Instantons, AGT and the PN computation of the luminosity of a binary system

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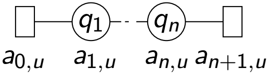
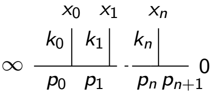
Introduction

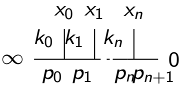
- $\mathcal{N} = 2$ non perturbative computations and black holes
- QNM, Love number...
- A new application: comparison with PN computations in binary systems

AGT





$$\mathcal{F}_{p_0}{}^{k_0}{}_{p_1}\,..{}^{k_n}{}_{p_{n+1}}(x_i) = \left\langle p_0 \right| V_{k_0}(x_0)\ldots V_{k_n}(x_n) \left| p_{n+1} \right\rangle_{p_1\ldots p_n} =$$



$$p_i = \frac{a_{i1} - a_{i2}}{2} \quad , \quad b\frac{Q}{2} - k_i = \sum_{u=1}^{2} \frac{a_{i+1,u} - a_{i,u}}{2} \quad , \quad \frac{x_i}{x_{i-1}} = q_i \quad , \quad \epsilon_1 = b^2 \quad , \quad \epsilon_2 = 1$$

AGT and BPZ

$$\mathcal{F}_{p_0}{}^{k_0..k_n}{}_{p_{n+1}}(x_i)=$$

$$\prod_{i=0}^{n} x_i^{\Delta_{p_i}-\Delta_{p_{i+1}}-\Delta_{k_i}} \prod_{\substack{i,j=0\\ i<j}}^{2}\left(1-\frac{x_j}{x_i}\right)^{\frac{(2k_i-bQ)(2k_j+bQ)}{2b^2}} Z_{\mathrm{inst}\,p_0}{}^{k_0..k_n}{}_{p_{n+1}}(x_i)$$

- A descendant in the correlator $L_{-n}|h>=L_{-n}\phi(0)|0>$
- If the descendant is null $\sum_Y \mathcal{L}_Y<\phi(w)X>=0$
- $\mathcal{L}_Y=L_{-r_1}L_{-r_2}\dots L_{-r_n}$
- taking in particular $L_{-1}^2+b^2L_{-2}$ leads to

$$\left[\partial_{z_i}^2+b^2\sum_{s\neq i}^{n+3}\left(\frac{\Delta_{P_s}}{(z_i-z_s)^2}+\frac{1}{z_i-z_s}\partial_{z_s}\right)\right]\Psi_{i\alpha}(z_s)=0$$

Braiding and Fusion

- In CFT braiding and fusion are given by the 4 point correlator whose solutions are

$$\Psi^R_\alpha(z)=(1-z)^{\frac{1}{2}+k+\frac{b^2}{2}}\,z^{\frac{1}{2}+\alpha p_2+\frac{b^2}{2}}\,{}_2F_1\left(\frac{1}{2}+k+p_0+\alpha p_2,\frac{1}{2}+k-p_0+\alpha p_2,1+2\alpha p_2;z\right)$$

$$\widetilde{\Psi}^L_\alpha(z)=(z-1)^{\frac{1}{2}+\alpha k+\frac{b^2}{2}}\,z^{\frac{1}{2}+p_2+\frac{b^2}{2}}\,{}_2F_1\left(\frac{1}{2}+p_0+p_2+\alpha k,\frac{1}{2}-p_0+p_2+\alpha k,1+2\alpha k,1-z\right)$$

$$\Psi^L_\alpha(z)=(z-1)^{\frac{1}{2}+k+\frac{b^2}{2}}\,z^{\alpha p_0-k+\frac{b^2}{2}}\,{}_2F_1\left(\frac{1}{2}+k+p_2-\alpha p_0,\frac{1}{2}+k-p_2-\alpha p_0,1-2\alpha p_0;\frac{1}{z}\right)$$

and

$$\Psi^L_\alpha(z)=\sum_\alpha B_{\alpha\alpha'}\left[\begin{smallmatrix}k_{12} & k\\ p_0 & p_2\end{smallmatrix}\right]\Psi^R_{\alpha'}(z)$$

$$\Psi^L_\alpha(z)=\sum_\alpha F_{\alpha\alpha'}\left[\begin{smallmatrix}k_{12} & k\\ p_0 & p_2\end{smallmatrix}\right]\widetilde{\Psi}^L_{\alpha'}(z)$$

$$\widetilde{\Psi}^R_\alpha(z)=\sum_\alpha F^{-1}_{\alpha\alpha'}\left[\begin{smallmatrix}k & k_{12}\\ p_0 & p_2\end{smallmatrix}\right]\Psi^R_{\alpha'}(z)$$

The connection to black holes

- Now the BPZ equation can always be put in the normal form

$$\left[\frac{d^2}{dx^2} + Q(x)\right]\Psi(x) = 0$$

- The evolution of a massless scalar in a $g_{\mu\nu}$ background is given by

$$\nabla^2\phi = \frac{1}{\sqrt{g}}\partial_\mu\left[\frac{g^{\mu\nu}\partial_\nu}{\sqrt{g}}\right]\phi$$

- This is (almost) the same result (radial part) you get if $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$ and solve (given boundary conditions)

$$R_{\alpha\beta} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\alpha\beta}$$

The connection to black holes

- In the bh perturbation approach we compute the GW from a test particle of mass μ orbiting around a bh of mass $M \ll \mu$
- Circular orbit: t, ϕ plane, $r_0 = const, \theta = \pi/2 \quad \Longrightarrow \quad ds^2 = fdt^2 - r_0^2 d\phi^2$

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^\mu u^\nu = \frac{1}{2}(f\dot{t}^2 - r_0^2\dot{\phi}^2)$$

- Solving the eom we find $\Omega = \left(\frac{M}{r_0^3}\right)^{\frac{1}{2}}$
- The eq. of the perturbations can be solved for $z \sim \omega r_0 = m(M/r_0)^{1/2} \sim v$ and a parameter $\epsilon = 2M\omega \sim 2m(M/r_0)^{3/2} \sim v^3$ with $v \equiv (M/r_0)^{1/2}$

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Boundary conditions

- A non homogeneous differential eq can be solved using the Green function

$$G(\omega,x,x')=\frac{1}{W(\omega)}[\theta(x-x')\phi_-(\omega,x')\phi_+(\omega,x)+\theta(x'-x)\phi_-(\omega,x)\phi_+(\omega,x')]$$

  leading to the solution

$$\phi(\omega,x)=\frac{1}{W(\omega)}[\phi_+(\omega,x)\int_{-\infty}^{x}dx'\phi_-(\omega,x')T(\omega,x')+$$
$$\phi_-(\omega,x)\int_{x}^{\infty}dx'\phi_+(\omega,x')T(\omega,x')]$$

- For $x\to\pm\infty$ the potential $\to 0$ and the solutions are

$$\begin{cases} a_1 e^{i\omega x}+a_2\, e^{-i\omega x} & x\to\infty \\ b_1 e^{i\omega x}+b_2\, e^{-i\omega x} & x\to-\infty \end{cases}$$

Boundary conditions

- We can select a wave left moving to $-\infty$ ($b_1 = 0$) and right moving to ∞ ($a_2 = 0$)
- A wave with $b_1 = 0$ when evolved to ∞ will be of the form with both coefficients $\neq 0$
- Now if T has a compact support $[x_1, x_2]$ for $x > x_2$ the solution will be

$$\phi(\omega, x) = \frac{\phi_+(\omega, x)}{W(\omega)} \int_{-\infty}^{x} dx' \phi_-(\omega, x') T(\omega, x')$$

and the solution is a right moving wave at ∞ if $\phi_+(\omega, x)$ is of this type leading to

$$\phi_+(\omega, x) = \begin{cases} e^{i\omega x} & x \to \infty \\ b_1 e^{i\omega x} + b_2 e^{-i\omega x} & x \to -\infty \end{cases}$$

$$\phi_-(\omega, x) = \begin{cases} a_1 e^{i\omega x} + a_2 e^{-i\omega x} & x \to \infty \\ e^{-i\omega x} & x \to -\infty \end{cases}$$

Emitted Energy

- The solution of the differential equation with no waves going out of the horizon and no waves coming from ∞ is therefore

$$\lim_{x\to\infty} 2i\omega a_2\phi(\omega,x) \sim e^{i\omega x}\int_{2M}^{\infty} dx'\phi_-(\omega,x')T(\omega,x')$$

- Let's define

$$2i\omega a_2\mu Z_{lm}\delta(\omega-m\Omega)=\int_{2M}^{\infty} dx'\phi_-(\omega,x')T(\omega,x')$$

- Finally the energy radiated per unit time is

$$\frac{dE}{dt}=\mu^2\sum_{lm}\frac{|Z_{lm}|^2}{4\pi\omega^2}$$

Computation

- In CFT language language

$$Q_{CHE}(z) = -\frac{x^2}{4z^4} + \frac{x\,c}{z^3} + \frac{U}{(z-1)z^2} + \frac{\delta_{k_0}}{(z-1)^2 z} + \frac{\delta_{p_0}}{(z-1)z}$$

- The (radial) dictionary with gravity ($q_1 = z, q_2 = 4i\omega r$)

$$\hat{x} = 8i\omega M \;,\; c = -2 - 2i\omega M \;,\; p_0 = 1 - \frac{i\omega M}{2} \;,\; k_0 = -1 + \frac{3i\omega M}{2}$$

$$U = (\ell + \tfrac{1}{2})^2 - \tfrac{1}{4} - 4iM\omega - 28M^2\omega^2 \;,\; z = \frac{2M}{r} \;,\; p_2 = \ell + \tfrac{1}{2} + O((M\omega)^2$$

and

$$Q_{\text{near horizon}}(z) \underset{z\approx 1}{\approx} \frac{p_2^2 - \frac{1}{4}}{(z-1)z^2} + \frac{\delta_{k_0}}{(z-1)^2 z} + \frac{\delta_{p_0}}{(z-1)z}$$

$$Q_{\text{infinity}}(z) \underset{z\approx 0}{\approx} -\frac{x^2}{4z^4} + \frac{x\,c}{z^3} + \frac{p_2^2 - \frac{1}{4}}{(z-1)z^2}$$

Computation

- The solutions are

$$\Psi_{\text{near horizon}}(z) = c_\alpha \Psi^1_\alpha(z) = D_\alpha \tilde{\Psi}^1_\alpha(z)$$

$$\Psi_{\text{infinity}}\left(\frac{x}{z}\right) = \tilde{c}_\alpha \Psi^0_\alpha\left(\frac{x}{z}\right) = C_\alpha \tilde{\Psi}^0_\alpha\left(\frac{x}{z}\right)$$

where

$$\frac{\tilde{\Psi}^1_\alpha(z)}{z^{\frac{1}{2}+p_2}} = (1-z)^{\frac{1}{2}+\alpha k_0} {}_2F_1\left(\frac{1}{2}+\alpha k_0+p_0+p_2, \tfrac{1}{2}+\alpha k_0-p_0+p_2; 1+2\alpha k_0; 1-z\right)$$

$$\Psi^1_\alpha(z) = (1-z)^{\frac{1}{2}+k_0} z^{\frac{1}{2}+\alpha p_2} {}_2F_1\left(\frac{1}{2}+k_0+p_0+\alpha p_2, \tfrac{1}{2}+k_0-p_0+\alpha p_2; 1+2\alpha p_2; z\right)$$

$$\Psi^0_\alpha\left(\frac{x}{z}\right) = \left(\frac{z}{x}\right)^{\frac{1}{2}+\alpha p_2} e^{-\frac{x}{2z}} {}_1F_1\left(\frac{1}{2}-c-\alpha p_2; 1-2\alpha p_2; \frac{x}{z}\right)$$

$$\tilde{\Psi}^0_\alpha\left(\frac{x}{z}\right) = \left(\frac{z}{x}\right)^{1-\alpha c} e^{\frac{\alpha x}{2z}} {}_2F_0\left(\frac{1}{2}-\alpha c+p_2; \frac{1}{2}-\alpha c-p_2; \frac{\alpha z}{\hat{x}}\right)$$

Computation

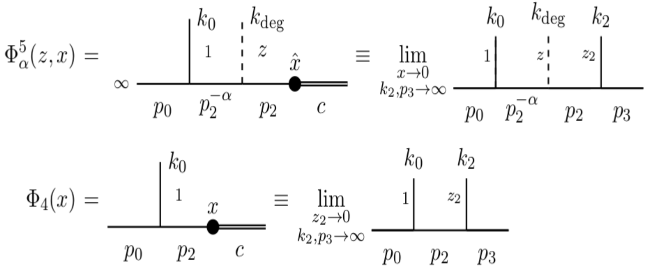
- Matching $c_\alpha \Psi^1_\alpha$ against $\tilde{c}_\alpha \Psi^0_\alpha$ in the region where $x << z << 1$ one finds $\tilde{c}_\alpha = x^{\frac{1}{2}+\alpha p_2} = D_{\alpha'} F^{-1}_{\alpha'\alpha}$, $C_{\alpha'} = \tilde{c}_\alpha B_{\alpha\alpha'}$ with

$$F^{-1}_{\alpha'\alpha} = \frac{\Gamma(1+2\alpha k_0)\Gamma\left(-2\alpha' p_2\right)}{\Gamma\left(\frac{1}{2}+\alpha k_0+p_0-\alpha' p_2\right)\Gamma\left(\frac{1}{2}+\alpha k_0-p_0-\alpha' p_2\right)}$$

$$B^{\rm conf}_{\alpha\alpha'} = e^{-i\pi\delta_{\alpha'+}\left(\frac{1}{2}-c-\alpha p_2\right)} \frac{\Gamma\left(1-2\alpha p_2\right)}{\Gamma\left(\frac{1}{2}-\alpha p_2+\alpha' c\right)}$$

Computation

- Next order in x the solution is $\Psi_\alpha(z) = \lim_{b\to 0} \frac{\Phi^5_\alpha(z)}{\Phi_4}$

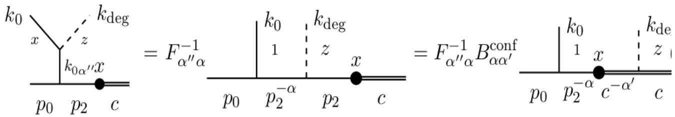


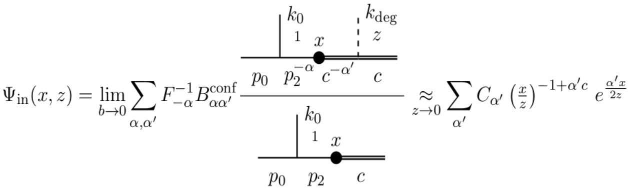
- $\Psi_\alpha(z)$ satisfies the CHE with $u = \lim_{b\to 0} b^2 x \partial_x \ln \Phi_4$ and

$$\mathcal{F} = \lim_{b\to 0} b^2 \ln \Phi^4(x) = -p_2^2 \ln x + \frac{cx\left(4k_0^2 - 4p_0^2 + 4p_2^2 - 1\right)}{2(1-4p_2^2)} + \ldots$$

Computation

- we can repeat the strategy of the static limit identifying the $z \approx 1, z \approx 0$ regions





$$C_{\alpha'}(x) = x^{\frac{1}{2}} \sum_\alpha F^{-1}_{-\alpha} B^{\rm conf}_{\alpha\alpha'} e^{-\frac{1}{2}(\alpha\partial_{p_2} + \alpha'\partial_c)\mathcal{F}}$$

Computation

- The solution of the homogeneous eq. with the right boundary conditions and coefficient is

$$R_{\rm in} = (1-z)^{\frac{1}{2}} z^{-2} \sum_{\alpha=\pm} F^{-1}_{-\alpha} \Psi_\alpha \underset{x,z\to 0}{\approx} (1-z)^{\frac{1}{2}} z^{-\frac{3}{2}-p_2} F^{-1}_{--} \Psi_- \left[1+O(z^{2p_2})\right]$$

$$C'_- = x^{-\frac{1}{2}+c} \sum_\alpha F^{-1}_{-\alpha} B^{\rm conf}_{\alpha -} e^{\frac{1}{2}(\partial_c - \alpha\partial_{p_2})\mathcal{F}}$$

$$\underset{x,z\to 0}{\approx} x^{-\frac{1}{2}+c} F^{-1}_{--} B^{\rm conf}_{--} e^{\frac{1}{2}(\partial_{p_2}+\partial_c)\mathcal{F}} \left[1+O(x^{2p_2})\right]$$

in GR variables

$$\frac{R_{\rm in}}{2\omega C'_-} = 2M(\omega r)^{\ell+2} \frac{\Gamma(\ell-1)}{\Gamma(\ell+2)} \left[1 + \frac{2ir\omega}{l+1} - \frac{(l+9)\omega^2 r^2}{2(l+1)(2l+3)} - \frac{(l+2)M}{r} - \frac{i(l+4)r^3\omega^3}{(l+1)(l+2)(2l+3)}\right.$$

$$\left. + M\omega(\pi - 3i - 2i\log(4M\omega) + 2i\psi_0(l-1)) - \frac{4iM\omega}{l+1} + \ldots\right]$$