Rotating Metrics from Scattering Amplitudes in Arbitrary Dimensions

“JENAS Initiative: Gravitational Wave Probes of Fundamental Physics”

Claudio Gambino, University of Rome “La Sapienza”

Based on [CG, P. Pani, F. Riccioni, 2403.XXXXX]

February 13th, 2024

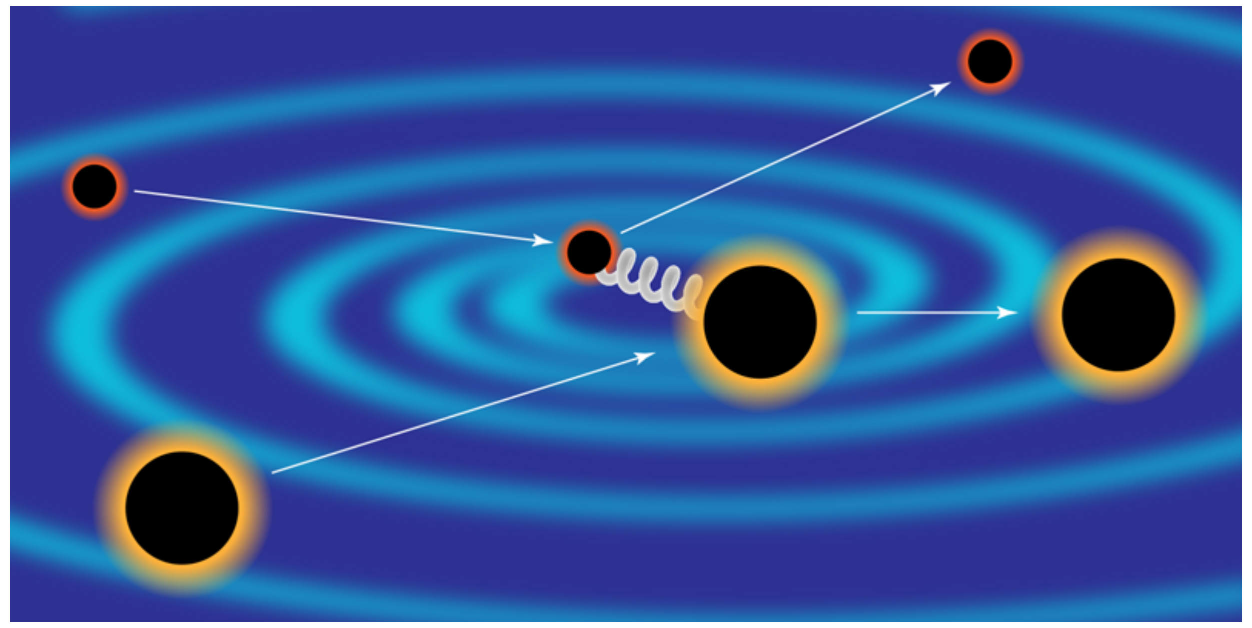
Sapienza Università di Roma

INFN Istituto Nazionale di Fisica Nucleare

Introduction

[Donoghue, 2211.09902]
[Bjerrum-Bohr, Planté, Vanhove, 2212.08957]

Nowadays we accept the idea that **GR is the classical limit of an effective quantum theory of gravity**, in which the Einstein-Hilbert action is the leading order in a higher-derivative expansion.

$\lim_{\hbar \to 0}$  $= \textit{Gravity}$

It has been proven that a static metric in arbitrary dimensions is recovered from 3-point amplitudes of massive scalars emitting gravitons.

We extend this program in the case of **spinning geometries at quadrupole order**:

- **No Birkhoff theorem for stationary objects.**
- **No black-hole uniqueness in $D > 4$.**

Metrics from scattering amplitudes

$\kappa^2 = 32\pi G$ & $D = d+1$

Consider a spin-s field coupled to gravity $\quad S = \int d^{d+1}x \left(-\frac{2}{\kappa^2}\sqrt{-g}R + \mathcal{L}_m(\Phi_s, g_{\mu\nu}) \right)$

Imposing the harmonic gauge we can rewrite the Einstein equations as

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} = \eta_{\mu\nu} + \kappa \sum_{n=1}^{+\infty} h^{(n)}_{\mu\nu} \quad \longrightarrow \quad h^{(n)}_{\mu\nu}(x) = -\frac{\kappa}{2}\int \frac{d^d\vec{q}}{(2\pi)^d}\frac{e^{i\vec{q}\cdot\vec{x}}}{\vec{q}^{\,2}}\left(T^{(n-1)}_{\mu\nu}(q) - \frac{1}{d-1}\eta_{\mu\nu}T^{(n-1)}(q)\right)$$

nPM order ($h^{(n)}_{\mu\nu}$); graviton ($h_{\mu\nu}$)

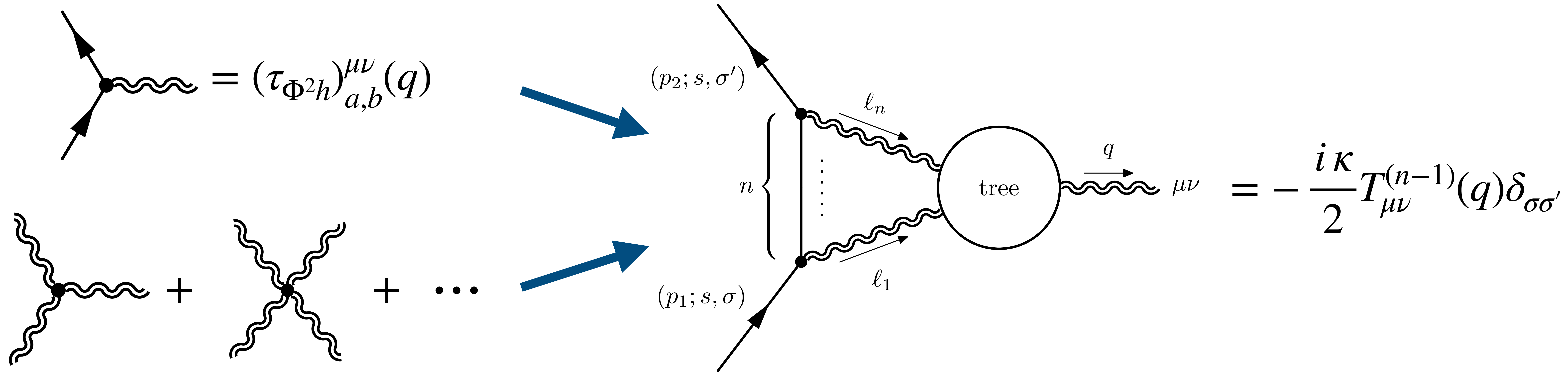
We want to compute $T^{(n)}_{\mu\nu}(q)$ through scattering amplitudes

- $T^{(0)}_{\mu\nu}$ is the stress-energy tensor of the matter source.
- $T^{(n)}_{\mu\nu}$ for $n > 0$ contains gravitational self-interactions.

Loop amplitudes are classical

[Donoghue, gr-qc/9310024]
[Mougiakakos, Vanhove, 2010.08882]

Quantizing the theory we can define the interaction vertices and prove that



The mantra of GR from scattering amplitudes is to ***consider the classical limit as soon as possible,*** in order to gain efficiency in the actual calculations.

We have to define a classical vertex!

[Bern, Luna, Roiban, Shen, Zeng, 2005.03071]

Dressed vertex

In the **stationary** and **classical limit** it is verified that $|p_1\rangle = |p_2\rangle + O(\hbar)$

Leading to the definition of the spin tensor $S^{\mu\nu}$ as the classical limit of the Lorentz generators $M^{\mu\nu}$

normalization coefficient

$$\langle p_2; s, \sigma' | M^{\mu\nu} | p_1; s, \sigma\rangle = S^{\mu\nu}\langle p_1; s, \sigma' | p_1; s, \sigma\rangle + O(\hbar) = S^{\mu\nu} C(s)\delta_{\sigma\sigma'} + O(\hbar^0)$$

$$\langle p_2; s, \sigma' | (\tau_{\Phi^2 h})^{\mu\nu} | p_1; s, \sigma\rangle = \langle p_1; s, \sigma' | (\tau_{\Phi^2 h})^{\mu\nu} | p_1; s, \sigma\rangle + O(\hbar) = \hat{\tau}^{\mu\nu}_{\Phi^2 h}(q, S)\delta_{\sigma\sigma'} + O(\hbar)$$

In $D > 4$ the angular momentum is an anti-symmetric rank-2 tensor

Spin-1

Consider a massive spin-1 field minimally coupled to gravity, *i.e.* Proca field

$$S_{\text{min}} = \int d^{d+1}x\sqrt{-g}\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}m^2 V_\mu V^\mu\right)$$

We can compute the minimal dressed vertex associated to such field as

$$\hat{\tau}^{\mu\nu}_{V^2h,\text{min}}(q,S) = -\frac{i\kappa}{2}\left(2m^2\delta^\mu_0\delta^\nu_0 - i\,m\,q_\lambda\left(S^{\mu\lambda}\delta^\nu_0 + S^{\nu\lambda}\delta^\mu_0\right) - q_\lambda q_\sigma S^{\mu\lambda}S^{\nu\sigma}\right)$$

$$T^{(0)}_{\mu\nu,\text{min}} = \frac{2\,i}{\kappa}\left(\hat{\tau}_{V^2h,\text{min}}\right)_{\mu\nu} \longrightarrow$$

The "simplest" metric

(metric associated to a minimally coupled field)

Non-minimal action

Defining a spin operator such that $\mathbb{S}^{\mu\nu}_{a,b} = S^{\mu\nu}\delta_{ab} + O(\kappa)$, we can build a non-minimal action as

$$S_{\text{non-min}} = \int d^D x \sqrt{-g}\left(K_1 \; R \, V^\alpha \left(\mathbb{S}^{\mu\nu}\mathbb{S}_{\mu\nu}\right)_{\alpha\beta} V^\beta + K_2 \; R_{\mu\nu} \, V^\alpha \left(\mathbb{S}^{\mu\lambda}\mathbb{S}_\lambda{}^{\nu}\right)_{\alpha\beta} V^\beta + \cdots \right)$$

$S_{\text{min}} + S_{\text{non-min}}$

$$\hat{\tau}^{\mu\nu}_{V^2 h}(q,S) = -\frac{i\kappa}{2}\Big(2\,m^2\,\delta^\mu_0\delta^\nu_0 - i\,m\,q_\lambda\left(S^{\mu\lambda}\delta^\nu_0 + S^{\nu\lambda}\delta^\mu_0\right) - H_1\,q_\rho q_\sigma S^{\mu\rho}S^{\nu\sigma} + H_2\,\delta^\mu_0\delta^\nu_0 q_\rho q_\sigma S^{\rho\lambda}S_\lambda{}^\sigma$$

$$+ C_1\,S^{\rho\sigma}S_{\rho\sigma}q^\mu q^\nu + C_2\left(\eta^{\mu\nu}q_\rho q_\sigma S^{\rho\lambda}S^\sigma{}_\lambda - q^\lambda\left(q^\mu S_{\lambda\sigma}S^{\nu\sigma} + q^\nu S_{\lambda\sigma}S^{\mu\sigma}\right)\right)\Big)$$

Minimal Limit

$H_1 = 1 \quad H_2 = 0$

$C_1 = 0 \quad C_2 = 0$

We obtain the stress-energy tensor of the most generic stationary matter distribution at quadrupole order which is spherically symmetric in the non-rotating limit.

[Bjerrum-Bohr, Donoghue, Holstein, hep-th/0211071]

Expanding the metric in a multipole series, guided by the ***spin universality*** idea, we get

$$h_{\mu\nu}^{(n)} = \sum_{j=0}^{2s} h_{\mu\nu}^{(n,j)} = h_{\mu\nu}^{(n,\text{monopole})} + h_{\mu\nu}^{(n,\text{dipole})} + h_{\mu\nu}^{(n,\text{quadrupole})} + \cdots$$

Spin-1  Quadrupole

The monopole and the dipole of any metric in asymptotically cartesian coordinates are unique!

$$h_{00}^{(1,0)}(r) = -\frac{4(d-2)}{d-1} Gm\,\rho(r)$$

$$h_{0i}^{(1,0)}(r) = 0$$

$$h_{ij}^{(1,0)}(r) = -\frac{4\delta_{ij}}{d-1} Gm\,\rho(r)$$

&

$$h_{00}^{(1,1)}(r) = 0$$

$$h_{0i}^{(1,1)}(r) = -\frac{2(d-2)x^k S_k^i}{r^2} G\,\rho(r)$$

$$h_{ij}^{(1,1)}(r) = 0$$

Where $\rho(r) = \dfrac{\Gamma(\frac{d}{2}-1)\pi^{1-d/2}}{r^{d-2}}$

The first non-trivial multipole order to look for to resolve the structure of different matter configurations is the *quadrupole*

Metric of a generic source at quadrupole order

$$h_{00}^{(1,2)}(r) = \frac{2(d-2)\Big(H_2(d-2)+H_1\Big)}{d-1}\,\frac{r^2 S_{k_1k_2}S^{k_1k_2} - d\,x^{k_1}x^{k_2}S_{k_1}{}^{k_3}S_{k_2k_3}}{mr^4}G\rho(r)$$

$$h_{0i}^{(1,2)}(r) = 0$$

$$h_{ij}^{(1,2)}(r) = -\frac{2(d-2)}{(d-1)mr^4}\Bigg(-C_1(d-1)d\,x_ix_jS_{k_1k_2}S^{k_1k_2} - r^2(d-1)\Big(2C_2+H_1\Big)S_{ik}S_j{}^k$$

$$+r^2\Big(C_1(d-1)+H_1-H_2\Big)S_{k_1k_2}S^{k_1k_2}\delta_{ij} + d\,C_2(d-1)x^{k_1}S_{k_1k_2}\Big(x_jS_i{}^{k_2}+x_iS_j{}^{k_2}\Big)$$

$$+d\,x^{k_1}x^{k_2}\Big((d-1)H_1S_{ik_1}S_{jk_2} + (H_2-H_1)S_{k_1}{}^{k_3}S_{k_2k_3}\delta_{ij}\Big)\Bigg)G\rho(r)$$

This metric depends on four different arbitrary parameters. Are they all physical?

Spoiler, no!

Eliminating gauge parameters

Consider an infinitesimal coordinate transformation as $x' = x + \xi(x)$ such that the metric transforms like $h'_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi \mu)$.

By definition in the harmonic gauge $\Box x^\mu = 0$, we can define a coordinate transformation inside the gauge if

$$\Box x'^\mu = 0 \quad \rightarrow \quad \Box \xi^\mu = 0$$

harmonic function in arbitrary dimensions

$$\Box \rho(r) = 0 \quad \longrightarrow \quad \xi^i = \frac{G}{m}\left(2C_2\, S^{ik} S_k{}^j + C_1\, S^{lm} S_{lm} \delta^{ij}\right) \partial_j \rho(r) \quad \& \quad \xi^0 = 0$$

With this coordinate transformation we can cancel the C_i's from the metric

H_1 and H_2 are physical parameters $\longleftrightarrow$ **In $D > 4$ there are two independent quadrupole moments**

$$h_{00}^{(1,2)} = \frac{2(d-2)\Big(H_2(d-2)+H_1\Big)}{d-1}\,\frac{r^2 S_{k_1k_2}S^{k_1k_2} - d\, x^{k_1}x^{k_2}S_{k_1}{}^{k_3}S_{k_2k_3}}{mr^4}\,G\rho(r)$$

$$h_{ij}^{(1,2)} = -\frac{2(d-2)}{(d-1)mr^4}\Bigg(-r^2(d-1)H_1 S_{ik}S_j^{\ k} + r^2\Big(H_1-H_2\Big)S_{k_1k_2}S^{k_1k_2}\delta_{ij}$$

$$+d\, x^{k_1}x^{k_2}\Big((d-1)H_1 S_{ik_1}S_{jk_2} + (H_2-H_1)S_{k_1}{}^{k_3}S_{k_2k_3}\delta_{ij}\Big)\Bigg)G\rho(r)$$

Two different quadrupole tensors enter in the metric, both in h_{00} and in h_{ij}

$D = 4$ is a special case since we can rewrite $S^{ij} = \epsilon^{ijk}S_k$ and then through a coordinate transformation we can show that the metric depends on H_1 and H_2 only by the combination $H_1 + H_2$.

In $D > 4$ there are two independent quadrupole moments

but

In $D = 4$ there is only one quadrupole moment

[Thorne, Rev.Mod.Phys. 52 (1980) 299-339]

ACMC coordinates

The "asymptotically Cartesian mass-centered" coordinates were introduced by Thorne and allow us to define gauge-invariant multipole moments for asymptotically flat stationary and vacuum solutions.

In $D = 4$

$$g_{00} = 1 - \frac{2mG}{r} + \frac{2m^2G^2}{r^2} + \frac{1}{r}\sum_{\ell=1}^{+\infty}\frac{1}{r^\ell}\mathscr{M}_{a_1\cdots a_\ell}\frac{x_{a_1}\cdots x_{a_\ell}}{r^\ell} + \cdots$$

Any angular dependence

$g_{ij} \sim$ mass multipoles $\qquad g_{0i} \sim$ current multipoles

The statement is that in $D = 4$ there exists only a mass quadrupole $\mathscr{M}_{a_1a_2}$ and we only need g_{00} to read it.

Our claim is that in $D > 4$ there exist two independent quadrupole tensors and to read them one needs both g_{00} and g_{ij}

Mass quadrupole in $D > 4$

From the generic 1PM metric at quadrupole order we can extrapolate the following structure

$$g_{00} = 1 - \frac{4(d-2)}{d-1} Gm\,\rho(r) + \frac{2(d-2)}{d-1}\frac{1}{m^2 r^2}\mathcal{Q}^{(1)}_{a_1a_2}\frac{x_{a_1}x_{a_2}}{r^2} Gm\,\rho(r) + \cdots$$

$$g_{ij} = -\delta_{ij} - \frac{4}{d-1}\delta_{ij} Gm\,\rho(r) + \frac{2(d-2)}{d-1}\frac{1}{m^2 r^2}\mathcal{Q}^{(2)}_{ij,a_1a_2}\frac{x_{a_1}x_{a_2}}{r^2} Gm\,\rho(r) + \cdots$$

New mass quadrupole!

$$\mathcal{Q}^{(1)}_{a_1a_2} = \Big(H_2(d-2) + H_1\Big)\Big(\delta_{a_1a_2} S_{k_1k_2} S^{k_1k_2} - d\, S_{a_1}{}^{k} S_{a_2k}\Big)$$

→ **Thorne's quadrupole in $D > 4$**

$$\mathcal{Q}^{(2)}_{ij,a_1a_2} = \delta_{a_1a_2}(d-1)H_1 S_{ik}S_j{}^{k} - \delta_{a_1a_2}\Big(H_1 - H_2\Big)S_{k_1k_2}S^{k_1k_2}\delta_{ij} - d\Big((d-1)H_1 S_{ia_1}S_{ja_2} + (H_2 - H_1)S_{a_1}{}^{k}S_{a_2k}\delta_{ij}\Big)$$

[Hartle, Thorne, Astrophys.J. 153 (1968) 807]

Matching the Hartle-Thorne metric in $D = 4$

To test our formalism we match the amplitude-based metric with the one associated with the most generic rotating solution at quadrupole order (HT) in *harmonic gauge*.

$$g_{tt}^{(HT)} = -1 + \frac{2GM}{r} - \frac{a^2 GM\kappa(3\cos(2\theta)+1)}{2r^3} + \cdots$$

$$g_{t\phi}^{(HT)} = -\frac{2aGM\sin^2(\theta)}{r} + \cdots$$

$$g_{rr}^{(HT)} = 1 + \frac{2GM}{r} - \frac{(a^2 GM\kappa(3\cos(2\theta)+1))}{2r^3} + \frac{4G^2M^2}{r^2} + \cdots$$

$$g_{\theta\theta}^{(HT)} = r^2 - \frac{a^2 GM\kappa(3\cos(2\theta)+1)}{2r} + \cdots$$

$$g_{\phi\phi}^{(HT)} = r^2\sin^2(\theta) - \frac{\left(a^2 GM\kappa(3\cos(2\theta)+1)\sin^2(\theta)\right)}{2r} + \cdots$$

$$S_{ij} = \begin{pmatrix} 0 & J & 0 \\ -J & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad a = J/m$$

This metric has only one quadrupole moment, parametrized by κ, and we are able to reproduce it by fixing

$$H_1 + H_2 = \kappa$$

We do not need to fix H_1 and H_2 independently!

Kerr limit of the Hartle-Thorne metric

For a specific value of the quadrupole we can recover the Kerr metric.

Kerr Limit

$\kappa = 1$

corresponds to → $H_1 = 1 - H_2$

The Kerr metric, ***the only black hole solution in*** $D = 4$, is reproduced by an infinite number of non-minimal actions and by the **minimally coupled theory**.

Minimal limit → $(H_1 = 1) + (H_2 = 0) = 1$

"Simplest" metric in $D = 4$ $=$ **Kerr black hole**

Simplest metric in higher dimension

If the simplest metric in $D = 4$ is the Kerr black hole, to what it does correspond in higher dimensions?

The simplest metric in arbitrary dimension

$$h_{00}^{(1,2)} = \frac{2(d-2)}{d-1} \frac{r^2 S_{k_1 k_2} S^{k_1 k_2} - d\, x^{k_1} x^{k_2} S_{k_1}{}^{k_3} S_{k_2 k_3}}{m r^4} G\rho(r)$$

$$h_{ij}^{(1,2)} = -\frac{2(d-2)}{(d-1) m r^4} \Bigg(-r^2 (d-1) S_{ik} S_j{}^k + r^2 S_{k_1 k_2} S^{k_1 k_2} \delta_{ij}$$

$$+ d\, x^{k_1} x^{k_2} \left((d-1) S_{i k_1} S_{j k_2} - S_{k_1}{}^{k_3} S_{k_2 k_3} \delta_{ij} \right) \Bigg) G\rho(r)$$

[Myers, Perry, Annals Phys. 172 (1986) 304]

Myers-Perry black holes in $D = 5$

Myers-Perry solutions are a class of black holes defined in arbitrary dimensions constructed in such a way that the limit $D = 4$ corresponds to Kerr.

$$ds^2 = -dt^2 + \frac{\mu}{\Sigma}\left(dt + a\,\sin^2\theta\, d\phi_1 + b\,\cos^2\theta\, d\phi_2\right)^2 + \frac{r^2\Sigma}{\Pi - \mu r^2}dr^2$$
$$+\Sigma d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\phi_1^2 + (r^2 + b^2)\cos^2\theta\, d\phi_2^2$$

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta$$
$$\Pi = (r^2 + a^2)(r^2 + b^2)$$

The solution now has two independent angular momenta since the group of the rotation $SO(4)$ has two Casimir

$$\frac{1}{m}S_{ij} = \frac{2}{3}\begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}$$

We need to fix H_1 and H_2 independently!

$$H_1 = \frac{3}{8} \quad H_2 = \frac{15}{16}$$

The Myers-Perry solution is not the "simplest" one

Conclusions

- It is possible to recover stationary rotating metrics at $2s$-pole order from the classical limit of massive spin-s particles.
- The most generic stationary spherically-symmetric stress-energy tensor can be related to a non-minimally coupled theory, and the "simplest" solution can be defined.
- At quadrupole order the metric depends on two physical parameters, meaning that in higher dimensions there exist two independent quadrupole moments.
- We have shown that in $D = 4$ there is only one quadrupole moment and that the "simplest" metric corresponds to the Kerr black hole.
- In $D > 4$ to match exact solutions we need to independent quadrupole moments, and we have seen that Myers-Perry black holes are not associated to the "simplest" metric.

Further directions & open questions

- Repeat the argument for $s = 3/2, 2, \ldots$ to see current multipoles in higher dimensions and to extend the definition of the “simplest” metric for higher multipoles.
- What is the mass multipole structure beyond the quadrupole?
- What are the phenomenological implications of having independent quadrupole moments in $D > 4$?
- Does the fact that the “simplest” metric corresponds to a Kerr black hole hold for higher multipoles?
- What is the interpretation of the “simplest” metric? Do exact solutions exist in $D = 5$ that satisfy this requirement?