Topology of the large-N expansion in SU(N) YM theory and spin-statistics theorem

Marco Bochicchio
INFN - Roma1

based on M. Bochicchio, M. Papinutto, F. Scardino
On the structure of the large-N expansion in
SU(N) Yang-Mills theory
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It has been known for almost 50 years that

U(N) YM theory admits the ’t Hooft large-N topological expansion, where Feynman diagrams contributing to n-point correlators of gauge-invariant single-trace operators are classified by n-punctured closed Riemann surfaces

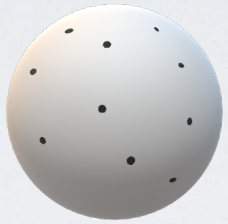
obtained by gluing reversely oriented lines in ’t Hooft double-line representation of the gluon propagator and vertices in perturbation theory

More precisely,

planar diagrams with the topology of a punctured sphere dominate the 't Hooft expansion

χ is the Euler characteristic

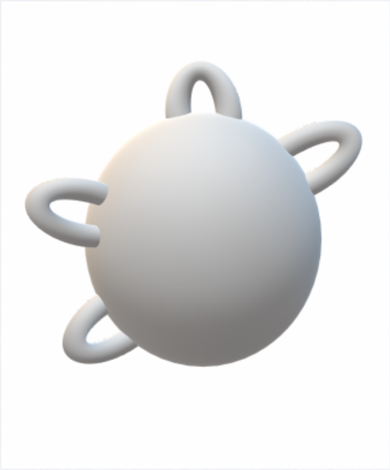
$$\langle \mathcal{G}_1 \ldots \mathcal{G}_n \rangle \sim N^{2-n}$$



Punctured sphere

$$\chi = 2 - n$$

while nonplanar diagrams, with the topology of higher-genus punctured Riemann surfaces and weight N^{χ}, are subleading



$$\chi = 2 - 2g - n$$

These are exactly the weights of a closed string theory ('t Hooft 1974, Veneziano 1976), with string coupling $g_s=1/N$,

whose excitations are nonperturbatively interpreted as glueballs,
and the handles of the Riemann surfaces as glueball loops,

the new ingredient of the canonical string theory being the conformal structure on the Riemann surfaces, interpreted as the string world-sheets

The aim of the present talk is to work out the large-N topological expansion in the SU(N)YM theory, as opposed to the U(N) theory

Naively, both expansions should be essentially the same, since the extra U(1) in the U(N) theory decouples and defines a free theory

As we will demonstrate, the naive expectation is wrong, and the 't Hooft expansion in terms of smooth closed Riemann surfaces is actually incomplete in the SU(N) case

Besides, this incompleteness has remarkable consequences for the existence of a nonperturbative solution of large-N SU(N) YM theory

In fact, we have discovered the aforementioned incompleteness because of the following computation, involving a puzzle about the compatibility of the 't Hooft expansion with the spin-statistics theorem for the glueballs

The generating functional of large-N connected correlators reads

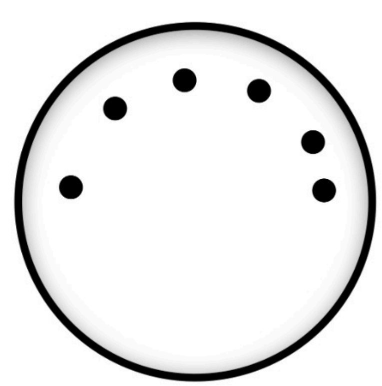
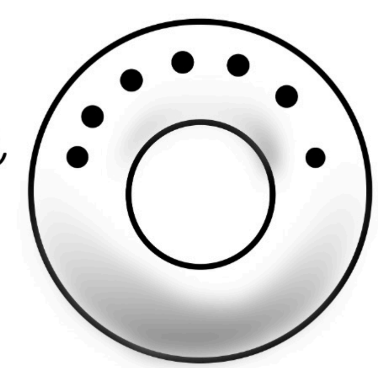
$$\mathcal{W}^E[J_{\mathcal{O}}] = \log \mathcal{Z}[J_{\mathcal{O}}]$$ with

$$\mathcal{Z}[J_{\mathcal{O}}] = \frac{1}{Z}\int \mathcal{D}A\, e^{-\frac{N}{2g^2}S_{YM} + \sum_k \int J_{\mathcal{O}_k}\mathcal{O}_k}$$

By assuming the 't Hooft expansion,

$$\mathcal{W} = \sum_n N^{2-n}\mathcal{W}_{\text{sphere}}(g,n) + N^{-n}\mathcal{W}_{\text{torus}}(g,n) + \ldots$$

where n stands for the number of operator insertions and the ellipses for higher-genus contributions. Graphically,

$$\mathcal{W} = \sum_n N^{2-n}$$  $$+ N^{-n}$$  $$+ \ldots$$

Nonperturbatively, in the effective theory involving an infinite number of glueball fields, the very same object, $\mathcal{W}^E[J_{\mathcal{O}}]$, reads schematically in Euclidean space-time to one loop of glueballs

$$\mathcal{W}^E_{\text{glueball}}[j] = \log \mathcal{Z}^E_{\text{glueball}}[j] = -\frac{1}{2}\int \phi_j *_2 (-\Delta + M^2)\phi_j$$

$$-\frac{1}{N}\frac{1}{3!}\int \phi_j *_3 \phi_j *_3 \phi_j + \int \phi_j *_1 j$$

$$-\frac{1}{2}\log \mathrm{Det}\left(*_2(-\Delta + M^2) + \frac{1}{N} *_3 \phi_j *_3 + \dots \right) + \dots$$

with $\frac{\delta S}{\delta \phi}\Big|_{\phi_j} = *_1 j$ and $\phi_j = (*_2(-\Delta + M^2))^{-1} *_1 j - \frac{1}{2N}(*_2(-\Delta + M^2))^{-1} *_3 \phi_j *_3 \phi_j + \dots$

where * stands for a presently unknown operator structure on the glueball fields that for N= ∞ is fixed by the spectral representation of 2-point correlators of the operators $\mathcal{O}$

The minus sign in front of the logDet arises from the spin-statistics theorem, since all the gauge-invariant glueball interpolating fields have integer spin, and thus the glueballs should be bosons.

Until recently nothing has been known quantitatively on

$\mathcal{W}^E[J_\mathcal{O}]$ and, correspondingly, on $\mathcal{W}^E_{\text{glueball}}[j]$

but recently we have computed its short-distance (UV) asymptotics for all the twist-2 operators by means of the renormalization group (RG), in 3 steps that we will describe momentarily [M. B., M. Papinutto, F. Scardino, JHEP 08 (2021), Phys. Rev. D 108 (2023)]

For simplicity we only report the computation involving the even-spin balanced operators, with maximal-spin component in the + light-cone direction. They read in the light-cone gauge:

$$\mathbb{O}_s = \frac{1}{2N} \bar{A}^a(x) \mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+) A^a(x)$$

with

$$\mathcal{Y}^{\frac{5}{2}}_{s-2}(\overrightarrow{\partial}_+, \overleftarrow{\partial}_+)$$
$$= \overleftarrow{\partial}_+ (i\overrightarrow{\partial}_+ + i\overleftarrow{\partial}_+)^{s-2} C^{\frac{5}{2}}_{s-2}\left(\frac{\overrightarrow{\partial}_+ - \overleftarrow{\partial}_+}{\overrightarrow{\partial}_+ + \overleftarrow{\partial}_+}\right) \overrightarrow{\partial}_+$$

1) We have computed the lowest-order conformal generating functional by a Gaussian integral, as the logarithm of a functional determinant

$$\mathcal{Z}_{\text{conf}}[J_{\mathbb{O}}] = \frac{1}{Z} \int \mathcal{D}A \mathcal{D}\bar{A} \, e^{\int -i\bar{A}^a \Box A^a + \sum_s J_{\mathbb{O}_s} \mathbb{O}_s \, d^4x}$$

$$\mathcal{W}_{\text{conf}}[J_{\mathbb{O}}]$$
$$= -(N^2 - 1) \log \text{Det} \left(\mathbb{I} + \frac{1}{2} i \Box^{-1} \frac{J_{\mathbb{O}_s}}{N} \otimes \mathcal{Y}^{\frac{5}{2}}_{s-2} \right)$$

2) We have constructed of a renormalization scheme where the twist-2 operators become multiplicatively renormalizable

$$Z_{\mathbb{O}_s}(\lambda) = \left(\frac{g(\mu)}{g(\frac{\mu}{\lambda})} \right)^{\frac{\gamma_{0\mathbb{O}_s}}{\beta_0}}$$

3) Correspondingly, after analytic continuation to Euclidean space-time, we have computed the asymptotic generating functional by RG improvement, as all the coordinates are uniformly rescaled by a factor $\lambda \to 0$

$$\mathcal{W}^E_{\text{asym}}[J_{\mathbb{O}^E}, \lambda]$$
$$= -(N^2 - 1) \log \operatorname{Det} \left(\mathbb{I} + \frac{1}{2} \frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}} \Delta^{-1} \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E \frac{5}{2}}_{s-2} \right)$$

Therefore, the leading-nonplanar contribution is

$$\mathcal{W}^{E}_{\text{asym torus}}[J_{\mathbb{O}^E}, \lambda]$$
$$= +\log \operatorname{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)$$

Remarkably, the asymptotic result reproduces the logDet structure of the glueball one-loop generating functional

$$-\frac{1}{2}\log \operatorname{Det}\left(*_2(-\Delta + M^2) + \frac{1}{N} *_3 \phi_j *_3 + \dots\right)$$

Yet, surprisingly, the sign is opposite to the one that follows from the spin-statistics theorem for the glueballs. The main aim of this talk is to solve this sign puzzle.

We should notice that there is no possibility that our computation of the above + sign be wrong, since it descends from the corresponding - sign entering the log of the perturbative gluon one-loop determinant, according to the spin-statistics theorem for gluons in the unitary light-cone gauge, and the fact that in the SU(N) theory there are exactly $N^2 - 1$ gluons

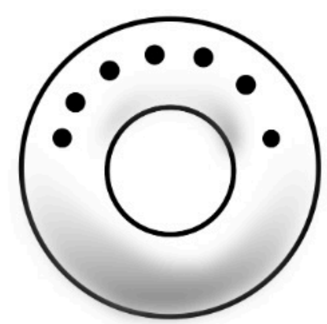
$$\mathcal{W}^{E}_{\text{conf}}[J_{\mathbb{O}^E}]$$
$$= -(N^2-1)\log \operatorname{Det}\left(\mathbb{I} + \frac{1}{2}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)$$

Hence, there are 2 alternatives:

Either the spin-statistics theorem is violated in large-N YM theory

or the 't Hooft expansion and the glueball effective action should be considerably refined, by possibly adding new topologies, so that

$$\mathcal{W}_{1-\text{loop}} = \sum_n N^{-n}$$

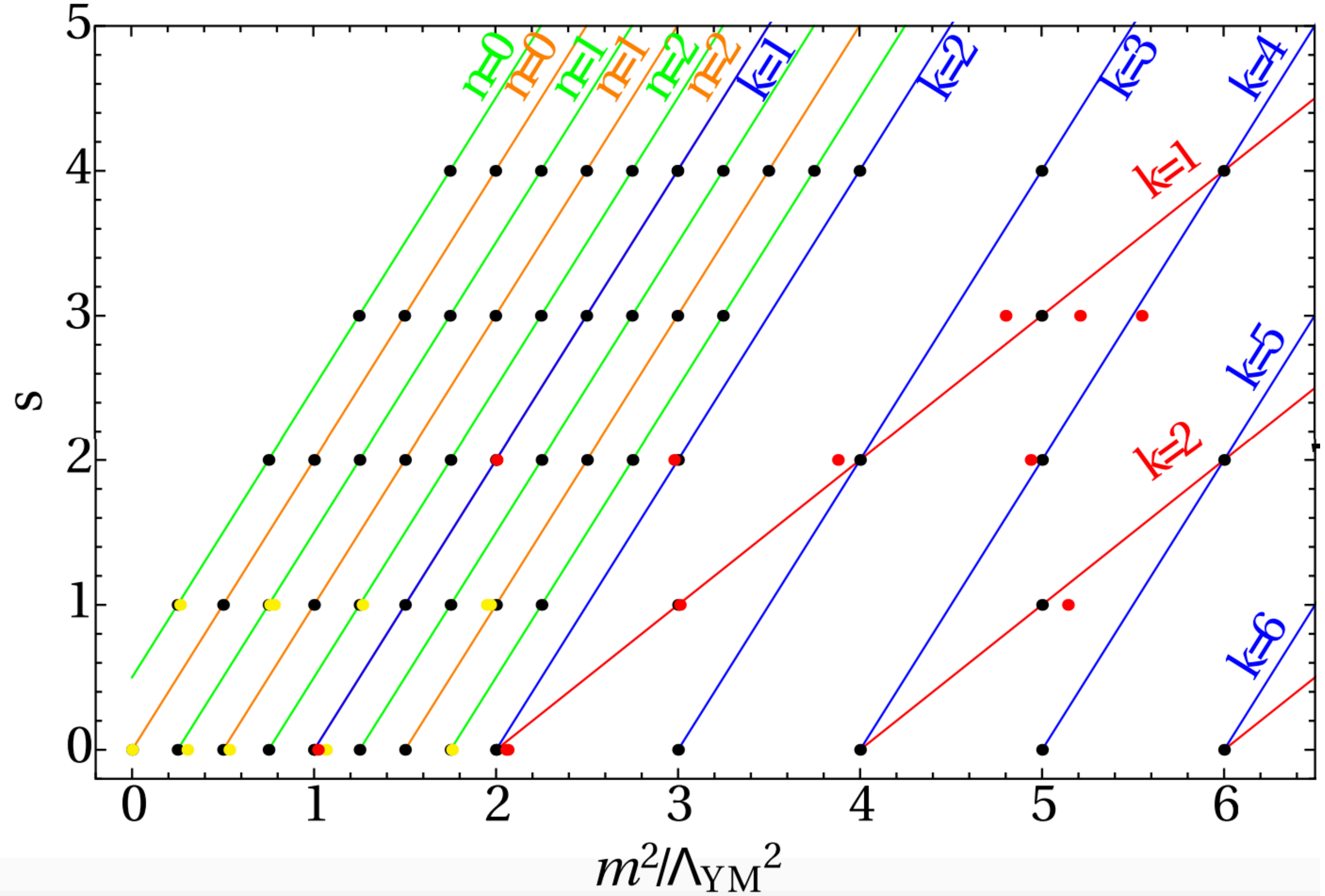


$$+\,\text{new topologies} \quad \sim + \log \mathrm{Det}$$

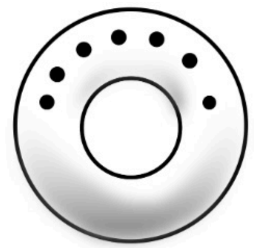
As we will show momentarily only the second alternative applies in large-N YM theory !

In fact, the spin-statistics theorem applies in full generality only in theories with a finite number of fields. If the number of fields is infinite, there exist rigorous counterexamples to the spin-statistics theorem [Streater, Comm. Math. Phys. 5 (1967)]
These counterexamples are based on massive infinite dimensional representations of the Lorentz group, which are the relevant ones for YM theory because of its mass gap.

They have infinite mass degeneracy, since each one decomposes – according to the Wigner theorem - into the sum of irreducible representations of the Poincaré group corresponding to an infinite number of particles of any spin having the same mass.
This would correspond to vertical Regge trajectories that is not acceptable both theoretically and numerically, as evidence from lattice calculations shows.



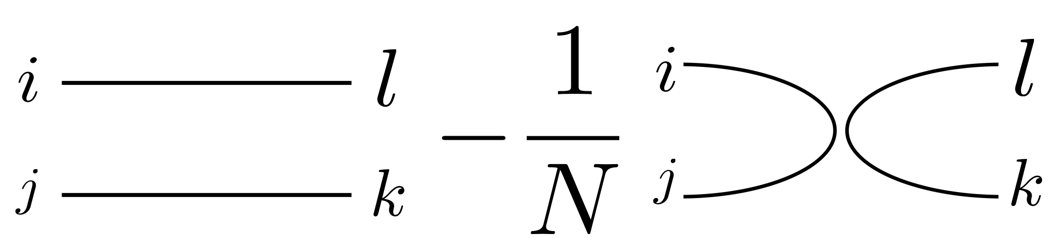
Hence, only the second alternative remains. Indeed, we have discovered that the ‘t Hooft topological expansion in the SU(N) theory is incomplete for twist-2 operators and must be refined by the addition of new topologies

$$\mathcal{W}_{1-\mathrm{loop}} = \sum_n N^{-n}$$  $$+\,\text{new topologies} \quad \sim + \log \mathrm{Det}$$

In order to understand why the new topologies actually arise, we should reconsider the proof of the ‘t Hooft topological expansion.
The delicate point of the proof in the SU(N) Yang-Mills theory, as opposed to the U(N) case, is that the color dependence of the propagator has a leading and subleading contribution

$$\langle A_{ij} A_{lk} \rangle \propto \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{lk}\right)$$

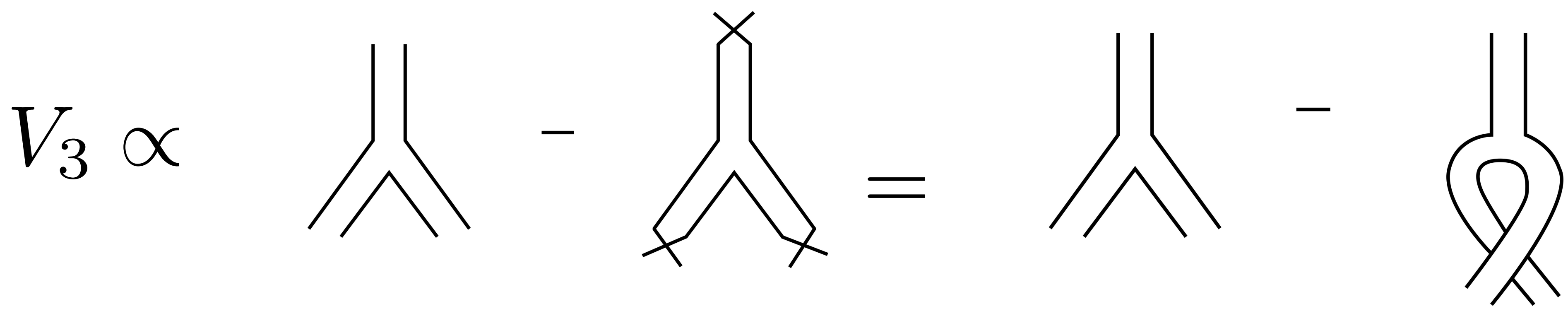
Graphically, in the double-line representation:



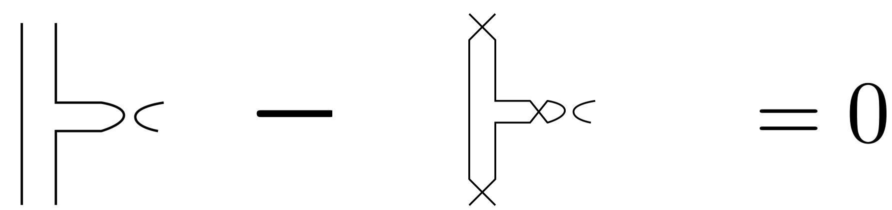
First, let us consider vacuum diagrams

The vertices of the Lagrangian are $V_3 \propto \frac{g}{\sqrt{N}} f^{abc}$ and $V_4 \propto \frac{g^2}{N} f^{abe} f^{ecd}$

Pictorially (Marino 2015):



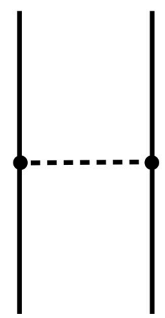
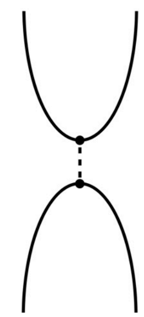
Now, if it is attached to the vertices of the Lagrangian, the subleading part of the propagator does not contribute, since it corresponds to a U(1) contribution, while the vertices are purely nonabelian.
Graphically:



Therefore, the topology of the above vacuum diagrams in SU(N) YM theory coincides with the ‘t Hooft expansion in the double-line representation. The same holds for those local composite operators in the adjoint representation whose local vertices are proportional to the product of some $(T^a)^b_c = -if^{abc}$, $(T^{a_2} \dots T^{a_{n-1}})^{a_1}_{a_n} = 2Tr(\lambda^{a_1}[\lambda^{a_2}, \dots [\lambda^{a_{n-1}}, \lambda^{a_n}] \dots])$
that require at least 3-gluon operators, as for the Lagrangian vertices.

Therefore, we conclude that in these cases we may drop the subleading contribution to the gluon propagator and the proof of ‘t Hooft topological expansion holds in the SU(N) theory exactly as in the U(N) theory, as we may have naively expected !

Yet, we point out that the above proof does not apply to 2-gluon operators - and specifically, to our twist-2 operators in the light-cone gauge. Indeed, even in the adjoint representation, their local vertex involves δ^{ab} as opposed to f^{abc} and contains a 1/N correction, exactly as the gluon propagator does, that spoils the topology of 't Hooft expansion:

 $-\frac{1}{N}$ 

Moreover, even limiting us to the U(N) theory does not cure the aforementioned sign problem. Indeed, the U(N) asymptotic generating functional is the sum of the SU(N) result and of the free U(1) part that obviously decouples

$$\mathcal{W}_{U(N)}[J_{\mathbb{O}}] \sim -(N^2-1)\log\mathrm{Det}\left(\mathbb{I}+\frac{1}{2N}\sum_s \frac{Z_{\mathbb{O}_s(\lambda)}}{\lambda^{s+2}}\Delta^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$
$$-\log\mathrm{Det}\left(\mathbb{I}+\frac{1}{2N}\sum_s \frac{1}{\lambda^{s+2}}\Delta^{-1}J_{\mathbb{O}_s}\otimes\mathcal{Y}^{\frac{5}{2}}_{s-2}\right)$$

and the above asymptotics is dominated by the first term for all the twist-2 operators but the stress-energy tensor that has zero anomalous dimension. This is due to the fact that, even in the U(N) theory, the SU(N) part of the twist-2 operators renormalizes in general with positive one-loop anomalous dimensions, differently from the U(1) part that does not renormalize, so that the topology of the bare 't Hooft expansion does not apply to the renormalized twist-2 operators

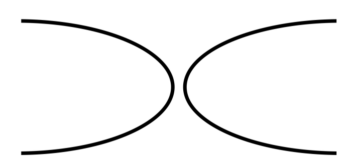
To understand what happens in the SU(N) theory, we may rewrite identically the conformal generating functional as

$$\mathcal{W}^{E}_{\mathrm{conf}}[J_{\mathbb{O}^E}]$$

$$= -\log Det\left(\mathcal{I} + \frac{1}{2}(I-P)\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right)$$

where I-P is graphically

$$-\frac{1}{N}$$



and the color traces in the loop expansion of the determinant produce the aforementioned factor

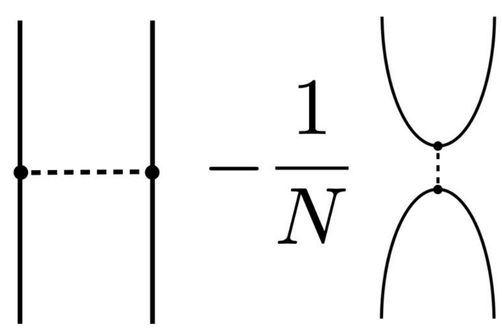
$$Tr(I-P)^n = Tr(I-P) = N^2 - 1$$

We get identically

$$
\begin{aligned}
&\mathcal{W}^{E}_{\mathrm{conf}}[J_{\mathbb{O}^E}] \\
&= -\log Det\left(\mathcal{I} + \frac{1}{2}\Delta^{-1} I \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right) \\
&- \log Det\left(\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\Delta^{-1} I \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)^{-1} \Delta^{-1} P \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\frac{5}{2}}_{s-2}\right)
\end{aligned}
$$

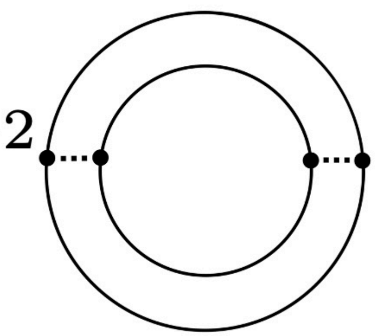
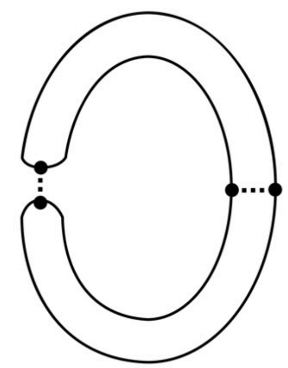
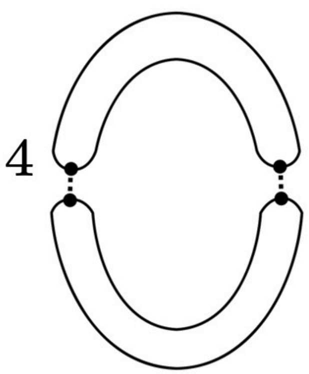
where the first term is the planar contribution,

while the second one is the subleading contribution involving the projector P associated to the aforementioned subleading local vertex

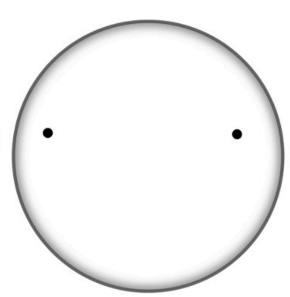
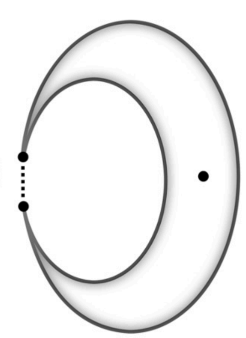
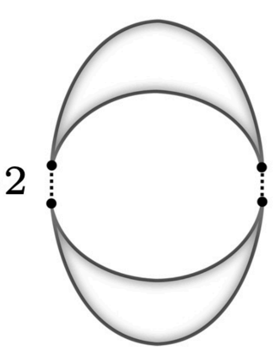


that gives rise to new topologies

For example, for the 2-point correlators of twist-2 operators to the leading perturbative order we get

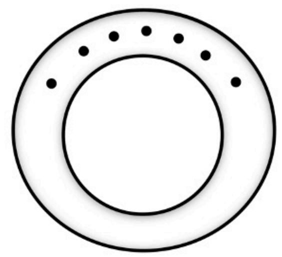
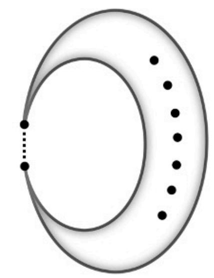
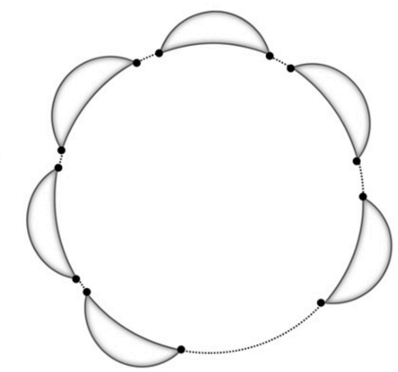
$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle = N^{-2}$  $-2N^{-3}$  $+N^{-4}$ 

Hence, new planar subleading diagrams arise - in addition to the first one - corresponding to punctured disks with at least two punctures identified in pairs. Following ‘t Hooft prescription, the first one corresponds to a 2-punctured sphere, and the other ones to possibly disconnected punctured spheres with at least two punctures identified in pairs:

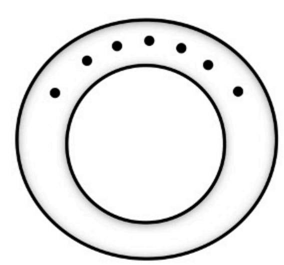
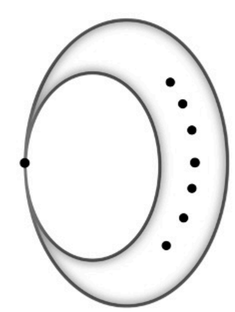
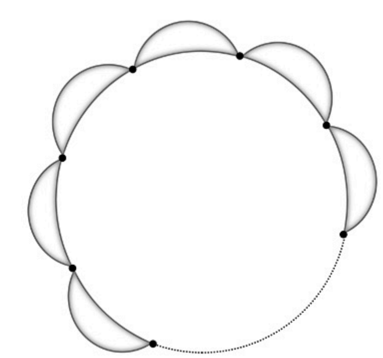
$\langle \mathbb{O}_s(x)\mathbb{O}_s(0)\rangle = N^{0}$  $+N^{-2}$  $+N^{-2}$ 

Indeed, in the new diagrams we have identified the doubly punctured disk with an infinite strip. Then, according to ’t Hooft prescription, we have glued the two sides of the infinite strip to get an infinite cylinder, i.e., a sphere with 2 punctures.

More generally, our refined topological expansion of the glueball one-loop generating functional reads for n-point correlators of twist-2 operators:

$\mathcal{W}_{1-\text{loop}} = \sum_n N^{-n}$  $+ N^{-n}$  $+ \ldots + N^{-n}$ 

Remarkably, the smooth punctured tori are now corrected by a sum of diagrams that can be thought of as the normalization of pinched/punctured tori:

$\mathcal{W}_{1-\text{loop}} = \sum_n N^{-n}$  $+ N^{-n}$  $+ \ldots + N^{-n}$ 

that are singular surfaces obtained by contracting to a point some nontrivial cycles of smooth tori - possibly with punctures. Importantly, for twist-2 operators no pinched Riemann surfaces of higher genus than the torus occur, so that the only correction to ’t Hooft expansion is one-loop exact !

Analytically, the sum of the above topologies reads asymptotically

$$\mathcal{W}^{E}_{\text{asym nonplanar}}[J_{\mathbb{O}^E}, \lambda]$$

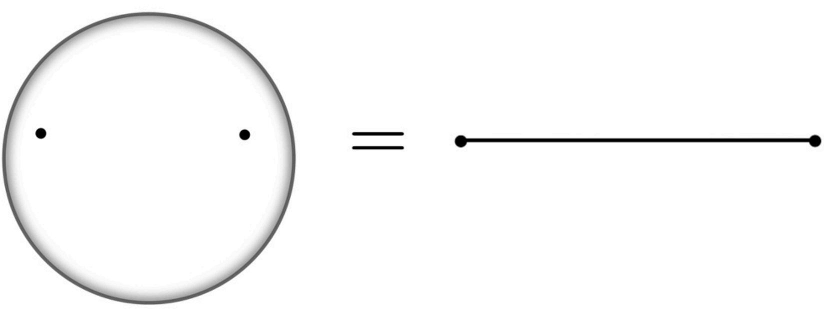
$$= -\log Det\left(\mathcal{I} - \frac{1}{2}\left(\mathcal{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} I \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right)^{-1} \times \frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} P \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right)$$

$$= +\log \mathrm{Det}\left(\mathbb{I} + \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1}\frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right)$$

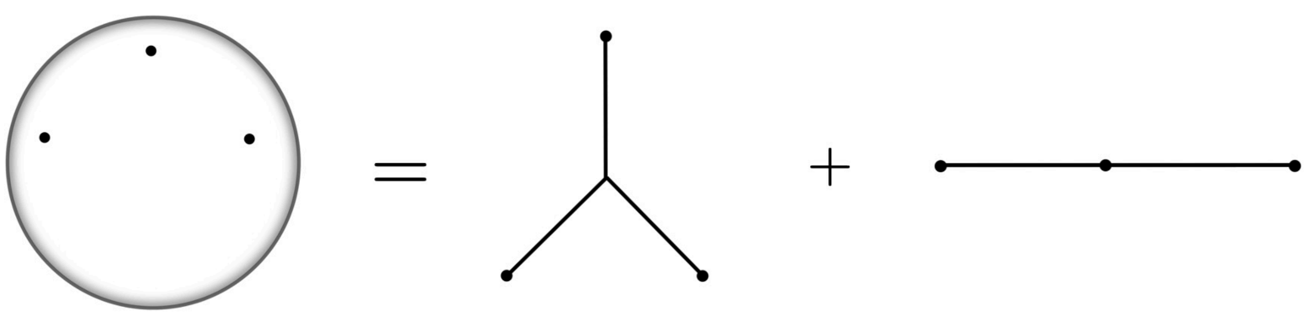
where now the sign of the first logDet agrees with the spin-statistic theorem, thus solving the sign puzzle, but at the price of introducing the aforementioned singular topologies !

We now provide a nonperturbative interpretation of our refined topological expansion in terms of the effective theory of glueballs.

It has been known for more than 40 years that punctured spheres correspond to glueball tree diagrams



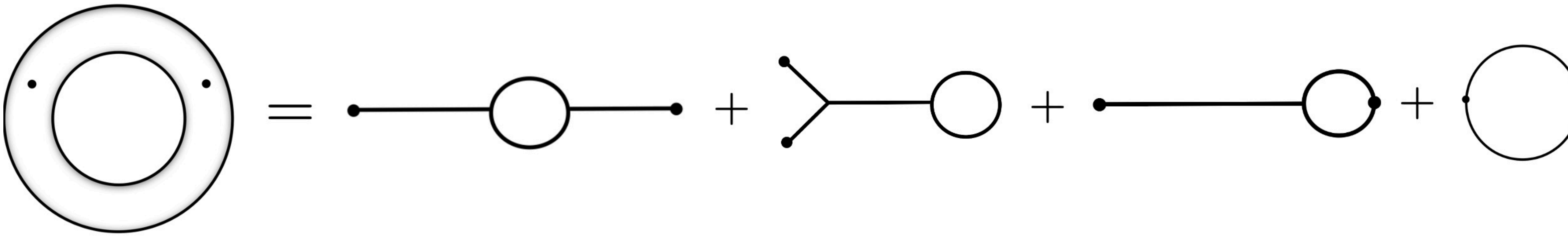
The sphere with two punctures corresponds to an infinite sum of glueball propagators [Migdal, (1977)]



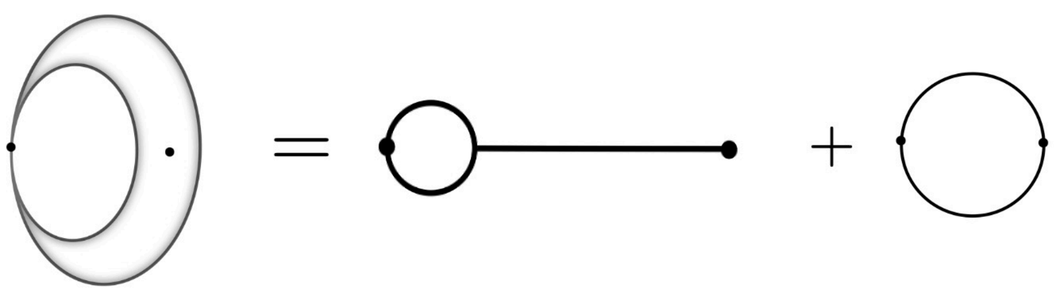
The sphere with three punctures corresponds to vertices that may carry sums of 3 or 2 poles in Minkowskian space-time [Witten, (1979)]

We observe that the last diagram contributes zero to the S matrix, since one external leg is missing.

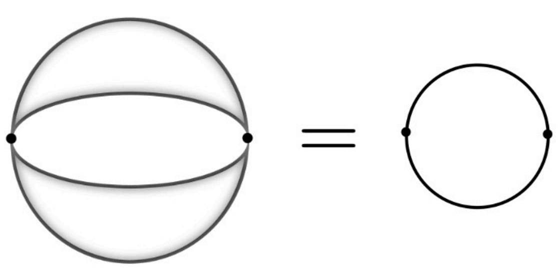
Moreover, the punctured tori correspond to glueball one-loop diagrams:



Interestingly, we observe that in the glueball effective theory on the right-hand side of the above picture also diagrams without external glueball legs occur. Moreover, for our new topologies we obtain



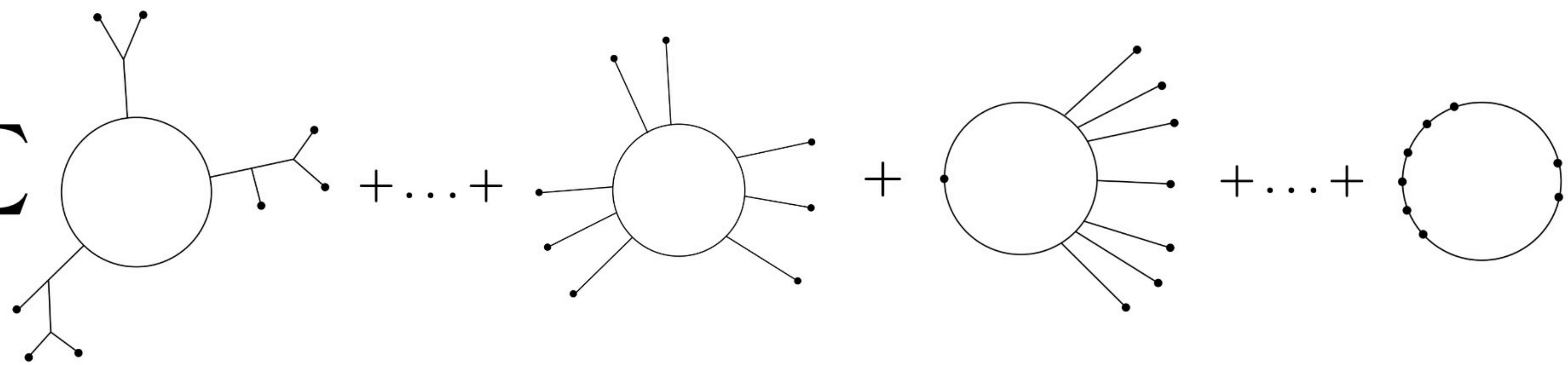
Both diagrams contribute zero to the S matrix because at least one external leg is missing.



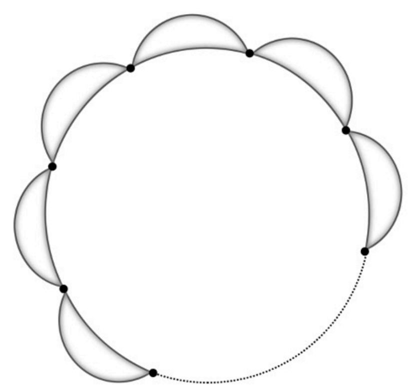
Besides, the maximally pinched diagrams are in 1-to-1 correspondence with the ones of the effective theory, since they do not carry any external leg.

Hence, in general

$$\mathcal{W}_{\text{1-loop}} = \sum_n$$



Besides, for the maximally pinched diagrams

$$\sum_n N^{-n}$$ 

we get asymptotically by explicit computation

$$\begin{aligned}
&\mathcal{W}^{E}_{\text{asym maximally pinched}}[J_{\mathbb{O}^E}, \lambda] \\
&= -\log \mathrm{Det}\left(\mathcal{I} - \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} P \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right) \\
&= -\log \mathrm{Det}\left(\mathbb{I} - \frac{1}{2}\frac{Z_{\mathbb{O}_s}(\lambda)}{\lambda^{s+2}}\Delta^{-1} \frac{J_{\mathbb{O}^E_s}}{N} \otimes \mathcal{Y}^{E\,\frac{5}{2}}_{s-2}\right)
\end{aligned}$$

that again agrees with the spin-statistics theorem

Finally, the occurrence of pinched tori with punctures resembles the Deligne-Mumford (DM) compactification of the moduli space of punctured closed Riemann surfaces that arises in canonical string theories due to their underlying conformal structure. Yet, in general our pinched tori do not occur in the DM compactification of n-punctured tori, whose components only involve punctured closed Riemann surfaces with $\chi < 0$ and, in particular, spheres with at least 3 punctures -- because in the DM compactification punctures and pinches never collide, according to the conformal structure of canonical string theories -- while the components of our pinched tori may contain 2-punctured spheres with $\chi = 0$

Our conclusions follow

In the large-N SU(N) YM theory a new topological sector exists that refines the 't Hooft topological expansion for the correlators of twist-2 operators, both perturbatively and nonperturbatively.

The new topologies solve the sign puzzle, specifically in the maximally pinched sector.

The new topological sector dominates the UV asymptotics of the correlators of twist-2 operators, but contributes zero to the nonperturbative S matrix, since nonperturbatively it consists of tori with at least one pinch, corresponding to glueball one-loop diagrams with at least one glueball external leg missing.

As in general the new topologies do not arise from the DM compactification of the moduli space of punctured tori, no canonical string theory admitting it may exist for the correlators of large-N YM theory in the new topological sector, but it may exist for the S-matrix amplitudes.

Finally, the existence of the new topological sector -- specifically, the maximally pinched one -- being one-loop exact and contribuing 0 to the S-matrix -- opens the way to an exact solution limited to the new sector:

Conjecturally, by a topological field/string theory -- noncanonical in the sense of the present talk -- along the lines foreseen in [M. B., HADRON 2015], by reinterpreting the logDet that arises ibidem from the coupling to D-branes as the generating functional of correlators in the new sector