Stefano and Marcello @ Work

QCD@Work

Trani, June 18th, 2024

Giovanni Ridolfi

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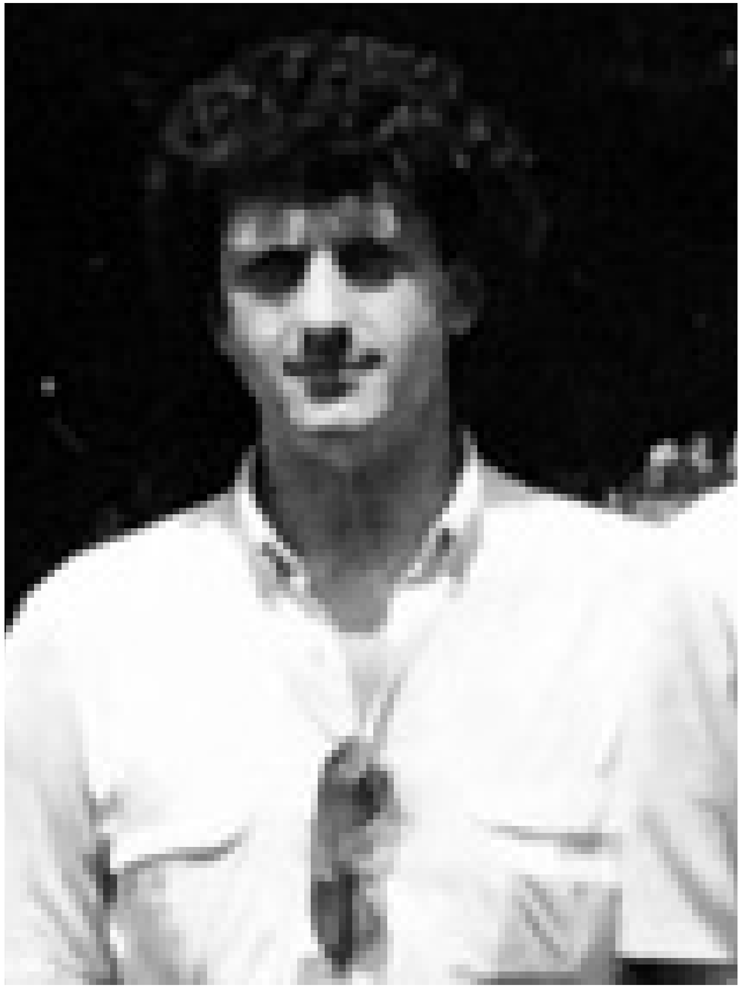
A few personal recollections

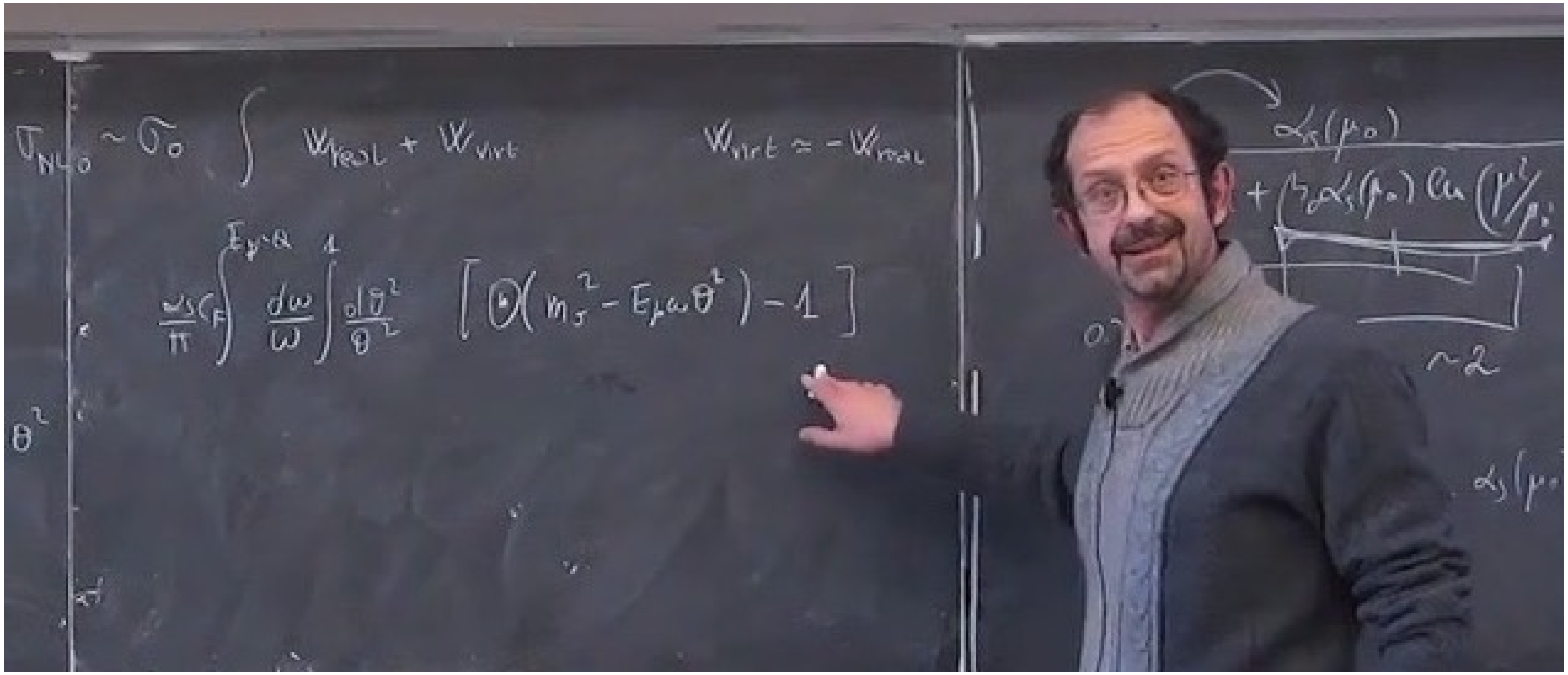
International School of Physics "Enrico Fermi"

Villa Monastero, Varenna, June 26 - July 6, 1984





NAVIGATOR


σNLO ~ σ0 ∫ Wreal + Wvirt
Wvirt ≃ - Wreal
[Θ(mJ² - Ejω θ²) - 1]
αs(μ0)
~2

Stefano's work I'm most familiar with: Threshold resummations

A short (and beautiful) review: S. Catani, hep-ph/9610413

Physical cross sections are always inclusive over arbitrarily soft particles in the final state, because of finite detector resolution.

Inclusiveness plays a crucial role in QCD: infrared divergences from virtual corrections are cancelled by radiation of undetected real gluons.

The finite left-over of these cancellations give large contributions if the tagged final state is forced to take most of the available energy.

Schematically: $(1-z)\sqrt{s}$ total energy carried by unobserved radiation. Virtual and real emission corrections at order $\alpha_{\rm S}$:

$$\frac{dw_{\rm virtual}}{dz} = -2C\,\alpha_{\rm S}\,\delta(1-z)\int_0^{1-\epsilon}\frac{dy}{1-y}\,\log\frac{1}{1-y}$$

$$\frac{dw_{\rm real}}{dz} = +2C\,\alpha_{\rm S}\,\frac{1}{1-z}\,\log\frac{1}{1-z}\,\theta(1-\epsilon-z)$$

because of the bremsstrahlung spectrum $d\omega/\omega$ and the spectrum for collinear emission, dk_T^2/k_T^2.

The sum of the two contributions is finite as the infrared cut-off ϵ goes to 0:

$$\frac{dw(z)}{dz} = \frac{dw_{\rm virtual}}{dz} + \frac{dw_{\rm real}}{dz} = 2\,C\,\alpha_{\rm S}\left[\frac{1}{1-z}\,\log\frac{1}{1-z}\right]_+$$

where

$$\int_0^1 dz\,[D(z)]_+ f(z) = \int_0^1 dz\,D(z)\,[f(z)-f(1)]$$

- **Virtual contribution concentrated at $z = 1$**
- **Real emission contribution spread over the interval $x < z < 1$, where x is the fraction of total energy carried by the observed final state.**

Hence, the contribution of soft emission to the cross section is proportional to

$$\int_x^1 dz \, \frac{dw}{dz} = -C \, \alpha_{\rm S} \, \log^2(1 - x),$$

a finite left-over of the cancellation of infrared divergences.

As $x \to 1$ in the final state, the phase space for real emission is suppressed, and the finite left-over becomes large.

At order n, at most two powers of $\log(1-x)$ for each power of α_s appear in the perturbative coefficients:

$$C_n(x)\alpha_s^n = \alpha_s^n \sum_{m=1}^{2n} c_{nm} \log^m(1-x) + \text{non singular terms}$$

The perturbative expansion becomes unreliable; logarithmically enhanced contributions must be resummed to all orders.

Examples:

1. **lepton-nucleon scattering in the quasi-elastic limit:**

$$x \to x_{\mathrm{Bj}} = \frac{Q^2}{2p \cdot q}, \qquad x \to 1$$

2. **production of heavy systems (Drell-Yan pairs, Higgs) close to threshold:**

$$x \to \tau = \frac{Q^2}{s}, \qquad s \gtrsim Q^2$$

3. **transverse momentum spectra in the small-q_T region:**

$$1 - x \to \frac{q_{\mathrm{T}}^2}{Q^2}, \qquad q_{\mathrm{T}}^2 \ll Q^2$$

Threshold resummation performed in the space of Mellin moments

$$\hat{F}(N) = \int_0^1 dx\, x^{N-1} F(x); \qquad F(N) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, x^{-N} \hat{F}(N)$$

(Fourier transform in the case of transverse momentum).

$$\hat{C}_n(N) = \hat{C}_n^{\mathrm{LL}}(N) + \hat{C}_n^{\mathrm{NLL}}(N) + \ldots$$

$$C_n^{\mathrm{LL}}(N) = \sum_{k=n+1}^{2n} \hat{c}_{nk} \log^k N$$

$$C_n^{\mathrm{NLL}}(N) = \hat{c}_{nn} \log^n N$$

Three non trivial points

1. Eikonal emission exponentiates in QED because soft photons are emitted independently. Gluon correlations are shown to cancel in the soft limit.

Nuclear Physics B236 (1984) 61–89

MANY-GLUON CORRELATIONS AND THE QUARK FORM FACTOR IN QCD

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Received 27 June 1983

On the basis of an eikonal approximation in configuration space we compute both the large-N behaviour of the quark fragmentation function and explicit soft gluon emission probabilities up to three loops. The results exhibit the exponentiation mechanism of the infrared contributions to the form factor. On the other hand, gluon correlations are present for any angular configurations, but are found important only in the strong-ordering region recently emphasized in the coherent branching scheme for jet evolution.

2. The argument of the running coupling is set at the transverse momentum of emitted gluons:

$$\alpha_{\rm S} \to \alpha_{\rm S}(k_T^2)$$

$$k_T^2 \le Q^2(1-x)^a; \qquad a = 1,2$$

very different from Q^2 in the threshold limit.

An important point: resummation of leading log terms of order $\alpha_{\rm S}^k \log^{k+1} N$.

Nuclear Physics B173 (1980) 429–455

A TREATMENT OF HARD PROCESSES SENSITIVE TO THE INFRARED STRUCTURE OF QCD

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M. CIAFALONI

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G. VENEZIANO

CERN, Geneva, Switzerland

Received 30 April 1980

We propose a modified jet evolution equation which resums large corrections to the usual leading logarithmic approximation when phase-space constraints expose the singular infrared structure of QCD. The modification, which consists simply of a rescaling of the argument of the running coupling constant, is based on perturbative arguments verified at the fourth-order level. Processes analyzed by this method include the quark (Sudakov) form factor, the large moments of structure and fragmentation functions, the asymptotic behaviour of multiplicities and the clustering of final quanta in colourless systems which occupy finite regions of (momentum and position) phase space.

EXPONENTIATION OF LARGE N SINGULARITIES IN QCD

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and

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Received 1 April 1981

Large N singularities of parton densities are related by a gauge transformation to a class of collinear singularities. We prove on this basis that the effective anomalous dimension shows $[\alpha_S(Q^2) \log N]^p$ singularities which are calculable to all orders by a change of scale in $\alpha_S(Q^2)$.

3. Exponentiation of next-to-leading logs

Nuclear Physics B327 (1989) 323–352
North-Holland, Amsterdam

RESUMMATION OF THE QCD PERTURBATIVE SERIES FOR HARD PROCESSES

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L. TRENTADUE

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Received 2 March 1989

We study the region of inhibited radiation in hard hadronic processes, as for jet cross sections and heavy flavour production near threshold. The cases of deep inelastic scattering and Drell–Yan annihilation are explicitly considered. A general method to exponentiate leading and next-to-leading logarithms to all orders in perturbation theory is developed. A complete formula for the large N-moments is given and shown to agree with previous two-loop calculations. The resummation procedure suggests how to connect the perturbative and nonperturbative regions. The natural limit within the perturbative phase is shown to be the intrinsic transverse momentum.

$$\frac{C_N(Q^2)}{C_N^{\mathrm{LO}}(Q^2)} = g_0(Q^2)\exp G_N(Q^2) + O\left(\frac{\log^k N}{N}\right)$$

$$G_N^{\mathrm{DIS}} = \log\Delta_q(Q^2,\mu^2) + \log J_q(Q^2) + \log\Delta_{\mathrm{int}}^{\mathrm{DIS}}(Q^2)$$

$$G_N^{\mathrm{DY}} = 2\log\Delta_q(Q^2,\mu^2) + \log\Delta_{\mathrm{int}}^{\mathrm{DY}}(Q^2)$$

$$\Delta_q(Q^2,\mu^2) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\int_{\mu^2}^{Q^2(1-z)^2}\frac{dq^2}{q^2}A(\alpha_{\mathrm{S}}(q^2))$$

$$J_q(Q^2,\mu^2) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}\left[\int_{Q^2(1-z)^2}^{Q^2(1-z)}\frac{dq^2}{q^2}A(\alpha_{\mathrm{S}}(q^2)) + B(\alpha_{\mathrm{S}}(Q^2(1-z)))\right]$$

$$\Delta_{\mathrm{int}}(Q^2,\mu^2) = \int_0^1 dz\,\frac{z^{N-1}-1}{1-z}D(\alpha_{\mathrm{S}}(Q^2(1-z)^2))$$

Since then, an impressive amount of work on this subject:

1. S. Catani and L. Trentadue, "Resummation of the QCD Perturbative Series for Hard Processes," Nucl. Phys. B 327 (1989), 323-352
2. S. Catani, L. Trentadue, G. Turnock and B. R. Webber, "Resummation of large logarithms in e+ e- event shape distributions," Nucl. Phys. B 407 (1993), 3-42
3. S. Catani, M. L. Mangano, P. Nason and L. Trentadue, "The Resummation of soft gluons in hadronic collisions," Nucl. Phys. B 478 (1996), 273-310
4. S. Catani, "Higher order QCD corrections in hadron collisions: Soft gluon resummation and exponentiation," Nucl. Phys. B Proc. Suppl. 54 (1997), 107-113
5. S. Catani, "Soft gluon resummation: A Short review," [arXiv:hep-ph/9709503 [hep-ph]].
6. R. Bonciani, S. Catani, M. L. Mangano and P. Nason, "NLL resummation of the heavy quark hadroproduction cross-section," Nucl. Phys. B 529 (1998), 424-450 [erratum: Nucl. Phys. B 803 (2008), 234]
7. S. Catani, M. L. Mangano and P. Nason, "Sudakov resummation for prompt photon production in hadron collisions," JHEP 07 (1998), 024
8. S. Catani, M. L. Mangano, P. Nason, C. Oleari and W. Vogelsang, "Sudakov resummation effects in prompt photon hadroproduction," JHEP 03 (1999), 025
9. M. Cacciari and S. Catani, "Soft gluon resummation for the fragmentation of light and heavy quarks at large x," Nucl. Phys. B 617 (2001), 253-290
10. S. Catani, D. de Florian, M. Grazzini and P. Nason, "Soft gluon resummation for Higgs boson production at hadron colliders," JHEP 07 (2003), 028

11. R. Bonciani, S. Catani, M. L. Mangano and P. Nason, "Sudakov resummation of multiparton QCD cross-sections," Phys. Lett. B 575 (2003), 268-278
12. G. Bozzi, S. Catani, D. de Florian and M. Grazzini, "Transverse-momentum resummation and the spectrum of the Higgs boson at the LHC," Nucl. Phys. B 737 (2006), 73-120
13. G. Bozzi, S. Catani, D. de Florian and M. Grazzini, "Higgs boson production at the LHC: Transverse-momentum resummation and rapidity dependence," Nucl. Phys. B 791 (2008), 1-19
14. G. Bozzi, S. Catani, G. Ferrera, D. de Florian and M. Grazzini, "Transverse-momentum resummation: A Perturbative study of Z production at the Tevatron," Nucl. Phys. B 815 (2009), 174-197
15. G. Bozzi, S. Catani, G. Ferrera, D. de Florian and M. Grazzini, "Production of Drell-Yan lepton pairs in hadron collisions: Transverse-momentum resummation at next-to-next-to-leading logarithmic accuracy," Phys. Lett. B 696 (2011), 207-213
16. S. Catani and M. Grazzini, "QCD transverse-momentum resummation in gluon fusion processes," Nucl. Phys. B 845 (2011), 297-323
17. S. Catani, M. Grazzini and A. Torre, "Soft-gluon resummation for single-particle inclusive hadroproduction at high transverse momentum," Nucl. Phys. B 874 (2013), 720-745
18. S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, "Universality of transverse-momentum resummation and hard factors at the NNLO," Nucl. Phys. B 881 (2014), 414-443
19. S. Catani, L. Cieri, D. de Florian, G. Ferrera and M. Grazzini, Nucl. Phys. B 888 (2014), 75-91

20. S. Catani, M. Grazzini and A. Torre, "Transverse-momentum resummation for heavy-quark hadroproduction," Nucl. Phys. B 890 (2014), 518-538

21. S. Catani, D. de Florian, G. Ferrera and M. Grazzini, "Vector boson production at hadron colliders: transverse-momentum resummation and leptonic decay," JHEP 12 (2015), 047

22. S. Catani, M. Grazzini and H. Sargsyan, "Transverse-momentum resummation for top-quark pair production at the LHC," JHEP 11 (2018), 061

A difficulty immediately arises. Define $\tilde{\Sigma}(L, \alpha_{\rm S})$ by

$$\frac{C_N(Q^2)}{C_N^{\rm LO}(Q^2)} = 1 + \tilde{\Sigma}(L, \alpha_{\rm S}(Q^2)) = 1 + \sum_{k=1}^{\infty} h_k(\alpha_{\rm S}(Q^2)) L^k$$

$\tilde{\Sigma}$ arises as an expansion in powers of $\alpha_{\rm S}(Q^2)$ of a function of $\alpha_{\rm S}(Q^2/N^a)$. To NLL we have

$$\alpha_{\rm S}\left(\frac{Q^2}{N^a}\right) = \frac{\alpha_{\rm S}(Q^2)}{1+L}\left[1 - \alpha_{\rm S}(Q^2)\frac{\beta_1}{\beta_0}\frac{\log(1+L)}{1+L}\right]; \qquad L = a\alpha_{\rm S}(Q^2)\beta_0 \log\frac{1}{N}$$

which has a branch cut on the real positive N axis for $L \leq -1$, or

$$N \geq N_L \equiv e^{\frac{1}{\bar{\alpha}}}; \qquad \bar{\alpha} = a\beta_0\alpha_{\rm S}(Q^2)$$

because of the Landau singularity.

The inverse Mellin transform of $C_N(Q^2)/C_N^{\rm LO}(Q^2)$ does not exist.

One possible way out: take the term-by-term inverse Mellin transform of $\tilde{\Sigma}(L, \alpha_s)$:

$$\Sigma(z, \alpha_s(Q^2)) = \sum_{k=1}^{\infty} h_k \bar{\alpha}^k \frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N} \log^k \frac{1}{N}$$

but the series is divergent! Proof:

$$\frac{1}{2\pi i} \oint_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N} \log^k \frac{1}{N} = \frac{k!}{2\pi i} \left[\oint \frac{d\xi}{\xi^{k+1}} \frac{\log^{\xi-1} \frac{1}{z}}{\Gamma(\xi)} \right]_+$$

$$\Sigma(z, \alpha_s(Q^2)) = \frac{1}{2\pi i} \left[\frac{1}{\log \frac{1}{z}} \oint \frac{d\xi}{\xi} \frac{\log^{\xi} \frac{1}{z}}{\Gamma(\xi)} \sum_{k=1}^{\infty} k! h_k \left(\frac{\bar{\alpha}}{\xi} \right)^k \right]_+$$

A second possible way out: taking the inverse Mellin transform of each $\log^k N$ term at the relevant (leading, next-to-leading...) logarithmic level, the perturbative series converges. For example, to leading log accuracy one has

$$\frac{1}{2\pi i}\int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, z^{-N}\, \log^k \frac{1}{N} = k\left[\frac{\log^{k-1}(1-z)}{1-z}\right]_+ + \mathrm{NLL}$$

The series now converges to

$$\Sigma_{\mathrm{LLx}}(z,\alpha_{\mathrm{S}}(Q^2)) = \bar{\alpha}\left[\frac{1}{1-z}\tilde{\Sigma}'(\bar{\alpha}\log(1-z),\alpha_{\mathrm{S}}(Q^2))\right]_+$$

but only for $z < z_L = 1 - e^{-\frac{1}{\bar{\alpha}}}$ again because of the Landau pole at $z = z_L$.

Similar situation in the case of the resummation of large logarithms of q_T^2/Q^2 in the small-q_T region of the spectrum.

In this case

$$\text{Mellin tr.} \quad \int_0^1 dz\, z^{N-1}\, f(z) \rightarrow \text{Fourier tr.} \quad \frac{1}{2\pi} \int d^2b\, e^{-i\vec{b}\cdot\vec{q}_T}\, f(\vec{q}_T)$$

The resummed cross section in $\vec{b}$ space has no inverse Fourier transform, again because of the Landau pole of the running coupling.

A solution (now universally adopted) was found by Stefano and collaborators:

ELSEVIER

Nuclear Physics B 478 (1996) 273–310

NUCLEAR PHYSICS B

The resummation of soft gluons in hadronic collisions

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[b] *CERN, TH Division, Geneva, Switzerland*
[c] *Univ. di Parma and INFN, Gruppo Collegato di Parma, Parma, Italy*

Received 26 April 1996; accepted 25 July 1996

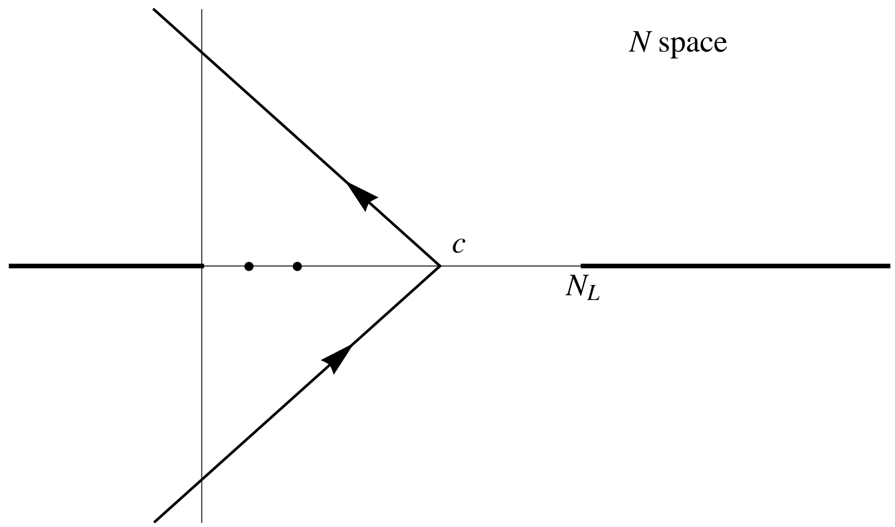
Abstract

We compute the effects of soft-gluon resummation for the production of high mass systems in hadronic collisions. We carefully analyse the growth of the perturbative expansion coefficients of the resummation formula. We propose an expression consistent with the known leading and next-to-leading resummation results, in which the coefficients grow much less than factorially. We apply our formula to Drell–Yan pair production, heavy flavour production, and the production of high invariant mass jet pairs in hadronic collisions. We find that, with our formula, resummation effects become important only fairly close to the threshold region. In the case of heavy flavour production we find that resummation effects are small in the experimental configurations of practical interest.

The minimal prescription. A very simple recipe: just take

$$\sigma(x,Q^2) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} dN\, x^{-N}\, \mathcal{L}(N,Q^2)\, C(N,\alpha_s(Q^2))$$

with $0 < c < N_L$**.**



Not a true inverse Mellin: the integrand is not analytical in any right half-plane, because of the branch cut due to the Landau pole.

The MP has a number of good properties:

- **it is well defined for all values of** x
- **it is an asymptotic sum of the original, divergent perturbative expansion**
- **the difference between the original series, truncated at the best-approximation term, and the minimal prescription, is suppressed more strongly than any power of** Λ^2/Q^2**.**

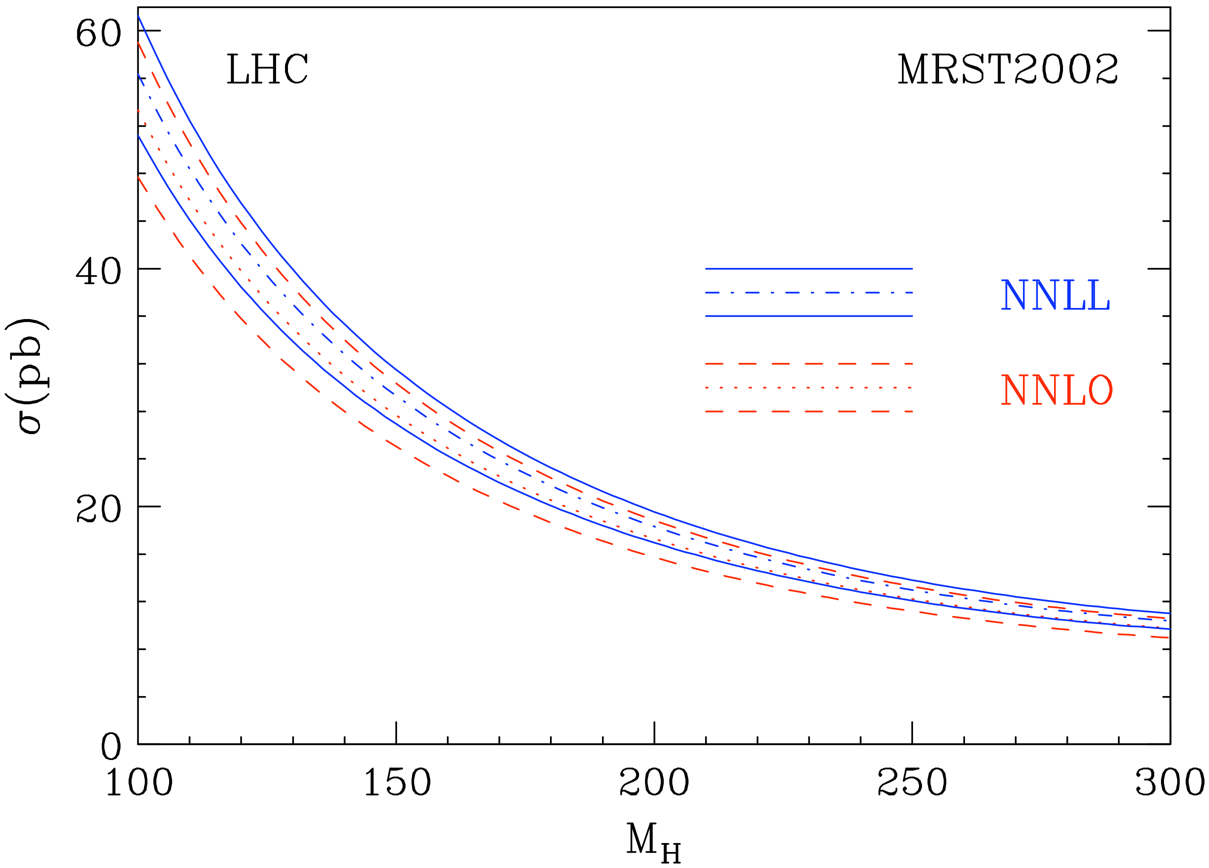
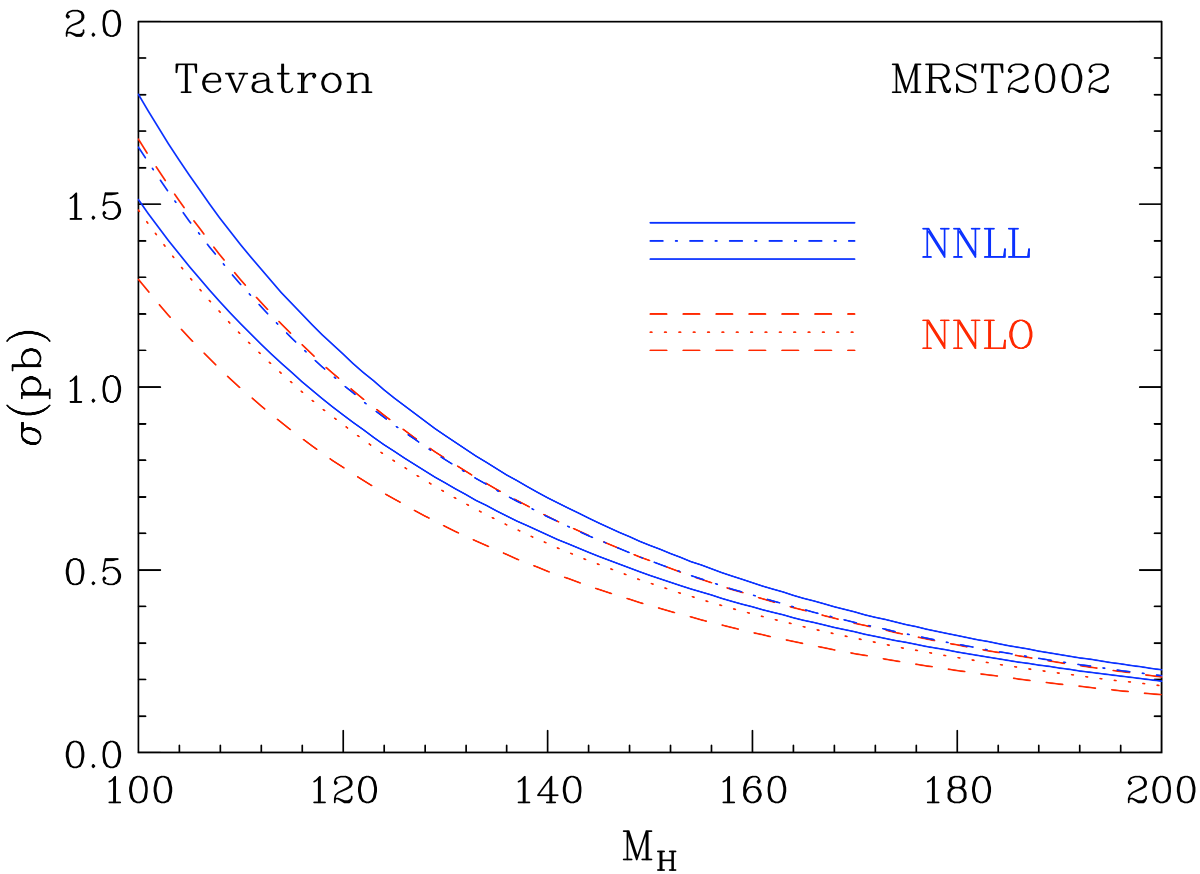


Figure 1: Higgs production at the LHC.

Catani, De Florian, Grazzini, Nason, JHEP 0307(2003)028, arXiv:hep-ph/0306211v1



Higgs production at the Tevatron.

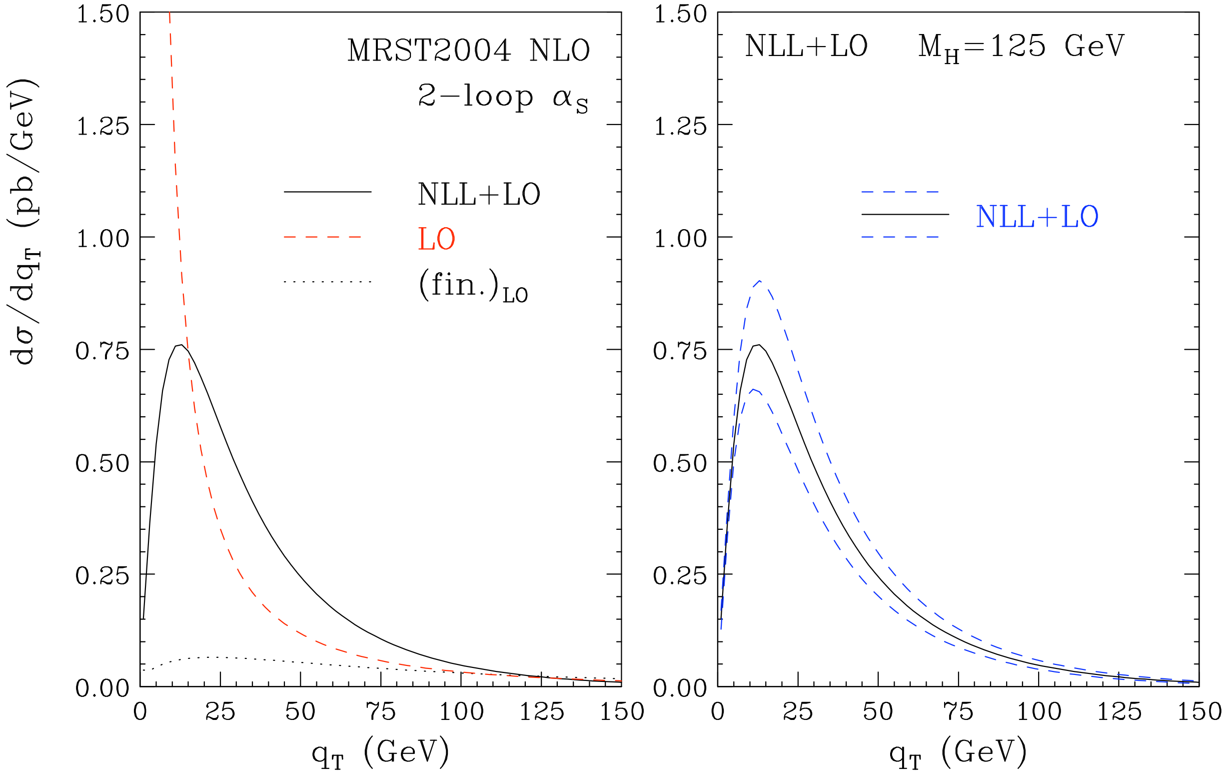
Catani, De Florian, Grazzini, Nason, JHEP 0307(2003)028, arXiv:hep-ph/0306211v1.

A minimal prescription for Fourier inversion in the case of $\vec{q}_T$ distributions: a deformation of the integration contour away from the real axis.

Minimal in the sense that it gives back the right result when applied to functions of $\vec{b}$ which do have a Fourier inverse.

E. Laenen, G. Sterman and W. Vogelsang, PRL 84(2000)4296;
A. Kulesza, G. Sterman and W. Vogelsang, PRD 66(2002)014011

Resummations can have a sizable impact:

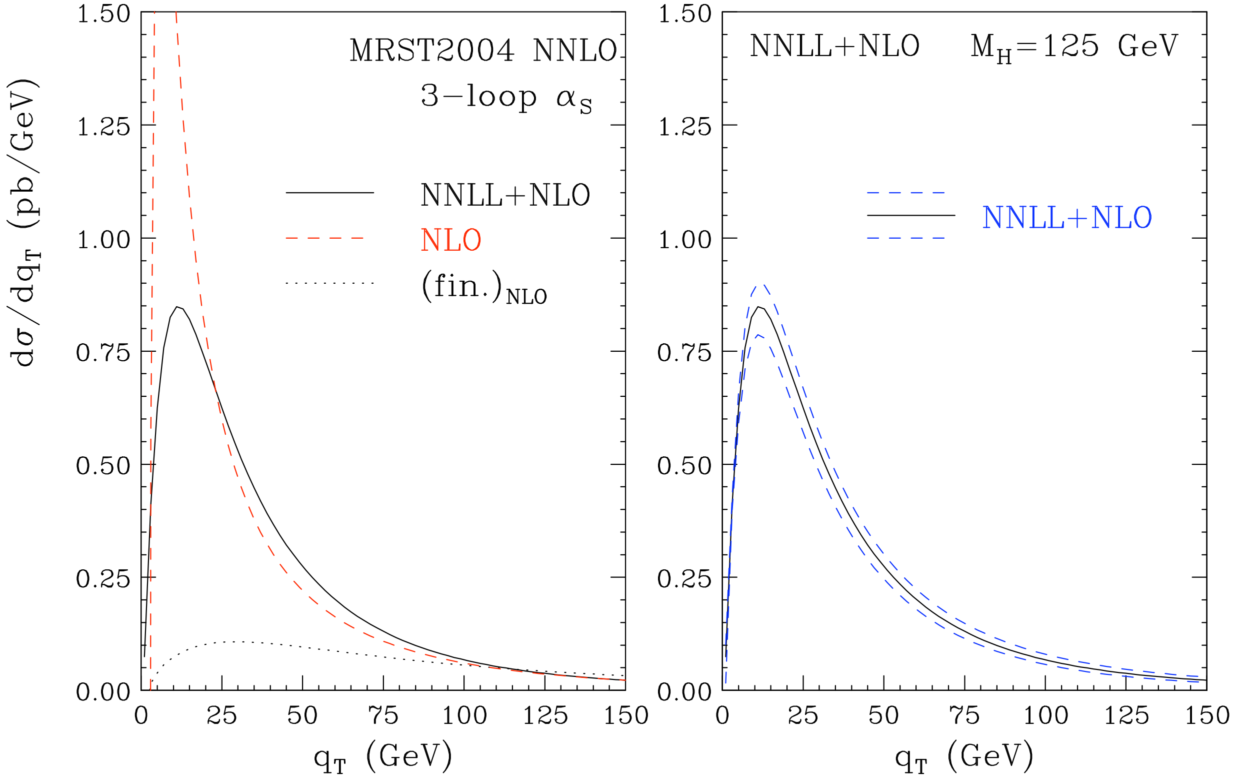


The q_T spectrum of Higgs production at the LHC

Left: NLL+LO compared with the LO spectrum

Right: uncertainty band from scale variations.

Bozzi, Catani, de Florian, Grazzini, NPB737(2006)73, hep-ph/0508068



The q_T spectrum of Higgs production at the LHC

Left: NNLL+NLO compared with the LO spectrum

Right: uncertainty band from scale variations.

Bozzi, Catani, de Florian, Grazzini, NPB737(2006)73, hep-ph/0508068

The same subject was addressed later by Stefano, Pino Marchesini and Bryan Webber from a different point of view, suitable for implementation in shower Monte Carlo Codes:

Nuclear Physics B349 (1991) 635–654
North-Holland

QCD COHERENT BRANCHING AND SEMI-INCLUSIVE PROCESSES AT LARGE x*

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Received 22 June 1990

Another chapter of Stefano's and Marcello's research:
perturbative QCD at high energy (small x)

Perturbative QCD predictions for hadronic processes at high p_T remarkably accurate:

- **non-perturbative contributions suppressed by powers of $\frac{\Lambda}{p_T}$**
- **logarithmic corrections to the naive partion model systematically computable as a powes series in**

$$\alpha_{\mathrm{S}}(p_T^2) \sim \frac{1}{\beta_0 \log \frac{p_T^2}{\Lambda^2}} \ll 1$$

(asymptotic freedom)

OK for p_t^2 (or any other relevant scale) of order s and much larger than Λ^2.

In the regime $s \gg p_t^2 \gg \Lambda^2$, powers of $\log x = \log \frac{p_t^2}{s}$ appear in the perturbative coefficients (small-x, or high-energy logarithms) and spoil the convergence of the perturbative expansion.

Parton distribution functions $f(z, p_t^2)$ poorly known in the relevant region $z \sim x$.

Resummation needed.

1. **S. Catani, M. Ciafaloni and F. Hautmann, "GLUON CONTRIBUTIONS TO SMALL x HEAVY FLAVOR PRODUCTION," Phys. Lett. B 242 (1990), 97-102 doi:10.1016/0370-2693(90)91601-7**
2. **S. Catani, M. Ciafaloni and F. Hautmann, "Small x structure functions and heavy flavor production," Nucl. Phys. B Proc. Suppl. 18 (1991), 220-225 doi:10.1016/0920-5632(91)90109-R**
3. **S. Catani, M. Ciafaloni and F. Hautmann, "High-energy factorization and small x heavy flavor production," Nucl. Phys. B 366 (1991), 135-188 doi:10.1016/0550-3213(91)90055-3**
4. **S. Catani, M. Ciafaloni and F. Hautmann, "Heavy flavor production at high energies," Nucl. Phys. B Proc. Suppl. 27 (1992), 33-38 doi:10.1016/0920-5632(92)90030-V**
5. **S. Catani, M. Ciafaloni and F. Hautmann, "Leptoproduction of heavy flavor at high energies," Nucl. Phys. B Proc. Suppl. 29 (1992), 182-191 doi:10.1016/0920-5632(92)90441-T**
6. **S. Catani, M. Ciafaloni and F. Hautmann, "Production of heavy flavors at high-energies," CERN-TH-6398-92.**
7. **S. Catani, M. Ciafaloni and F. Hautmann, "High-energy factorization in QCD and minimal subtraction scheme," Phys. Lett. B 307 (1993), 147-153**

A beautiful series of papers by Stefano in collaboration with Mike Seymour:

S. Catani and M. H. Seymour
The Dipole formalism for the calculation of QCD jet cross-sections at next-to-leading order
Phys. Lett. B 378 (1996), 287-301

S. Catani and M. H. Seymour
A General algorithm for calculating jet cross-sections in NLO QCD
Nucl. Phys. B 485 (1997), 291-419 [erratum: Nucl. Phys. B 510 (1998), 503-504]

S. Catani, S. Dittmaier, M. H. Seymour and Z. Trocsanyi
The Dipole formalism for next-to-leading order QCD calculations with massive partons
Nucl. Phys. B 627 (2002), 189-265

A generalization of the subtraction method

The cancellation of infrared singularities is not easy to implement numerically, because it takes place between processes with different final states:

$$d\sigma^{\rm NLO} = d\sigma^{\rm R}_{\rm m+1 partons} + d\sigma^{\rm v}_{\rm m\,partons}$$

Both infrared divergent.

The subtraction method:

$$d\sigma^{\rm NLO} = \left[d\sigma^{\rm R}_{\rm m+1 partons} - d\sigma^{\rm A}_{\rm m+1 partons}\right] + d\sigma^{\rm A}_{\rm m+1 partons} + d\sigma^{\rm v}_{\rm m\,partons}$$

with $d\sigma^{\mathrm{A}}$ chosen so that

1. **it is observable-independent**
2. **it cancels the singularities in $d\sigma^{\mathrm{R}}$**
3. **it can be integrated analytically in the singular region**

The Catani-Seymour dipole formalism: a choice of $d\sigma^{\mathrm{A}}$ which is completely general, i.e. not only observable-independent for a given process, but also process independent.

Suitable for the implementation of QCD corrections in an event generator.

More achievements in Marcello's scientific work are worth mentioning:

- **early work on relativistic bound states**
- **large logarithms in electroweak radiative corrections**
- **gravitational scattering**

Back-up slides

Resummed cross sections: a schematic derivation

Typical expression of an observable in QCD (eg, Drell-Yan cross section):

$$\begin{aligned}\sigma(x,Q^2) &= \int_x^1 \frac{dy}{y}\,\mathcal{L}(y,Q^2)\,C\left(\frac{x}{y},\alpha_{\rm S}(Q^2)\right)\\ &= \mathcal{L}\otimes C\end{aligned}$$

The function $\mathcal{L}(y,Q^2)$ is a parton luminosity, e.g.

$$\mathcal{L}(y,Q^2) = \int_y^1 \frac{dy'}{y'}\,f_1(y',Q^2)\,f_2\left(\frac{y}{y'},Q^2\right)$$

in hadron-hadron collisions.

The coefficient function C is essentially a partonic cross section.

Resummation usually (but not always) perfomed in the space of Mellin transformed quantities:

$$f(N) = \int_0^1 dx\, x^{N-1}\, f(x); \qquad f(x) = \frac{1}{2\pi i} \int_{\bar{N}-i\infty}^{\bar{N}+i\infty} dN\, x^{-N}\, f(N)$$

(a Laplace transform with respect to $t = \log \frac{1}{x}, x = e^{-t}$).

- **well defined and analytic in the half-plane $\mathrm{Re\,N} > N_0$ if $f(x)$ is at most as singular as x^{-N_0} in $x = 0$**
- **In the space of the Mellin-conjugate variable N, convolution products are turned into ordinary products:**

$$\sigma(N, Q^2) = \mathcal{L}(N, Q^2)\, C(N, \alpha_{\mathrm{S}}(Q^2))$$

- **The region $x \to 1$ is mapped in the region $N \to \infty$:**

$$\int_0^1 dx\, x^{N-1} \left[\frac{\log^k(1-x)}{1-x}\right]_+ = \frac{1}{k+1} \log^{k+1} \frac{1}{N} + O(\log^k N)$$

Why Mellin moments?

$$C(z, \alpha_S) = \delta(1-z) + \sum_{n=1}^{\infty} \int_0^1 dz_1 \dots dz_n \, \frac{dw_n(z_1, \dots, z_n)}{dz_1 \dots dz_n} \, \Theta_{PS}(z; z_1, \dots, z_n)$$

The multi-gluon emission probability factorizes in the soft limit,

$$\frac{dw_n(z_1, \dots, z_n)}{dz_1 \dots dz_n} \simeq \frac{1}{n!} \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i}$$

(easily seen in QED in the eikonal approximation) but the phase space factor

$$\Theta_{PS}(z; z_1, \dots, z_n) = \delta(z - z_1 z_2 \cdots z_n)$$

does not ...

... unless one goes to Mellin moments:

$$
\begin{aligned}
C(N,\alpha_{\rm S}) &= \int_0^1 dz\, z^{N-1}\, C(z,\alpha_{\rm S}) \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_0^1 dz_1 \cdots dz_n \prod_{i=1}^{n} \frac{dw(z_i)}{dz_i} \int_0^1 dz\, z^{N-1} \delta(z - z_1 \cdots z_n) \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^1 dz_1\, z_1^{N-1} \frac{dw(z_1)}{dz_1} \right] \ldots \left[\int_0^1 dz_n\, z_n^{N-1} \frac{dw(z_n)}{dz_n} \right]
\end{aligned}
$$

Hence

$$
C(N,\alpha_{\rm S}) = \exp \int_0^1 dz\, z^{N-1}\, \frac{dw}{dz}
$$

Multigluon emission exponentiates in the soft limit.

Some details: consider the production of a heavy object with energy Q_0, plus n gluons of energies $\omega_1, \ldots, \omega_n$. The differential cross section in the eikonal approximation takes the form

$$dC \sim \frac{d\omega_1}{\omega_1}\frac{d\theta_1}{\theta_1}\ldots\frac{d\omega_n}{\omega_n}\frac{d\theta_n}{\theta_n}\delta(\sqrt{s} - Q_0 - \omega_1 \ldots - \omega_n)$$

Define $z = \frac{Q_0}{\sqrt{s}}$ **and**

$$\omega_1 = \sqrt{s}(1 - z_1)$$
$$\omega_2 = \sqrt{s}z_1(1 - z_2)$$
$$\ldots$$
$$\omega_n = \sqrt{s}z_1 \ldots z_{n-1}(1 - z_n)$$

We have

$$J = \left|\frac{d(\omega_1, \ldots, \omega_n)}{d(z_1, \ldots, z_n)}\right| = s^{n/2} z_1^{n-1} z_2^{n-2} \ldots z_{n-1}$$

Furthermore

$$\omega_1 \ldots \omega_n = \sqrt{s} z_1^{n-1} z_2^{n-2} \ldots z_{n-1} (1 - z_1) \ldots (1 - z_n)$$

and

$$\begin{aligned} \omega_1 + \ldots + \omega_n &= \sqrt{s}(1 - z_1 + z_1 - z_1 z_2 + \ldots - z_1 z_2 \ldots z_n) \\ &= \sqrt{s}(1 - z_1 z_2 \ldots z_n) \end{aligned}$$

Hence, after angular integration,

$$dC \sim \left[\frac{dz_1}{1 - z_1} \log \frac{1}{1 - z_1} \right] \ldots \left[\frac{dz_n}{1 - z_n} \log \frac{1}{1 - z_n} \right] \delta(z - z_1 \ldots z_n)$$

Recalling that

$$\frac{dw}{dz} = 2\, C\, \alpha_{\rm S} \left[\frac{1}{1-z} \, \log \frac{1}{1-z} \right]_{+}$$

and using the leading-log result

$$\begin{aligned} \int_0^1 dz\, z^{N-1} \left[\frac{\log^p(1-z)}{1-z} \right]_{+} &= \frac{1}{p+1} \, \log^{p+1} \frac{1}{N} + O(\log^p N) \\ &= -\int_0^{1-\frac{1}{N}} \frac{dz}{1-z} \log^p(1-z) + O(\log^p N) \end{aligned}$$

we find

$$C(N, \alpha_{\rm S}) = \exp \left[C \alpha_{\rm S} \log^2 N + O(\log N) \right]$$

Strictly valid in QED; in QCD, complications arise because of gluon emission from gluon lines and because of color structure, but the essential features remain the same.

Extension to QCD

QCD corrections essentially amount to the replacement $\alpha_S \to \alpha_S(k_T^2)$ in the computation of the single-gluon emission probability:

$$2C\alpha_S\left[\frac{1}{1-z}\log\frac{1}{1-z}\right]_+ \to 2C\left[\frac{1}{1-z}\int_{Q^2(1-z)}^{Q^2}\frac{dk_T^2}{k_T^2}\alpha_S(k_T^2)\right]_+$$

The running coupling can then be expanded in powers of $\alpha_S(Q^2)$

$$\alpha_S(k_T^2) = \frac{\alpha_S(Q^2)}{1+\alpha_S(Q^2)\beta_0\log\frac{k_T^2}{Q^2}} = \alpha_S(Q^2)\sum_{n=0}^{\infty}(-\alpha_S(Q^2)\beta_0)^n\,\log^n\frac{k_T^2}{Q^2}$$

and the expansion integrated term by term. One gets

$$C(N,\alpha_S) = \exp[\log N\, g_1(\alpha_S\log N)]$$

where the function g_1 has a Taylor expansion in its argument, starting at order 1.

$$2C\left[\frac{1}{1-z}\int_{Q^2(1-z)}^{Q^2}\frac{dk_T^2}{k_T^2}\alpha_S(k_T^2)\right]_+ = 2C\left[\frac{1}{1-z}\int_{\alpha_S(Q^2(1-z))}^{\alpha_S(Q^2)}\frac{d\alpha}{\beta(\alpha)}\alpha_S(k_T^2)\right]_+$$

For a generic process one can prove the generalized result

$$\begin{aligned} C^{\rm res}(N,\alpha_S(Q^2)) &= g_0(\alpha_S)\,\exp\mathcal{S}\,(L,\bar\alpha) \\ \mathcal{S}(L,\bar\alpha) &= \frac{1}{\bar\alpha}\,g_1(L)+g_2(L)+\bar\alpha\,g_3(L)+\bar\alpha^2\,g_4(L)+\ldots \\ \bar\alpha &= a\,\alpha_S(Q^2)\,\beta_0; \qquad a=1,2; \qquad L=\bar\alpha\log\frac{1}{N} \end{aligned}$$

which defines an improved expansion (in powers of α_S with $\alpha_S\log N$ fixed) for $C^{\rm res}(N,\alpha_S)$: g_1 gives the leading-log (LL) approximation, g_1 and g_2 give the next-to-leading-log approximation (NLL), and so on.