Black holes as point particles: from amplitudes to self-force

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THE UNIVERSITY of EDINBURGH

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Content

Motivation and introduction (I)

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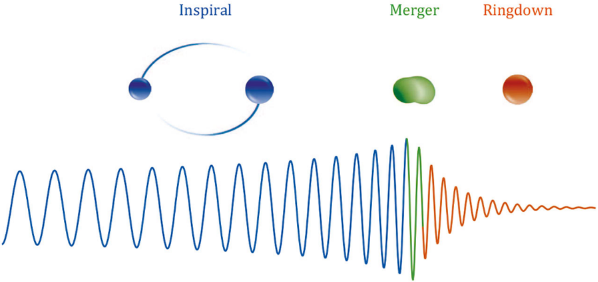

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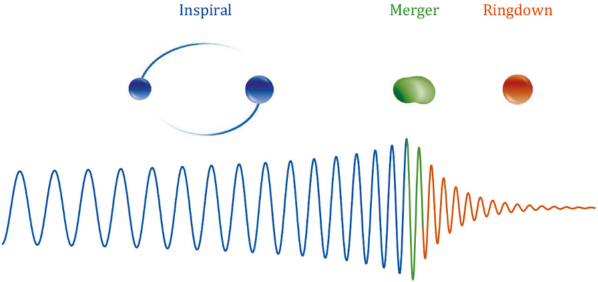

- Today: focus on the inspiral phase, where we can model compact objects as point particles in the spirit of effective field theory [Goldberger,Rothstein]

Motivation and introduction (II)

- Idea: use particle field theory tools ($\rightarrow$ scattering amplitudes)

Real world	EFT of point particles
Compact objects of mass M	Point particles of mass M
Spin effects of magnitude a	Spinning particles of classical spin a
Tidal effects, GR curvature corrections	Higher-dimensional operators
Absorption effects	Non-unitary absorption dofs

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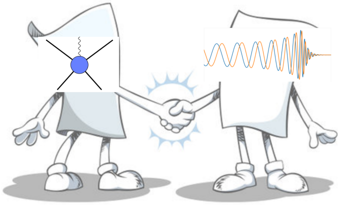
- Why amplitudes? (adapted to scattering orbits... bound orbits? Stay tuned!)

Amplitudes are gauge-invariant, universal objects which encode in a compact way the perturbative scattering dynamics for point particles in a QFT.
New perspective on GR!



KMOC formalism for the two-body problem (I)

- Two-body scattering in GR: Consider as initial state two massive particles separated by an impact parameter b^μ [Kosower,Maybee,O'Connell=KMOC]

$$|\psi_{\text{in}}\rangle = \int \mathrm{d}\Phi\,(p_1, p_2)\,\psi_1(p_1)\psi_2(p_2)e^{i(b\cdot p_1)/\hbar}\,|p_1 p_2\rangle$$

with some wavefunctions ψ_1,ψ_2 localized on classical trajectories.

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- The dynamics of the evolution is determined by the action

$$S = -\frac{1}{16\pi G_N}\int \mathrm{d}^4x\sqrt{-g}R + \sum_{j=1}^{2}\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(g^{\mu\nu}\partial_\mu\phi_j\partial_\nu\phi_j - m_j^2\phi_j^2\right) + S_{\mathrm{GF}}$$

where we perform the perturbative expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}\,, \quad \kappa = \sqrt{32\pi G_N} \rightarrow \text{Post-Minkowskian expansion in } G_N\,.$$

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- See Gambino's talk if you are interested in the metric from point particles!

KMOC formalism for the two-body problem (II)

- We can compute classical observables $\mathcal{O}$ with expectation values

$$\langle\psi_{\text{in}}|\mathcal{S}^\dagger\mathcal{O}\mathcal{S}|\psi_{\text{in}}\rangle\Big|_{\hbar\to0} = 2\Re i\langle\psi_{\text{in}}|\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to0} + \langle\psi_{\text{in}}|T^\dagger\mathcal{O}T|\psi_{\text{in}}\rangle\Big|_{\hbar\to0}$$

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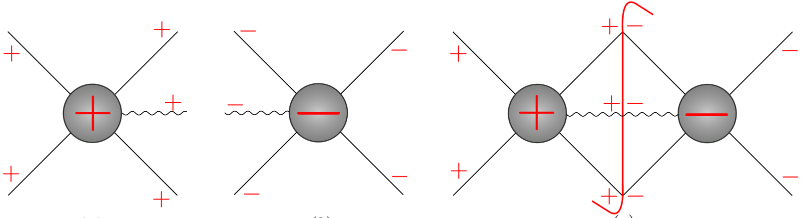
- Connection with the "classical" on-shell reduction of the in-in approach in the $(+)/(-)$ basis [Britto, RG, Jehu]



(a) (b) (c)

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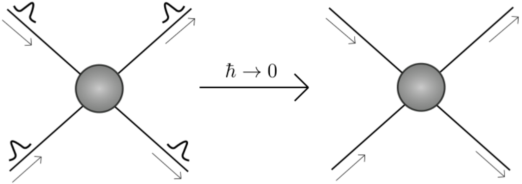

$$\psi(p) = \mathcal{N} m^{-1} \exp\left[-\frac{p \cdot u}{\hbar \ell_c / \ell_w^2}\right] \xrightarrow{\text{rest frame}} \mathcal{N}' \exp\left(-\frac{p^2}{2m^2 \ell_c^2/\ell_w^2}\right)$$

where p^μ is the momentum, $\ell_{c,j} = \hbar/m_j$ is the Compton wavelength, ℓ_w the intrinsic spread of the wavefunction. If b^μ is the impact parameter we require,

$$\ell_{c,j} \ll \ell_w \ll b = \sqrt{-b^2}\,.$$

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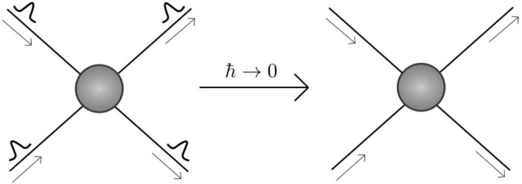

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- Massless particles: use coherent states! [Cristofoli,RG,Kosower,O'Connell]

Classical limit of scattering amplitudes

- Conservative 4-pt amplitude $\mathcal{M}_4(p_1, p_2; p_1', p_2')$: in the classical limit $\hbar \to 0$

$$p_1^\mu := p_A^\mu + \hbar\frac{\bar{q}^\mu}{2}\,, \qquad (p_1')^\mu := p_A^\mu - \hbar\frac{\bar{q}^\mu}{2}\,, \qquad s = (p_A + p_B)^2\,,$$
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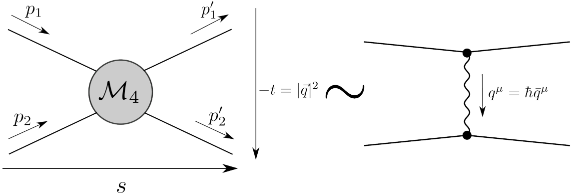
where p_A, p_B are the classical momenta and q is the momentum transfer.



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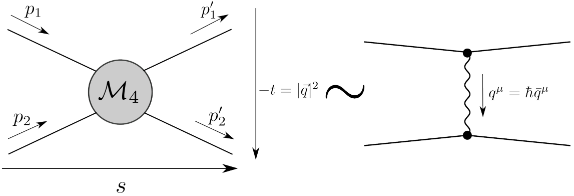

- Generalization for the $4 + M$-pt amplitude $\mathcal{M}_{4+M}(p_1, p_2; p_1', p_2', k_1, \ldots, k_M)$

$$q_{1,2}^\mu = p_{1,2}^\mu - (p_{1,2}')^\mu = \hbar \bar{q}_{1,2}^\mu\,, \qquad k_j^\mu = \hbar \bar{k}_j^\mu\,, j = 1, \ldots, M\,.$$

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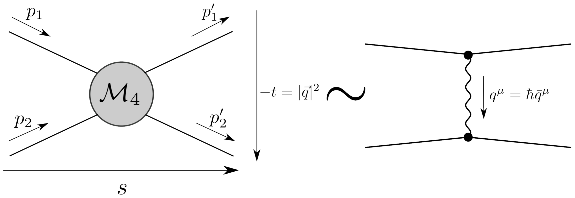

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- Main lesson: only wavevectors $\bar{q}_{1,2}^\mu, \bar{k}_j$ are classical, need to restore $\hbar$!

Waveforms from KMOC formalism: an example

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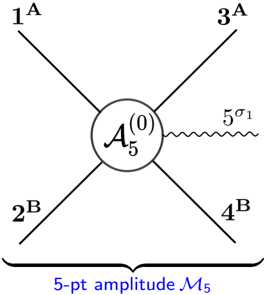
Waveforms from KMOC formalism: an example

- How is the waveform derived from scattering amplitudes?
- The on-shell expectation value of the time-domain waveform relevant for the inspiral phase is [Cristofoli,RG,Kosower,O'Connell]

$$\langle \psi_{\text{in}} | \mathcal{S}^\dagger h_{\mu\nu}(x) \mathcal{S} | \psi_{\text{in}} \rangle = \frac{1}{\hbar^{\frac{1}{2}}} 2\Re \sum_{\sigma=\pm} \int \mathrm{d}\Phi(k)\, \varepsilon_\mu^{*(\sigma)}(k) \varepsilon_\nu^{*(\sigma)}(k) \tilde{j}(b; k^\sigma) e^{-ik\cdot x/\hbar}$$

where at leading Post-Minkowskian order only the 5-pt amplitude is relevant

$$\tilde{j}(b; k^{\sigma_1}) \equiv \int \mathrm{d}\Phi(p_1' p_2' p_1 p_2) \psi^*(p_1', p_2') \psi(p_1, p_2) e^{-ib\cdot \bar{q}_1}$$



The Kovacs-Thorne waveform for hyperbolic encounters

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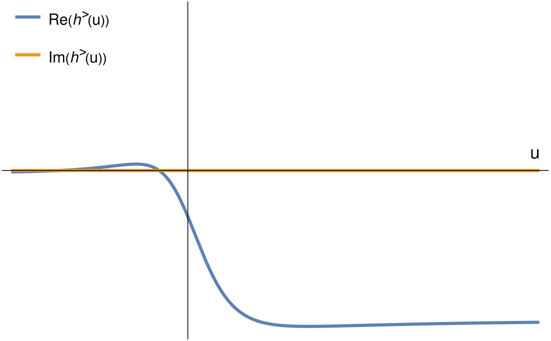

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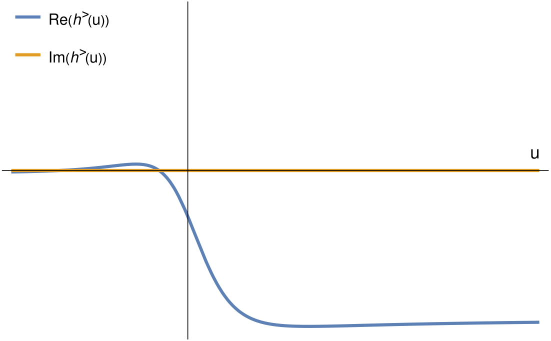

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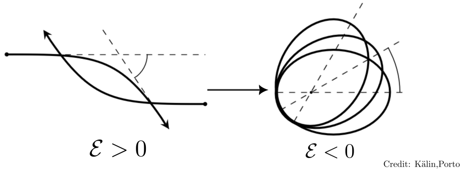
- How about bound waveforms for compact binaries coalescence?

From scattering to bound observables (I)

- Classical scattering amplitudes describe hyperbolic encounters. If we define

$$\mathcal{E} := \frac{E - m_1 - m_2}{\mu}, \qquad p_\infty^2 = -\tilde{p}_\infty^2 = \frac{E^2 - (m_1 + m_2)^2}{2 m_1 m_2},$$

we have $\mathcal{E}, p_\infty^2 > 0$ for scattering orbits and $\mathcal{E}, p_\infty^2 < 0$ for bound orbits.



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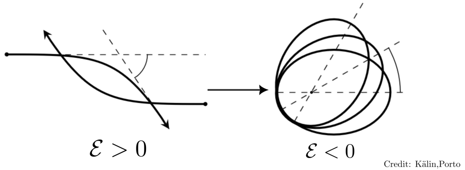

- Two powerful methods to extract bound state physics from amplitudes:
1) Extract perturbatively the classical potential ($\sim$ Hamiltonian) valid for arbitrary orbits [Niell,Rothstein;Cheung,Rothstein,Solon]
2) Gauge invariant map between scattering and bound observables [Kälin,Porto]

$$\boxed{\mathcal{O}^{>}(\mathcal{E} > 0, J, c_X, a_1, a_2, m_1, m_2) \to \mathcal{O}^{<}(\mathcal{E} < 0, J, c_X, a_1, a_2, m_1, m_2)}$$

which can be derived from the Bethe-Salpeter eq. [Adamo,RG; Adamo,RG,Ilderton].

From scattering to bound observables (II)

- For aligned-spin binaries where the motion remains on the equatorial plane there is a conjectural dictionary [Kälin,Porto;Saketh,Vines,Steinhoff, Buonanno;Cho,Kälin,Porto;Adamo,RG; Heissenberg;Adamo,RG,Ilderton]

Bound observable	Scattering observable
$\Delta\Phi(\tilde{p}_\infty, L, a, c_X)$	$\chi(-i\tilde{p}_\infty, L, a, c_X) + \chi(+i\tilde{p}_\infty, L, a, c_X)$
$\Delta E^{<}_{\rm rad}(\tilde{p}_\infty, L, a, c_X)$	$\Delta E^{>}_{\rm rad}(-i\tilde{p}_\infty, L, a, c_X) + \Delta E^{>}_{\rm rad}(+i\tilde{p}_\infty, L, a, c_X)$
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- New waveform map (up to 1PN and tree-level)[Adamo,RG,Ilderton]

$$\boxed{h^{<\rm dyn}(u;\tilde{p}_\infty, L, a, c_X) = h^{>\rm dyn}(u; +i\tilde{p}_\infty, L, a, c_X)}$$

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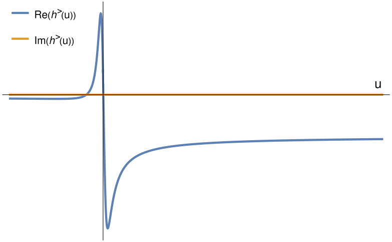
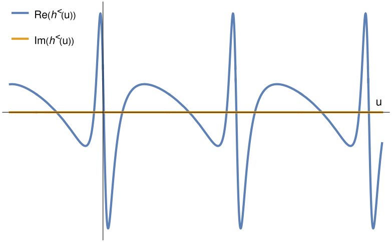
- Need to study tail effects appearing at higher orders! [Cho,Kälin,Porto]

From scattering to bound observables (III)

- The analytical continuation of the waveform computed for eccentric orbits requires a resummation in the eccentricity to recover the bound waveform periodicity in the time u [Adamo,RG,Ilderton]





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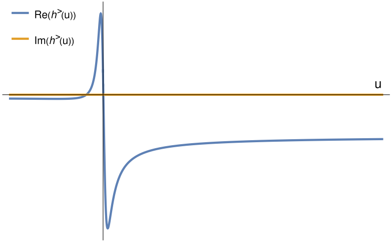
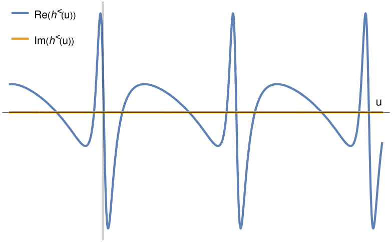




- At 1PN [Junker, Schäfer] this corresponds to a map between PN multipoles [Adamo, RG, Ilderton]

$$h^{>}(u; p_\infty, L) = \frac{4G_N}{c^2}\left(U_2^{>} + \frac{1}{c}(V_2^{>} + U_3^{>}) + \frac{1}{c^2}(V_3^{>} + U_4^{>})\right),$$

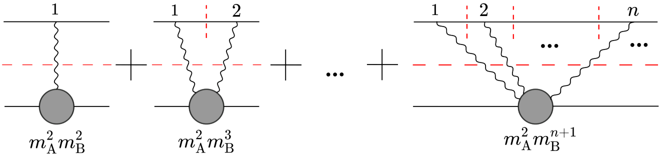
$$U_j^{<}(u; \tilde{p}_\infty) = U_j^{>}(u; +i\tilde{p}_\infty) \text{ for } j \leq 4\,, \quad V_k^{<}(u; \tilde{p}_\infty) = V_k^{>}(u; +i\tilde{p}_\infty) \text{ for } k \leq 3\,.$$

Need for resummation: from amplitude to self-force

- Need for resummation of perturbative contributions! Can we look at simpler examples? [Del Duca, RG, Sasank]
- Consider the geodesic equation,

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\tau}\frac{\mathrm{d}x^\beta}{\mathrm{d}\tau} = 0\,,$$

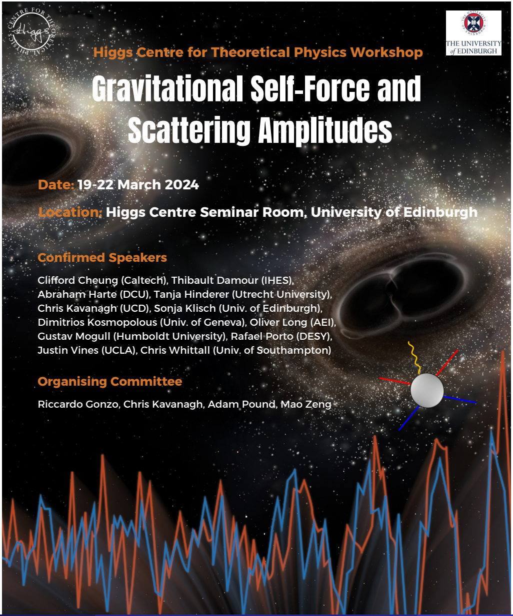
corresponding to the leading order in the mass ratio expansion: this requires to resum an infinite set of diagrams [Cheung, Solon, Shah; Brandhuber, Chen, Travaglini, Wen; Bjerrum-Bohr, Planté, Vanhove]



Interestingly, the scattering-to-bound map works for Kerr geodesics! [Shi,RG]

- Open problem: What can we learn by studying the self-force expansion? [Kosmopolous,Solon; Cheung,Parra-Martinez,Rothstein,Shah,Wilson-Gerow]

Self-force & Amplitudes workshop (19-22 March 2024)



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- Future directions: extend the scattering-to-bound map to include tail effects for the waveform and other observables, explore the resummation of PM contributions, extend the analytical continuation to generic spin alignments, …

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