Gravity and off-shell scattering amplitudes

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and work in progress

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Istituto Nazionale di Fisica Nucleare

SAPIENZA UNIVERSITÀ DI ROMA

Intro: classical physics from loops

Why gravity from scattering amplitudes?

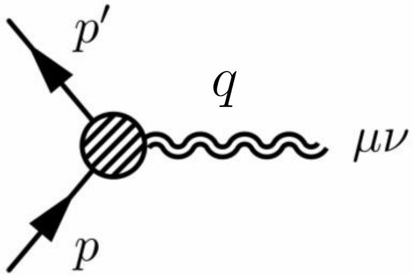
In the weak field regime, black holes and compact objects can be considered as point particles $\longrightarrow$ Massive fields coupled to gravity

From scattering amplitudes we can therefore recover the dynamics of unbound systems

Post-Minkowskian expansion, i.e. expansion in powers of G corresponds to a loop expansion

Crucial: loop contributions survive in the classical limit!

Intro: classical physics from loops

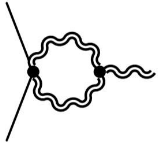


Massive propagator: $$\frac{i\hbar}{k^2 - (m/\hbar)^2 + i\epsilon}$$

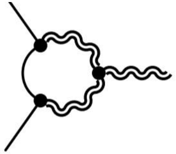
Analytic terms in amplitudes give rise to short-range quantum contributions

Non-analytic terms: signature of long range effects

Consider the one-loop diagrams



$$\int \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2(\ell+q)^2} = \frac{i}{32\pi^2}(-2\ln(-q^2)) + \dots$$



$$\int \frac{d^4\ell}{(2\pi)^2} \frac{1}{\ell^2(\ell+q)^2((p+\ell)^2 - m^2)} = \frac{i}{32\pi^2 m^2}\left(-\ln(-q^2) - \frac{\pi^2 m}{\sqrt{-q^2}}\right) + \dots$$

[Bjerrum-Bohr, Donoghue, Holstein, 2002]

"…we want to point out that there seems to exist an erroneous belief that only tree diagrams contribute to the classical process."
[Iwasaki, 1971]

Intro: classical physics from loops

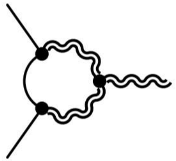
Modern approach: one can directly extract the classical contribution before computing the integral. By using

$$(p+\ell)^2 - m^2 = \ell^2 + 2\ell\cdot p \simeq 2\ell_0 m$$

2-scalar-1-graviton vertex in the static low energy limit:

$$(\tau_{\phi^2 h})_{\mu\nu} \simeq -i\,\kappa\, m^2 \delta^0_\mu \delta^0_\nu$$

we get

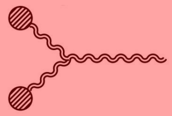


$$\frac{1}{2m}\int \frac{d^4\ell}{(2\pi)^2}\frac{1}{(\ell^2+i\epsilon)((\ell+q)^2+i\epsilon)(\ell_0+i\epsilon)}$$

Performing the ℓ_0 integral one gets

$$\int \frac{d^3\vec{\ell}}{(2\pi)^3}\frac{i}{4m}\frac{1}{\vec{\ell}^2}\frac{1}{(\vec{\ell}+\vec{q})^2} = -\frac{i}{32m|\vec{q}|}$$

[Bjerrum-Bohr, Damgaard, Festuccia, Planté, Vanhove, 2018]



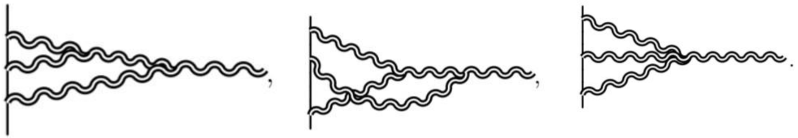
We recognize the 3dim integral of two static sources emitting massless fields [Duff, 1973]

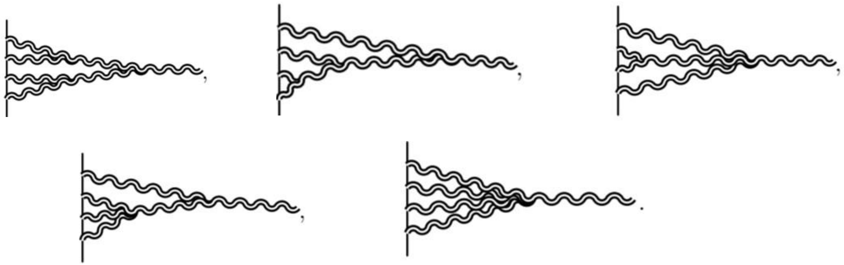
Extracting the classical limit from amplitudes made systematic using the KMOC procedure

[Kosower, Maybee, O'Connell, 2018]

BH solutions from scattering amplitudes

This has recently been exploited to derive the metric of a Schwarzschild-Tangherlini black hole up to 4PM (3 loops)
[Mougiakakos, Vanhove, 2020]





Computations performed in de Donder gauge:

$$\eta^{\mu\nu}\Gamma^{\alpha}_{\mu\nu} = \partial_\mu h^{\mu\alpha} - \frac{1}{2}\partial^\alpha h = 0$$

Metric is then derived from energy-momentum tensor

$$h_{\mu\nu}(\vec{x}) = -\frac{\kappa}{2}\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^2}\left(T_{\mu\nu}(\vec{q}^2) - \frac{1}{d-1}\eta_{\mu\nu}T(\vec{q}^2)\right)$$

$$\frac{\delta^3 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}} \to$$

$$\begin{aligned}
&\mathrm{Sym}\big[-\tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}) - \tfrac{1}{4}P_6(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}) + \tfrac{1}{4}P_3(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}) + \tfrac{1}{2}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}) + P_3(p^\sigma p^\lambda\eta^{\mu\nu}\eta^{\tau\rho})\\
&-\tfrac{1}{2}P_3(p^\tau p'^\mu\eta^{\nu\sigma}\eta^{\rho\lambda}) + \tfrac{1}{2}P_3(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}) + \tfrac{1}{2}P_6(p^\rho p^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}) + P_6(p^\sigma p'^\lambda\eta^{\tau\mu}\eta^{\nu\rho}) + P_3(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu})\\
&\qquad - P_3(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu})\big], \quad (2.6)
\end{aligned}$$

$$\frac{\delta^4 S}{\delta\varphi_{\mu\nu}\delta\varphi_{\sigma'\tau'}\delta\varphi_{\rho''\lambda''}\delta\varphi_{\iota'''\kappa'''}} \to$$

$$\begin{aligned}
&\mathrm{Sym}\big[-\tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\lambda}\eta^{\iota\kappa}) - \tfrac{1}{8}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\lambda}\eta^{\iota\kappa}) - \tfrac{1}{4}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa}) + \tfrac{1}{8}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\lambda}\eta^{\iota\kappa})\\
&+\tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\tau}\eta^{\rho\iota}\eta^{\lambda\kappa}) + \tfrac{1}{4}P_{12}(p^\sigma p^\tau\eta^{\mu\nu}\eta^{\rho\iota}\eta^{\lambda\kappa}) + \tfrac{1}{2}P_6(p^\sigma p'^\mu\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa}) - \tfrac{1}{4}P_6(p\cdot p'\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\rho\iota}\eta^{\lambda\kappa})\\
&+\tfrac{1}{4}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\sigma\rho}\eta^{\tau\lambda}\eta^{\iota\kappa}) + \tfrac{1}{4}P_{24}(p^\sigma p^\tau\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\iota\kappa}) + \tfrac{1}{4}P_{12}(p^\rho p'^\lambda\eta^{\mu\sigma}\eta^{\nu\tau}\eta^{\iota\kappa}) + \tfrac{1}{2}P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\lambda}\eta^{\iota\kappa})\\
&-\tfrac{1}{2}P_{12}(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\mu}\eta^{\iota\kappa}) - \tfrac{1}{2}P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\nu}\eta^{\iota\kappa}) + \tfrac{1}{2}P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\nu}\eta^{\iota\kappa}) - \tfrac{1}{2}P_{24}(p\cdot p'\eta^{\mu\nu}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\sigma})\\
&-P_{12}(p^\sigma p^\tau\eta^{\nu\rho}\eta^{\lambda\iota}\eta^{\kappa\mu}) - P_{12}(p^\rho p'^\lambda\eta^{\nu\iota}\eta^{\kappa\sigma}\eta^{\tau\mu}) - P_{24}(p_\sigma p'^\rho\eta^{\tau\iota}\eta^{\kappa\mu}\eta^{\nu\lambda}) - P_{12}(p^\rho p'^\iota\eta^{\lambda\sigma}\eta^{\tau\mu}\eta^{\nu\kappa})\\
&+P_6(p\cdot p'\eta^{\nu\rho}\eta^{\lambda\sigma}\eta^{\tau\iota}\eta^{\kappa\mu}) - P_{12}(p^\sigma p^\rho\eta^{\mu\nu}\eta^{\tau\iota}\eta^{\kappa\lambda}) - \tfrac{1}{2}P_{12}(p\cdot p'\eta^{\mu\rho}\eta^{\nu\lambda}\eta^{\sigma\iota}\eta^{\tau\kappa}) - P_{12}(p^\sigma p^\rho\eta^{\tau\lambda}\eta^{\mu\iota}\eta^{\nu\kappa})\\
&\qquad -P_6(p^\rho p'^\iota\eta^{\lambda\kappa}\eta^{\mu\sigma}\eta^{\nu\tau}) - P_{24}(p^\sigma p'^\rho\eta^{\tau\mu}\eta^{\nu\iota}\eta^{\kappa\lambda}) - P_{12}(p^\sigma p'^\mu\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\nu}) + 2P_6(p\cdot p'\eta^{\nu\sigma}\eta^{\tau\rho}\eta^{\lambda\iota}\eta^{\kappa\mu})\big]. \quad (2.7)
\end{aligned}$$

[De Witt, 1967]

BH solutions from scattering amplitudes

Master integral:

$$\int \prod_{i=1}^{l} \frac{d^d \vec{\ell}_i}{(2\pi)^d} \frac{\vec{q}^2}{\left(\prod_{i=1}^{l} \vec{\ell}_i^{\,2}\right)\left(\vec{q} - \vec{\ell}_1 - \ldots - \vec{\ell}_l\right)^2} = J_{(l)}(\vec{q}^2) \qquad J_{(l)}(\vec{q}^2) = \frac{\Gamma\left(l+1-\frac{ld}{2}\right)\Gamma\left(\frac{d-2}{2}\right)^{l+1}}{(4\pi)^{\frac{ld}{2}}\Gamma\left(\frac{(l+1)(d-2)}{2}\right)}(\vec{q}^2)^{\frac{l(d-2)}{2}}$$

The same computations can actually be performed in a generalized gauge

$$G^\alpha = (1-a)(\partial_\mu h^{\mu\alpha} - \frac{1}{2}\partial^\alpha h) + a g^{\mu\nu}\Gamma^\alpha_{\mu\nu} = 0$$

where a interpolates between de Donder and harmonic gauge

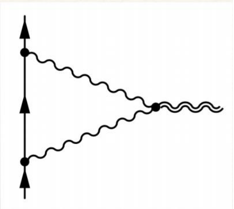
[Jakobsen, 2020]

If $a \neq 0$, the gauge fixing term in the lagrangian modifies the n-point vertex

Performed at one-loop (2PM) order in arbitrary dimension

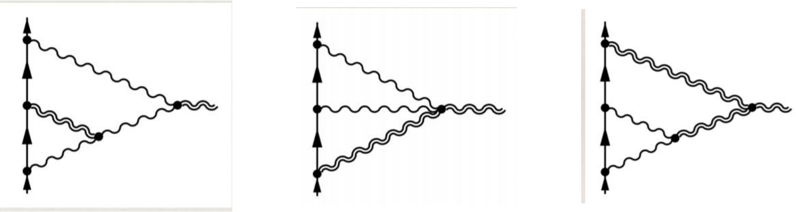
BH solutions from scattering amplitudes

We have extended the analysis to charged non-rotating black holes by considering charged scalars



2-scalar-1-photon vertex in the static low-energy limit:

$$\left(\tau_{\phi^2 A}\right)_\alpha \simeq -2\, i\, Q\, m\, \delta^0_\alpha$$



[Donoghue, Holstein, Garbrecht, Konstandin, 2001]

[D'Onofrio, Fragomeno, Gambino, FR, 2022]

Reissner-Nordstrom-Tangherlini metric in de Donder gauge (up to two loops):

$$h_0^{(3,2)}(r) = -\frac{4(d-2)^2(3d-11)}{(d-4)(d-1)^2} m\alpha G_N^2 \rho^3$$

$$h_1^{(3,2)}(r) = \frac{8(3d^3-25d^2+69d-65)}{3(d-4)(d-3)(d-1)^2} m\alpha G_N^2 \rho^3$$

$$h_2^{(3,2)}(r) = -\frac{4(d-2)^2(3d^3-19d^2+33d-17)}{(d-4)(d-3)(d-1)^3} m\alpha G_N^2 \rho^3$$

$$ds^2 = h_0(r)dt^2 - h_1(r)d\vec{x}^2 - h_2(r)\frac{(\vec{x}\cdot d\vec{x})^2}{r^2}$$

$$\rho(r) = \frac{\Gamma(\frac{d}{2}-1)\pi^{1-d/2}}{r^{d-2}}$$

Singular in four and five dimensions!

BH solutions from scattering amplitudes

In order to renormalize the divergences, we add higher-derivative coupling terms to the action

$$\delta^{(1)} S^{ct} = (G_N m)^{\frac{2}{d-2}} \alpha^{(1)}(d) \int d^{d+1}x \sqrt{-g}\, R\, D_\mu \phi D^\mu \phi$$

$$\alpha^{(1)}(3) = \frac{\omega(3)}{d-3} + \Omega(3)$$

$$\omega(3) = \frac{1}{12} - \frac{\alpha G_N}{6(G_N m)^2}$$

The renormalization procedure leaves a free parameter undetermined

$$\alpha^{(1)}(4) = \frac{\omega(4)}{d-4} + \Omega(4)$$

$$\omega(4) = \frac{5}{18\pi} + \frac{G_N \alpha}{(G_N m)^2} \frac{1}{24\pi}$$

The divergence occurs at one loop in five dimensions and at two loops in four.

There is a logarithmic term in the metric. The free parameter signals the fact that there is a redundancy in the gauge-fixing condition.

In five dimensions, the special gauge choice $a = \frac{5}{6}$ removes the pole

[Jakobsen, 2020]

Kerr black holes and spin universality

Radiative corrections to the energy momentum tensor of a massive spin ½ field: originally done at one loop.
[Bjerrum-Bohr, Donoghue, Holstein, 2002]

From the 2-spinor-1-graviton vertex, employing the Gordon identity in the low-energy limit one writes

$$\bar{u}(p_2,\sigma_2)\tau_{\mu\nu}u(p_1,\sigma_1) = -i\frac{\kappa}{2}\left[m\delta^0_\mu\delta^0_\nu - \frac{i}{2}(\delta^0_\mu S_{\nu\sigma} + \delta^0_\mu S_{\mu\sigma})q^\sigma\right]$$

Where the first term coincides with the scalar case.

From the tree-level energy-momentum tensor one then obtains the scalar and dipole terms of the Kerr metric at 1PM in harmonic gauge

$$h_{0i} = \frac{2G}{r^3}(\vec{S}\times\vec{r})_i$$

Trick: write vertex in terms of $P = 1/2(p_1+p_2)$ and q.
Then work out the powers of $\hbar$

Kerr black holes and spin universality

At one loop: we rewrite the massive propagator using

$$\sum u^s(\vec{p})\bar{u}^s(\vec{p}) = \not{p} + m$$

This allows to rearrange the amplitude as containing two ``tree-level'' vertices and a scalar propagator which is treated exactly as before.

The scalar part of the vertices reproduces the scalar part of the metric at 2PM, which obviously coincides with Schwarzschild.

The terms at first order in S give rise to the dipole term at 2PM of the Kerr metric.

This can naturally be extended to any loop order obtaining the dipole term at any PM order.

Computations have been performed up to two loops and in generalized gauge.

Kerr black holes and spin universality

For scalars, non minimal couplings do not give classical contributions, higher derivative couplings can only give counterterms. [Mougiakakos, Vanhove, 2020]

This corresponds to the fact that the Schwarzschild metric is the most general spherically symmetric vacuum solution of GR.

For spin ½ we expect the same to happen because the dipole term is universal. There is no non-minimal coupling. Same for higher derivative terms.

We expect that at quadrupole order we can add non-minimal couplings giving rise to classical contributions that differ from Kerr.

Kerr black holes and spin universality

$$\epsilon_\alpha\{J_{\mu\rho}, J_{\nu\sigma}\}^{\alpha\beta}\epsilon_\beta = -S_{\mu\rho}S_{\nu\sigma}$$

Consider a massive spin 1 field (Proca field).

At tree level, the vertex contracted with the polarizations can be rearranged in such a way that the scalar and dipole terms are exactly as before. Quadrupole term comes from anticommutator $\{J, J\}$.

Taking the classical and low-energy limit one gets the dressed vertex

Rest frame:
$S^{ij} = \epsilon^{ijk}S_k$

$$\hat{\tau}_{\mu\nu} = -i\frac{\kappa}{2}\left[2m^2\delta^0_\mu\delta^0_\nu - im(\delta^0_\mu S_{\nu\sigma} + \delta^0_\mu S_{\mu\sigma})q^\sigma - S_{\mu\alpha}S_{\nu\beta}q^\alpha q^\beta\right]$$

One can then compute the metric from the tree-level amplitude, obtaining the same contributions as before as far as scalar and dipole are concerned, plus the quadrupole term of Kerr.

Adding the operator $R_{\mu\nu}A^\mu A^\nu$ produces a term which is exactly of the form of the quadrupole term above.

Kerr black holes and spin universality

One loop: same as spinors. The completeness relation allows to rearrange the massive line in such a way that one ends up with two dressed vertices (scalar plus dipole plus quadrupole) and a scalar propagator.

Integrals involved (quadrupole terms): up to four momenta in the numerator at one loop. Momentum integrals computed using LiteRed.

The analysis has been performed in generalized gauge

In particular we have derived the four-graviton vertex in generic gauge, and we have performed the two loop calculation for massive spin 1

Spin 1 quadrupole term

Out[281]=

$$-\frac{3}{16}\kappa^2 l^a l^{a1} q^{a2} q^{b1} S_{ab1} S_{a1a2}+\frac{\kappa^2 l^a l^{a1} q^{a2} q^{b1} S_{ab1} S_{a1a2}}{4(-1+d)}-\frac{d\kappa^2 l^a l^{a1} q^{a2} q^{b1} S_{ab1} S_{a1a2}}{4(-1+d)}+\frac{1}{16}\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}-\frac{3\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}}{8(-1+d)^2}-\frac{\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}}{8(-1+d)}+$$

$$\frac{d\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}}{2(-1+d)^2}+\frac{d\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}}{8(-1+d)}-\frac{d^2\kappa^2 l^a l^{a1} q_{a2} q^{a2} S_{ab1} S_{a1}{}^{b1}}{8(-1+d)^2}+\frac{1}{16}\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}-\frac{\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}}{4(-1+d)^2}-\frac{\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}+$$

$$\frac{3d\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}+\frac{d\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}-\frac{d^2\kappa^2 l^a l_{a1} l^{a1} l^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{1}{16}\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}+\frac{\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}}{4(-1+d)^2}+\frac{\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}}{8(-1+d)}-$$

$$\frac{3d\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{d\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}}{8(-1+d)}+\frac{d^2\kappa^2 l^a l^{a1} l^{a2} q_{a1} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{1}{16}\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}+\frac{\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{4(-1+d)^2}+\frac{\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}-$$

$$\frac{3d\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{d\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}+\frac{d^2\kappa^2 l^a l_{a1} l^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{\kappa^2 l_{a1} l^{a1} q^a q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}-\frac{\kappa^2 l_{a1} l^{a1} q^a q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)}+\frac{3d\kappa^2 l_{a1} l^{a1} q^a q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}+$$

$$\frac{d\kappa^2 l_{a1} l^{a1} q^a q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)}-\frac{d^2\kappa^2 l_{a1} l^{a1} q^a q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}+\frac{1}{16}\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}-\frac{\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{4(-1+d)^2}-\frac{3\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}+\frac{3d\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}+$$

$$\frac{d\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}-\frac{d^2\kappa^2 l^a l^{a1} q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}+\frac{\kappa^2 l^{a1} q^a q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)^2}+\frac{3\kappa^2 l^{a1} q^a q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)}-\frac{3d\kappa^2 l^{a1} q^a q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}-\frac{d\kappa^2 l^{a1} q^a q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)}+$$

$$\frac{d^2\kappa^2 l^{a1} q^a q_{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}-\frac{3\kappa^2 q^a q_{a1} q^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}-\frac{\kappa^2 q^a q_{a1} q^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{8(-1+d)}+\frac{d\kappa^2 q^a q_{a1} q^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{4(-1+d)^2}+\frac{d\kappa^2 q^a q_{a1} q^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)}-\frac{d^2\kappa^2 q^a q_{a1} q^{a1} q^{a2} S_a{}^{b1} S_{a2b1}}{16(-1+d)^2}-$$

$$\frac{1}{16}\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}+\frac{3\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}}{8(-1+d)^2}+\frac{\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}}{4(-1+d)}-\frac{d\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}}{2(-1+d)^2}-\frac{d\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}}{8(-1+d)}+\frac{d^2\kappa^2 l^{a1} q_a q^a q^{a2} S_{a1}{}^{b1} S_{a2b1}}{8(-1+d)^2}$$

Numerator of the 1-loop integrand - 00 component

Spin 1 quadrupole term

$$\text{Out[295]=} -\frac{\kappa^2 q^i q_{i1} q^{i1} q^{i2} S_i{}^{j1} S_{i2j1}}{8(-1+d)^2}+\frac{23\, d\, \kappa^2 q^i q_{i1} q^{i1} q^{i2} S_i{}^{j1} S_{i2j1}}{128(-1+d)^2}-\frac{d^2 \kappa^2 q^i q_{i1} q^{i1} q^{i2} S_i{}^{j1} S_{i2j1}}{128(-1+d)^2}-\frac{5\kappa^2 q_i q^i q_{i1} q^{i1} S_{i2j1} S^{i2j1}}{128(-1+d)^2}-\frac{d\,\kappa^2 q_i q^i q_{i1} q^{i1} S_{i2j1} S^{i2j1}}{128(-1+d)^2}$$

The result of the loop integral (performed using LiteRed)

$$\text{Out[316]=} \frac{(-2+d)\,\kappa^2\left(\left(16-23\,d+d^2\right) R^i R^{i1} S_{ii2} S_{i1}{}^{i2}+13\, r^2 S_{i1i} S^{i1i}\right)\rho^2}{512(-1+d)\,\pi^2 r^4}$$

Fourier transform using master integrals

$$\rho(r)=\frac{\Gamma(\frac{d}{2}-1)\pi^{1-d/2}}{r^{d-2}}$$

Courtesy of C. Gambino

Kerr black holes and spin universality

In progress:

We want to compare with the multipole expansion of Kerr

Primary interest: non-Kerr metrics. We want to study the systematics of how non-minimal couplings producing classical terms occur and how to compare with exotic compact objects [Hartle, Thorne, 1968]

In higher dimensions: Mayers-Perry

Picture that emerges: Minimally coupled field of spin S reproduces Kerr in the limit $S \to \infty$

Spin universality: considering a field of spin S, one obtains the multipole expansion of Kerr up to $2S$ order

Kerr black holes and spin universality

Massive spinor helicity formalism (on-shell): most general 3pt amplitude of two massive spin S fields of mass m and a massless field of helicity h is
[Arkani-Hamed, Huang, Huang, 2017]

$$M_3^{h,s,s} = (mx)^h \left[g_0 \frac{\langle \mathbf{21} \rangle^{2s}}{m^{2s-1}} + g_1 x \frac{\langle \mathbf{21} \rangle^{2s-1} \langle \mathbf{23} \rangle \langle \mathbf{31} \rangle}{m^{2s}} + \cdots + g_{2s} x^{2s} \frac{\langle \mathbf{23} \rangle^{2s} \langle \mathbf{31} \rangle^{2s}}{m^{4s-1}} \right]$$

Consider for instance a gauge field coupled to a spin ½ massive spinor. The minimal coupling corresponds to the first term, while the Pauli coupling to the second.

Simplest amplitude (best UV behaviour): only first term (minimal coupling)

We apply this to gravity: minimal coupling reproduces Kerr in the $S \to \infty$ limit
[Chung, Huang, Kim, Lee, 2018]

Effective Kerr vertex at all orders in a

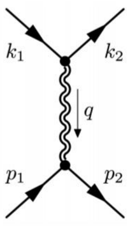
Simplest 3pt amplitude of a massive field coupled to gravity:

$$M_3^{+2,s,s} = x^2 \frac{1}{M_{Pl}} \frac{\langle \mathbf{12} \rangle}{m^{2s-2}} \qquad M_3^{-2,s,s} = \frac{1}{x^2} \frac{1}{M_{Pl}} \frac{[\mathbf{12}]}{m^{2s-2}}$$

Kerr: Simplest massive S-matrix. The fact that minimally coupled fields reconstruct BHs resembles the no-hair theorem in a QFT language

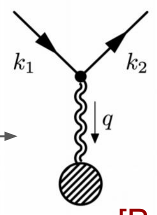
Starting from the Kerr metric it is possible to extract the effective vertex in the harmonic gauge exactly in a [Vines, 2017].

$$g_{\mu\nu}^{Kerr}(x) = \eta_{\mu\nu} + h_{\mu\nu}(x) \dots \qquad \tilde{h}_{\mu\nu}(q) = V^{\alpha\beta}(q) \frac{iP_{\mu\nu,\alpha\beta}}{q^2}$$



BH-BH scattering

$$\mathcal{M}_{2\to 2} = V_{\mu\nu}(q) \frac{iP^{\mu\nu\alpha\beta}}{q^2} V_{\alpha\beta}(q)$$



$$\mathcal{M}_{fix} = -\frac{i}{2} \tilde{h}^{\mu\nu}(q) T_{\mu\nu}(q)$$

[Bautista, Guevara, Kavanagh, Vines, 2023]

Kerr-Schild gauge

Idea: study the Kerr-Newman metric in Kerr-Schild gauge

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + \Phi K_\mu K_\nu \qquad \Phi = -\frac{2GMr + GQ^2}{r^2 + a^2\cos^2\theta} \qquad K_\mu = \left(1, \frac{rx + ay}{r^2 + a^2}, \frac{ry - ax}{r^2 + a^2}, \frac{z}{r}\right)$$

Crucial property:

$$g^{\mu\nu} K_\mu K_\nu = \eta^{\mu\nu} K_\mu K_\nu = 0$$

Implies

$$g^{\mu\nu} = \eta^{\mu\nu} - \Phi K^\mu K^\nu$$

$$\sqrt{-g} = 1$$

For Schwarzschild: $\Phi = -\frac{2GM}{r} \qquad K_\mu = (1, \hat{r})$

$$ds^2 = \left(1 - \frac{2GM}{r}\right) dt^2 - \left(1 + \frac{2GM}{r}\right) dr^2 - \frac{4GM}{r} dr dt - r^2 d\Omega^2$$

$$t = t_S + 2GM \log(r - 2GM)$$

Gauge potential:

$$A_\mu = V_A K_\mu \qquad V_A = \frac{Qr}{r^2 + a^2\cos^2\theta}$$

Oblate spheroidal coordinates:

$$x = \sqrt{r^2 + a^2} \sin\theta \cos\phi$$
$$y = \sqrt{r^2 + a^2} \sin\theta \sin\phi$$
$$z = r\cos\theta$$

Kerr-Schild gauge

Scalar-gravity interaction:

$$\mathcal{L}_{int} = \frac{1}{2} h^{\mu\nu}(x) T^{\phi}_{\mu\nu}(x)$$

No higher-order interaction terms. All information encoded in trilinear vertex.

Scattering amplitude: $$i\mathcal{M}_{KN}(p, p', \vec{q}\,) = i\tilde{h}^{\mu\nu}_{KN}(\vec{q}\,) p_\mu p'_\nu$$

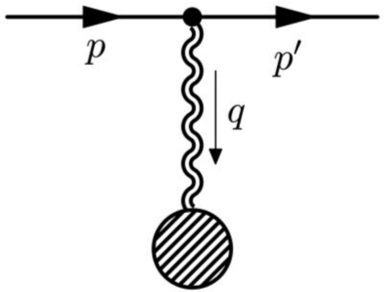
We have to compute the Fourier transform of $h_{\mu\nu}(x)$

Remarkably, we manage to compute this exactly thanks to another crucial property:

$d^3x\Phi(x) = dr d\Omega G(-2Mr + Q^2)$

The resulting metric in momentum space contains spherical Bessel (i.e. trigonometric) and Bessel functions.

Kerr-Schild gauge



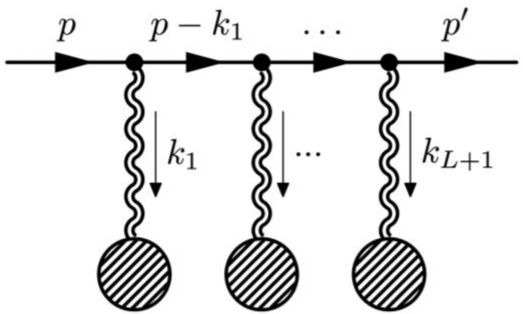
$$\tilde{h}_{00}(\vec{q}) = \frac{1}{|\vec{q}|^2} \cos|\vec{a}\times\vec{q}|$$

$$\tilde{h}_{0i}(\vec{q}) = -i\frac{q_i}{|\vec{q}|^3}\frac{\pi}{2}J_0(|\vec{a}\times\vec{q}|) + i\frac{(\vec{a}\times\vec{q})_i}{|\vec{q}|^2}j_0(|\vec{a}\times\vec{q}|)$$

$$\tilde{h}_{ij}(\vec{q}) = \frac{j_0(|\vec{a}\times\vec{q}|)}{|\vec{q}|^2}\left(\delta_{ij} - 2\frac{q_i q_j}{|\vec{q}|^2}\right) + \frac{1}{|\vec{q}|^3}\frac{\pi}{2}\frac{J_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}\Big(q_i(\vec{a}\times\vec{q})_j + q_j(\vec{a}\times\vec{q})_i\Big)$$
$$-\frac{1}{|\vec{q}|^2}\frac{j_1(|\vec{a}\times\vec{q}|)}{|\vec{a}\times\vec{q}|}(\vec{a}\times\vec{q})_i(\vec{a}\times\vec{q})_j$$

In the tree-level amplitude, putting on-shell the scalar momenta, the Bessel functions disappear in the term proportional to the mass of the BH, while the spherical Bessel functions disappear in the term proportional to the charge.

The eikonal expansion



S-matrix:

$$\widetilde{\mathcal{S}}(p,\vec{b}\,) = 1 + i\widetilde{\mathcal{T}}(p,\vec{b}\,) = e^{2i\delta(p,\vec{b}\,)}$$

where $\delta(p,\vec{b}\,)$ is the eikonal phase and $\vec{b}$ the impact parameter

$$i\widetilde{\mathcal{T}}(p,\vec{b}\,) = i\sum_{n=1}^{+\infty}\widetilde{\mathcal{M}}^{(n)}(p,\vec{b}\,) = \sum_{m=1}^{+\infty}\frac{1}{m!}\left(2i\sum_{n=1}^{+\infty}\delta^{(n)}(p,\vec{b}\,)\right)^m$$

where $i\widetilde{\mathcal{M}}^{(n)}$ is the amplitude in impact parameter space .

Tree level: classical

One loop: classical plus hyper-classical

The eikonal expansion

Let's focus on the one-loop case. The amplitude is

$$i\mathcal{M}^{(2)} = \int \frac{d^3k}{(2\pi)^3} i\mathcal{M}_{KN}(p, p-k, \vec{k}\,) \frac{i}{(p-k)^2 - m^2 + i\varepsilon} i\mathcal{M}_{KN}(p-k, p-q, \vec{q}-\vec{k}\,)$$

We want to compute the classical contribution arising from this amplitude. We write the ``tree-level'' amplitudes as

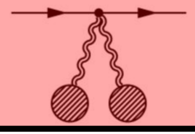
$$i\mathcal{M}^{KN} = i\mathcal{M}_0^{KN} + i\mathcal{M}_{extra}^{KN}$$

where the first terms contains the spherical Bessel functions and the extra term contains the Bessel functions.
Classical contribution:

$$i\mathcal{M}^{(2)}\Big|_{cl.} = i\mathcal{M}^{(2)}\Big|_{cl.}^{(0,0)} + i\mathcal{M}^{(2)}\Big|_{cl.}^{(extra,extra)} + 2i\mathcal{M}^{(2)}\Big|_{cl.}^{(extra,0)} .$$

In the (extra,extra) term the propagator drops and the amplitude looks like arising from a contact vertex!!

$$i\mathcal{M}^{(2)}\Big|_{cl.}^{(extra,extra)} \sim$$



$.$

The eikonal expansion

From the eikonal we can finally compute the deflection angle

$$\vartheta(p,\vec{b}\,) = -\frac{2}{|\vec{p}\,|}\frac{\partial\delta(p,\vec{b}\,)}{\partial b}$$

At tree level we get

$$\vartheta^{(1)} = \frac{GM}{v^2}\sum_{\pm}\frac{(1\pm v)^2(b\mp a\cos\beta)}{a^2\sin^2\alpha\sin^2\beta+(b\mp a\cos\beta)^2}$$

reproducing known results [Guevara, Ochirov, Vines, 2018]. We also derive the leading eikonal phase for the charged part of the metric.

We work out the Schwarzschild case at one loop, obtaining the well known expression

[Damour, 2017]

$$\vartheta^{(2)}\Big|_{a=0} = \frac{3G^2M^2\pi^2}{4v^2b^2}(4+v^2)$$

Summary and conclusions

Why is Kerr-Schild gauge so special? Might be more fundamental because it manifestly resembles the double copy [Monteiro, O'Connell, White, 2014]

Easier and complementary way to perform computations

What next?

- 2PM eikonal phase for KN
- Extend analysis to probes with spin
- Is there a propagator in Kerr-Schild gauge? Can we generalise this approach to 2-to-2 scattering of BHs?

THANK YOU!