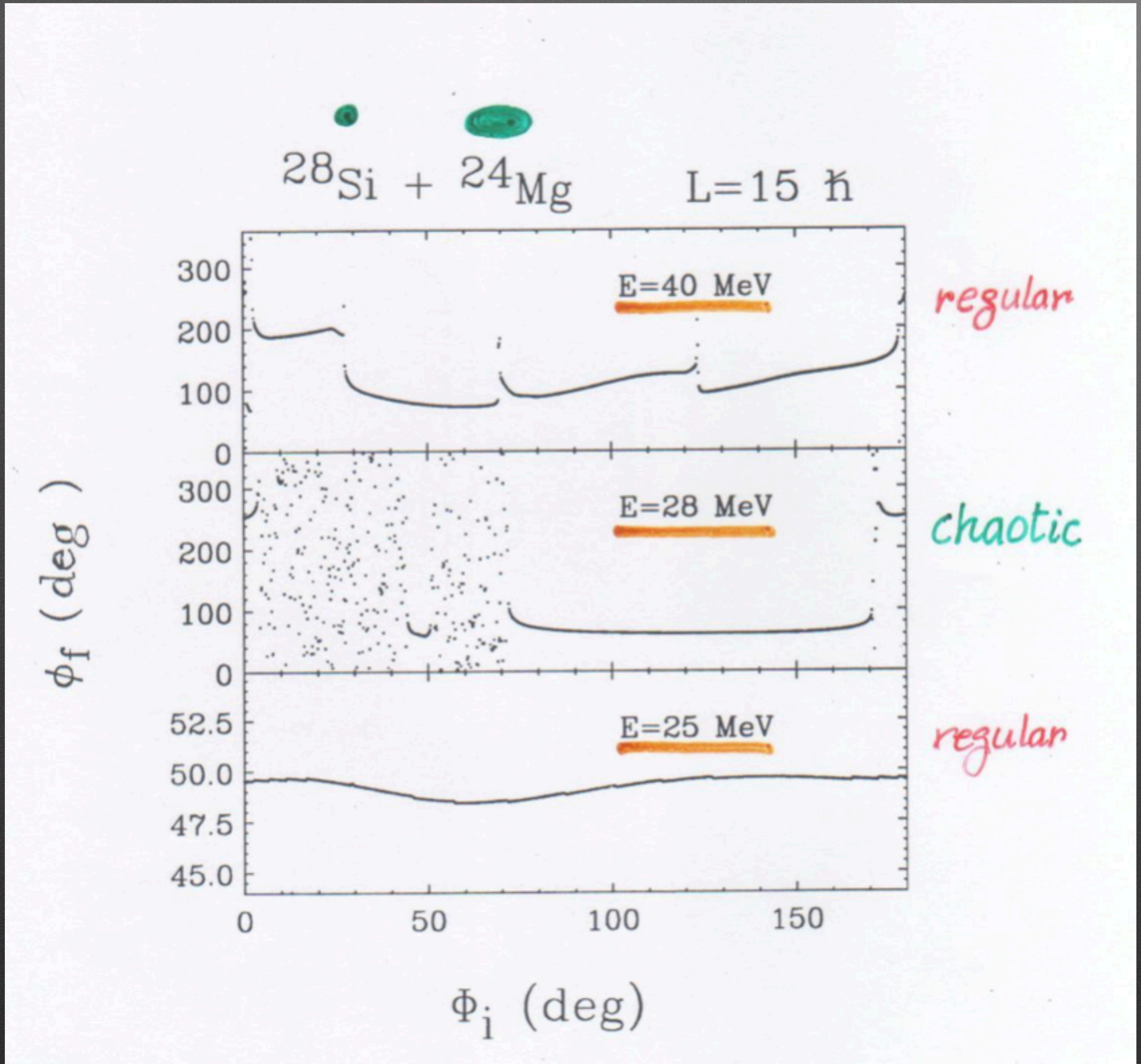
$$\mathcal{R}(\xi) = \frac{R_1^0 R_2^0}{R_1^0 + R_2^0}\left(1 - \frac{2R_i^0}{R_1^0 + R_2^0}\,\alpha_{20} Y_{20}(\xi)\right)$$

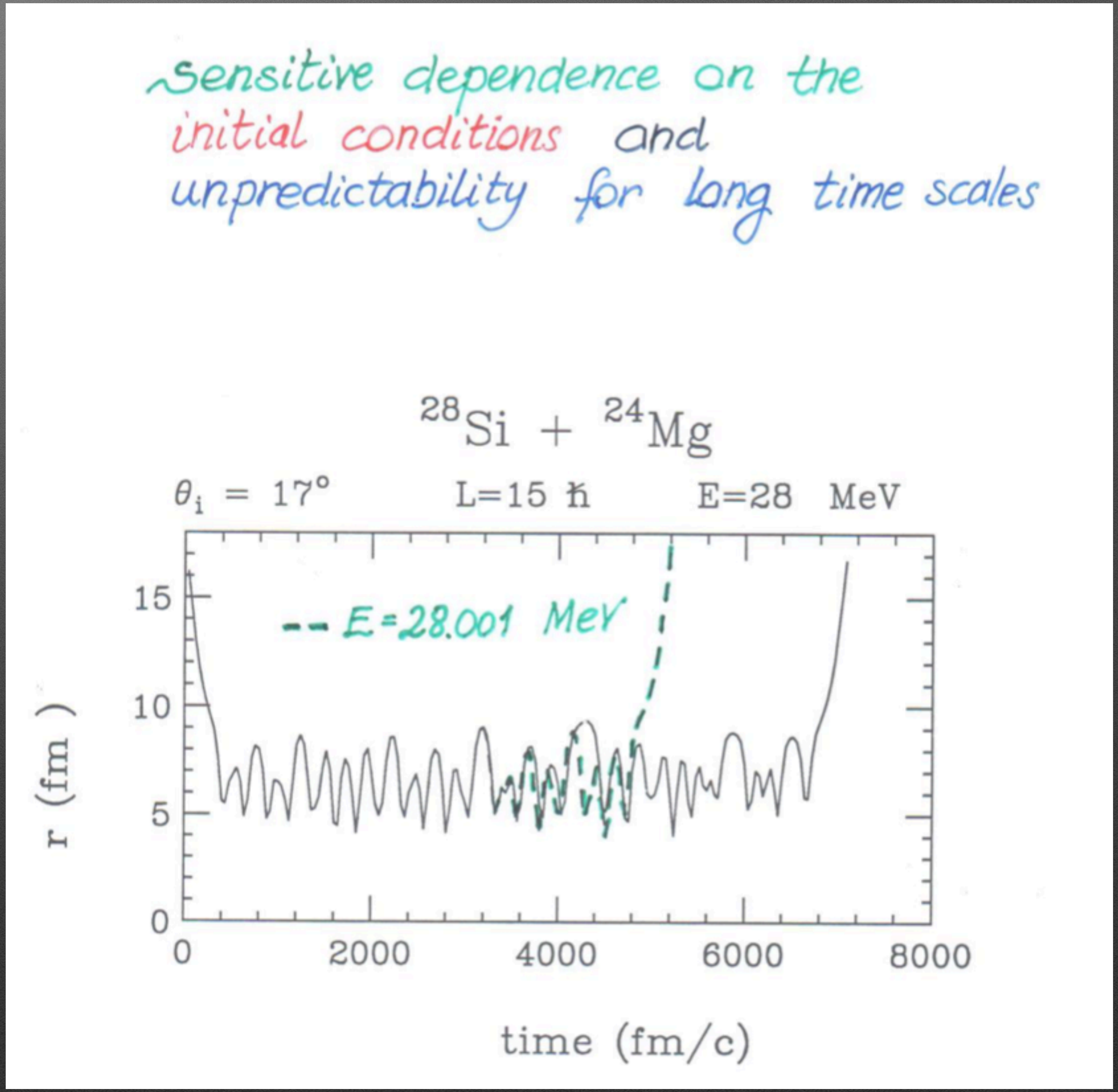
$$\gamma = 0.95\left[1 - 1.8\left(\frac{N_1 - Z_1}{A_1}\right)\left(\frac{N_2 - Z_2}{A_2}\right)\right] \text{ MeV fm}^{-2}$$

surface tension

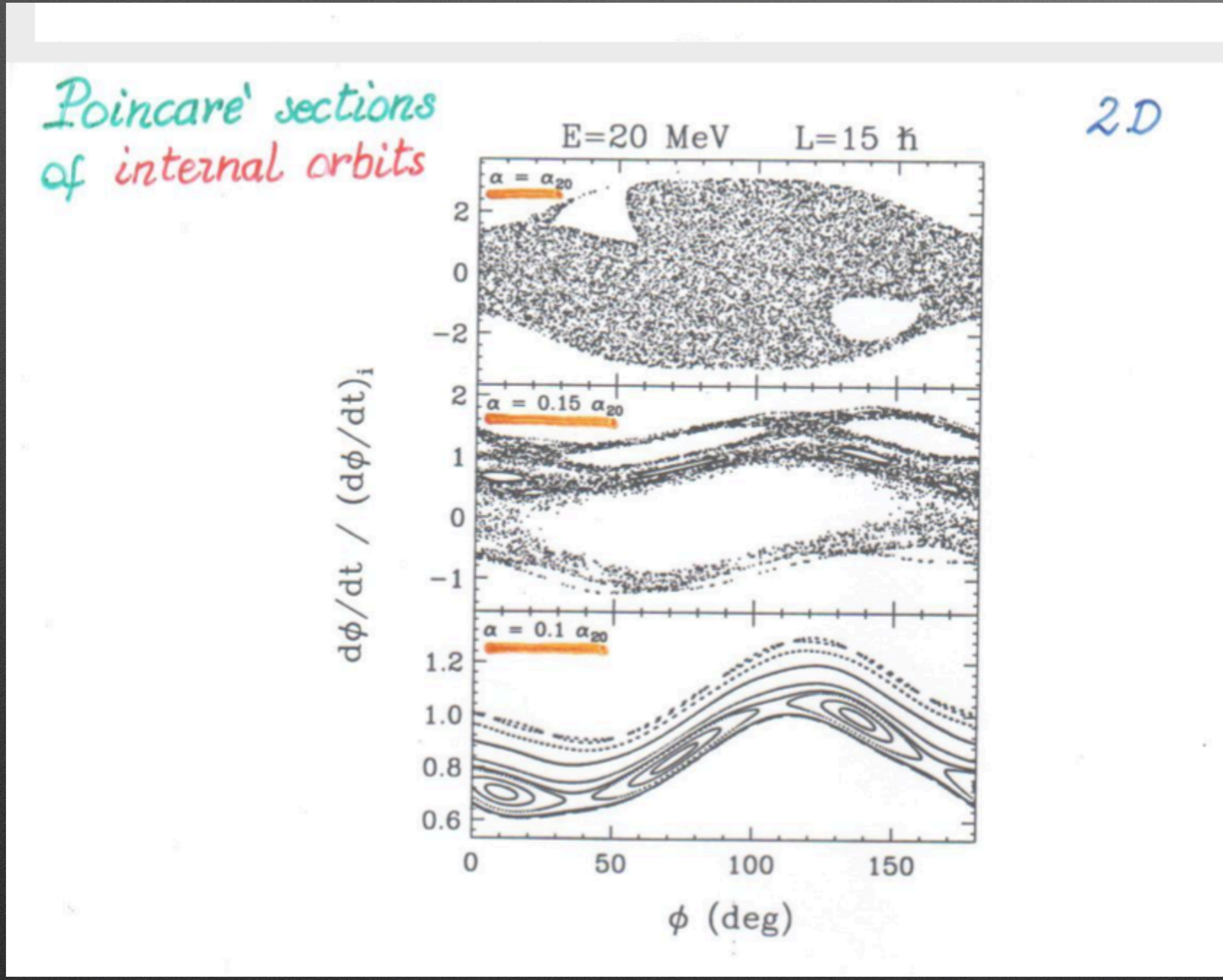
(*) Blocki, Randrup, Tsang & Swiatecki Ann. Phys. 105 (1977) 427

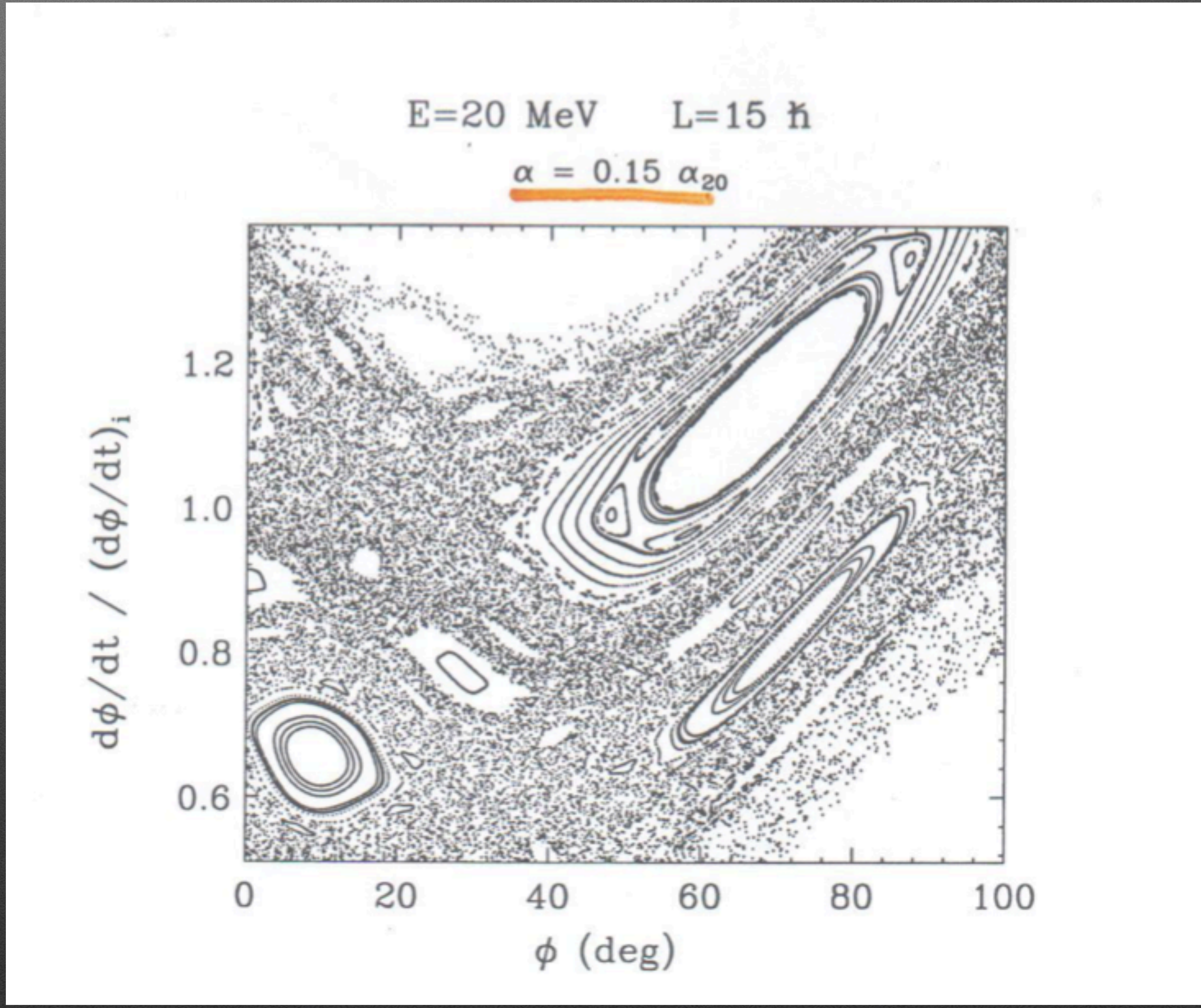
Chaotic scattering around the Coulomb barrier





Chaotic scattering around the Coulomb barrier

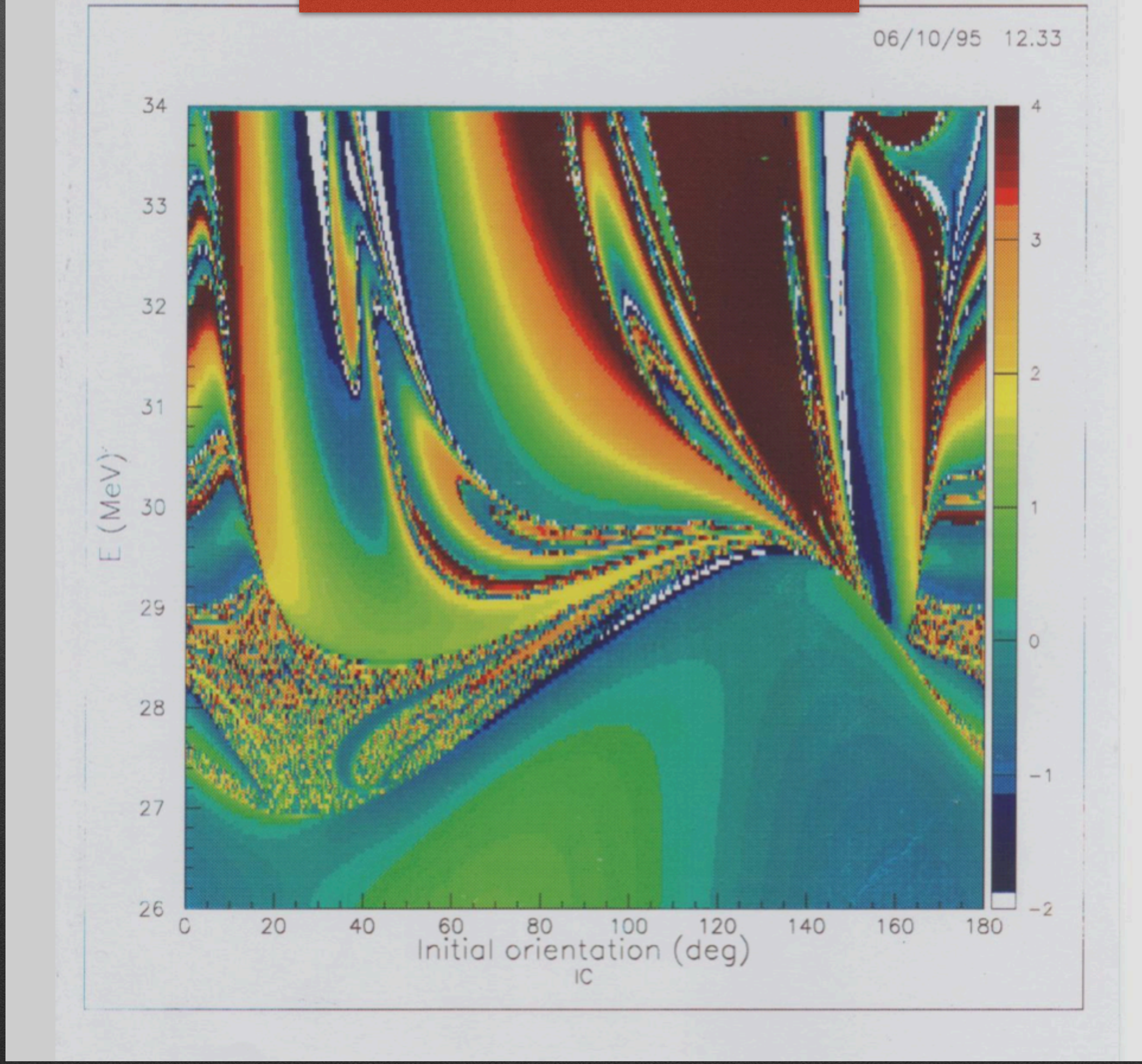




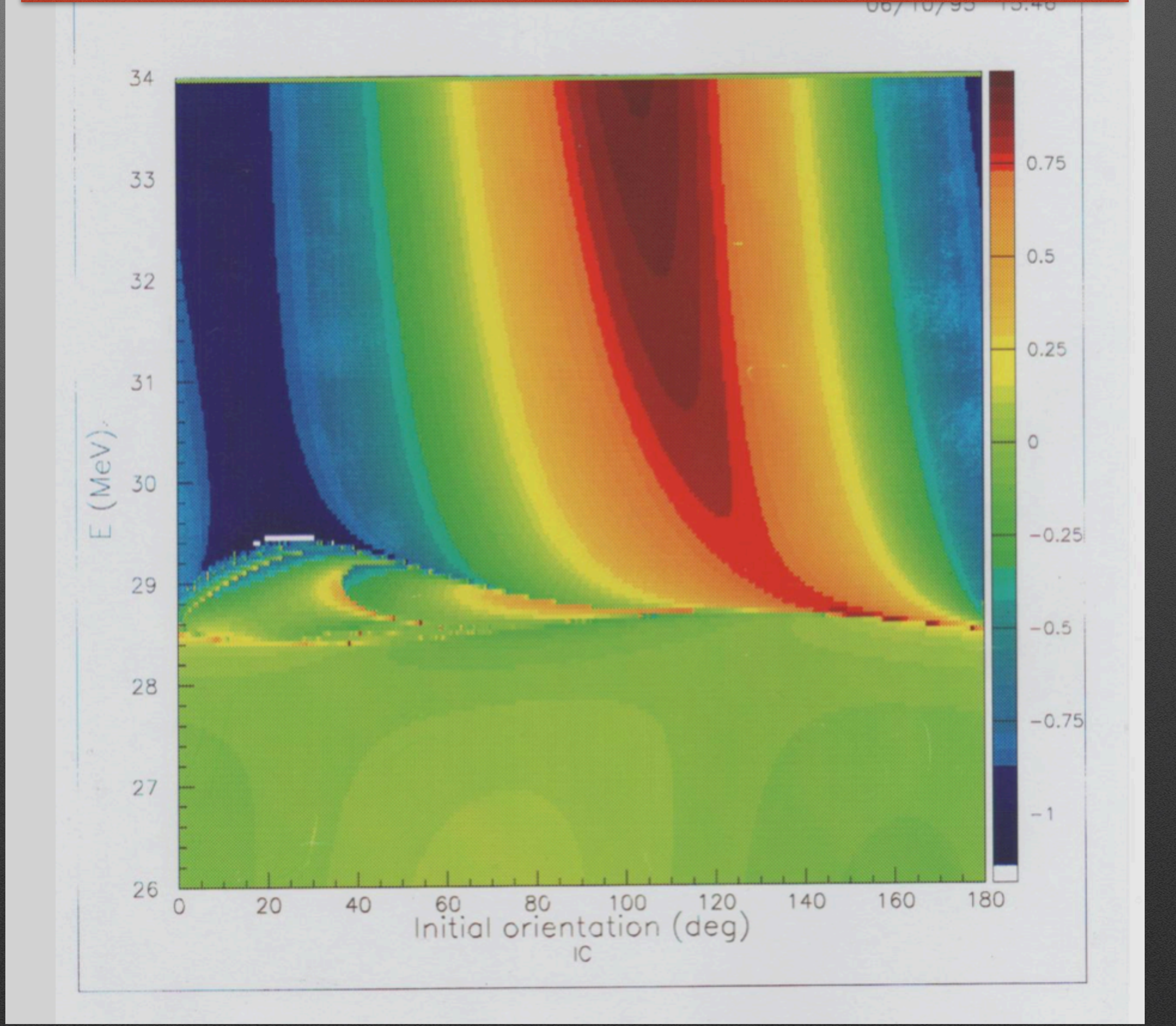
Chaotic scattering around the Coulomb barrier

$^{28}Si + ^{24}Mg$

Realistic deformation



deformation reduced to 10% of the real one



Review paper on Chaotic Scattering

Chaotic scattering in heavy-ion reactions

M. Baldo, E. G. Lanza, and A. Rapisarda
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(Received 14 June 1993; accepted for publication 13 September 1993)

We discuss the relevance of chaotic scattering in heavy-ion reactions at energies around the Coulomb barrier. A model in two and three dimensions which takes into account rotational degrees of freedom is discussed both classically and quantum mechanically. The typical chaotic features found in this description of heavy-ion collisions are connected with the anomalous behavior of several experimental data.

CHAOS **3** (4), 1993 1054-1500/93/3(4)/691/16/$6.00 691

Collaboration with the Chimera group (1995)… moving towards nuclear multi fragmentation

NUCLEAR PHYSICS A

ELSEVIER Nuclear Physics A583 (1995) 461–464

Chimera: a project of a 4Π detector for heavy ion reactions studies at intermediate energy

S.Aiello[b], A.Anzalone[a], M.Baldo[b], G.Cardella[b], S.Cavallaro[a,e], E.De Filippo[b], A.Di Pietro[a,e], S.Feminò[c,f], P.Figuera[a], P.Guazzoni[d,g], C.Iacono-Manno[a], G.Lanzanò[b], U.Lombardo[b,e], S.Lo Nigro[b,e], A.Musumarra[a,e], A.Pagano[b], M.Papa[b], S.Pirrone[b], G.Politi[b,e], F.Porto[a,e], A.Rapisarda[b], F.Rizzo[a,e], S.Sambataro[b,e], M.L.Sperduto[a,e], C.Sutera[b], L.Zetta[d,g].

INFN: [a]Lab. Naz. del Sud, [b]Sez. di Catania and [c]Gruppo coll. di Messina, [d]Sez. di Milano

Dip. di Fisica: [e]Univ. di Catania,[f]Univ. di Messina,[g]Univ. di Milano

NUCLEAR PHYSICS A

ELSEVIER Nuclear Physics A583 (1995) 343–346

Beyond linear response theory in multifragmentation

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[a]INFN sez. di Catania and Dipartimento di Fisica Universitá di Catania, Corso Italia 57, I-95129 Catania, Italy

Within the framework of the Vlasov equation, we discuss the validity of linear response theory in the dynamics of fragment formation. Considering a hot piece of nuclear matter inside the spinodal zone, we demonstrate by numerical simulations that after the first stages of the time evolution, nonlinear terms become important and cannot be neglected. Nonlinear and chaotic dynamics seems to characterize multifragmentation occurring in heavy-ion collisions.

PHYSICAL REVIEW C VOLUME 58, NUMBER 4 OCTOBER 1998

Generalized entropy and temperature in nuclear multifragmentation

A. Atalmi,* M. Baldo,† G. F. Burgio,‡ and A. Rapisarda§

Istituto Nazionale di Fisica Nucleare, Sezione di Catania and Dipartimento di Fisica, Università di Catania, Corso Italia 57, I-95129 Catania, Italy

(Received 2 February 1998)

Chaos and Phase transitions

VOLUME 80, NUMBER 4 PHYSICAL REVIEW LETTERS 26 JANUARY 1998

Lyapunov Instability and Finite Size Effects in a System with Long-Range Forces

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Andrea Rapisarda†

Istituto Nazionale di Fisica Nucleare, Sezione di Catania and Dipartimento di Fisica, Universitá di Catania, Corso Italia 57, I-95129 Catania, Italy

Stefano Ruffo‡

Centro Internacional de Ciencias, Cuernavaca, Morelos, Mexico

(Received 29 July 1997; revised manuscript received 29 October 1997)

We study the largest Lyapunov exponent λ and the finite size effects of a system of N fully coupled classical particles, which shows a second order phase transition. Slightly below the critical energy density U_c, λ shows a peak which persists for very large N values ($N = 20\,000$). We show, both numerically and analytically, that chaoticity is strongly related to kinetic energy fluctuations. In the limit of small energy, λ goes to zero with an N-independent power law: $\lambda \sim \sqrt{U}$. In the continuum limit the system is integrable in the whole high temperature phase. More precisely, the behavior $\lambda \sim N^{-1/3}$ is found numerically for $U > U_c$ and justified on the basis of a random matrix approximation. [S0031-9007(97)05121-1]

The Hamiltonian Mean Field Model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i,j=1}^{N} [1 - \cos(\vartheta_i - \vartheta_j)]$$

Antoni and Ruffo PRE 52 (1995) 2361

- The system has an infinite range force

- It is a useful paradigmatic model to study Hamiltonian long-range interacting (nonextensive) systems as for example astrophysical systems, but also fragmenting nuclei and atomic clusters

The Hamiltonian Mean Field Model

The model can be seen as N classical interacting spins or particles moving on the unit circle. One can define the total magnetization **M** as an **order parameter**

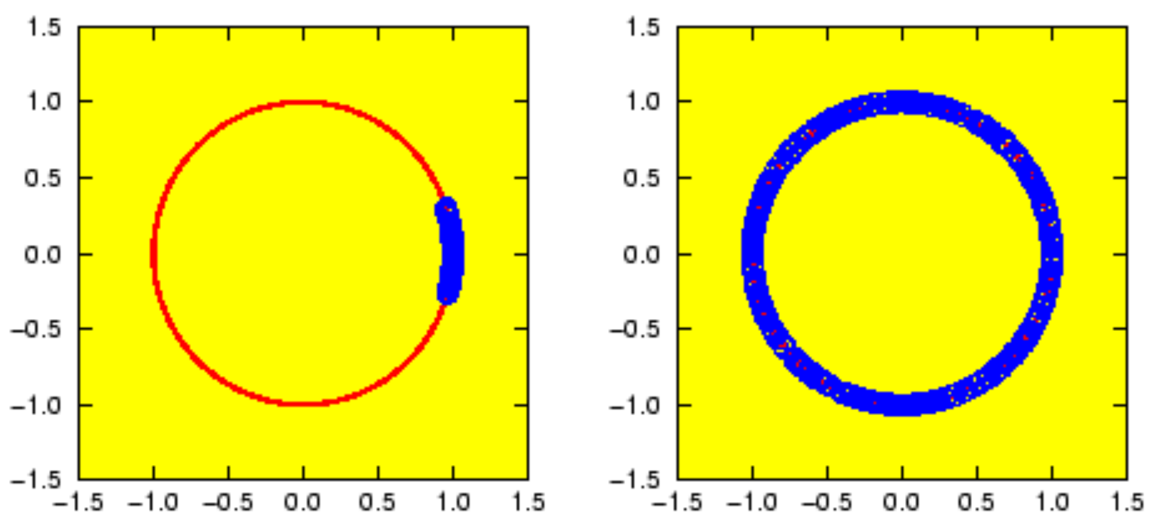
$$\vec{M} = \frac{1}{N}\sum_{i=1}^{N}\vec{m}_i$$

where the single spin is

$$\vec{m}_i = (\cos\vartheta_i, \sin\vartheta_i)$$

M=1 clustered phase for $U<U_c$

M=0 homogeneous phase for $U>U_c$



The model shows a ***second-order phase transition***, passing from a clustered phase to a homogeneous one as a function of energy

The Hamiltonian Mean Field Model

Critical behavior of the model

The model has a second order phase transition.
The critical point is at

$U_c = \frac{3}{4}$ and $T_c = \frac{1}{2}$

Close to the critical point one gets for $\beta \sim \beta_c$

$$M \approx \frac{4}{\beta}\sqrt{\frac{1}{2} - \frac{1}{\beta}}$$

$$U \approx \frac{1}{2\beta}\left[1 - \frac{8(\beta - 2)}{\beta}\right] + \frac{1}{2}$$

Hence M vanishes with the classical critical mean field exponent 1/2

On the other hand, the specific heat $C_V = \frac{\partial U}{\partial T}$ is

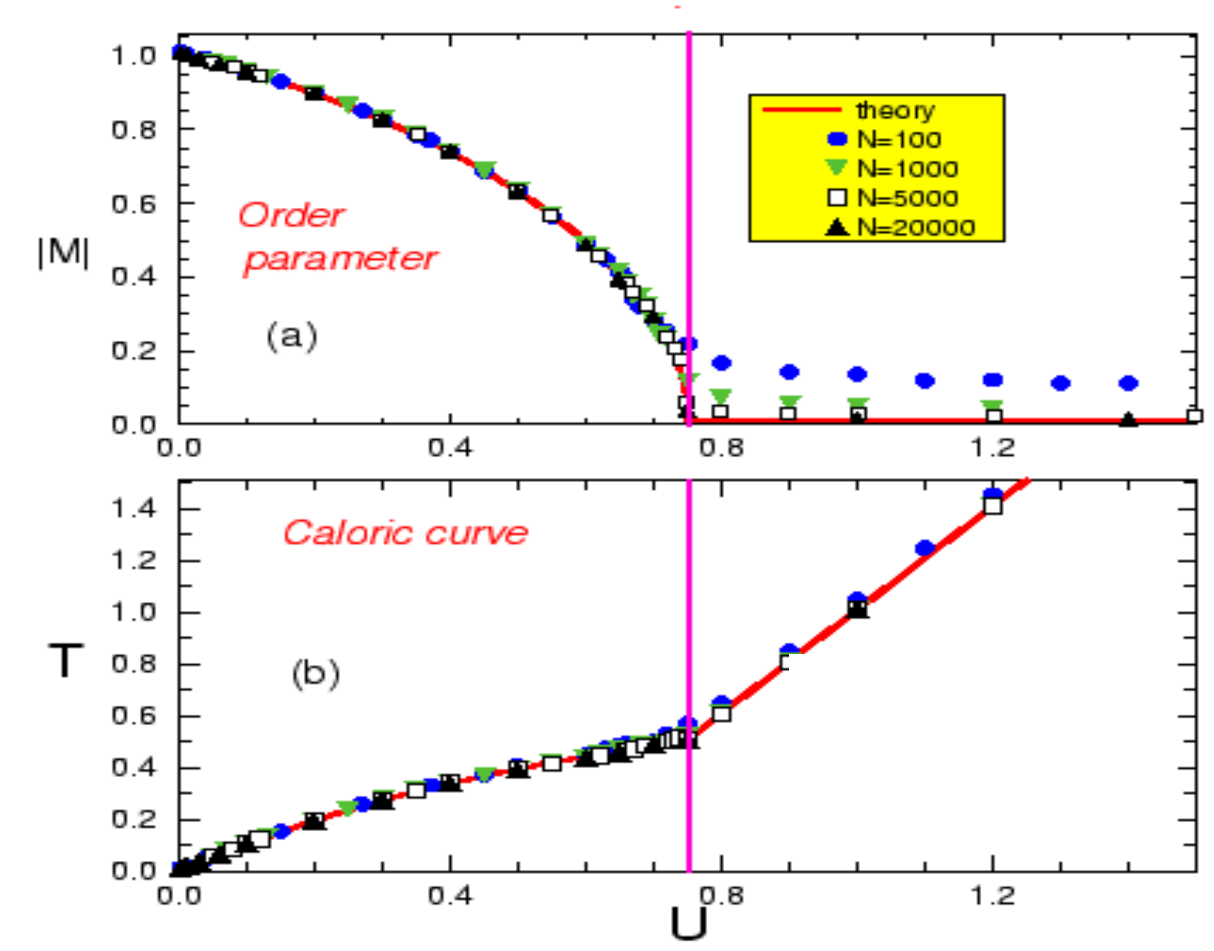
$C_V(T_c) = \frac{5}{2}$ and $C_V = \frac{1}{2}$ for $T > T_c$

Close to the critical point $C_V \approx (T_c - T)^{-\alpha}$ with $\alpha = 0$

The Hamiltonian Mean Field Model

Comparison with numerical simulations at equilibrium

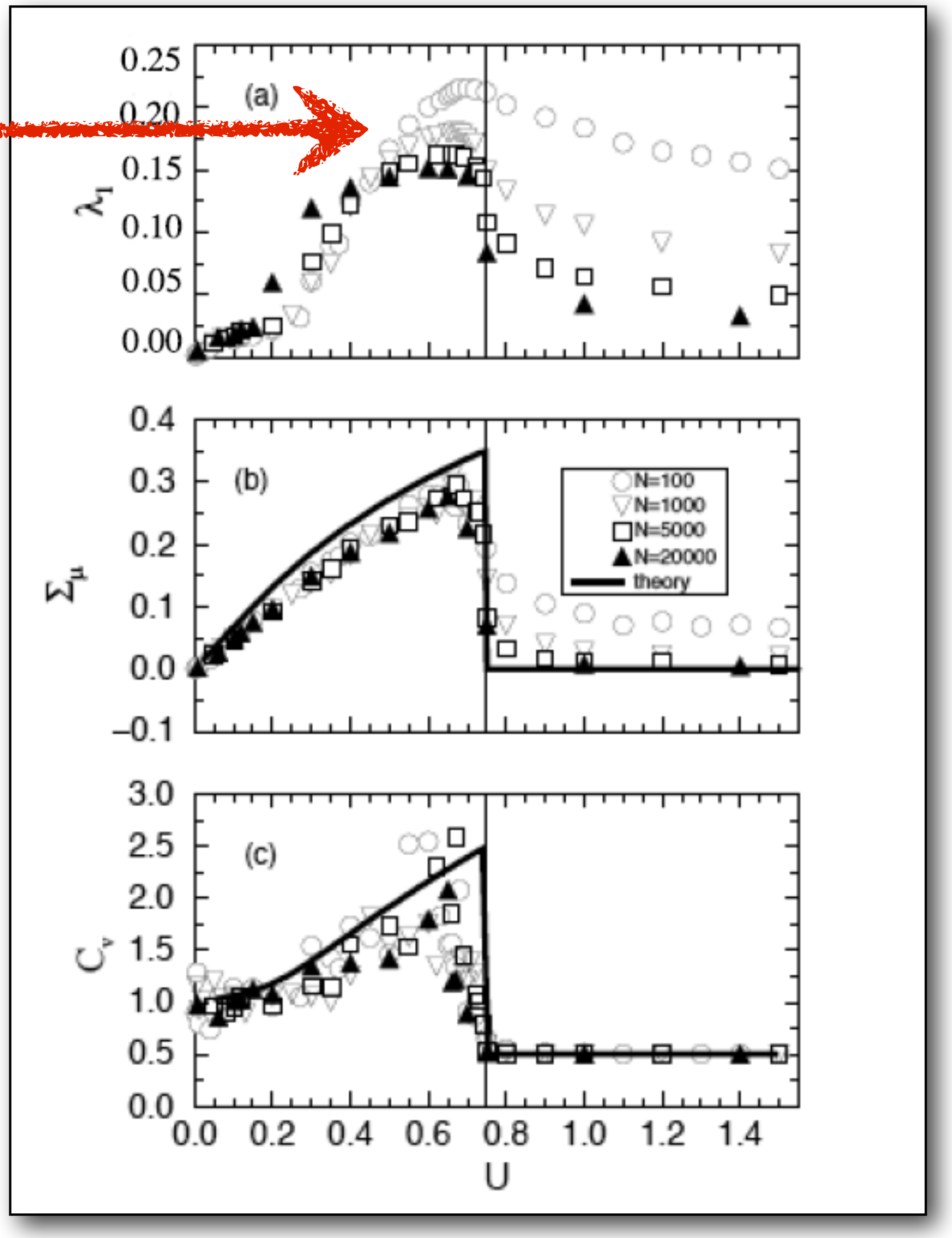
Good agreement between exact canonical solution and numerical microcanonical simulations at equilibrium for various sizes N of the system



Latora, Rapisarda and Ruffo PRL **80** (1998) 692,
Physica D **131** (1999) 38

The Hamiltonian Mean Field Model

One finds a maximum of the Largest Lyapunov Exponent in correspondence of the critical point, where fluctuations in kinetic energy and the specific heat have also a peak!



Latora, Rapisarda and Ruffo
Physica D **131** (1999) 38

The Hamiltonian Mean Field Model

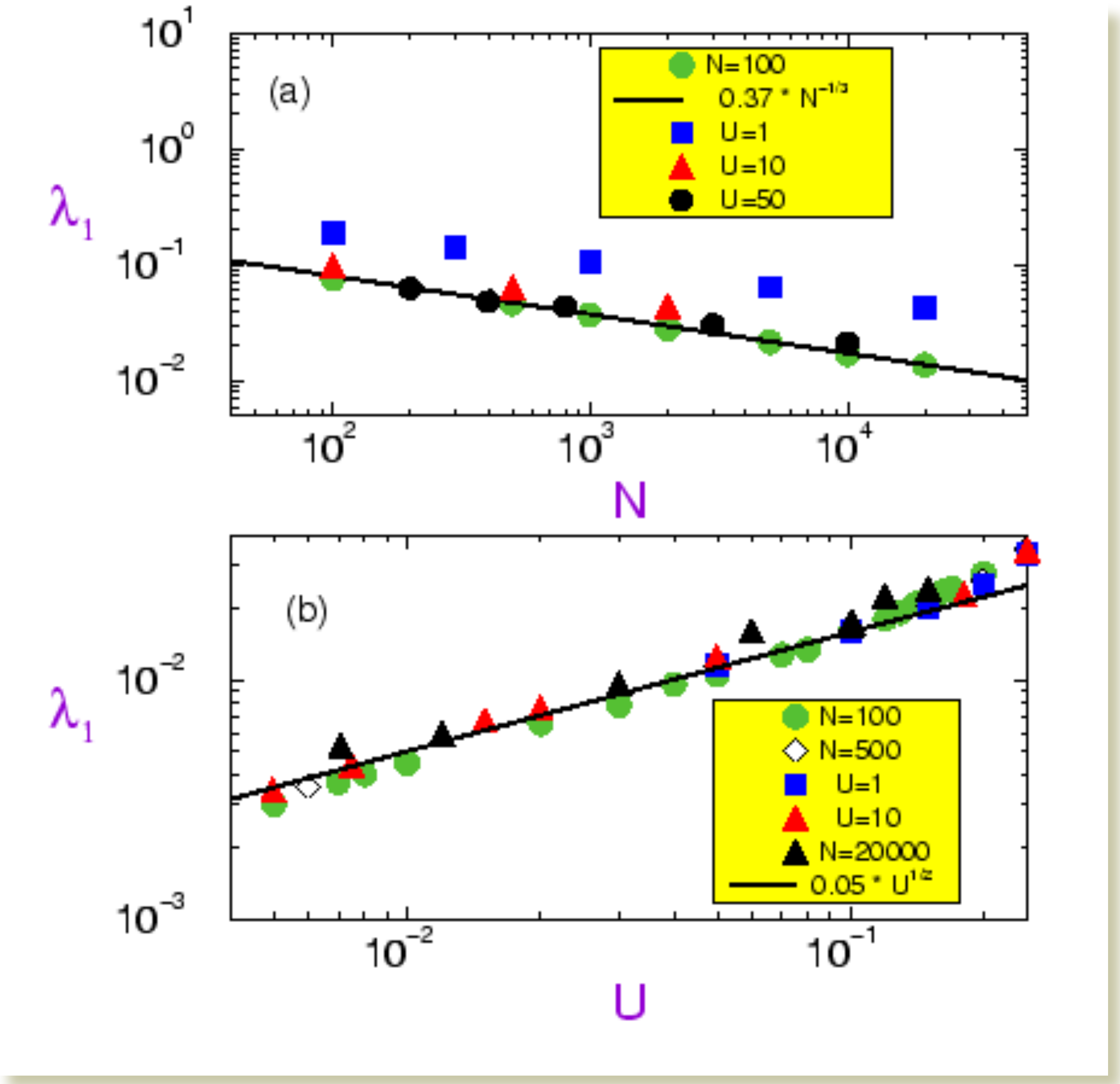
Dynamics at Equilibrium: scaling of the LLE

$$\lambda_1 \propto N^{-1/3}$$

for $U > U_c$

$$\lambda_1 \propto U^{1/2}$$

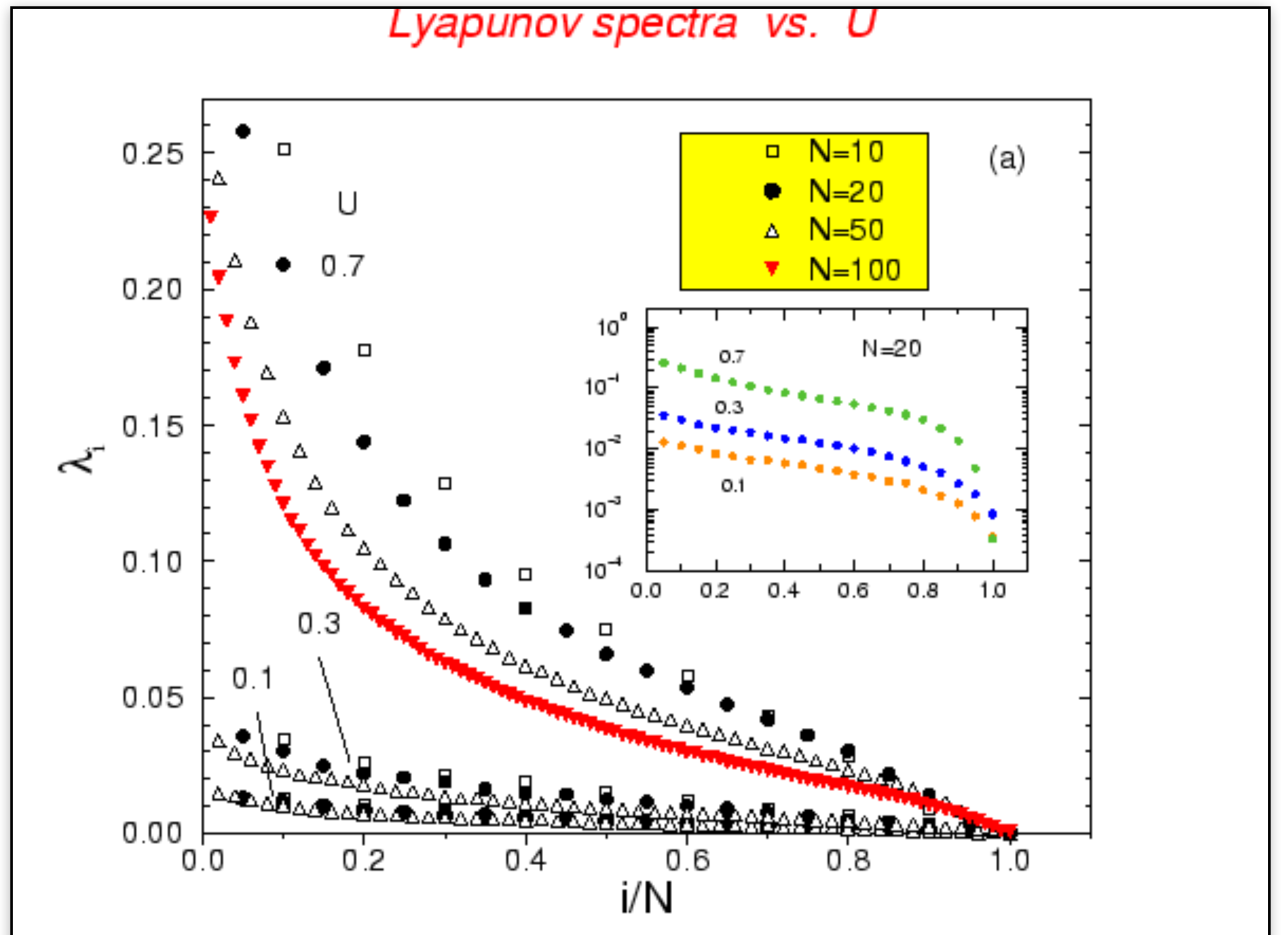
for $U << U_c$



The Hamiltonian Mean Field Model

Lyapunov spectra at Equilibrium

In Hamiltonian systems with N degrees of freedom



(positive part only)

$$\lambda_i = -\lambda_{2N-i+1}$$

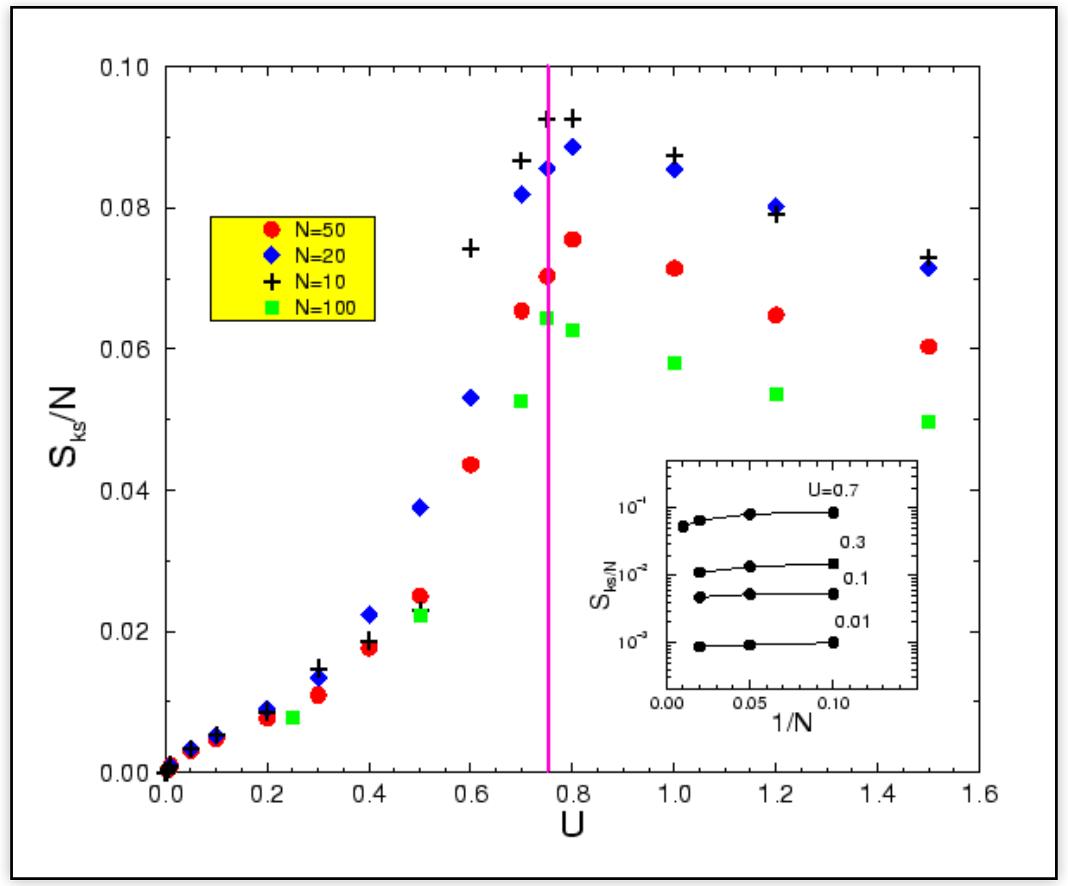
At low energy only a few degrees of freedom are active.

The Hamiltonian Mean Field Model

Kolmogorov Sinai entropy

$$S_{KS} = \sum_{i=1}^{N} \lambda_i \quad \text{with} \quad \lambda_i > 0$$

A peak close to the critical point is found also for S_{KS}



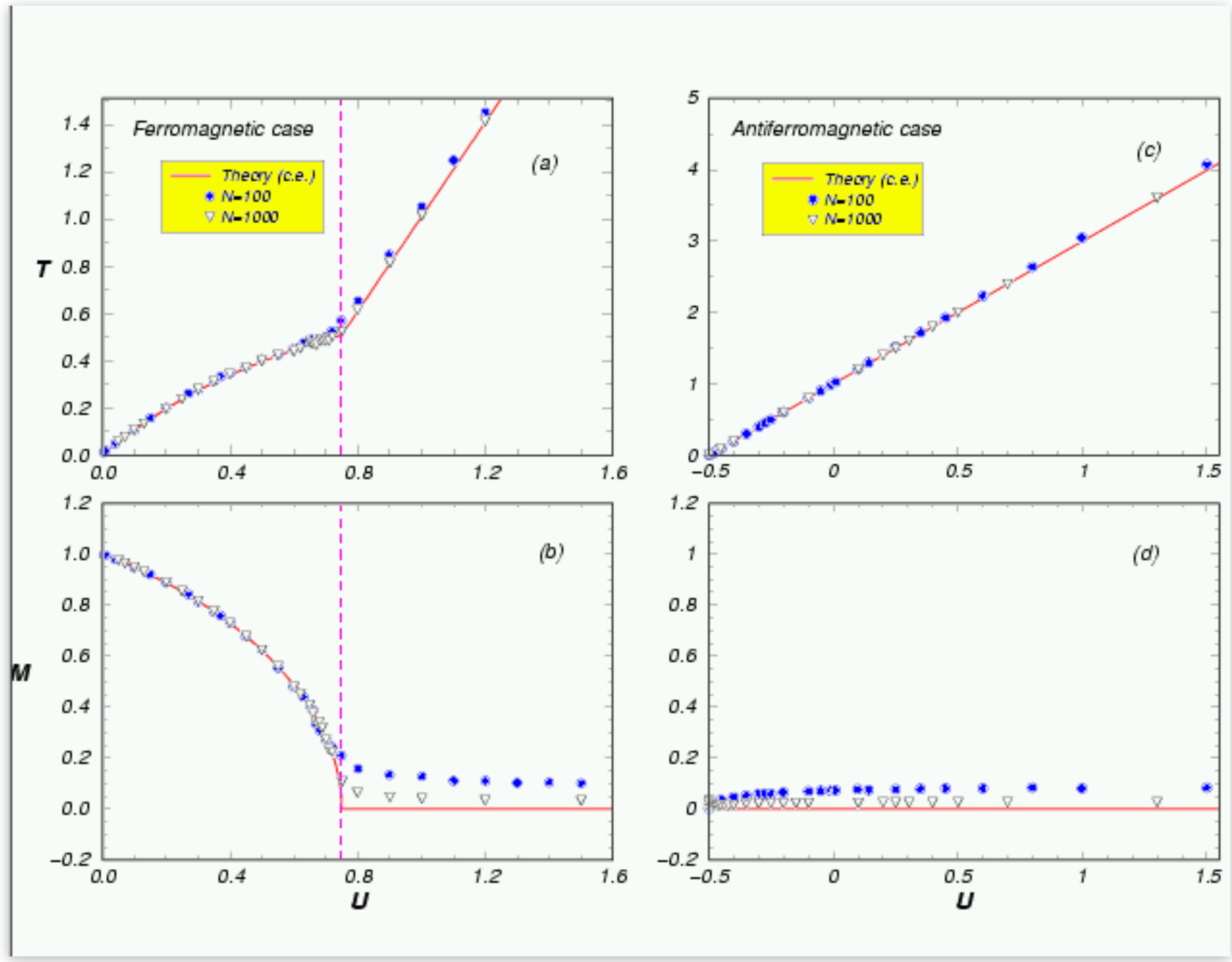
The Hamiltonian Mean Field Model

Antiferromagnetic behavior of HMF

The HMF model can have also an antiferromagnetic behavior if one considers

$$\boxed{H = K - V}$$

The general canonical solution for $\pm V$ is



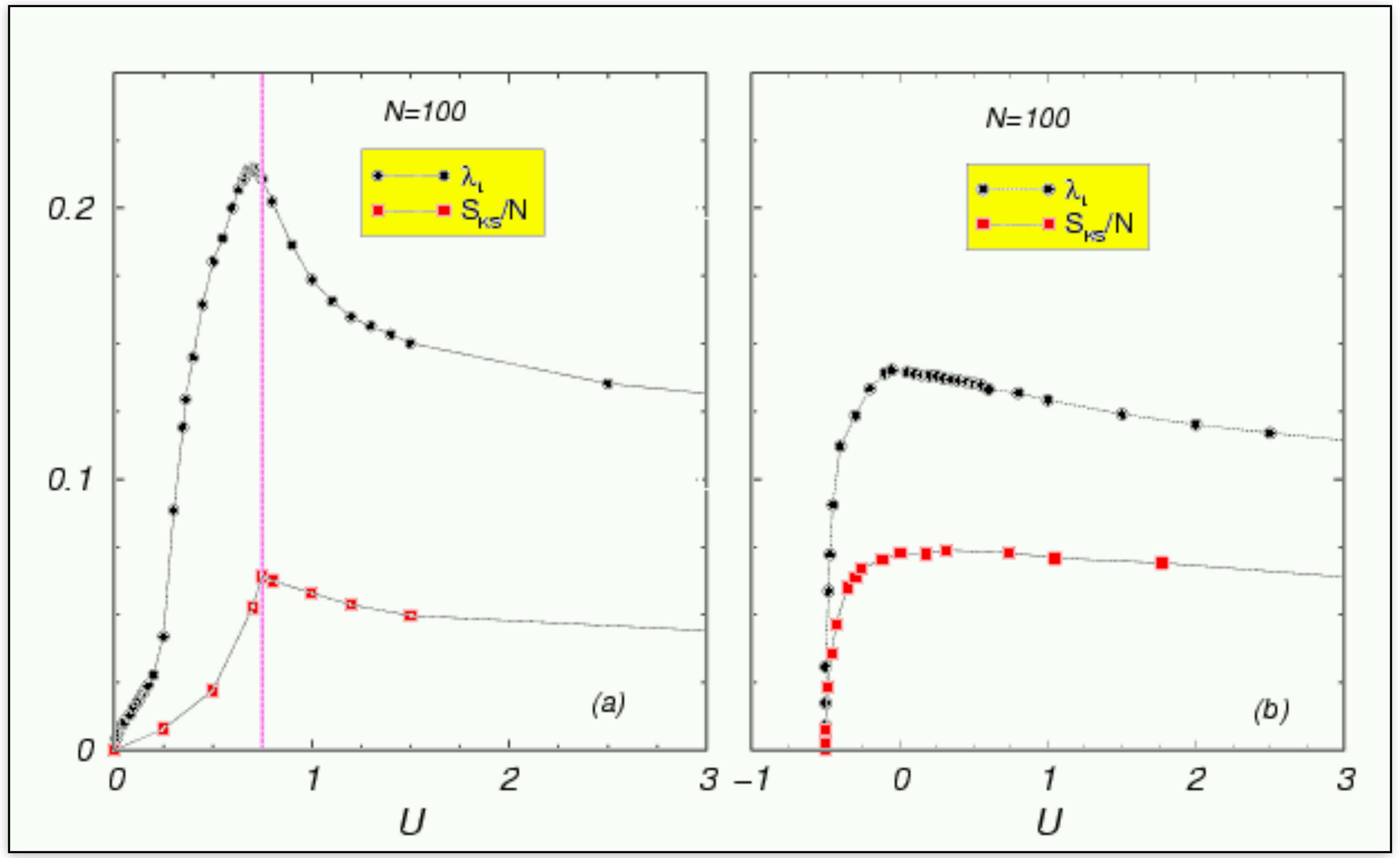
$$\boxed{U = \frac{1}{2\beta} + \frac{\varepsilon}{2}\left(1 - M^2\right)}$$

with $\varepsilon = \pm 1$

The Hamiltonian Mean Field Model

Dynamics at Equilibrium

One has a different behavior of the Largest Lyapunov exponent and the KS entropy in the ferromagnetic and antiferromagnetic case

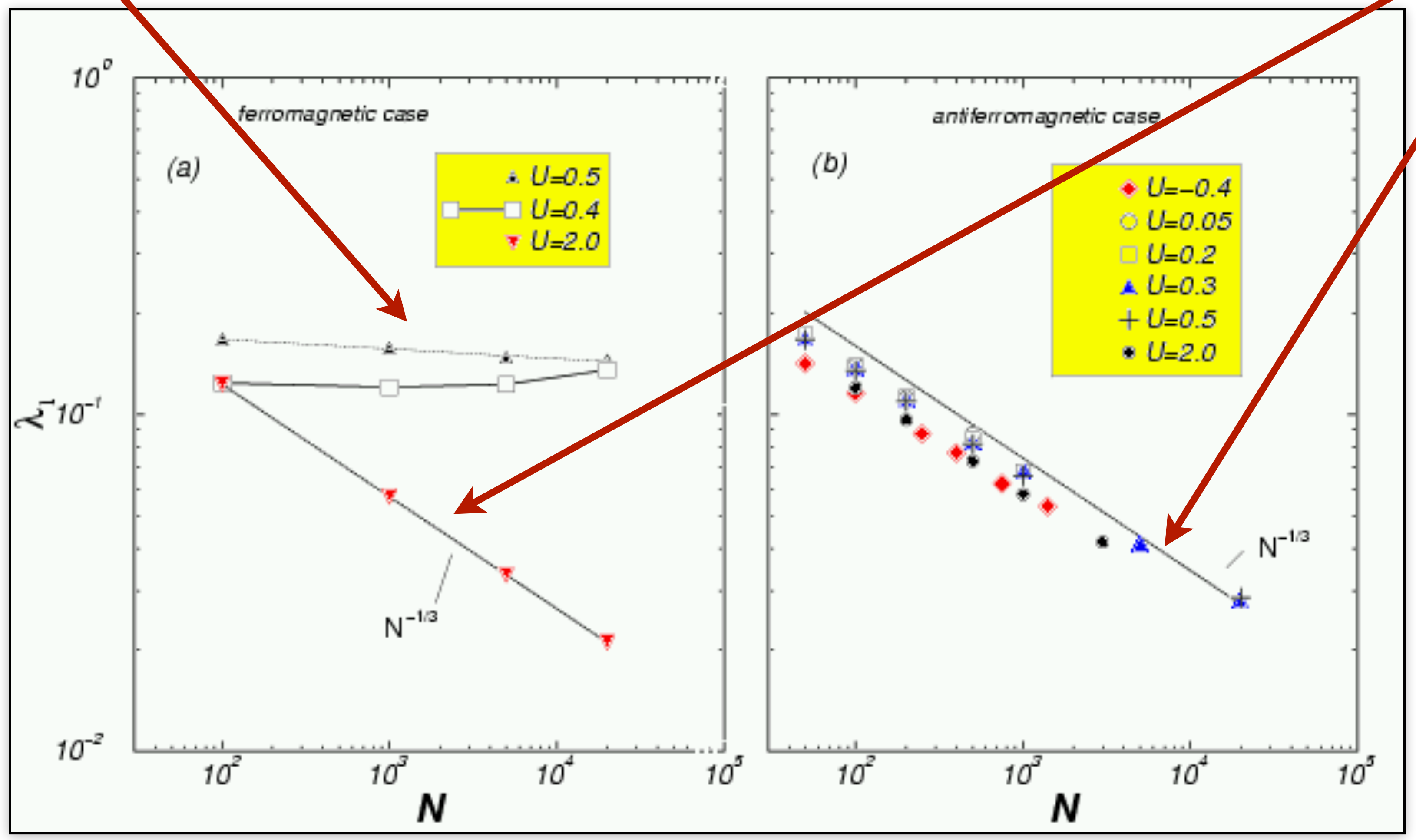


The Hamiltonian Mean Field Model

LLE in the thermodynamical limit

In the thermodynamic limit, the LLE λ_1 goes to zero for the whole energy range in the antiferromagnetic case, while it remains finite, for energies smaller than the critical one (Uc=0.75), in the ferromagnetic one. In the latter case it goes to zero for overcritical energies as

$\lambda_1 = constant$ $\lambda_1 \sim N^{-1/3}$



The Hamiltonian Mean Field Model

Equilibrium PDFs for the HMF model

In the continuum limit, considering the one-body distribution function F, the evolution of the HMF model is described by the Vlasov equation

$$\frac{\partial F}{\partial t} + p\frac{\partial F}{\partial \vartheta} - \frac{\partial V}{\partial \vartheta}\frac{\partial F}{\partial p} = 0$$

Supposing a factorization of the distribution function $F = f(p)g(\vartheta, t)$

One gets the stationary equilibrium solution

$$f = f_0 \frac{1}{\sqrt{2\pi T}} e^{-p^2/2T} \qquad g = g_0 e^{M\cos(\vartheta - \phi)/T}$$

where $g_0 = \frac{1}{2\pi I_0(M/T)}$, ϕ is the phase of M

and I_0 is the Bessel function

In the overcritical region $M = 0 \Rightarrow g = \frac{1}{2\pi}$

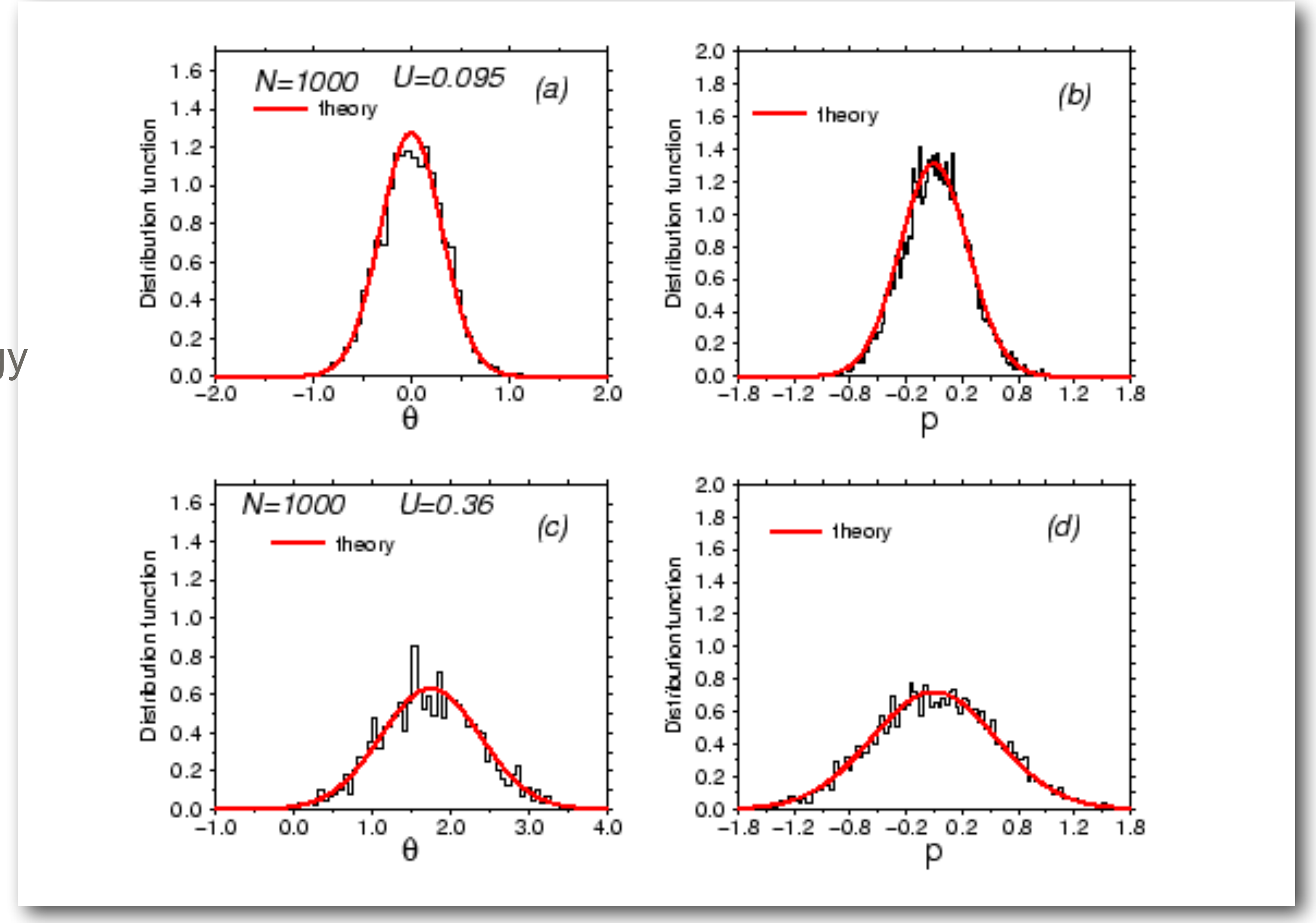
In the low energy region $I_0(z) \approx \frac{e^z}{\sqrt{2\pi z}}\left[1 + \frac{1}{8z} + \ldots\right] \Rightarrow g \approx \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\vartheta^2/2\sigma^2}$

Latora, Rapisarda and Ruffo Physica D **131** (1999) 38

The Hamiltonian Mean Field Model

Comparison with numerical pdfs

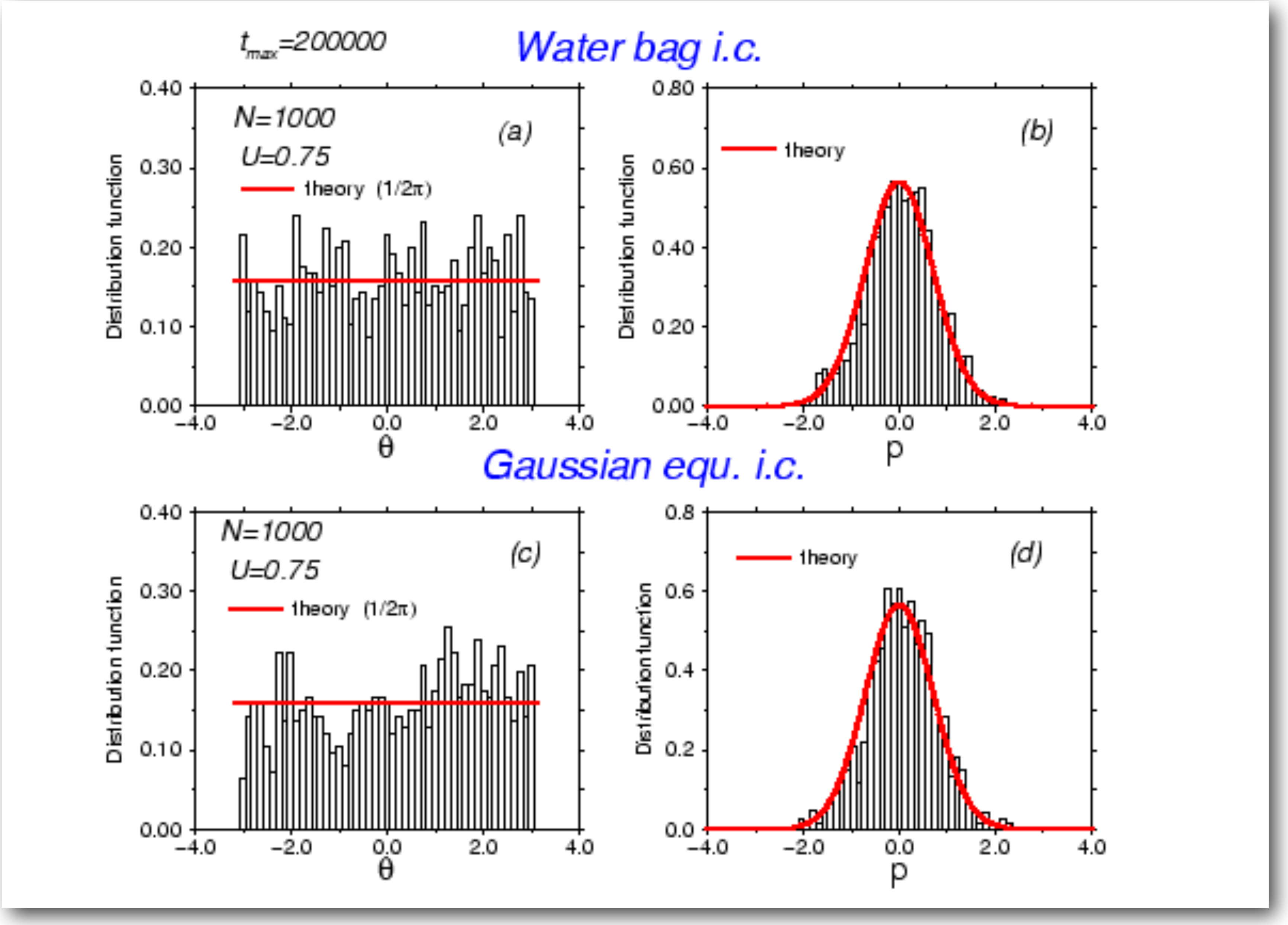
At low energy



The Hamiltonian Mean Field Model

Comparison with numerical pdfs at equilibrium

At the critical point



The Hamiltonian Mean Field Model

Out-of-equilibrium regime

When the system is started with initial conditions very far from equilibrium…..

…… one observes many dynamical anomalies, in particular in an energy range below the critical point.

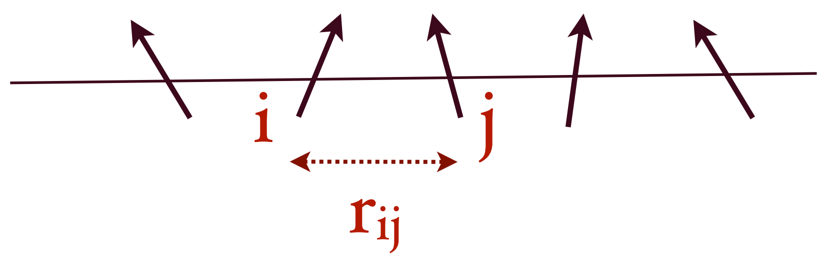
The Generalized Hamiltonian Mean Field Model

α-XY model

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} + \frac{1}{2N} \sum_{i \neq j}^{N} \frac{\left[1 - \cos(\theta_i - \theta_j)\right]}{r_{ij}^{\alpha}}$$

The HMF model has been generalized to study the dynamic and thermodynamic behavior as a function of the range of the interaction

spins are put on a lattice



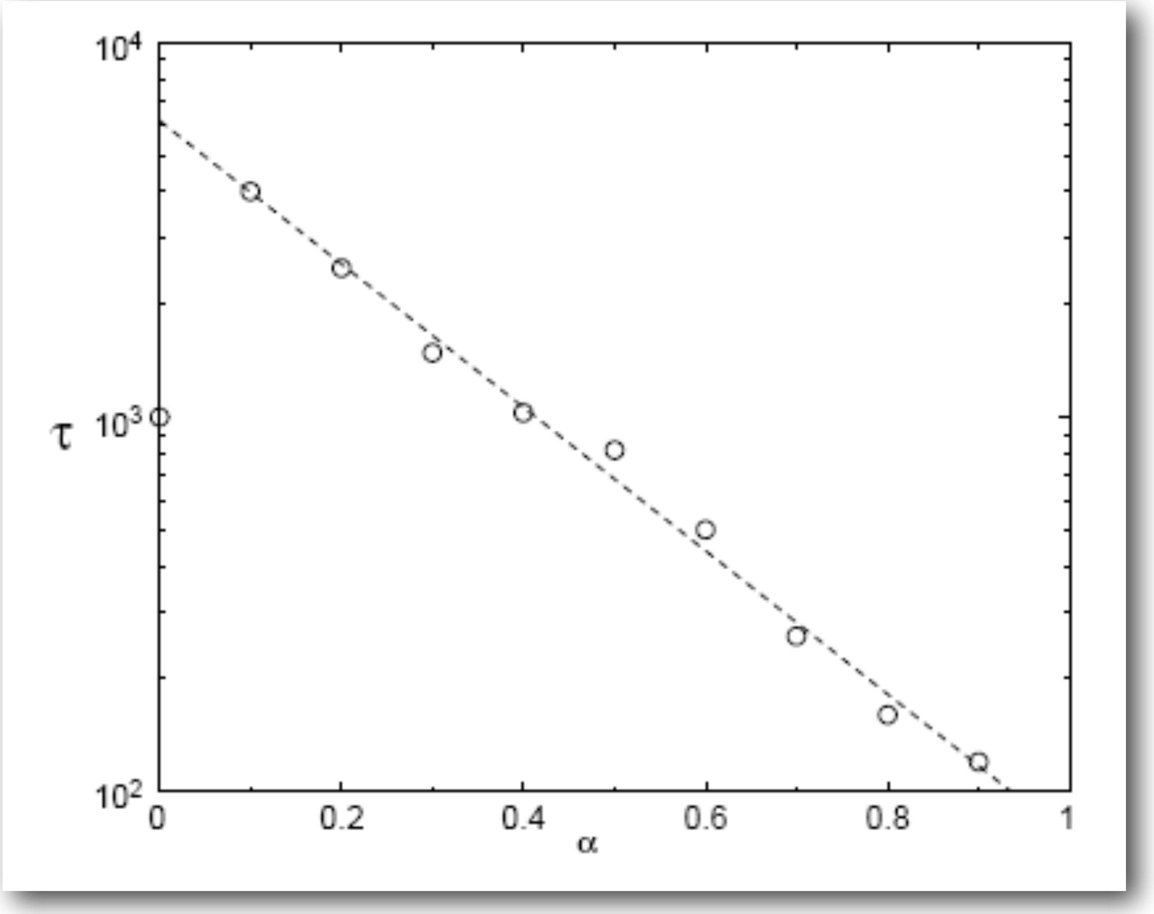
- Anteneodo and Tsallis, PRL 80 (1998) 5313
- Campa, Giansanti and Moroni, PRE 62 (2000) 303
- Tamarit and Anteneodo, PRL 84 (2000) 208
- Campa, Giansanti and Moroni, J. Phys. A 36 (2003) 6897

For $\alpha \leq d$ this generalized model reduces to HMF.

For $\alpha \rightarrow \infty$ one has interaction only among nearest neighbour spins.

The Generalized Hamiltonian Mean Field Model

α-XY model and nonextensive effects



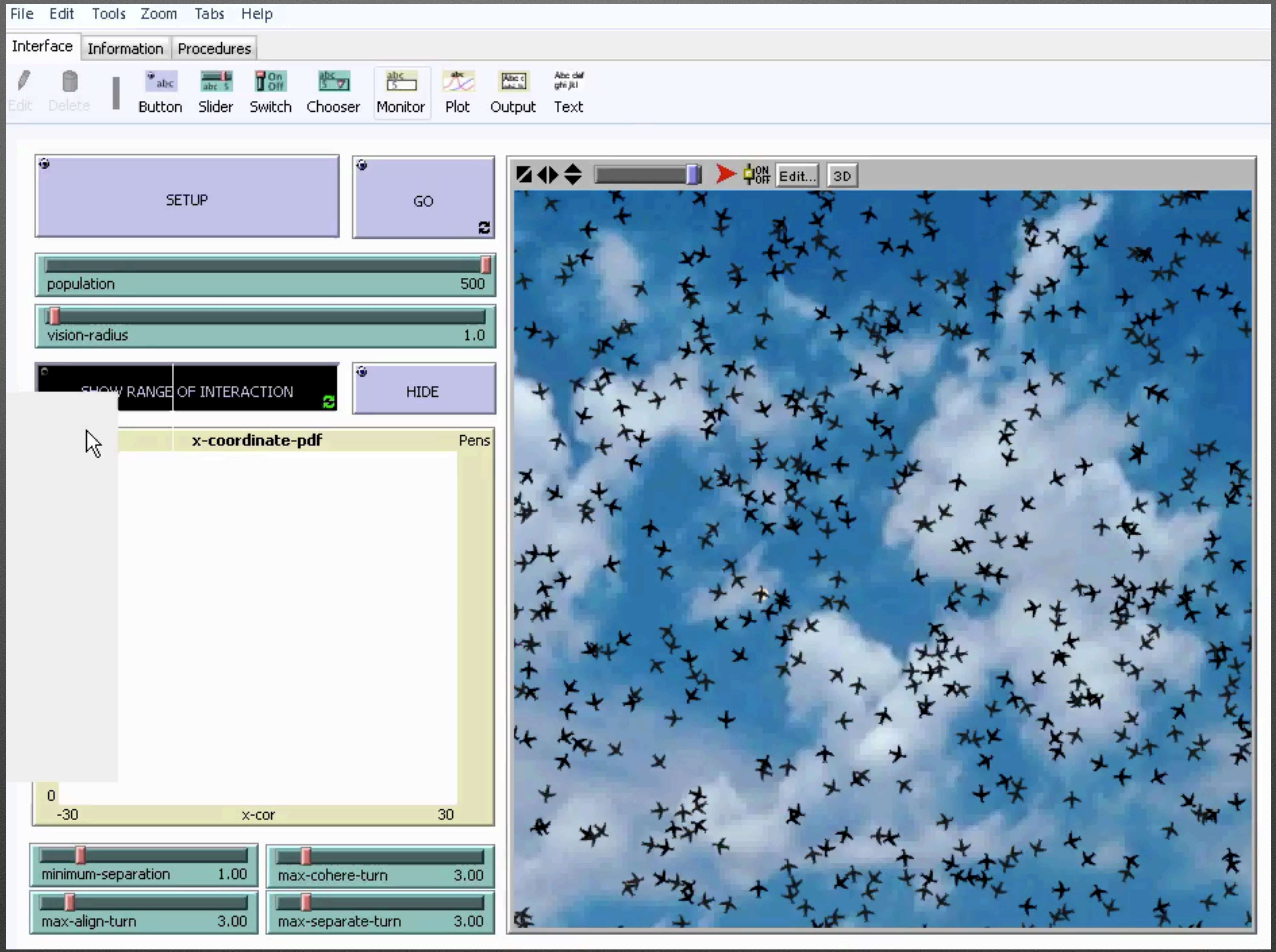
Anomalies depend in a crucial way on the range of the interaction

The lifetime τ of the QSS does not diverge for all values of α

see A. Campa et al. Physica A 305 (2002) 137

Decreasing the range of the interaction, i.e. diminishing nonextensivity $(\alpha > 0)$
anomalous behaviour disappears:
- Relaxation is very fast $(\tau \propto e^{-\alpha})$
- No negative specific heat is observed

Complex behavior for systems with long-range interactions



A simple and instructive example: birds flocking

Long range interactions slow down the dynamics and break ergodicity, inducing a complex behavior !!

The Hamiltonian Mean Field Model

Interpretation of QSS regime

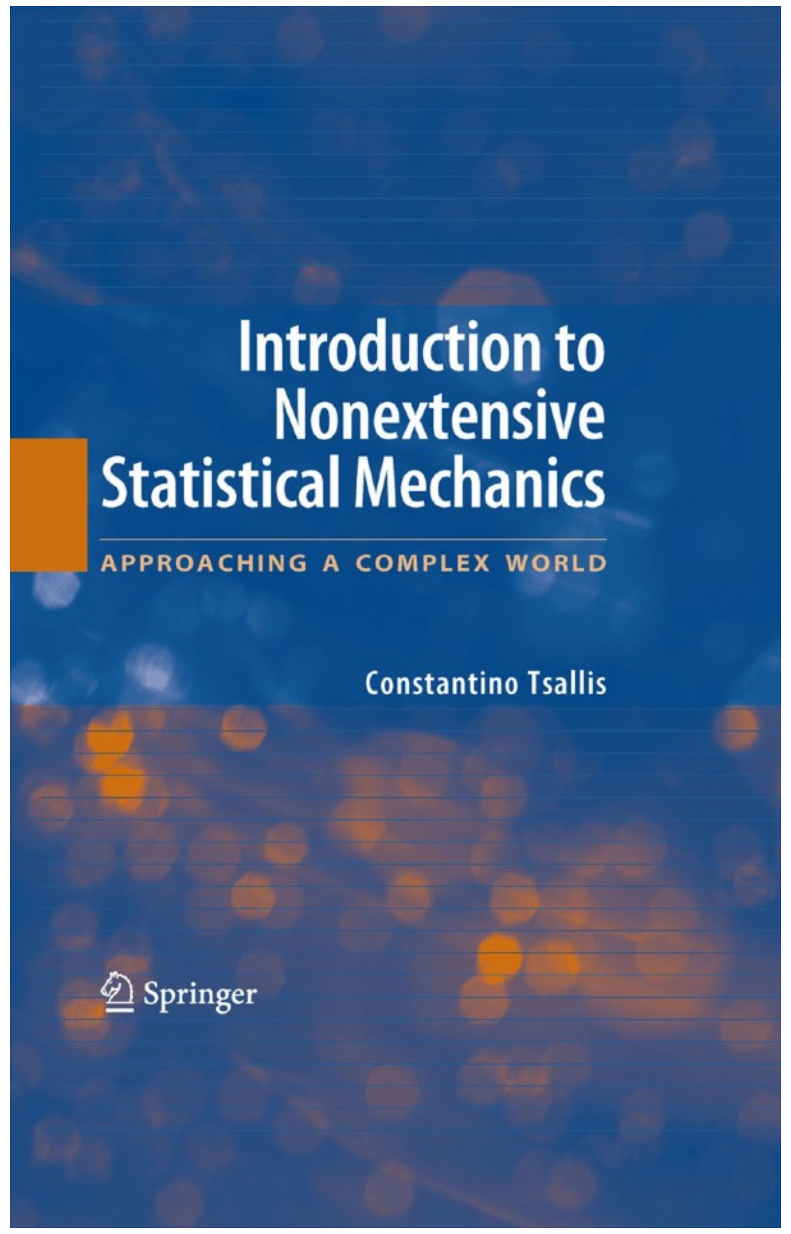
The anomalous QSS regime is the effect of non extensivity or, in other words, of the long-range character of the interaction.

These anomalies can be connected to Tsallis generalized thermostatistics

Tsallis generalized formalism

In the last decades a lot of effort has been devoted to understand if thermostatistics can be generalized to nonequilibrium complex systems

In particular one of these attempts is that one started by Constantino Tsallis with his seminal paper on J. Stat. Phys. 52 (1988) 479



For reviews see for example:

•*``Nonextensive Entropy - Interdisciplinary Applications"*, C. Tsallis and M. Gell-Mann eds., **Oxford University Press** (2003).

• Special issue of **Europhysics News** 36 (2005)

•Complexity, metastability and nonextensivity, CYNEXT07, AIP conference proceedings 965 (2007)

•**Introduction to Nonextensive Statistical Mechanics: Approaching a complex world** - Springer 2009 (2nd edition 2023)

For a regularly updated bibliografy: http://tsallis.cat.cbpf.br/biblio.htm

Tsallis generalized formalism

The generalized Tsallis entropy is $S_q = \dfrac{1 - \sum_i p_i^q}{q-1}$

S_q is non extensive, i.e. for two independent systems A and B one gets

$$S_q(A+B) = S_q(A) + S_q(B) + (1-q)S_q(A)S_q(B)$$

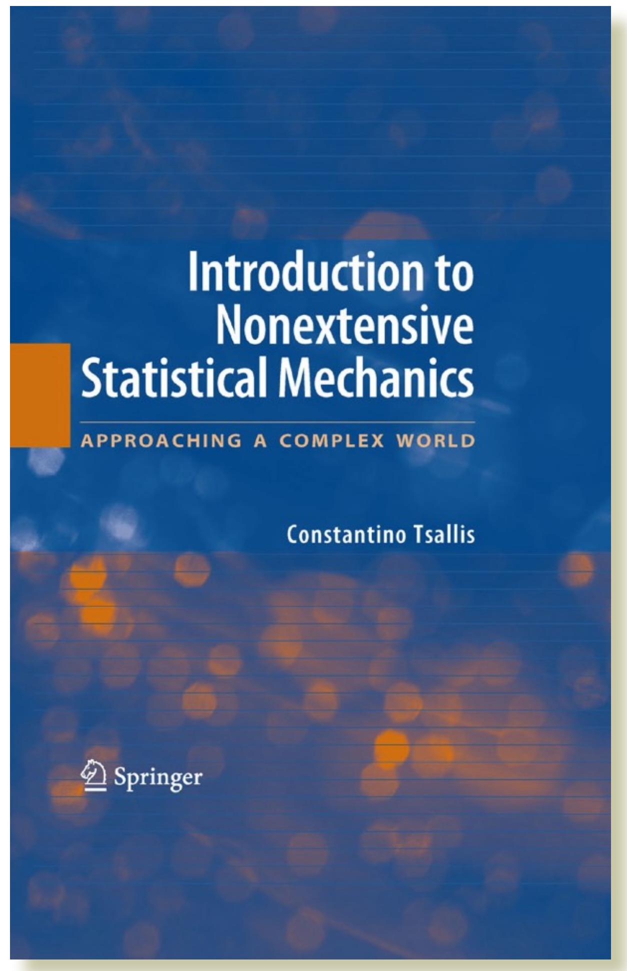
It is easy to show that reduces to the Boltzmann entropy for $q = 1$

The Boltzmann weight is also generalized (q-exponential) and reads

$$e_q(-E/kT) = \left[1-(1-q)\frac{E}{kT}\right]^{\frac{1}{1-q}} \rightarrow e^{-\frac{E}{kT}} \quad for \quad q \rightarrow 1$$

In general the standard statistical mechanics formalism is q-invariant

Foundations and applications



For recent studies on the theoretical foundations see

C Tsallis, Introduction to nonextensive statistical mechanics, Springer 2009 (2nd ed 2023)

S Umarov, C Tsallis, S Steinberg - Milan J. Math 76 (2008) 307

C. Tsallis, M. Gell-Mann and Y. Sato, PNAS 102 (2005) 15377

C. Tsallis, M. Gell-Mann, Y. Sato, Europhysics News 36 (2005)

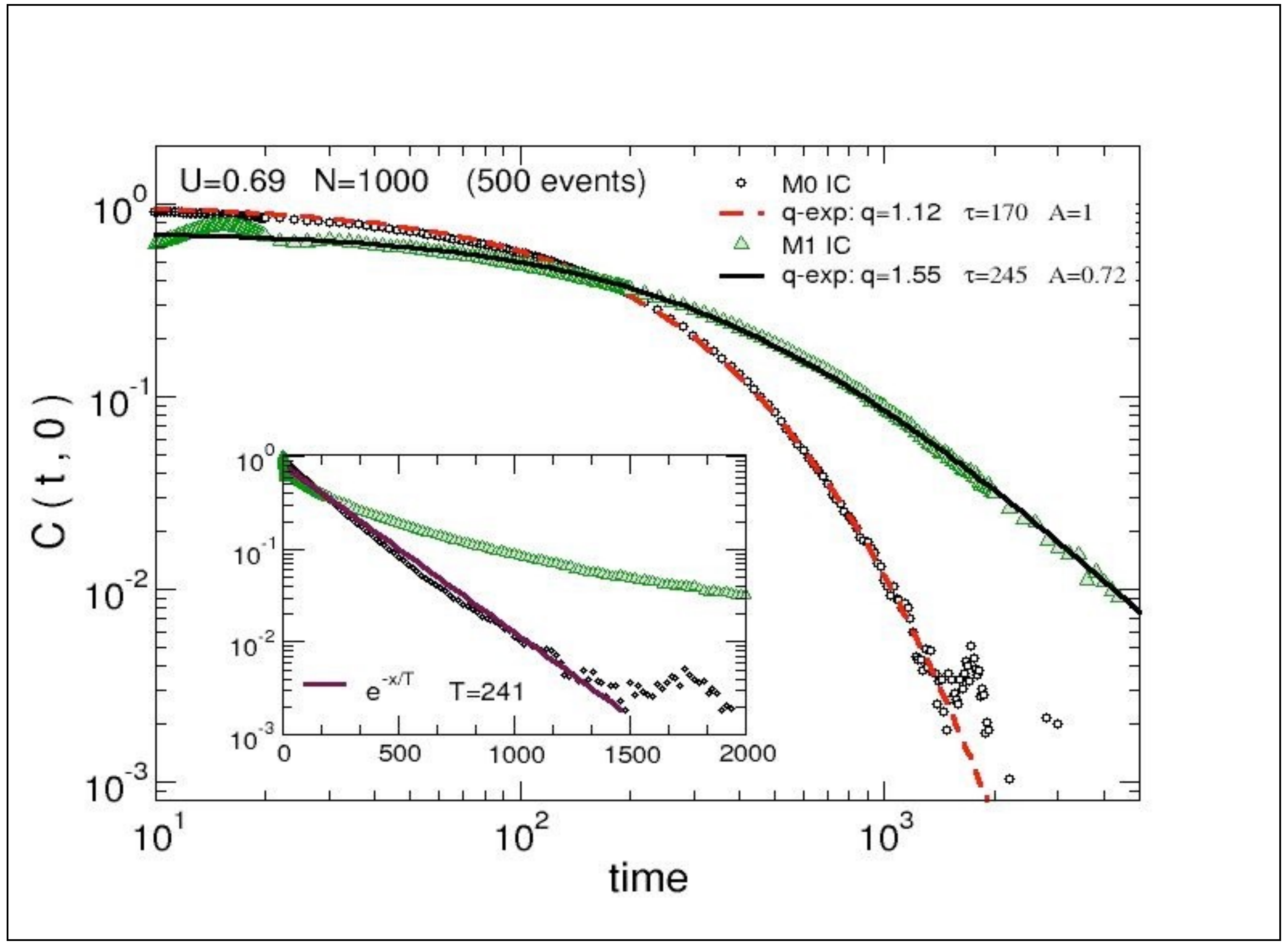
C. Beck and E.G.D. Cohen, *Superstatistics* Physica A 322 (2003) 267

F. Baldovin and A. Robledo, Phys. Rev. E 66 (2002) 045104(R) and Europhys. Lett. 60 (2002) 518

For recent successful applications of the generalized statistics, see for example

Cold atoms	P. Douglas, S. Bergamini and F. Renzoni, Phys Rev Lett 96, 110601
Turbulence	Beck, Lewis and Swinney Phys. Rev. E 63 (2001) 035303R ; Beck, Physica A 295 (2001) 195 and Phys. Rev. E **72**, 056133 (2005) K.E. Daniels, C. Beck and E. Bodenschatz, Physica D 193, 208 (2004)
High energy collisions	Wilk et al Phys. Rev. Lett. 84 (2000) 2770, Beck Physica A 286 (2000) 164; Bediaga, Curado, De Miranda , Physica A 286 (2000) 156; Depmann et al., Phys Rev. D **(2020)**
Cosmic rays	C. Tsallis, J.C. Anjos and E.P. Borges, Phys. Lett. A 310 (2003) 372.
Econophysics	L. Borland, Phys. Rev. Lett. 89 (2002) 098701 L Borland, Europhys News 36, 228 (2005)
Biological systems	Upaddhyaya et al Physica A 293 (2001) 549; R.J. Al-Azawi et al., Peer J Computer Science **(2021)**
Granular media	G. Combe et al Phys. Rev. Lett. 115 **(2015)** 238301
Maps at the edge of chaos	V.Latora, M.Baranger, A. Rapisarda and C.Tsallis, Phys. Lett. A 273 (2000) 97; U. Tirnakli, Phys. Rev. E 66;(2002) 066212; E.P. Borges, C. Tsallis,G.F.J. Ananos, P.M.C. de Oliveira, Phys. Rev. Lett. 25 (2002) 254103. Tirnakli, U.; Borges, E.P. *Sci. Rep.* 6 **(2016)** 23644
Complex Networks	N. Cinardi, A. Rapisarda, C. Tsallis, Jour. Stat Mech. **(2020)** 043404

q-exponential decay of the correlation function C(t,0) for the HMF model



Tsallis and Buckman PRE 54 (1996) R2197

The decay of the velocity correlation function can be reproduced very well by means of the generalized q-exponential

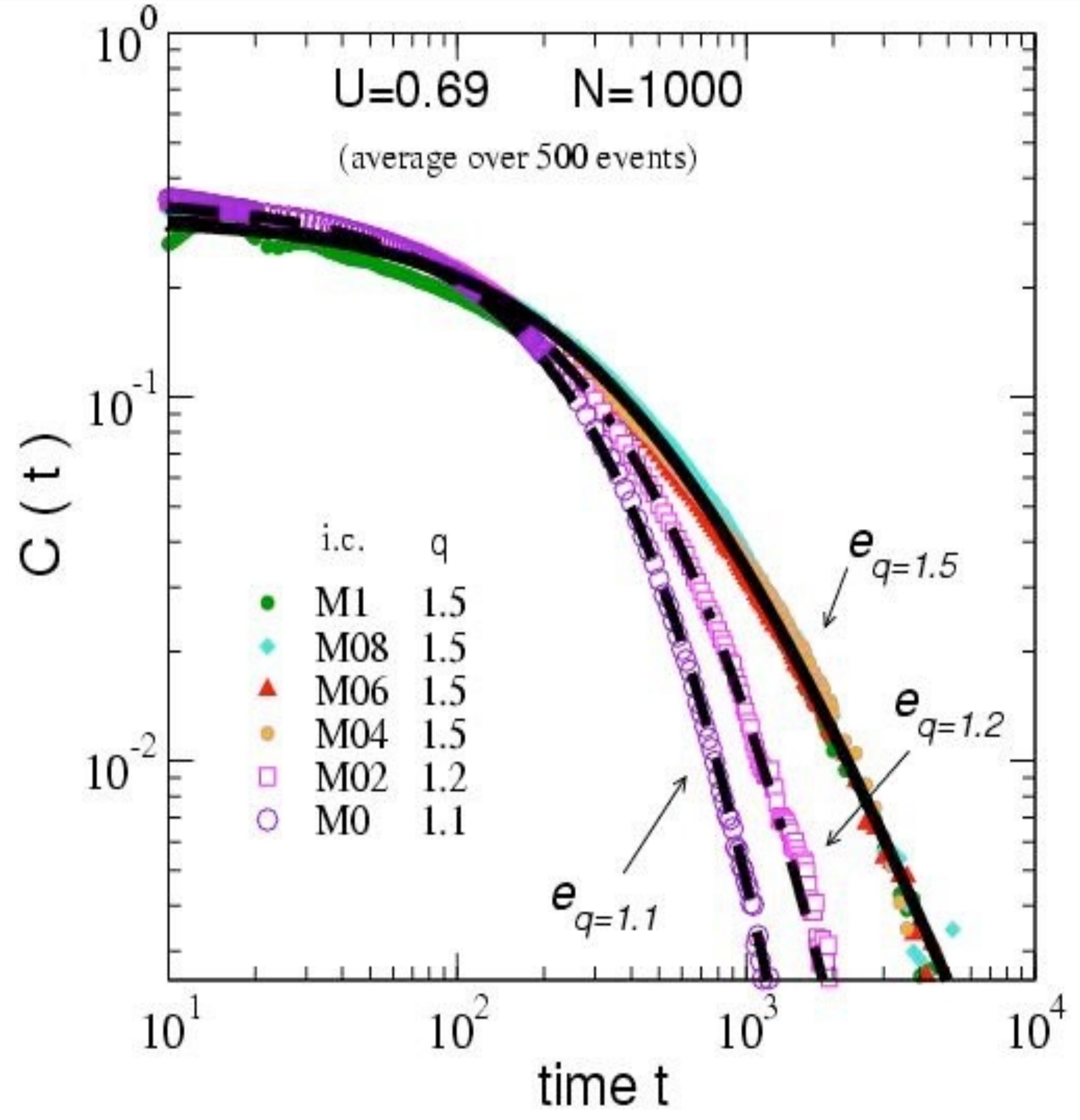
$$Ae_q(x) = A[1+(1-q)x]^{\frac{1}{1-q}}$$

In our case x=-t/τ . Within a generalized Fokker-Plank equation which generates Tsallis q-exponential pdfs [1], one can extract the following relation between the exponent of the anomalous diffusion and **q**

$$\gamma = \frac{2}{3-q}$$

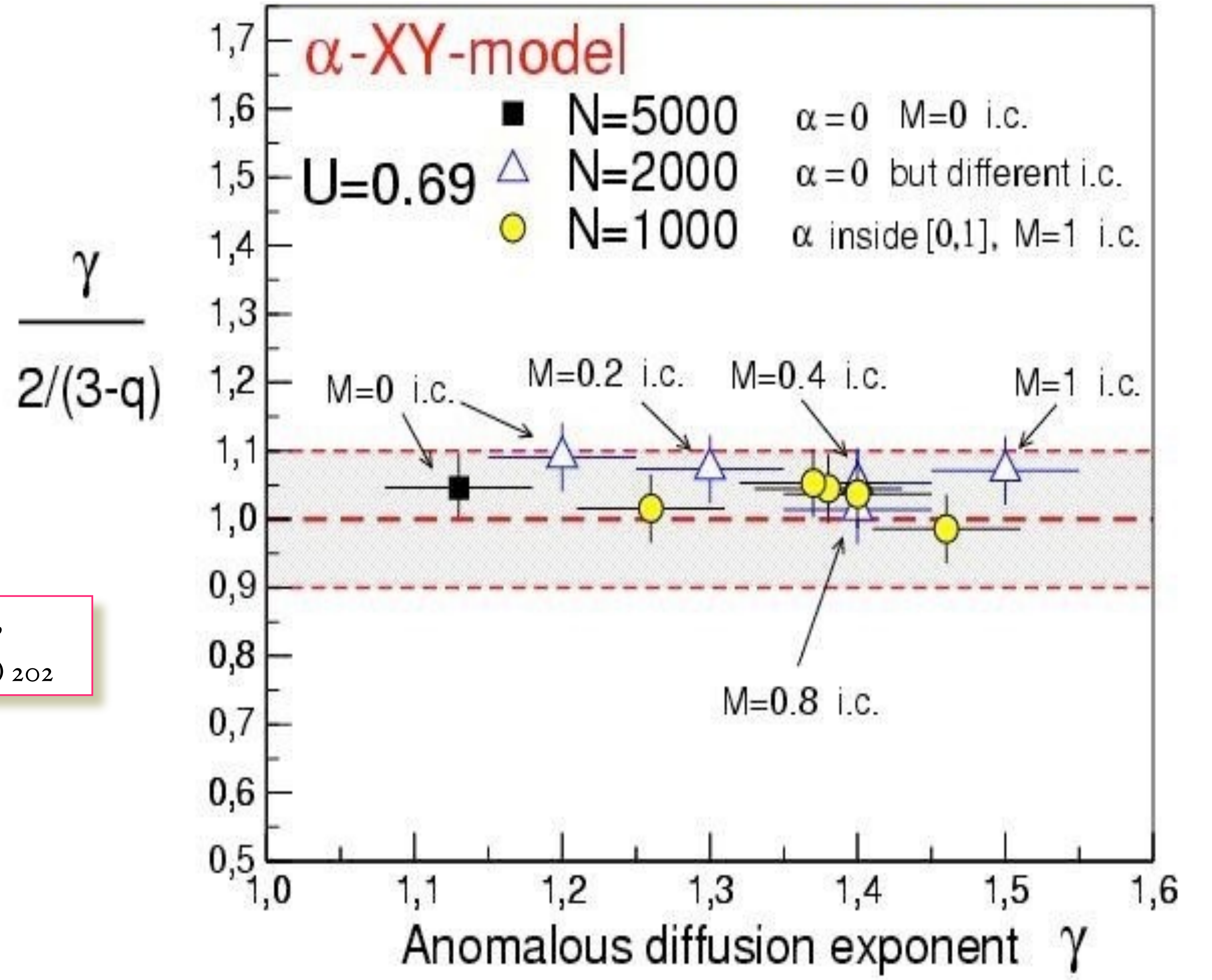
In our case γ**=1.38-1.4**, thus we expect **q=1.55-1.6**, which is confirmed by the fit in the figure for M1IC. On the other hand, for M0IC the decay is almost exponential.

q-exponential decay also for different initial conditions



HMF model and q-statistics

Anomalous diffusion vs q-exponential decay



Rapisarda and Pluchino,
Europhysics News 36 (2005) 202

The Kuramoto Model

(Kuramoto 1975)

Eqs. for the N coupled oscillators

$$\dot{\theta}_i(t) = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \qquad i = 1, \ldots, N$$

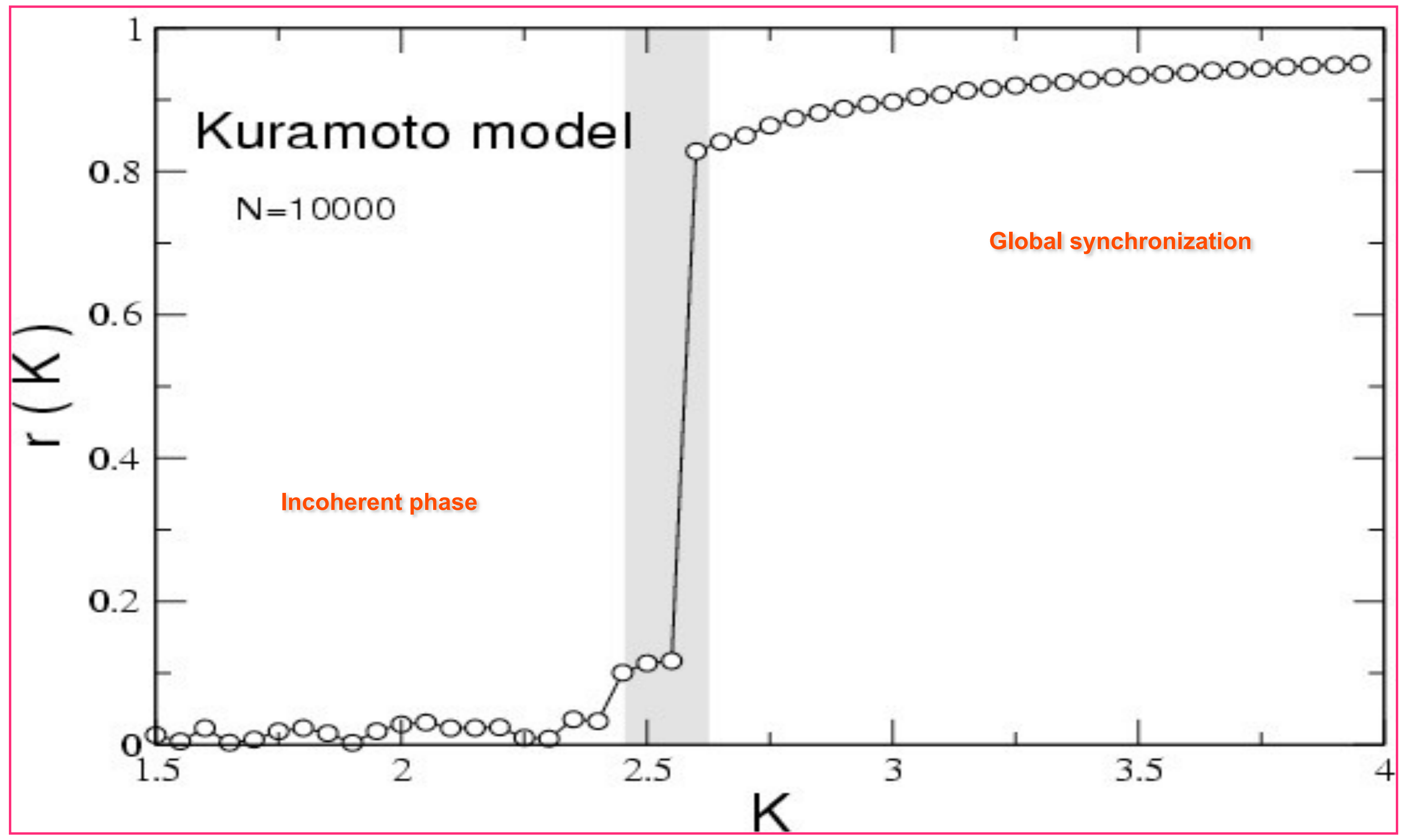
Order parameter

$$re^{i\Psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$$

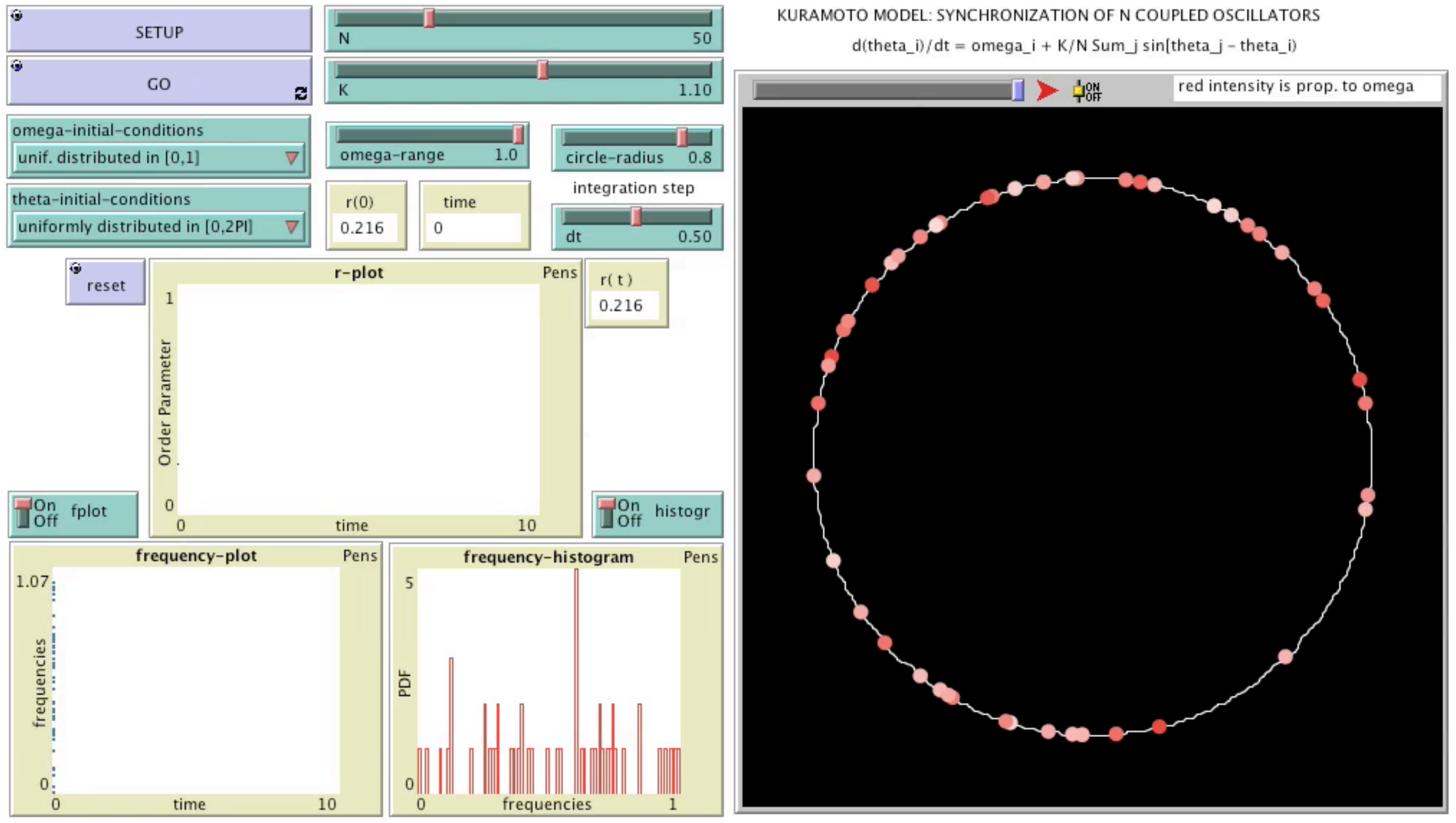
mean field equation

$$\dot{\theta}_i(t) = \omega_i + Kr\sin(\Psi - \theta_i)$$

Phase transition



Kuramoto dynamics



$$\dot{\theta}_i = \omega_i + M\,K\sin(\Psi - \theta_i) \qquad i = 1,\ldots,N$$

Similarities between Kuramoto and HMF model

✳ Both models can be derived from the more general one

$$\ddot{\vartheta}_i + B\dot{\theta}_i + M\,K\sin(\theta_i - \Psi) = \omega_i \qquad i = 1,\dots,N$$

Conservative case

$B = 0, K = 1, \omega_i = 0$

$$\ddot{\vartheta}_i = M\sin(\Psi - \theta_i) \qquad i = 1,\dots,N$$

HMF

Dissipative case

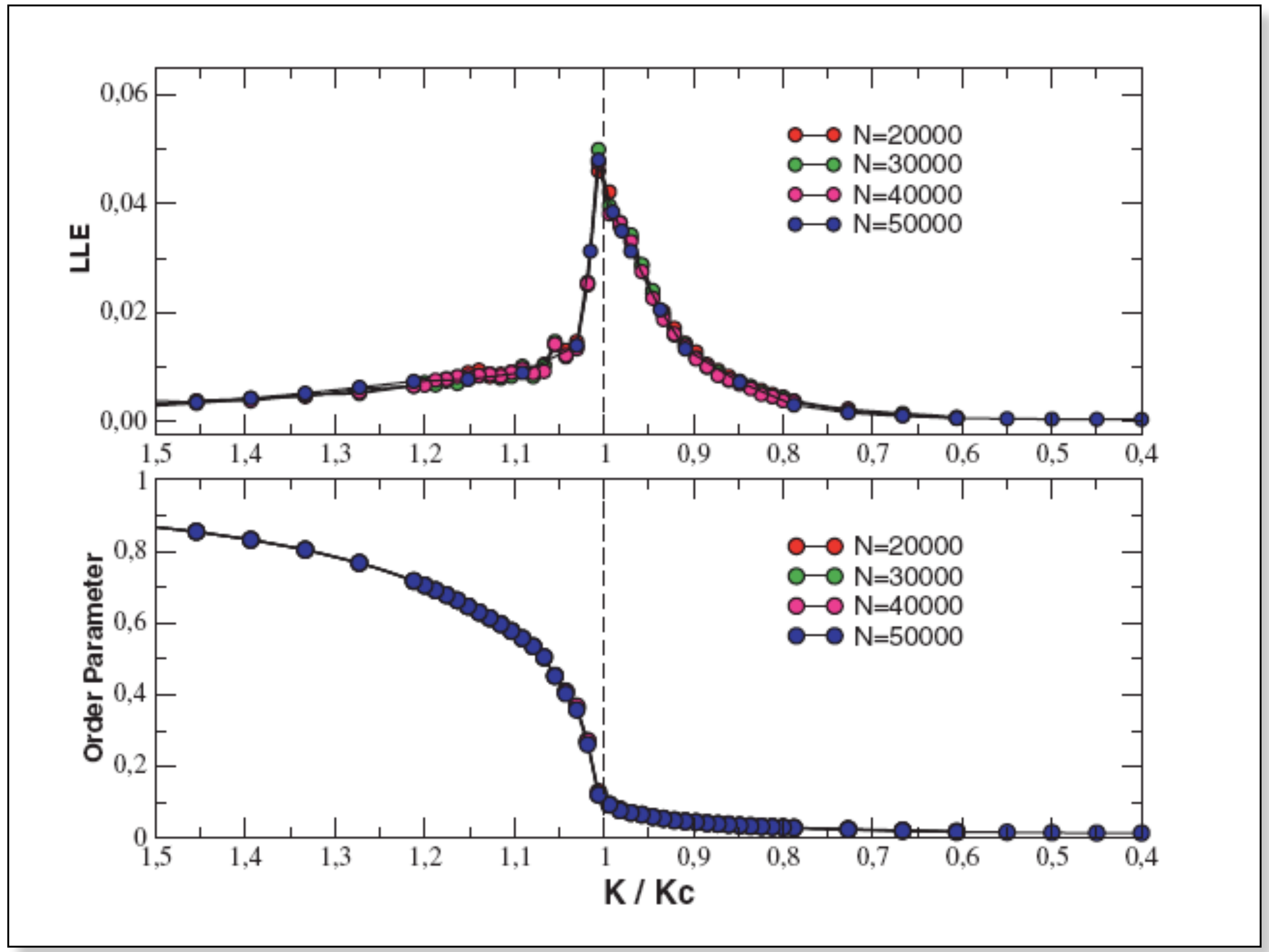
$B \gg 1, K \geq 0, \omega_i \neq 0$

$$\dot{\theta}_i = \omega_i + M\,K\sin(\Psi - \theta_i) \qquad i = 1,\dots,N$$

KURAMOTO

A. Pluchino, A. Rapisarda, Physica A 365 (2006) 184

As for HMF there is a peak in the LLE at the phase transition (no dependence on the size is observed) and the transition seems very similar to a second order one



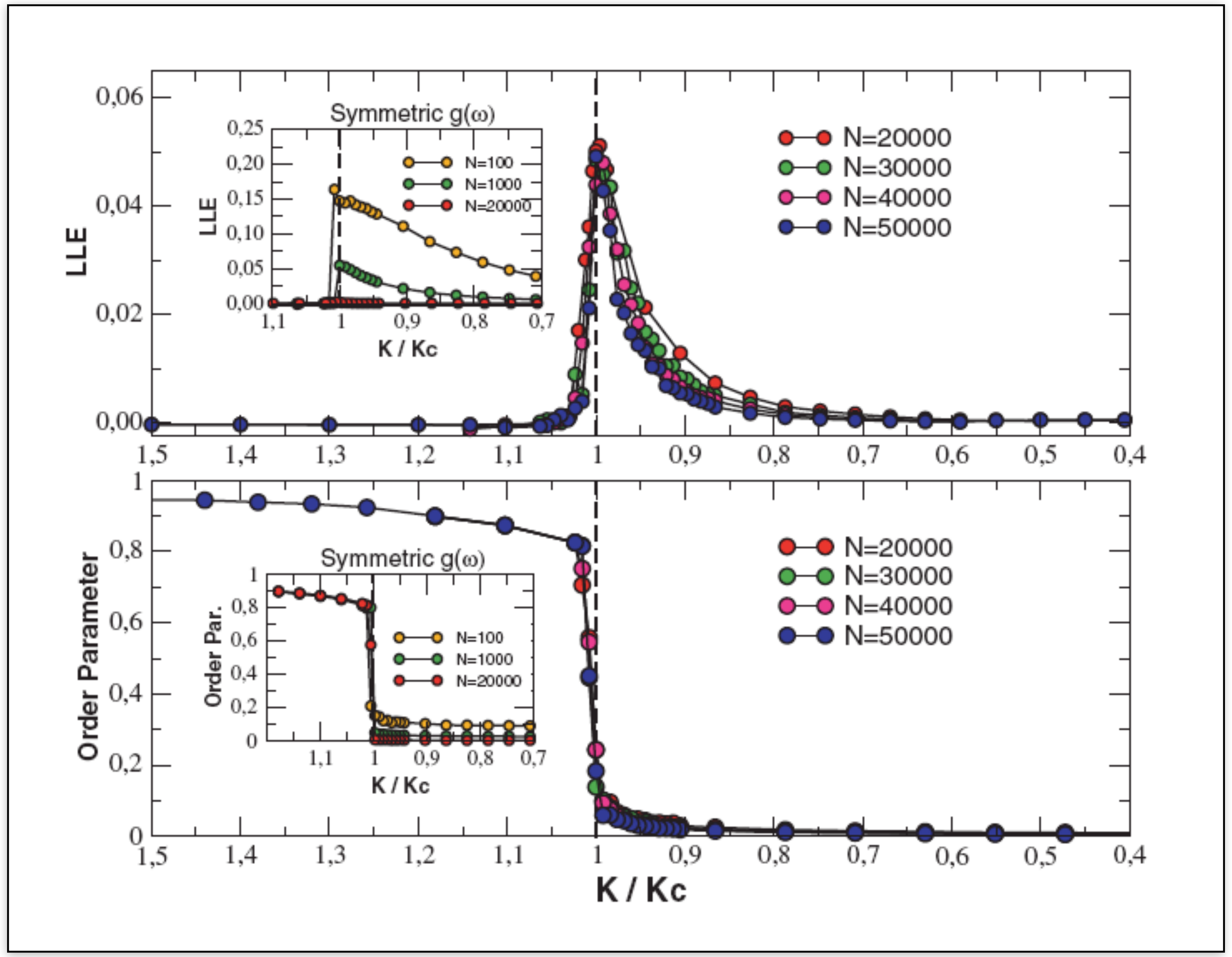
Gaussian g(w)

$$K_c = 1.6$$

$$K_c = \frac{2}{\pi\, g(0)}$$

Miritello, Pluchino, Rapisarda Epl 85 (2009) 10007

The same happen for a **uniform distribution of frequencies**, but now the transition is sharp and similar to a **first order one**



$$K_c = 2.55$$

phase diagram

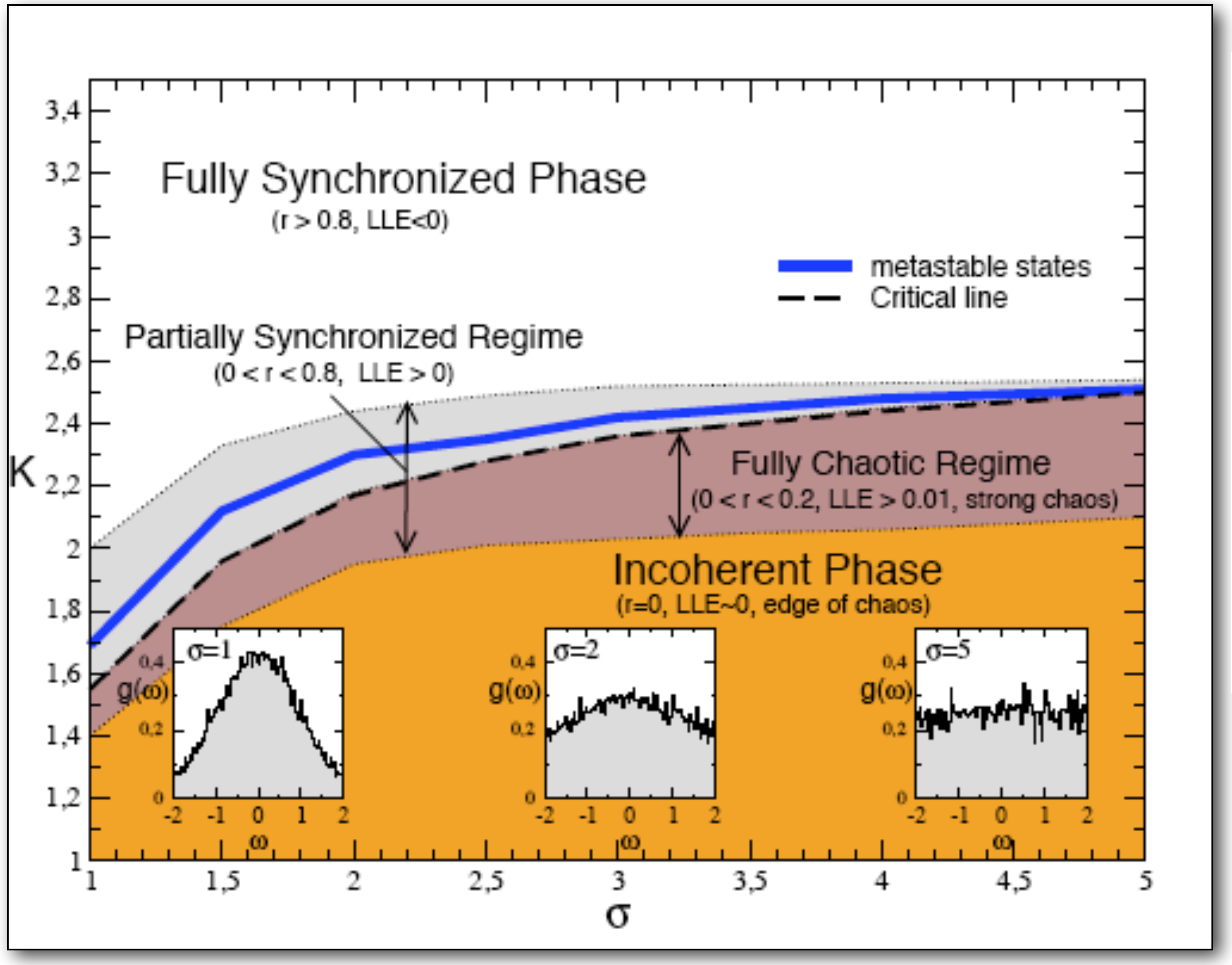
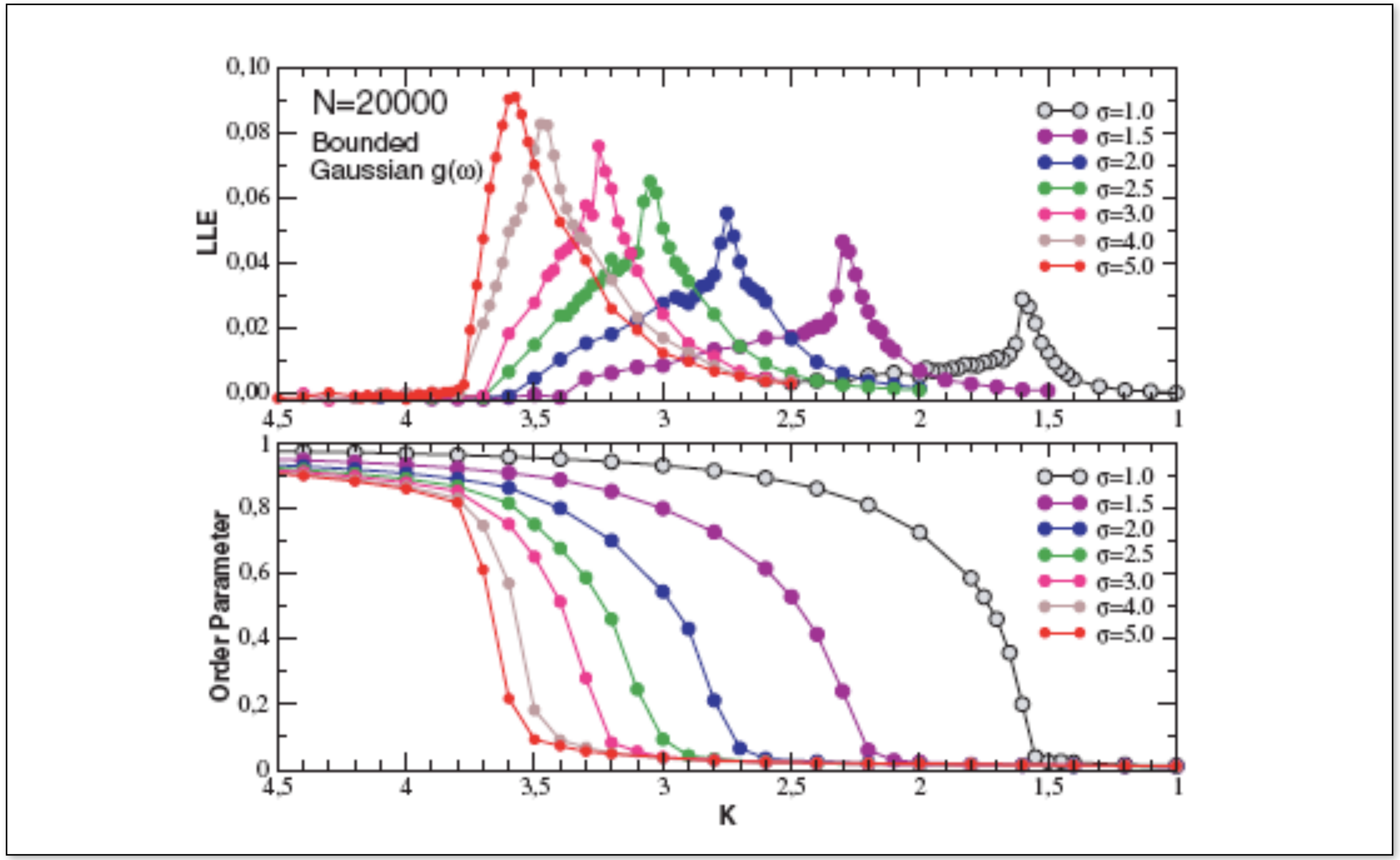


Fig. 2. Phase Diagram K versus σ for the Kuramoto model with $N = 20000$. The critical line separates the fully synchronized phase (characterized by $r > 0.8$ and a slightly negative LLE) from the "edge of chaos" incoherent one (characterized by $r = 0$ and a vanishing, but positive, LLE). The partially synchronized regime (characterized by $0 < r < 0.8$ and a positive LLE) is also visible between the two, around the phase transition. In particular, just below the critical line, we recognize a weakly synchronized sub-region of the partially synchronized regime, with values $0 < r < 0.2$ and $LLE > 0.01$, that we call "fully chaotic regime" (see text). In the insets, the bounded $g(\omega)$ distributions (with $\omega \in [-2, 2]$) used in the simulations are plotted for three increasing values of σ. See text.

Miritello, Pluchino, Rapisarda, **Physica A 388 (2009) 4818**

Smooth change from 2nd to 1st order phase transition vs the standard deviation of the initial distribution of frequencies



Miritello, Pluchino, Rapisarda Epl 85 (2009) 10007

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January 2009

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doi: 10.1209/0295-5075/85/10007

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Phase transitions and chaos in long-range models of coupled oscillators

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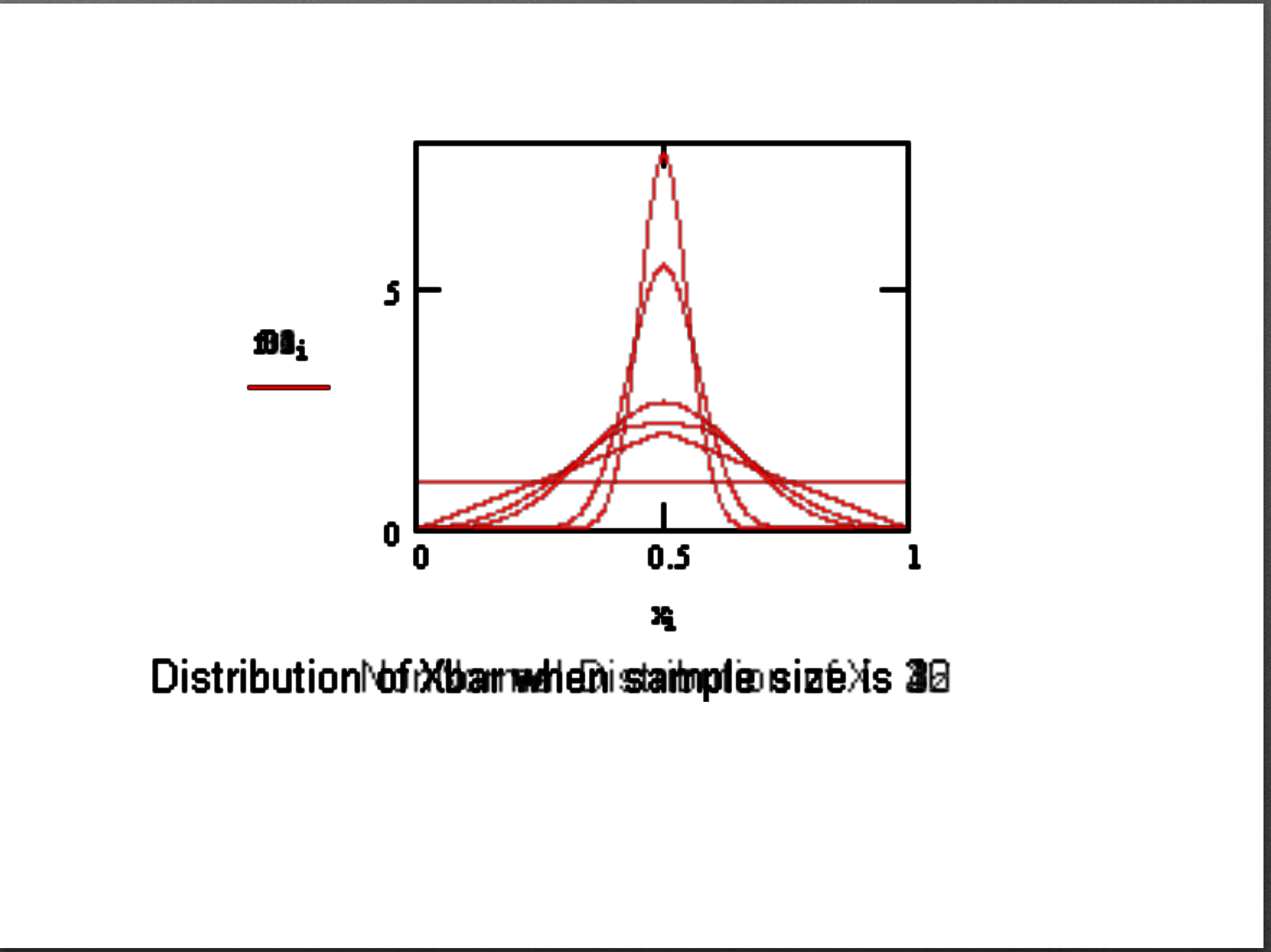
PACS 05.45.Xt – Synchronization; coupled oscillators
PACS 05.70.Fh – Phase transitions: general studies

Abstract – We study the chaotic behavior of the synchronization phase transition in the Kuramoto model. We discuss the relationship with analogous features found in the Hamiltonian mean-field (HMF) model. Our numerical results support the connection between the two models, which can be considered as limiting cases (dissipative and conservative, respectively) of a more general dynamical system of damped/driven coupled pendula. We also show that, in the Kuramoto model, the shape of the phase transition and the largest Lyapunov exponent behavior are strongly dependent on the distribution of the natural frequencies.

Central limit theorem

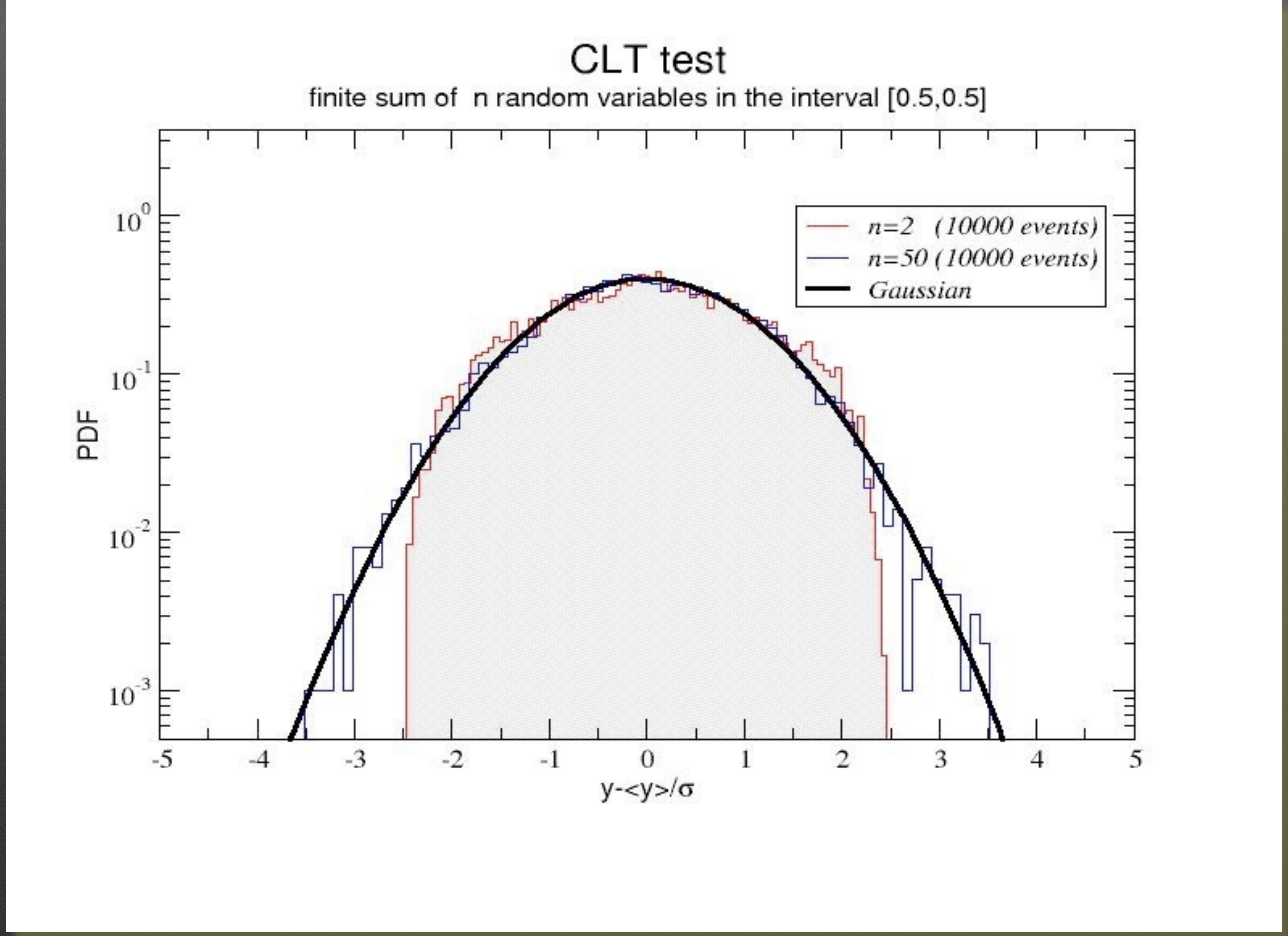
The Central Limit Theorem says that …….the distribution of an average will tend to be Normal as the sample size increases, regardless of the distribution from which the average is taken ***except*** when the moments of the parent distribution do not exist.

The distribution of the average converges quite fast to the final Gaussian attractor



Central limit theorem

Usually the convergence is quite fast
and begins in the central part of the PDF



CLT for the logistic map in the chaotic regime

CENTRAL LIMIT BEHAVIOR OF DETERMINISTIC… PHYSICAL REVIEW E 75, 040106(R) (2007)

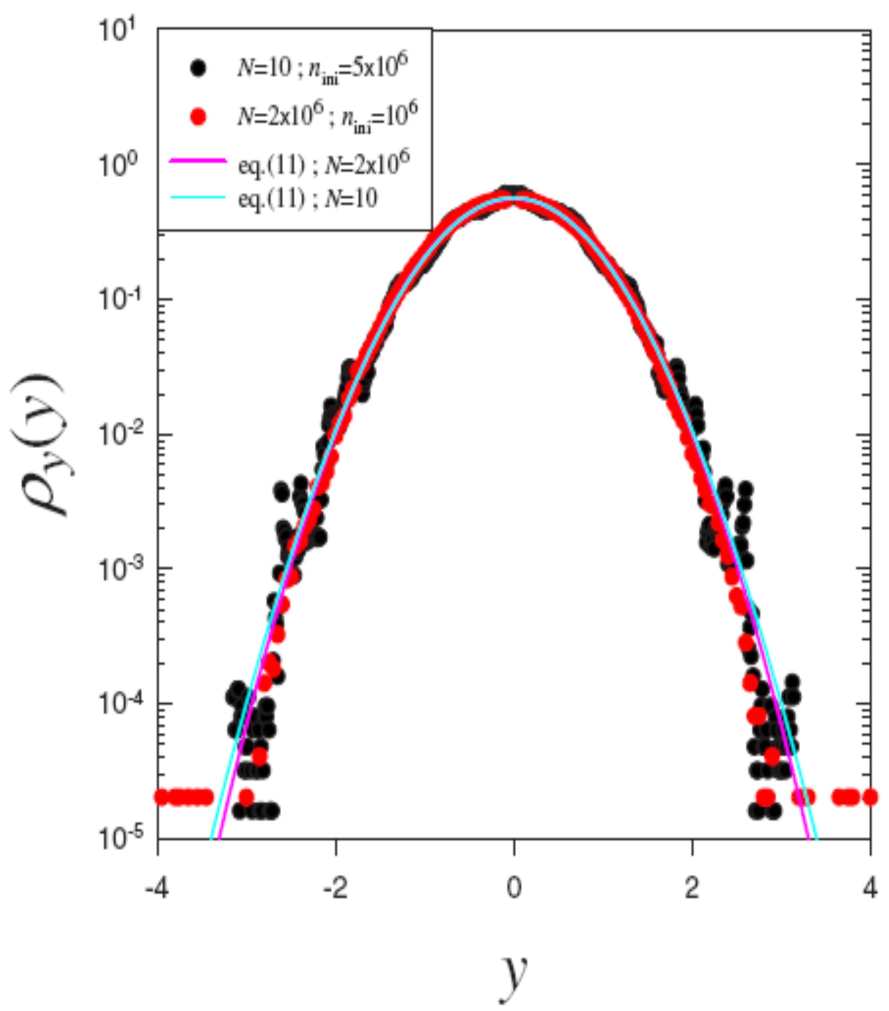


FIG. 2. (Color online) Probability density of rescaled sums of iterates of the cubic map (10) for $N=10^7$ and $N=10$. The number of initial values is $n_{ini}=10^6$, respectively $n_{ini}=5\times10^6$. The solid lines correspond to Eq. (11).

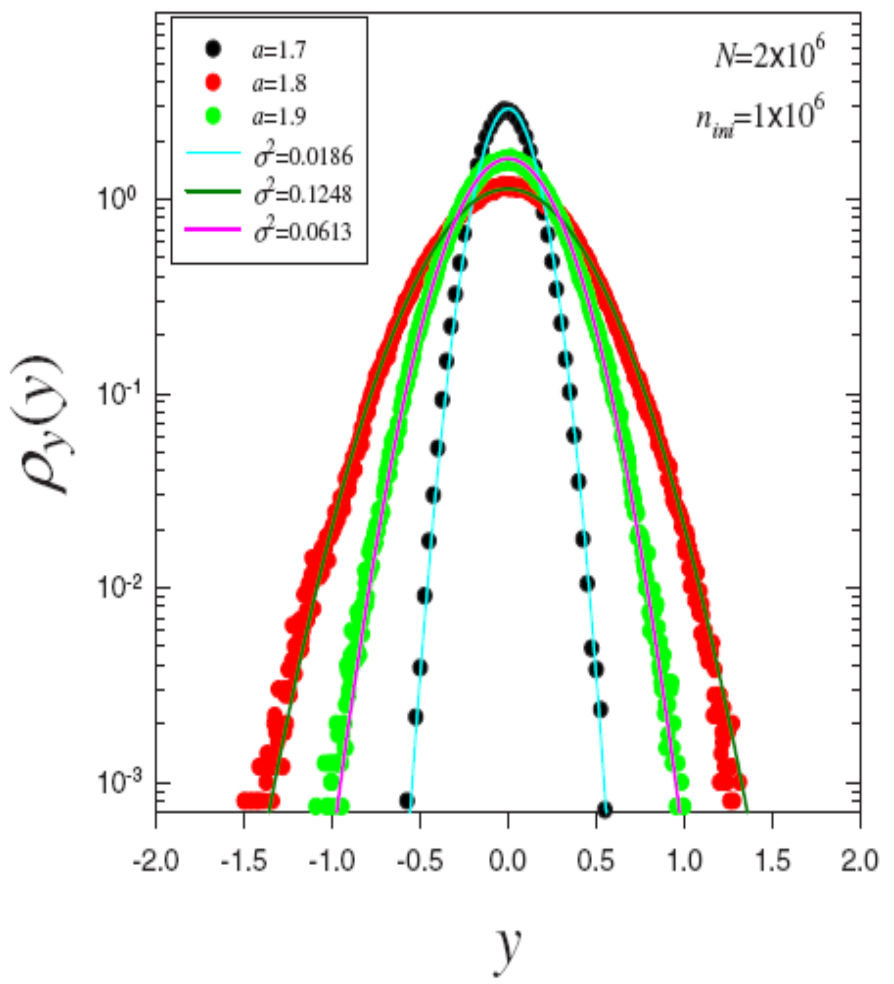


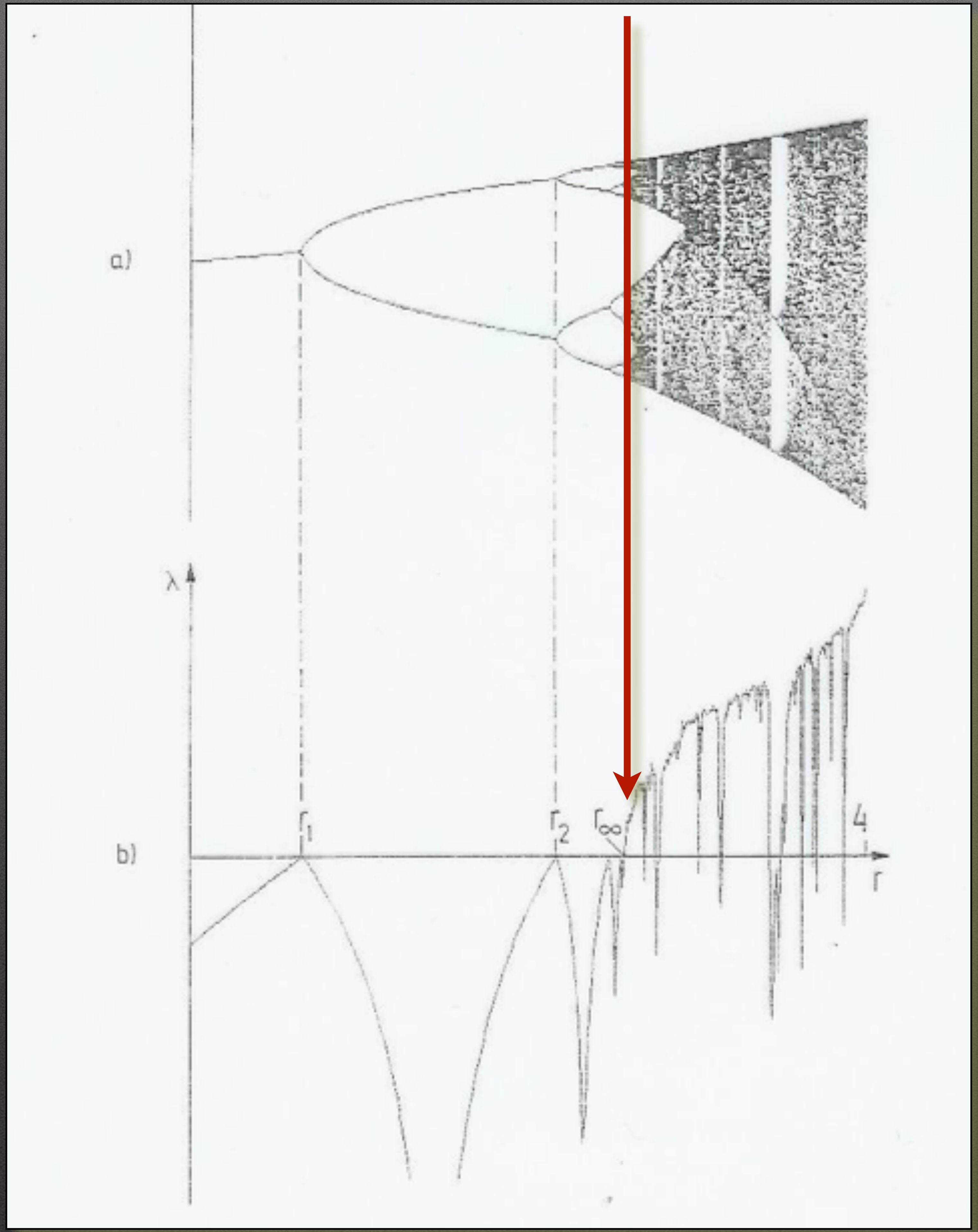
FIG. 3. (Color online) Probability density of rescaled sums of iterates of the logistic map as given by Eq. (8) for $a=1.7, 1.8, 1.9$ and $N=2\times10^6$, $n_{ini}=10^6$. The solid lines show Gaussians $e^{-y^2/(2\sigma^2)}/\sqrt{2\pi\sigma^2}$ with variance parameter σ^2 determined from Eq. (3).

where

$$y := \frac{1}{\sqrt{N}}\sum_{i=1}^{N}(x_i - \langle x\rangle)$$

$$x_{i+1} = T(x_i) = 1 - a x_i^2$$

CLT at the edge of chaos



Generalized Central limit theorem

At the edge of chaos q-Gaussians appear

$$e_q^{-\beta y^2} := \frac{1}{[1+\beta(q-1)y^2]^{1/(q-1)}}$$

$$x_{i+1} = 1 - a|x_i|^z,$$

RAPID COMMUNICATIONS

TIRNAKLI, BECK, AND TSALLIS

PHYSICAL REVIEW E 75, 040106(R) (2007)

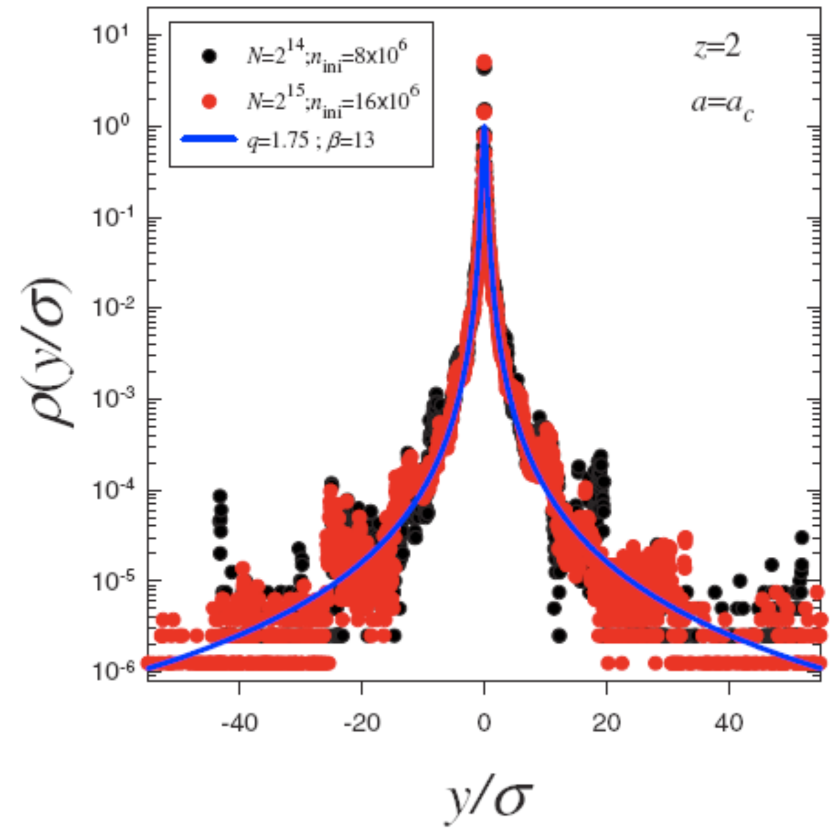


FIG. 4. (Color online) Probability density of the quantity y/σ at the critical point a_c for $z=2$, $N=2^{14}$, and $N=2^{15}$.

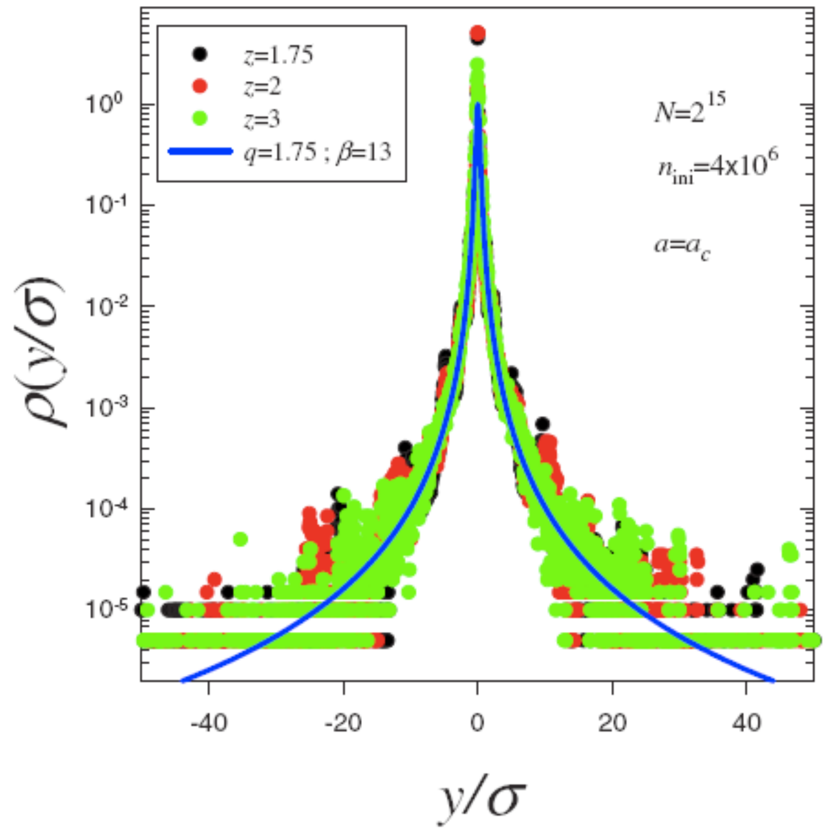


FIG. 5. (Color online) Probability density of the quantity y/σ at the critical point a_c for $z=1.75, 2, 3$.

Tirnakli, Beck, Tsallis PRE 75 (2007) 040106 (R)

Generalized Central limit theorem

The previous numerical results are supported by a generalization of the CLT within Tsallis q-statistics

Milan j. math. 76 (2008), 307–328

1424-9286/010307-22, *published online* 14.3.2008
DOI 10.1007/s00032-008-0087-y

Milan Journal of Mathematics

On a q-Central Limit Theorem Consistent with Nonextensive Statistical Mechanics

Sabir Umarov, Constantino Tsallis and Stanly Steinberg

Abstract. The standard central limit theorem plays a fundamental role in Boltzmann-Gibbs statistical mechanics. This important physical theory has been generalized [1] in 1988 by using the entropy $S_q = \frac{1-\sum_i p_i^q}{q-1}$ (with $q \in \mathcal{R}$) instead of its particular BG case $S_1 = S_{BG} = -\sum_i p_i \ln p_i$. The theory which emerges is usually referred to as *nonextensive statistical mechanics* and recovers the standard theory for $q = 1$. During the last two decades, this q-generalized statistical mechanics has been successfully applied to a considerable amount of physically interesting complex phenomena. A conjecture[2] and numerical indications available in the literature have been, for a few years, suggesting the possibility of q-versions of the standard central limit theorem by allowing the random variables that are being summed to be strongly correlated in some special manner, the case $q = 1$ corresponding to standard probabilistic independence. This is what we prove in the present paper for $1 \le q < 3$. The attractor, in the usual sense of a central limit theorem, is given by a distribution of the form $p(x) = C_q[1-(1-q)\beta x^2]^{1/(1-q)}$ with $\beta > 0$, and normalizing constant C_q. These distributions, sometimes referred to as q-Gaussians, are known to make, under appropriate constraints, extremal the functional S_q (in its continuous version). Their $q = 1$ and $q = 2$ particular cases recover respectively Gaussian and Cauchy distributions.

Central limit behavior in the HMF model

A Letters Journal Exploring the Frontiers of Physics

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Nonergodicity and central-limit behavior for long-range Hamiltonians

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[2] *Centro Brasileiro de Pesquisas Fisicas - Rua Xavier Sigaud 150, 22290-180 Rio de Janeiro-RJ, Brazil*
[3] *Santa Fe Institute - 1399 Hyde Park Road, Santa Fe, NM 87501, USA*

We have been studying the behavior of PDFs obtained considering time averages of the variables y so defined (along deterministic trajectories)

$$y_i = \frac{1}{\sqrt{n}} \sum_i^n p_j(i\delta) \qquad j = 1, 2, ...N$$

where p_j are the velocities of the *j-th* rotor taken at fixed intervals of time δ along the same trajectory

Central limit behavior in the HMF model

We consider deterministic trajectories in different regimes and for different energies and system sizes

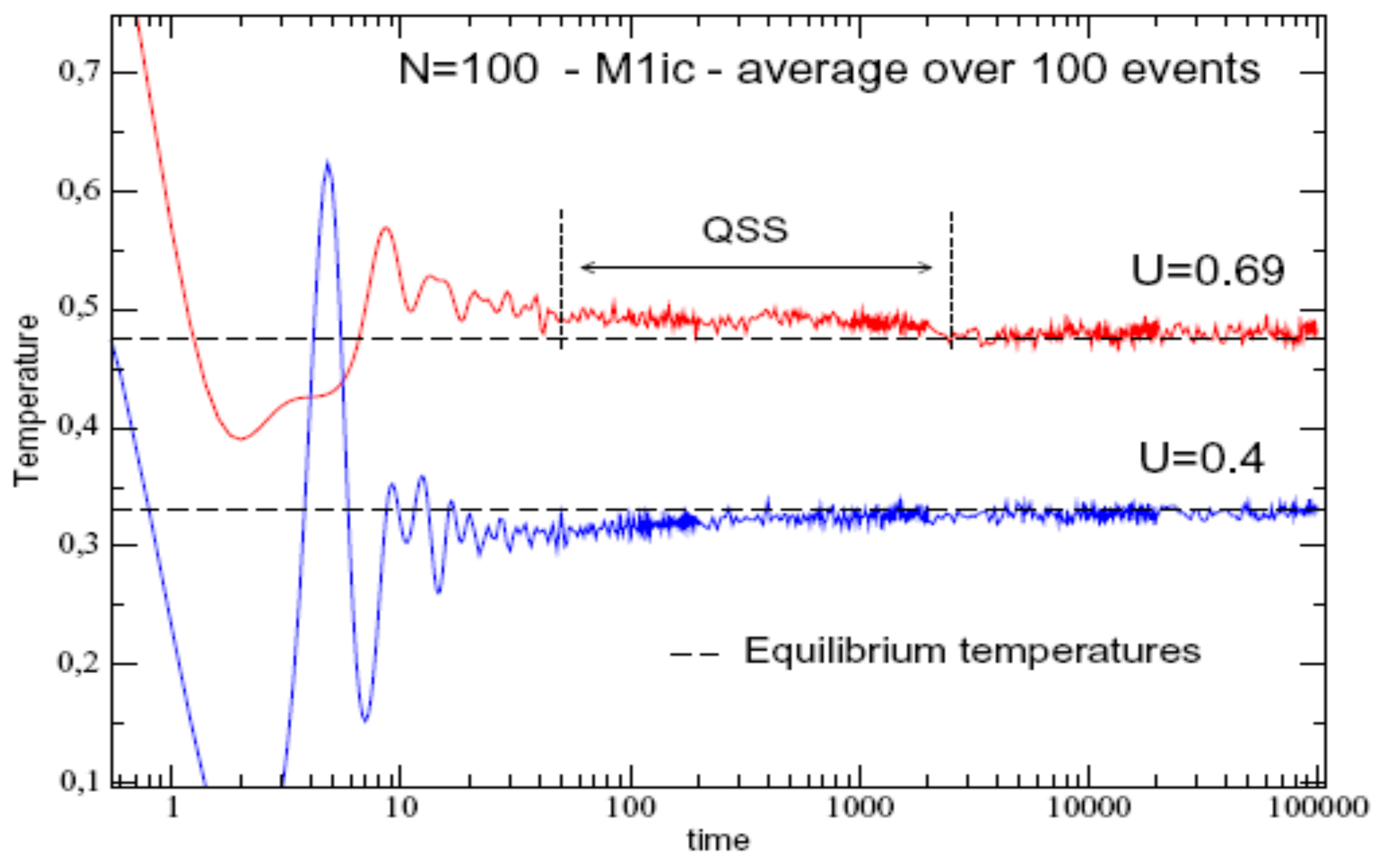
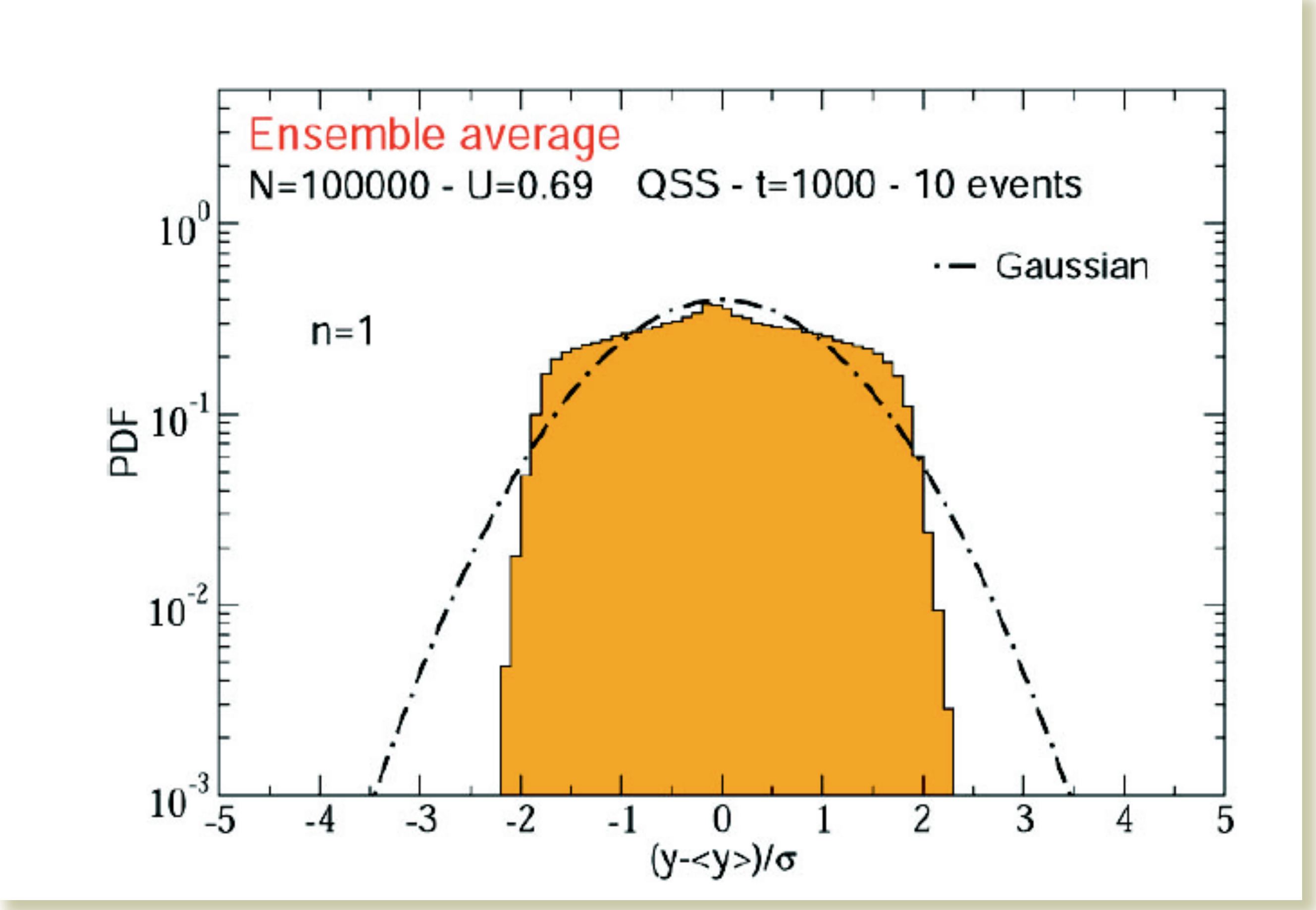
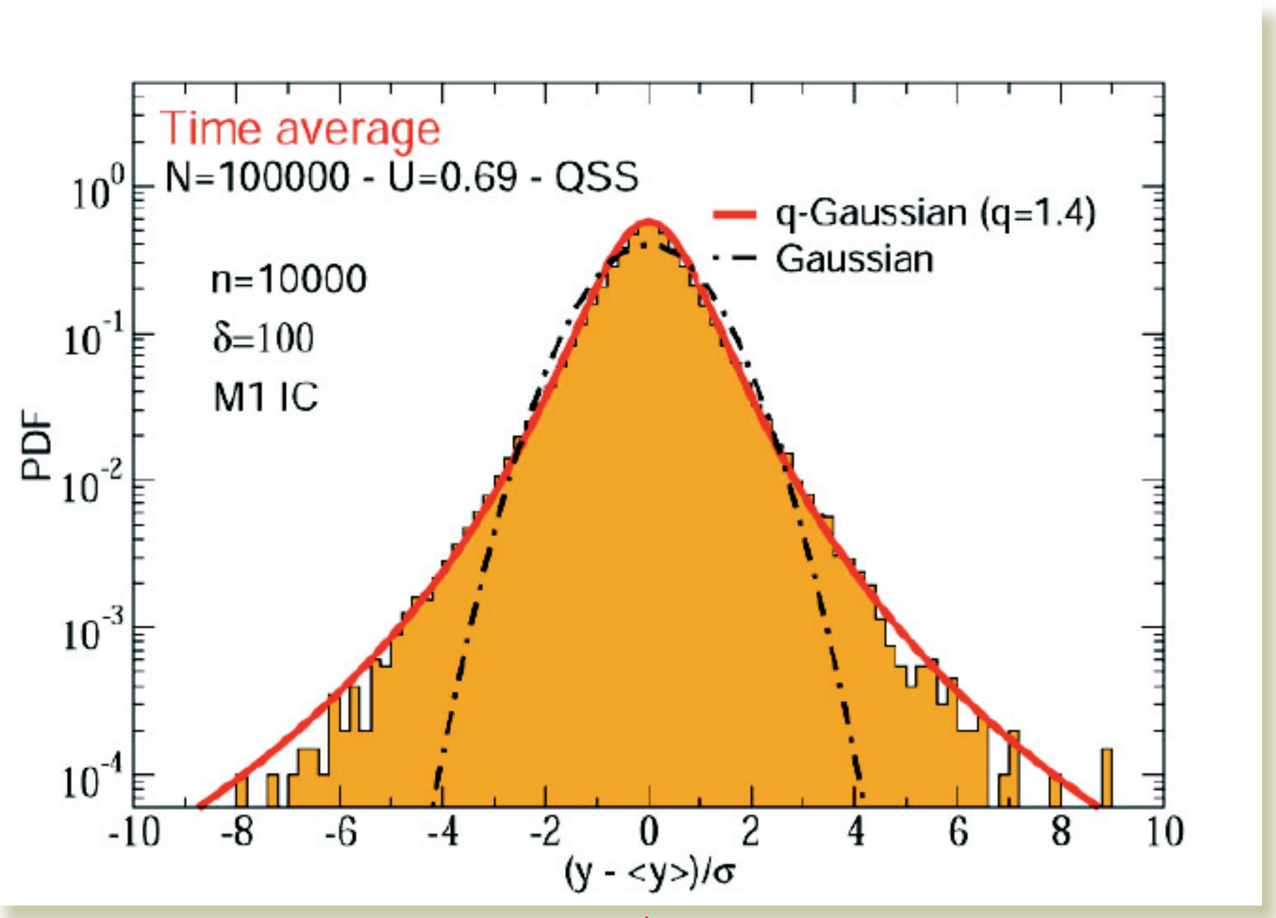


Fig. 1: Temperature evolution for the HMF system, with **N=100** and **M1 initial conditions**, for **U=0.69** (red line) and for **U=0.4** (blue line). A range of 100000 time steps is plotted. The presence of a QSS regime is visible only in the U=0.69 case, although a transient regime exist also for U=0.4.

CLT for the Hamiltonian Mean Field Model

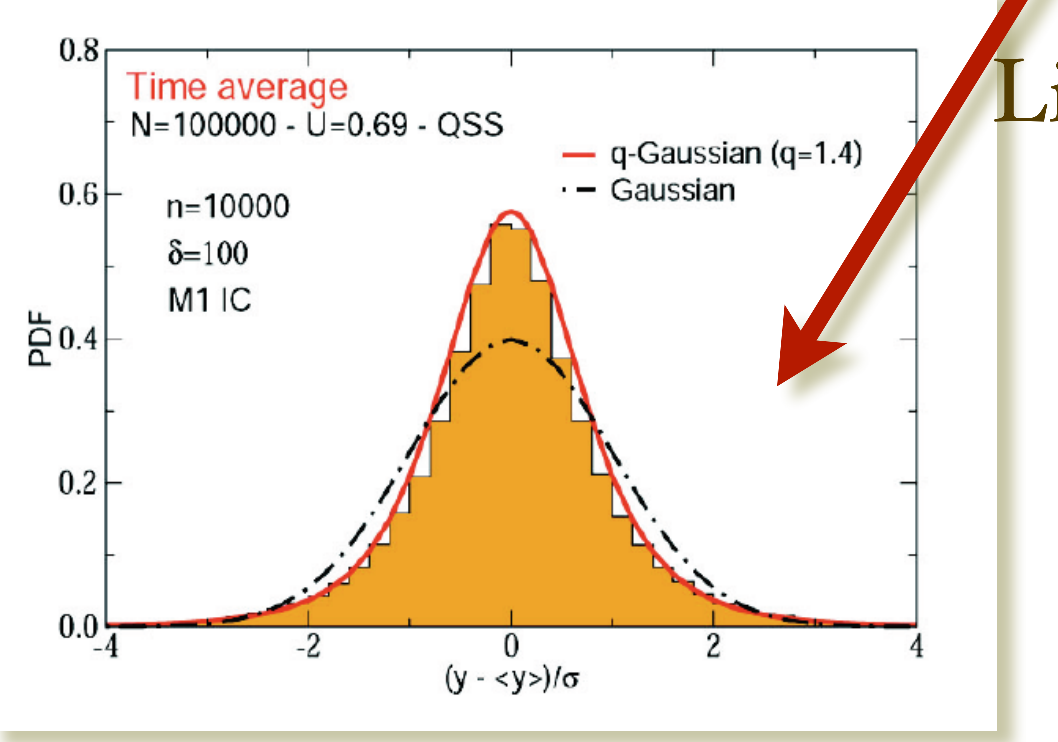
Inequivalence between ensemble average and time average
for N=100000



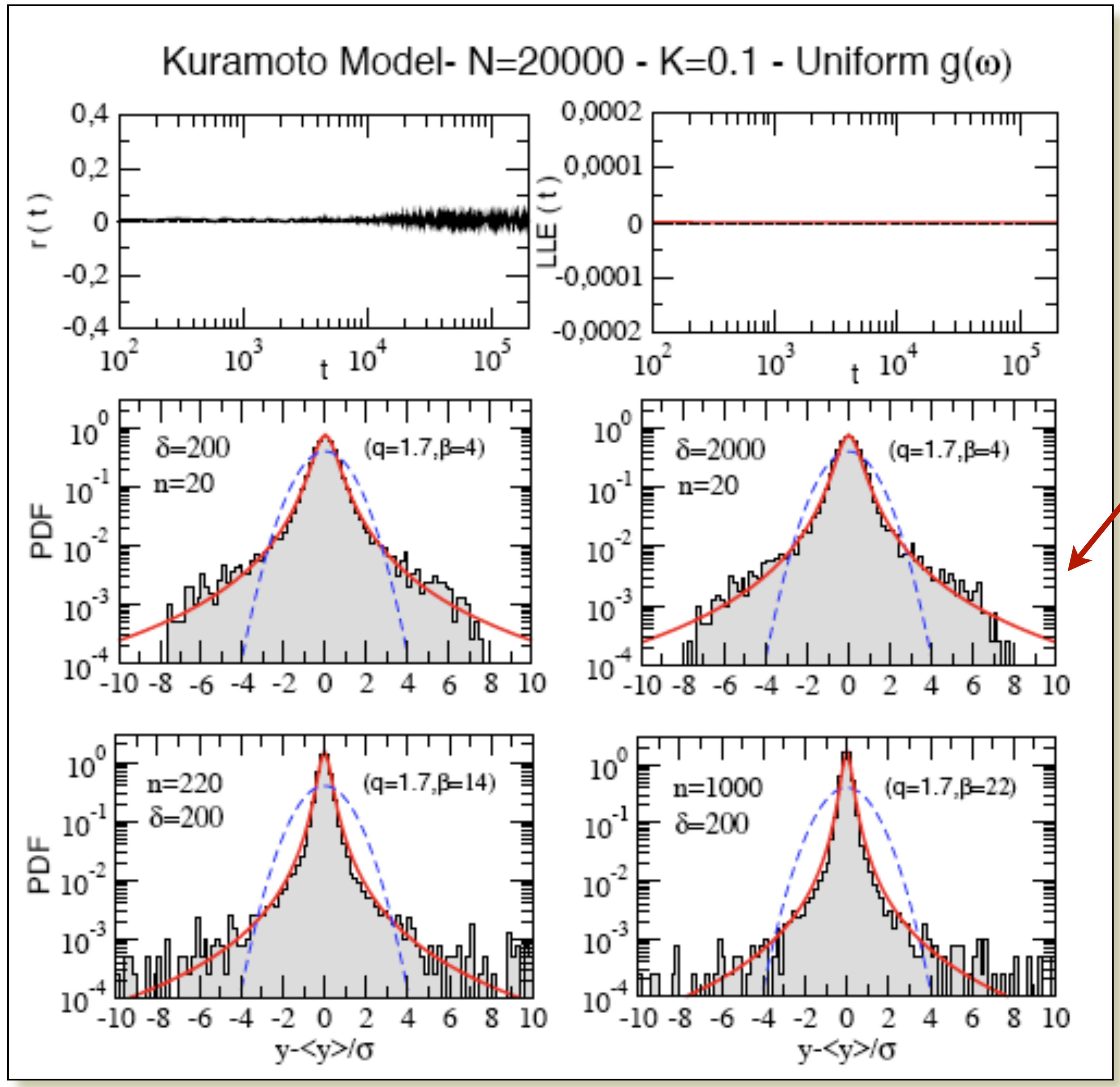


Linear scale

The q-Gaussian curve is able to reproduce well *not only the tail,* but also the *central part* of the PDF



CLT for the Kuramoto model



$$G_q(x) = A(1-(1-q)\beta x^2)^{1/1-q}$$

$$y_k = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta_k(i\delta) \qquad k = 1, \ldots, N$$

When the system is at the edge of chaos a q-Gaussian-like attractor emerges

A review of the latest results

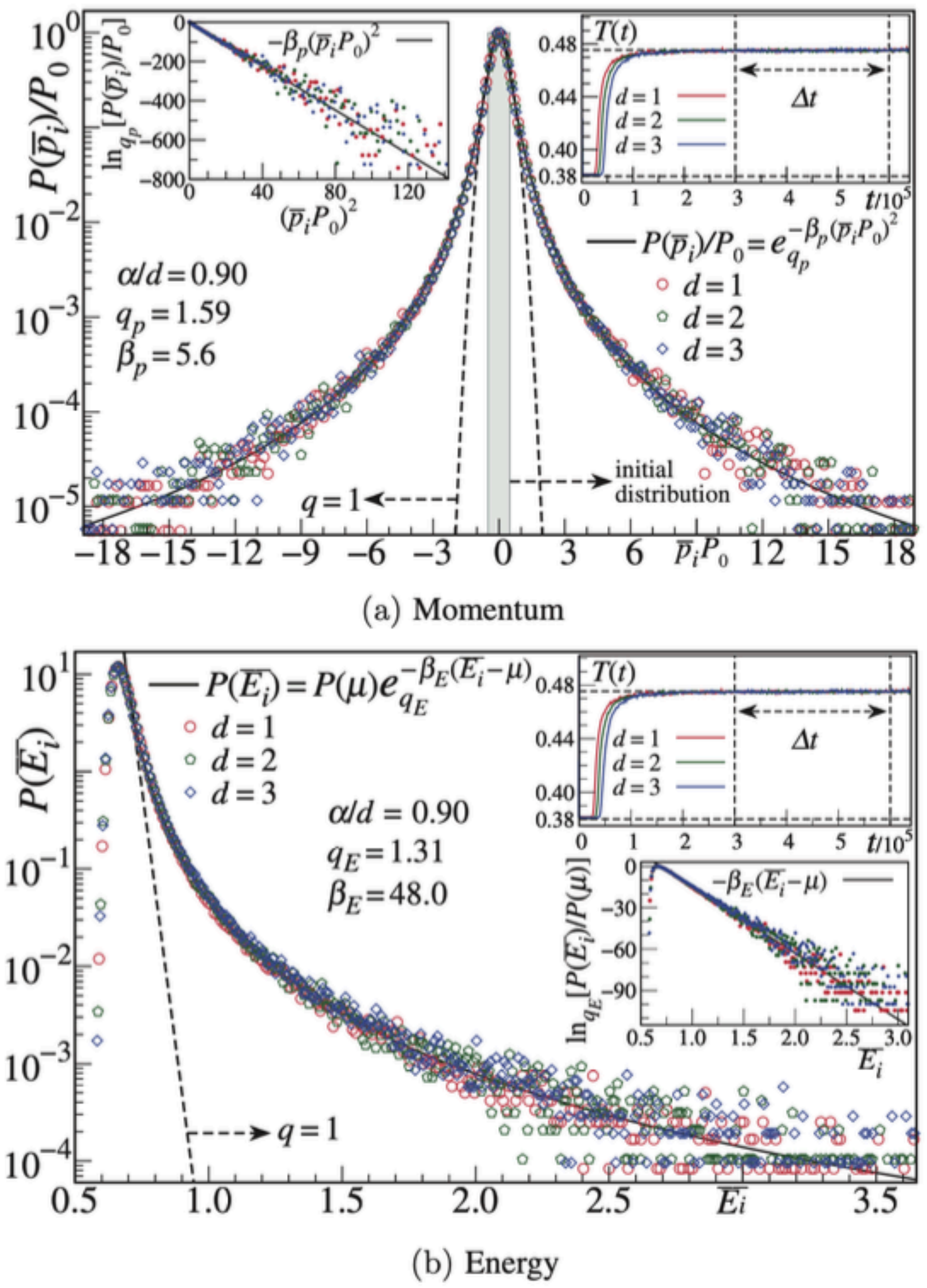


Figure 13. Inertial α-XY d-dimensional model (for $d = 1,2,3$) for $\alpha/d = 0.9$. **Left**: q_p-Gaussian distribution of momenta (for comparison, a Maxwellian distribution is indicated in dashed line). **Right**: q_E-exponential distribution of energies (for comparison, a BG distribution is indicated in dashed line). Both distributions are averaged along the very long-time interval indicated in the insets. Figure reproduced from Ref. [72].

symmetry MDPI

Review

Nonextensive Footprints in Dissipative and Conservative Dynamical Systems

Antonio Rodríguez [1,*,†], Alessandro Pluchino [2,3,†], Ugur Tirnakli [4,†], Andrea Rapisarda [3,5,†] and Constantino Tsallis [5,6,7,†]

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Abstract: Despite its centennial successes in describing physical systems at thermal equilibrium, Boltzmann–Gibbs (BG) statistical mechanics have exhibited, in the last several decades, several flaws in addressing out-of-equilibrium dynamics of many nonlinear complex systems. In such circumstances, it has been shown that an appropriate generalization of the BG theory, known as nonextensive statistical mechanics and based on nonadditive entropies, is able to satisfactorily handle wide classes of anomalous emerging features and violations of standard equilibrium prescriptions, such as ergodicity, mixing, breakdown of the symmetry of homogeneous occupancy of phase space, and related features. In the present study, we review various important results of nonextensive statistical mechanics for dissipative and conservative dynamical systems. In particular, we discuss applications to both discrete-time systems with a few degrees of freedom and continuous-time ones with many degrees of freedom, as well as to asymptotically scale-free networks and systems with diverse dimensionalities and ranges of interactions, of either classical or quantum nature.

Keywords: nonextensive statistical mechanics; long-range dynamical systems; entropy; complex systems

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1. Introduction

Statistical mechanics constitutes one of the pillars of contemporary theoretical physics. It was introduced in the 19th century by L. Boltzmann and J.W. Gibbs, and the name was coined by Gibbs himself. It is based on mechanics (classical, quantum, relativistic), electromagnetism, and theory of probabilities. Probabilities enter through the so-called entropic functional S, whose generic form for discrete stochastic variables is given by

$$S(\{p_i\}) = kF(\{p_i\}) \quad \left(\sum_{i=1}^{W} p_i = 1\right), \tag{1}$$

where $F(\{p_i\})$ is an appropriate generic functional, k being typically equal either to unity or to the Boltzmann constant k_B. Historically, Boltzmann and Gibbs used continuous variables ($p(x)$ instead of p_i). The corresponding discrete form is given by

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Happy 80th birthday Marcello !!!

Many thanks for your inspirational and invaluable guidance !

Thanks for your attention