Feynman Integral

Synergies Between Particle Physics and Gravitational Waves

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QCD@Work

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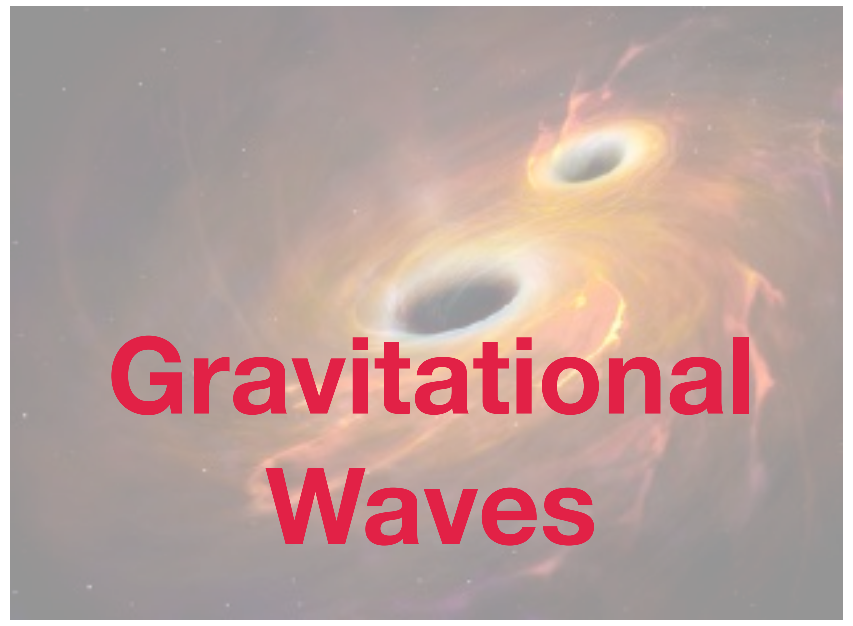
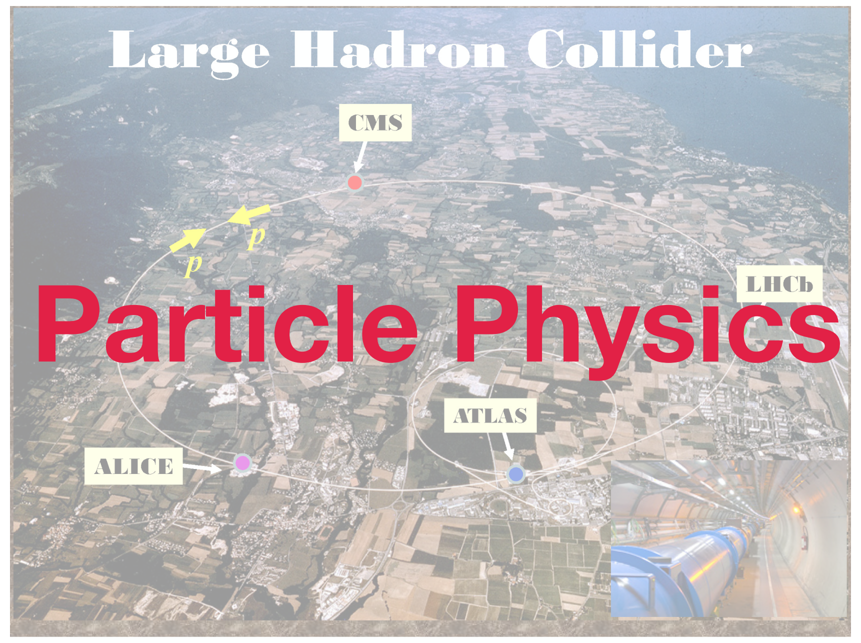
Feynman Integral





Feynman Integral

- Differential Equations
- Algebraic geometry
- Twisted De-Rham Cohomology
- Complex Analysis
- Differential geometry
- Tropical geometry
- Special Functions

Computation of the Loop Amplitude

Generation of Feynman Diagram

↓

Spin sums, Dirac Algebra, Trace

↓

Tensor Reduction → $$\mathcal{M}_{\rm b}^{(n)} = (S_\epsilon)^n \int \prod_{i=1}^{n} \frac{d^d k_i}{(2\pi)^d} \sum_G \frac{1}{\prod_{\sigma\in G} D_\sigma}$$

$\mathcal{O}(10^5)$

↓

Integral-By-Parts Reduction → $$\mathcal{M}_{\rm b}^{(n)} = \mathbb{C}^{(n)} \cdot \mathbf{I}^{(n)}$$

Master Integrals

$\mathcal{O}(10^2)$

↓

Evaluation of Master Integrals

Integration-By-Parts Identity

Chetyrkin, Tkachov

Loop momenta

$$\int \prod_{\alpha=1}^{l} d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = 0$$

Loop and external momenta

$$\int_{\alpha=1}^{l} \prod d^d k_\alpha \frac{\partial}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) = \int_{\alpha=1}^{l} \prod d^d k_\alpha \left[\frac{\partial v^\mu}{\partial k_{j,\mu}} \left(\frac{1}{D_1^{a_1} \cdots D_N^{a_N}} \right) - \sum_{j=1}^{N} \frac{a_j}{D_j} \frac{\partial D_j}{\partial k_{j,\mu}} \left(\frac{v^\mu}{D_1^{a_1} \cdots D_N^{a_N}} \right) \right]$$

$$C_1 \, I(a_1, \cdots a_N - 1) + \cdots + C_r \, I(a_1 + 1, \cdots a_N) = 0$$

- Gives relations between different scalar integrals with different exponents
- l(l+E) number of equations
- Solve the system symbolically : Recursion relations — LiteRed
- Solve for specific integer value of the exponents : Laporta Algorithm — Fire, Reduze, Kira,..

Intersection Theory and Feynman Integral

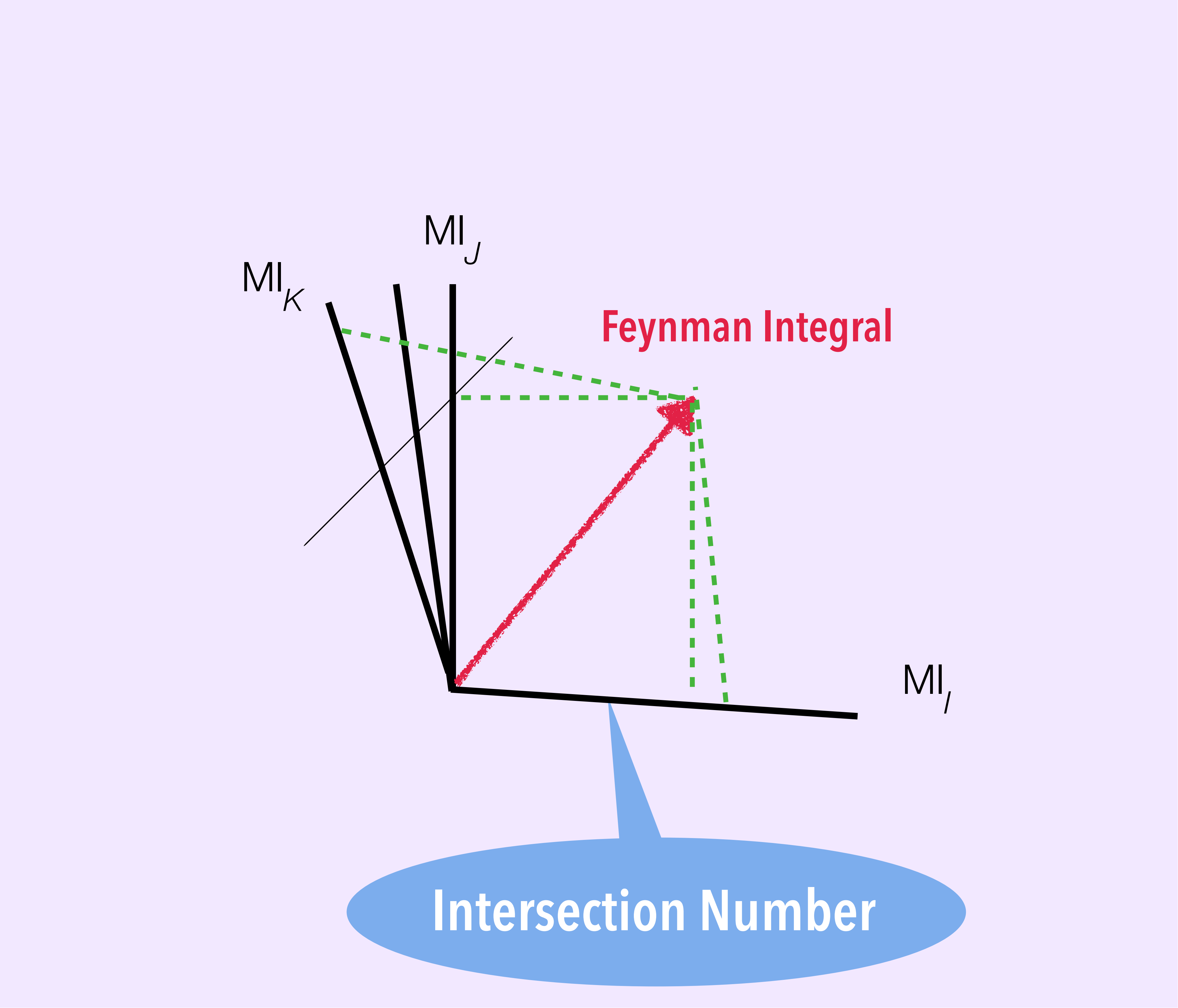
$$I = \sum_{i=1}^{\nu} c_i J_i$$

$$I \cdot J_i$$

$$J_i \cdot J_j = \delta_{ij}$$

$$I \cdot J_j \left(C^{-1}\right)_{ji}$$

$$J_i \cdot J_j = C_{ij} \neq \delta_{ij}$$



Intersection Theory **Feynman Integral**

Mastrolia, Mizera (2018)

Frellesvig, Gasparotto, **MKM**, Mastrolia, Mattiazzi, Mizera (2019)

Frellesvig, Gasparotto, Laporta, **MKM**, Mastrolia, Mattiazzi, Mizera (2019)

Frellesvig, Gasparotto, **MKM**, Mastrolia, Mattiazzi, Mizera (2020)

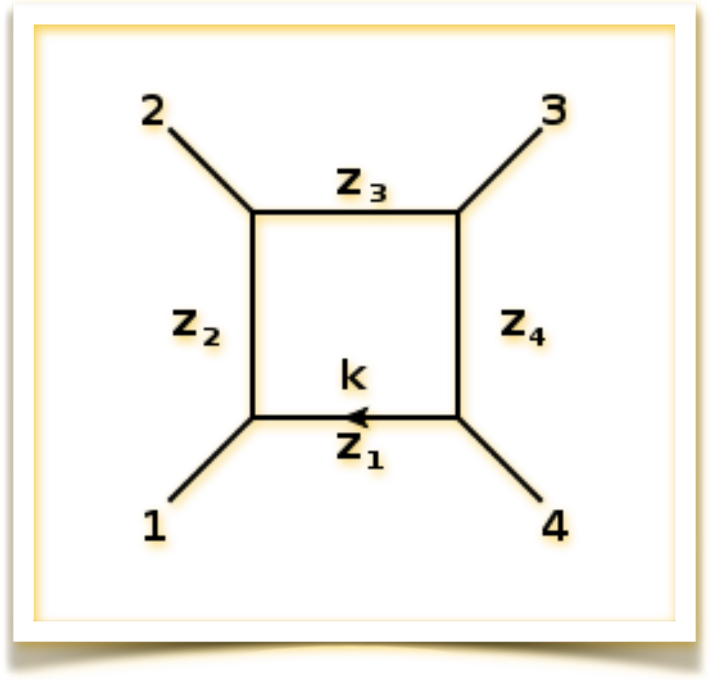
Chestnov, Frellesvig, Gasparotto, **MKM**, Mastrolia (2022)

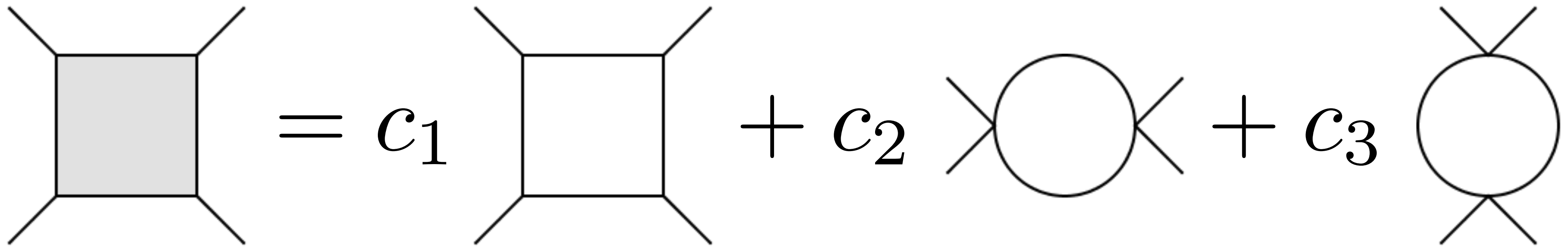
Chestnov, Gasparotto, **MKM**, Mastrolia, Matsubara-Heo, Munch, Takayama (2022)

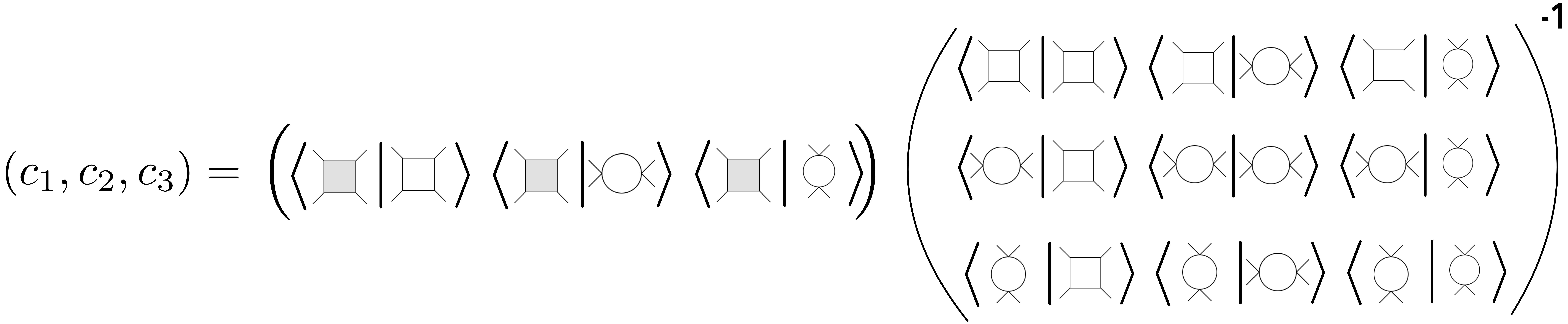
The Vector space is identified as the twisted co-homology group

The scalar product is Intersection Number

Examples of decomposition



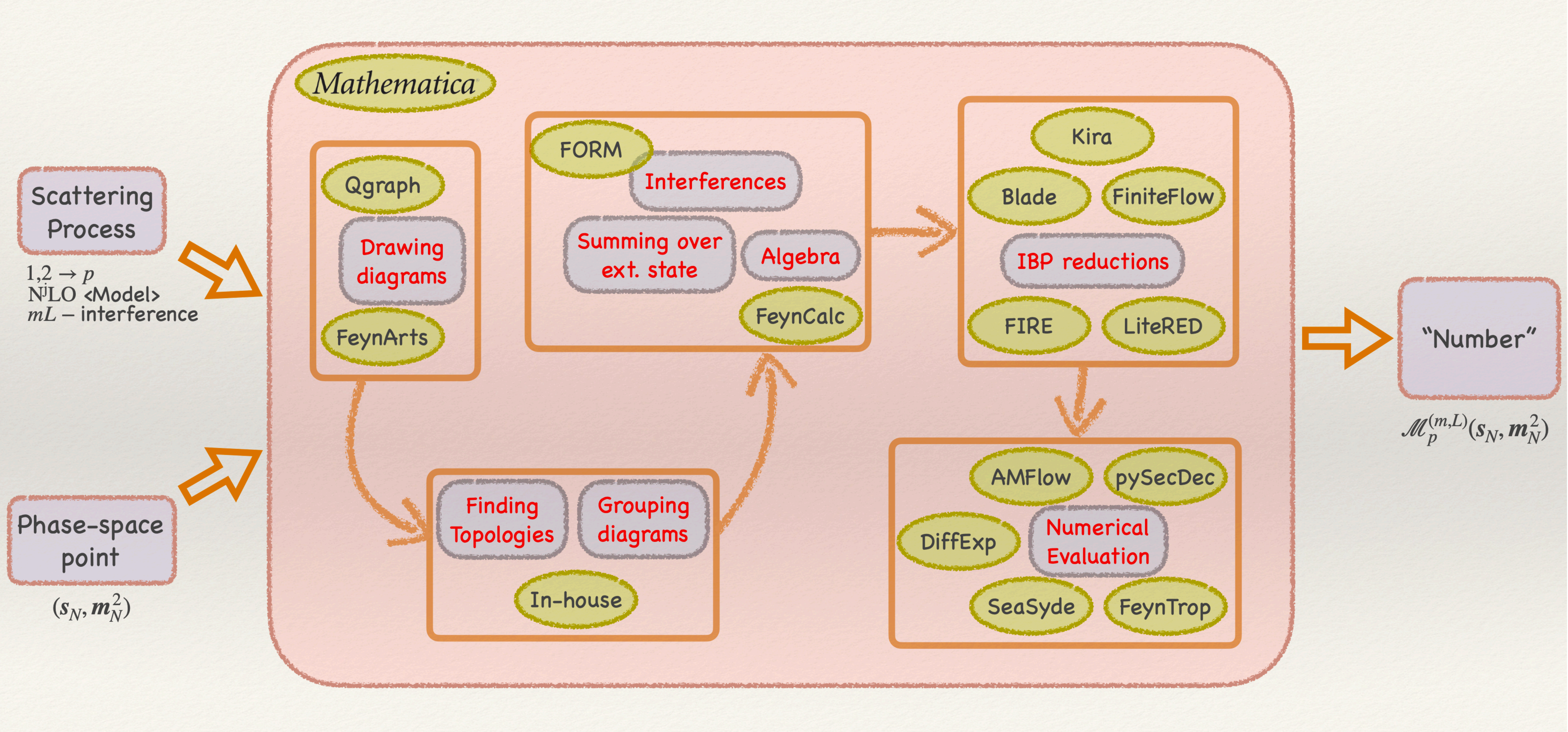




LOOPIN

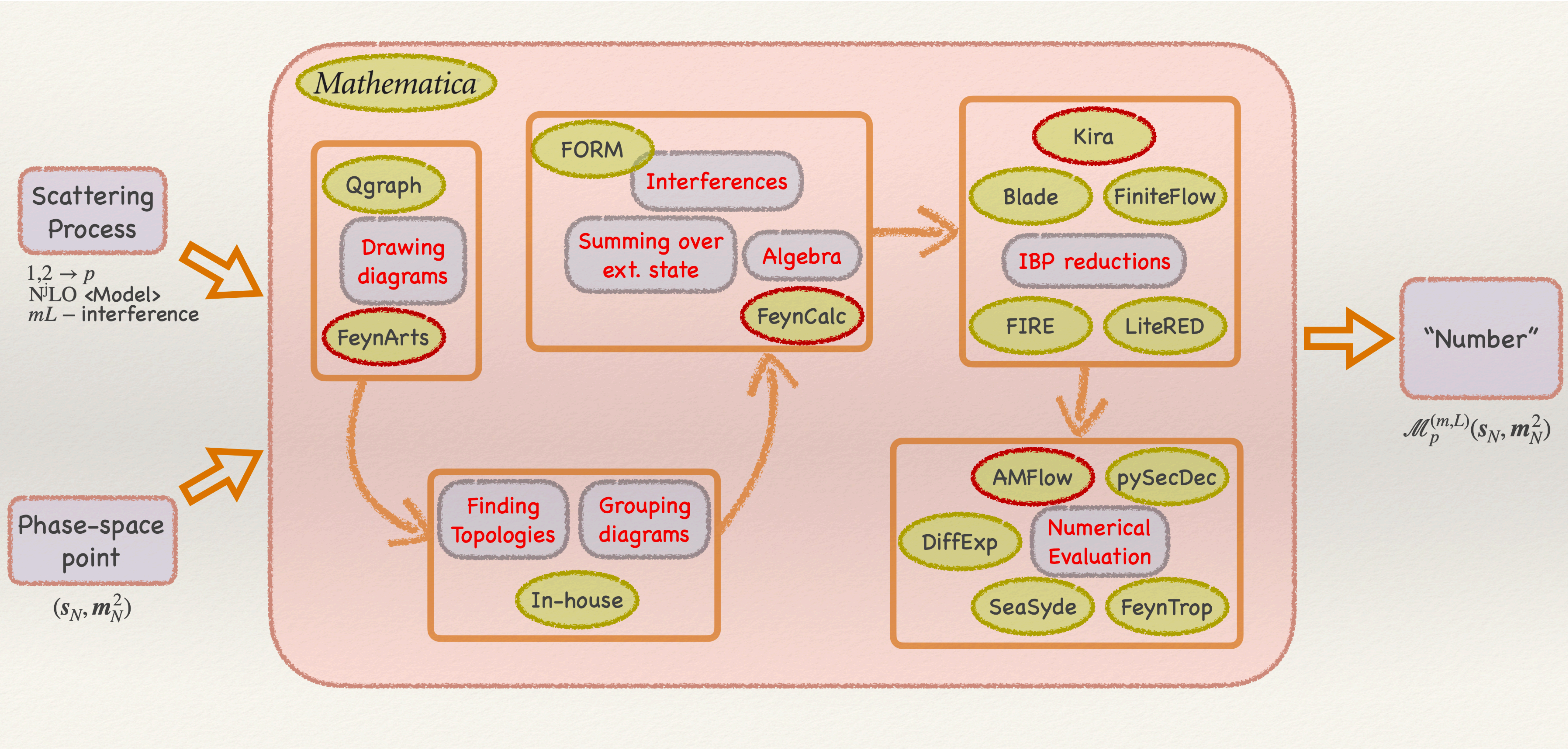
Bigazzi, Brunello, Crisanti, Dave, **MKM**, Mastrolia, Ronca, Smith, Torres Bobadilla

LoopIn



Scattering Process
$1,2 \to p$
N^jLO <Model>
mL – interference
Phase-space point
(s_N, m_N^2)
Mathematica
Qgraph
Drawing diagrams
FeynArts
FORM
Interferences
Summing over ext. state
Algebra
FeynCalc
Finding Topologies
Grouping diagrams
In-house
Kira
Blade
FiniteFlow
IBP reductions
FIRE
LiteRED
AMFlow
pySecDec
DiffExp
Numerical Evaluation
SeaSyde
FeynTrop
"Number"
$\mathcal{M}_p^{(m,L)}(s_N, m_N^2)$

LoopIn



Slides of Jonathan Ronca

Gravitational Wave Observables

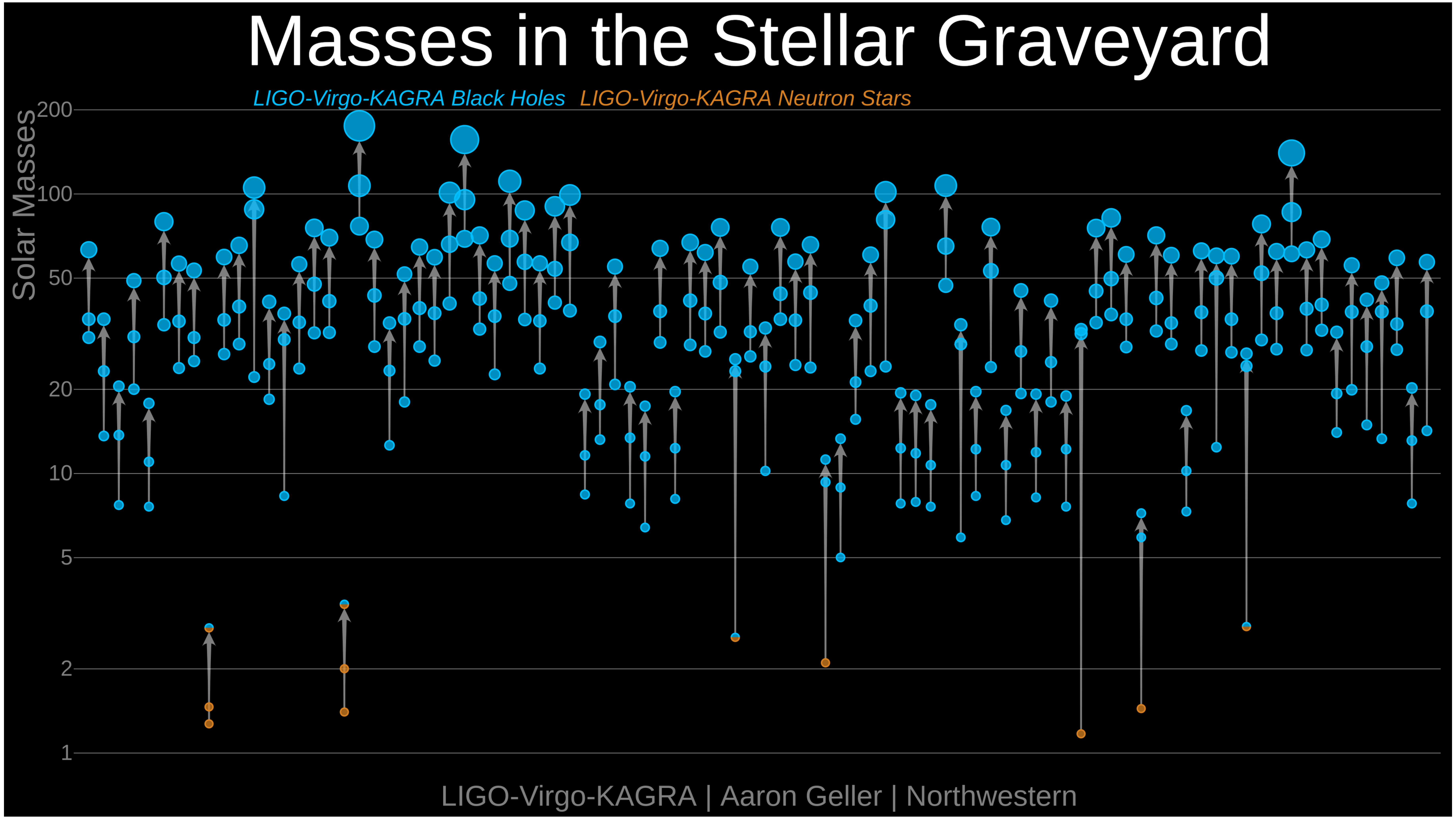
MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)

MKM, Mastrolia, O Silva, Patil, Steinhoff (2023)

GW observations

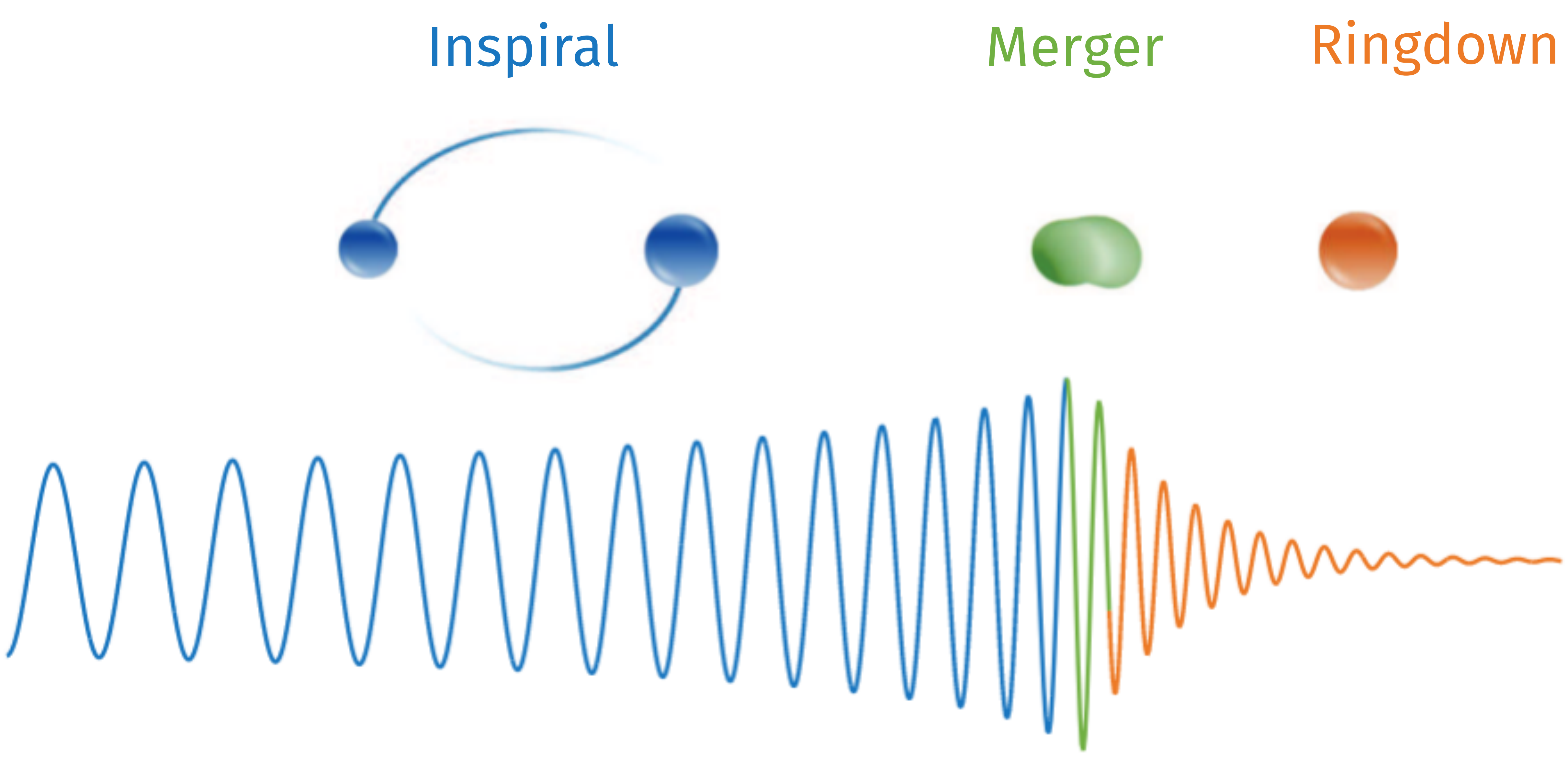


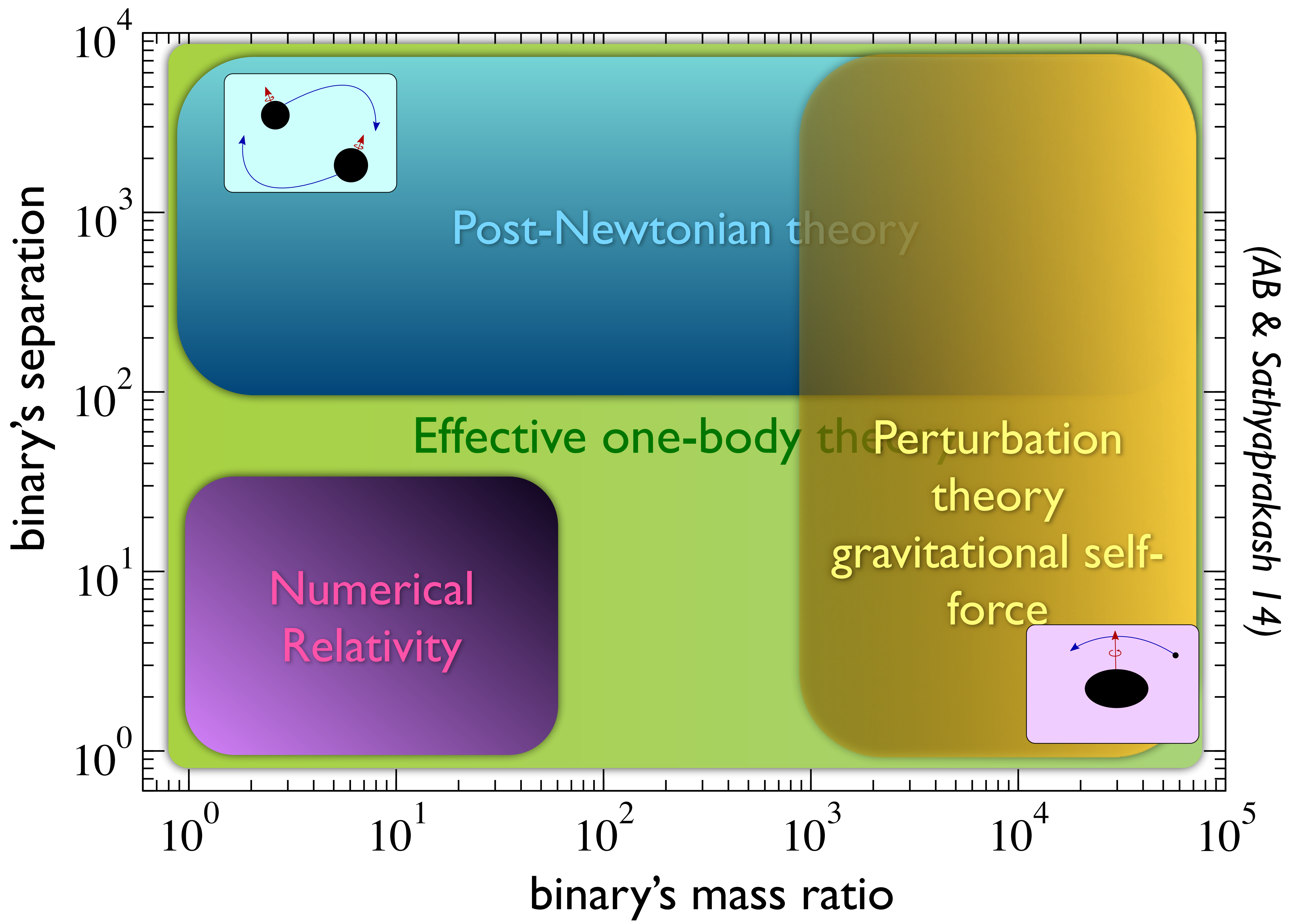
Tasks

- Supplement conventional Analysis
- Increase Theoretical Precision
- Perform Gravity phenomenology

Solving two-body problem in GR

Antelis, moreno (2016)





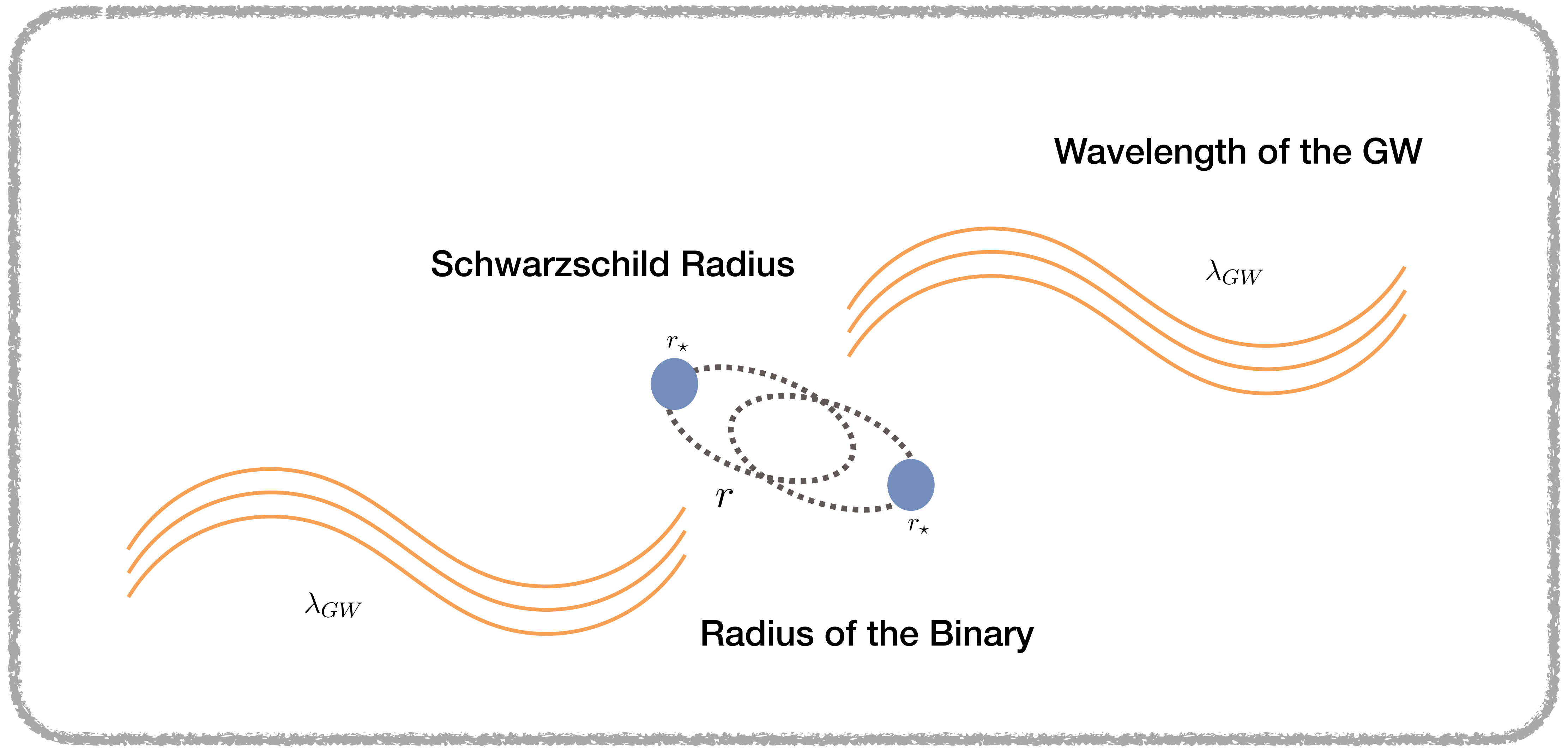
Post-Newtonian (PN)

Post-Minkowskian (PM)

Numerical Relativity

Perturbation Theory

Post-Newtonian Expansion EFT set up

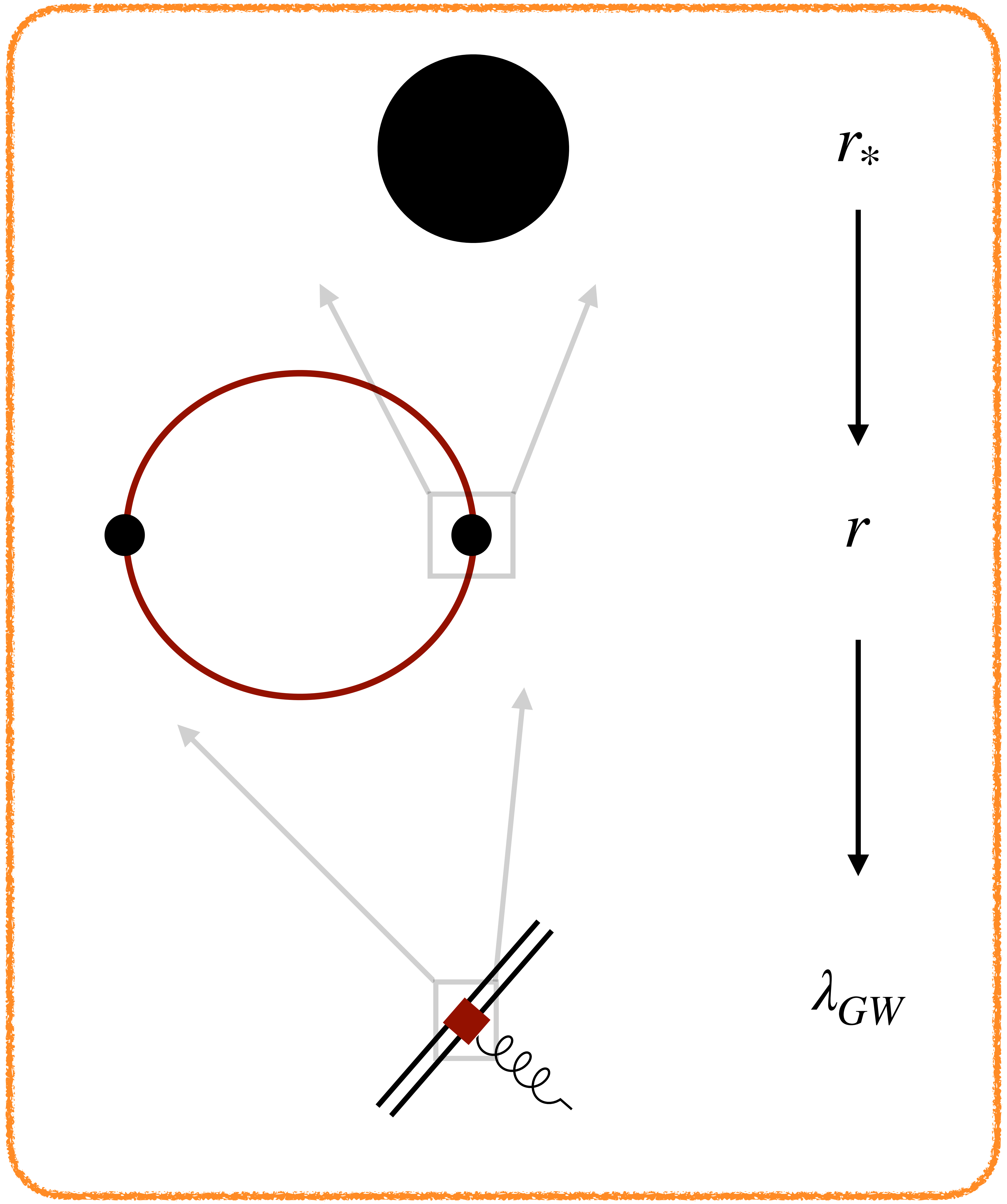


Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs Goldberger, Rothstein

1. One-Particle EFT for Compact Object
2. EFT of Composite Particle for Binary
3. Effective Theory of Dynamical Multipoles

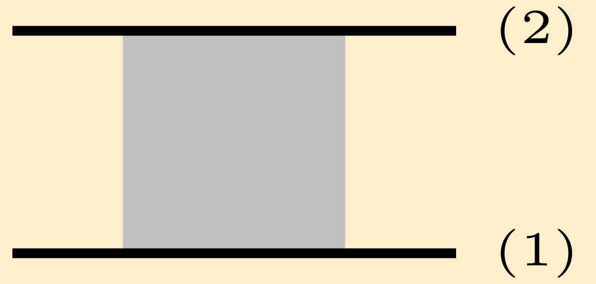


Potential for the 2-body system

Goldberger, Rothstein, Porto, Levi, …

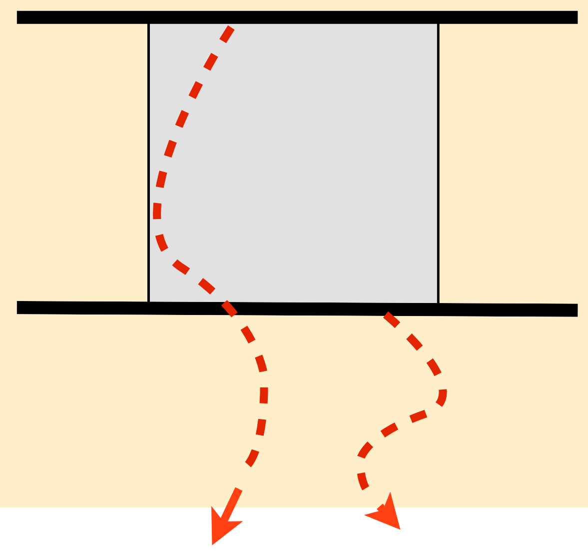
Foffa, Sturani, Sturm, Mastrolia (2016)

$$\mathcal{V}_{\rm eff} = \mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{i}\mathbf{p}\cdot(\mathbf{x}_{(1)}-\mathbf{x}_{(2)})}$$



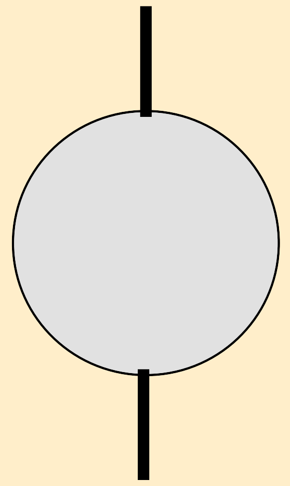
Key Observation

Diagrams in GR



$=$

Diagrams in QFT



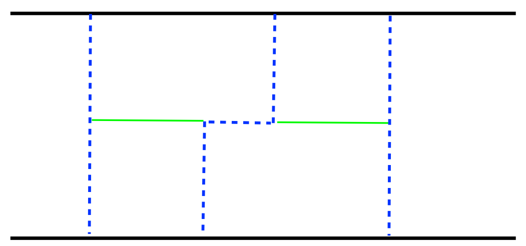
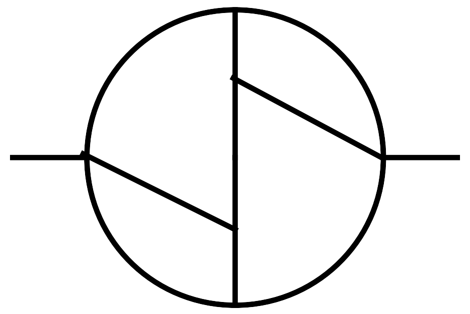
$$p^2 \equiv s \neq 0$$

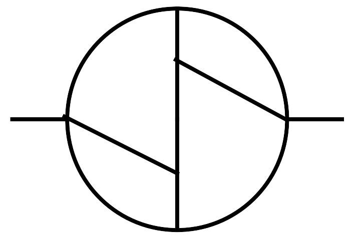
Static Source :: Not Propagating

An example of the Computation [4PN Static]

Foffa, Sturani, Sturm, Mastrolia (2016)

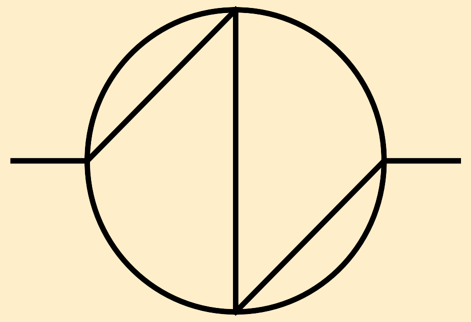
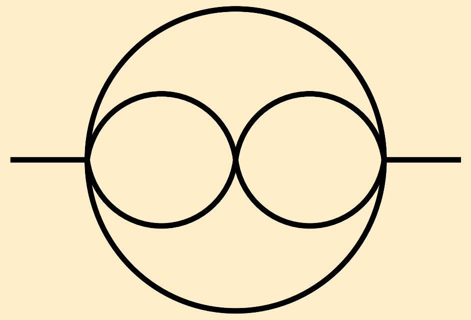
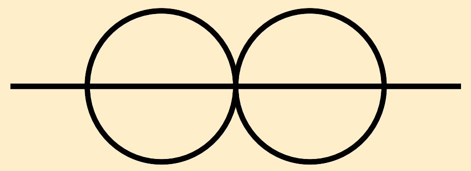
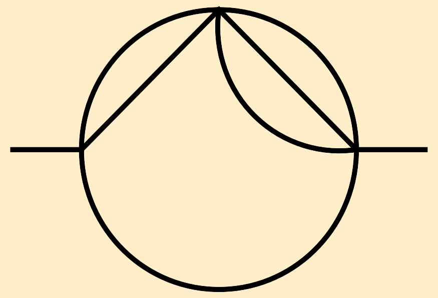
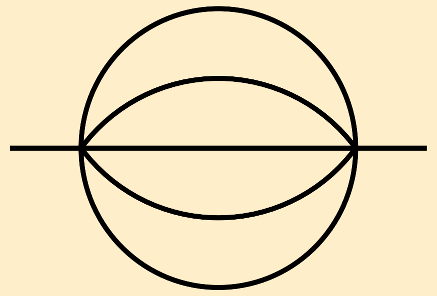
Amplitude

$\mathcal{A}_{49} =$  $= -2\,\mathrm{i}\,(8\pi G_N)^5 \left(\frac{(d-2)}{(d-1)}\, m_1 m_2 \right)^3$  $[N_{49}]$

 $[N_{49}] \equiv \int_{k_1,k_2,k_3,k_4} \frac{N_{49}}{k_1^2\ p_2^2\ k_3^2\ p_4^2\ k_{12}^2\ k_{13}^2\ k_{23}^2\ k_{24}^2\ k_{34}^2}\,,$

$$N_{49} \equiv (k_1 \cdot k_3\ k_{12} \cdot k_{23} - k_1 \cdot k_{12}\ k_3 \cdot k_{23} - k_1 \cdot k_{23}\ k_3 \cdot k_{12}) \times (p_2 \cdot k_{23}\ p_4 \cdot k_{34} + p_4 \cdot k_{23}\ p_2 \cdot k_{34} - p_2 \cdot p_4\ k_{23} \cdot k_{34})\ ,$$

IBPs

$= c_1$  $+ c_2$  $+ c_3$  $+ c_4$  $+ c_5$ 

MIs

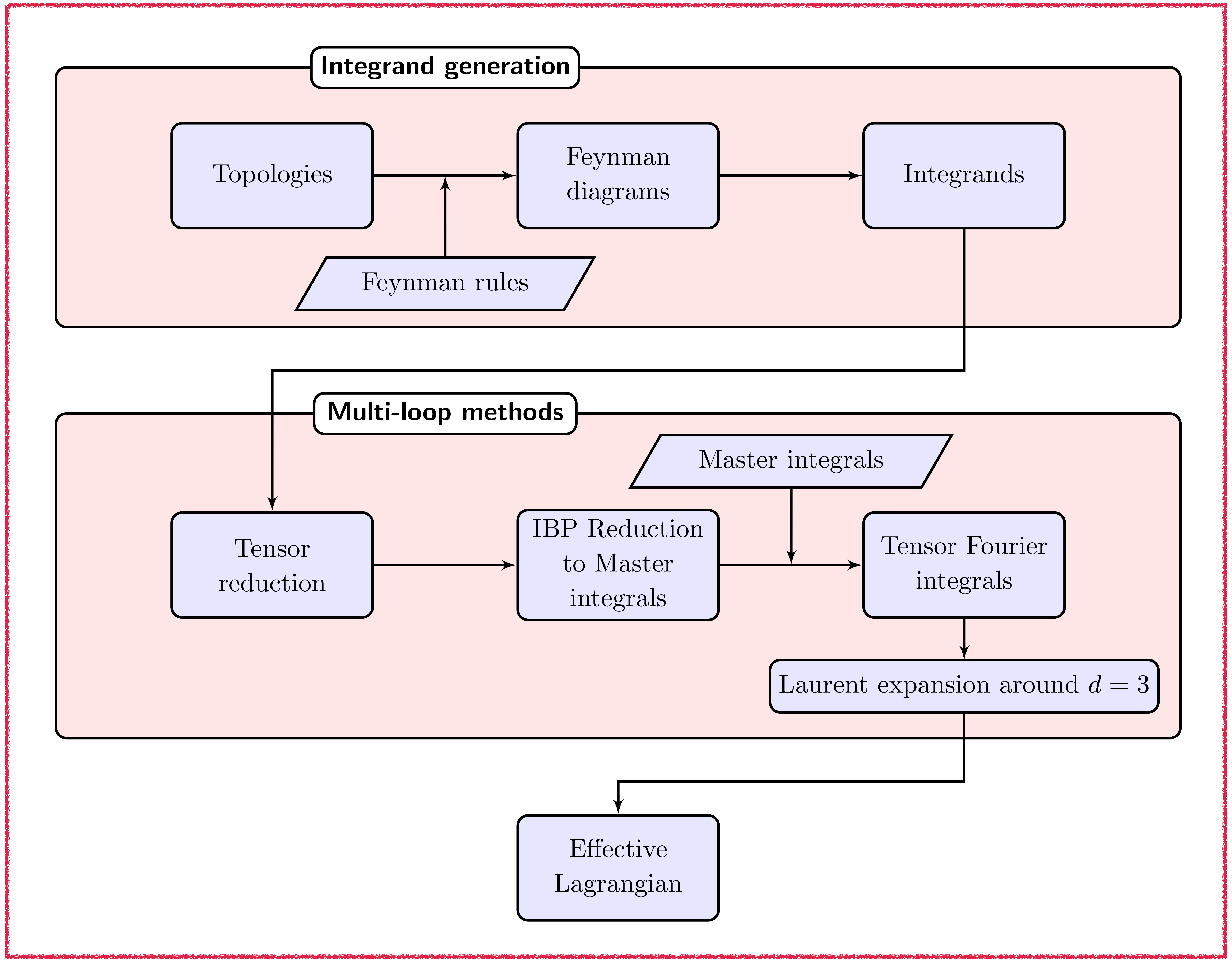
$$= -\mathrm{i}(8\pi G_N)^5 (m_1 m_2)^3 2^{-4} (4\pi)^{-(4+2\varepsilon)} e^{2\varepsilon\gamma_E} s^{(1+2\varepsilon)} \left[\frac{1}{\varepsilon}\left(\frac{\pi^2}{16} - \frac{2}{3} \right) + \frac{29}{18} - \frac{13}{144}\pi^2 - \frac{\pi^2}{8}\log 2 + \mathcal{O}(\varepsilon^1) \right]$$

Lagrangian

$$\mathcal{L}_{49} = -\mathrm{i} \lim_{d\to 3} \int_p \mathrm{e}^{ip\cdot r} \mathcal{A}_{49} = (32 - 3\pi^2) \frac{G_N^5 m_1^3 m_2^3}{r^5}$$

Fourier Transformation

Computational Algorithm : Towards Automation



☑ Automated in-house codes

- Aim to publish the code in future

☑ Inclusion of spin-effects

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

☑ Inclusion of Tidal-effects

MKM, Mastrolia, Silva, Patil, Steinhoff (2023)

MKM, Mastrolia, Silva, Patil, Steinhoff (2023)

Conclusion

- ☑ Novel Algebraic Property Unveiled
  - ☑ The algebra of Feynman Integrals is controlled by intersection numbers
  - ☑ Intersection Numbers : Scalar Product/Projection between Feynman Integrals
  - ☑ Useful for both Physics and Mathematics
- ☑ Automated framework for the evaluation of Loop Amplitudes
  - ☑ Focus on Parallelization
  - ☑ Modular and easily upgradable
  - ☑ Tested on a number of 1-loop and 2-loop processes in QED and QCD
- ☑ Applications to GW and Collider phenomenology
  - ☑ progress in understanding spin effects / tidal effects for the compact binaries
  - ☑ A number of observables e.g binding energy, scattering angle has been computed to high precision
  - ☑ muon-electron scattering at NNLO has been obtained
  - ☑ top-pair production from quark annihilation has been computed analytically

Thank You

Back Up

Notion of Loop Integral

Number of Loops

$$I\left(a_1 \cdots a_N\right) = \int \cdots \int d^d k_1 \cdots d^d k_l \frac{\mathcal{N}\left(\{k_i\}, \{p_j\}\right)}{D_1^{a_1} \cdots D_N^{a_N}}$$

Loop Momenta

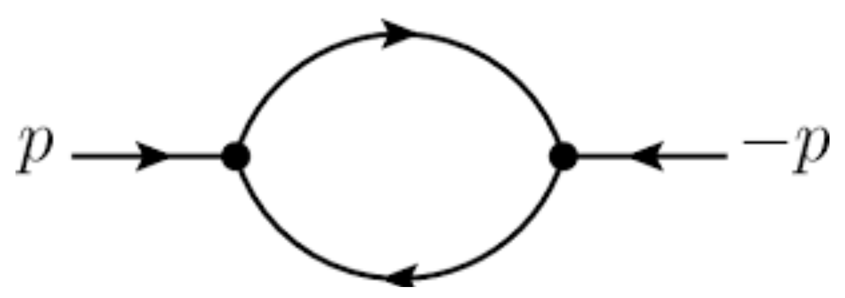
Number of Propagators

$$D_i = q_i^2 - m_i^2$$

$$q_i = \sum_j k_j + \sum_m p_m$$

Integration-By-Parts Identity (Example)

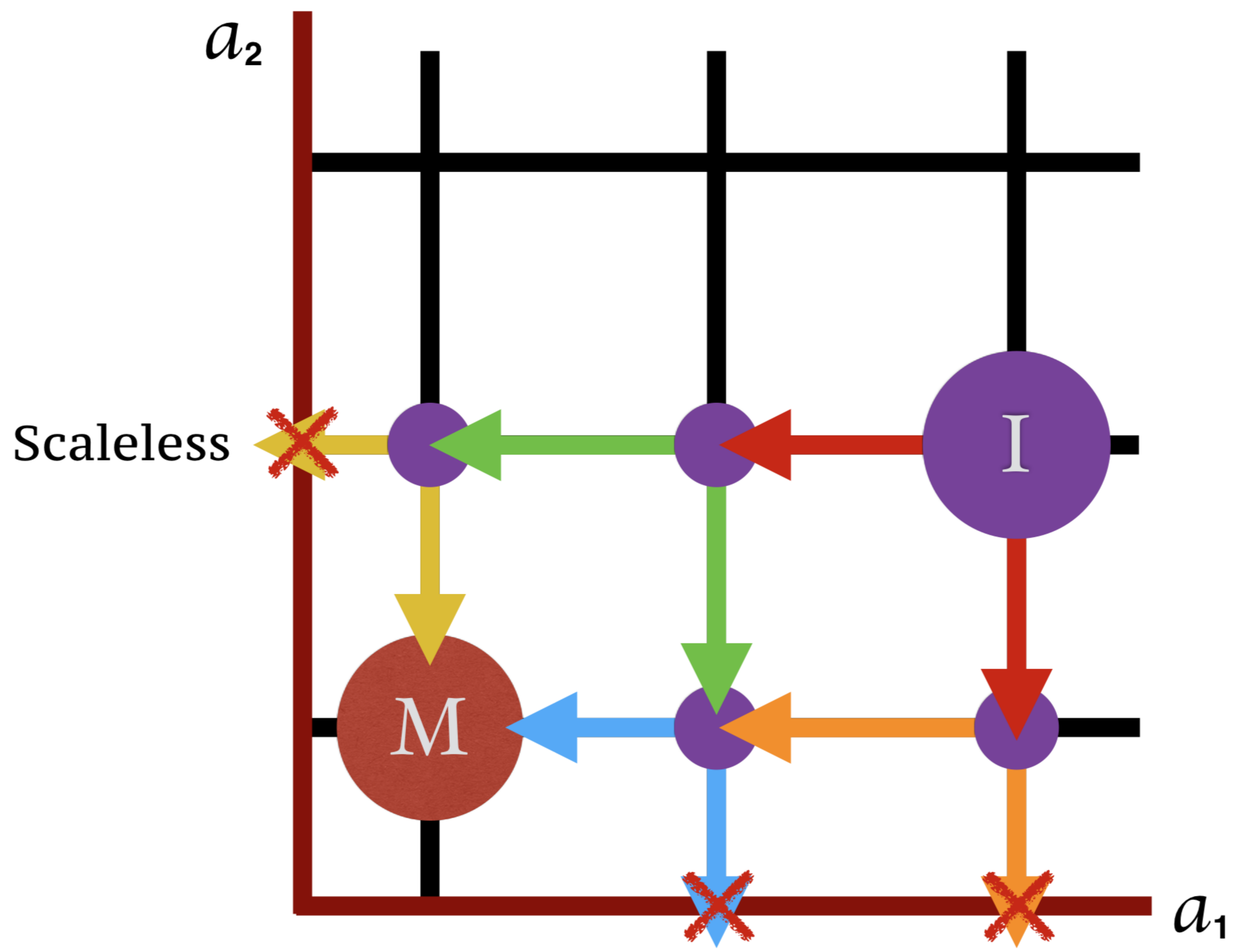
One Loop Massless Bubble



$$I(a_1, a_2) = \int \frac{d^d k_1}{(k_1^2)^{a_1}((k_1+p)^2)^{a_2}}$$

IBP Identity

$$I(a_1, a_2) = \frac{a_1 + a_2 - d - 1}{p^2(a_2 - 1)} I(a_1, a_2 - 1) + \frac{1}{p^2} I(a_1 - 1, a_2)$$



Intersection Theory

Aomoto, Gelfand, Kita, Cho, Matsumoto,
Mimachi, Mizera, Yoshida

$$I = \int_{\mathcal{C}} u(\mathbf{z})\, \varphi(\mathbf{z})$$

Twisted Co-cycle

Twisted Cycle

$\hat{\varphi}(\mathbf{z}) d^m \mathbf{z}$

Single-valued differential Form

$\langle \varphi | \mathcal{C} \otimes u]$

Pairing

$u(\mathbf{z})$ is a multi-valued function

$u(\mathbf{z})$ *vanishes on the boundaries* of $\mathcal{C}$, $u(\partial\mathcal{C}) = 0$

Basics of Intersection Theory

$$0 = \int_{\mathcal{C}} d\,(u\,\xi) = \int_{\mathcal{C}} (du \wedge \xi + u\, d\xi) = \int_{\mathcal{C}} u \left(\frac{du}{u} \wedge + d \right) \xi \equiv \int_{\mathcal{C}} u\, \nabla_\omega \xi.$$

$\omega \equiv d \log u$

$\nabla_\omega \equiv d + \omega \wedge$

Equivalence Class $\quad {}_\omega\langle \varphi | : \; \varphi \sim \varphi + \nabla_\omega \xi$

$$\int_{\mathcal{C}} u\, \varphi = \int_{\mathcal{C}} u\, (\varphi + \nabla_\omega \xi)$$

Vector Space of n-forms

$$H^n_\omega \equiv \{n\text{-forms } \varphi_n \,|\, \nabla_\omega \varphi_n = 0\} / \{\nabla_\omega \varphi_{n-1}\}$$

Twisted Cohomology Group

Dual space $\quad H^n_{-\omega}, \qquad \nabla_{-\omega} = d - \omega \wedge$

Dimension of the Vector Space: Number of MIs

$$\chi(X) = \sum_{k=0}^{2n} (-1)^k \dim H^k_\omega .$$

$H^{k\neq n}_\omega$ vanish.

Aomoto (1975)

$$\nu = (-1)^n \chi(X)$$

$$= (-1)^n \left(n+1 - \chi(\mathcal{P}_\omega)\right)$$

$$= \{\text{number of solutions of } \omega=0\}$$

Decomposition of differential forms

Number of Linearly independent forms (twisted co-cycle) is ν

Basis	$\langle e_i \vert$	$i = 1, 2, \ldots, \nu$
Dual Basis	$\vert h_j \rangle$	$j = 1, 2, \ldots, \nu$

Monomial Basis : $\langle e_i | = \langle \phi_i | \equiv z^{i-1} dz$

d-Log Basis : $\langle e_i | = \langle \varphi_i | \equiv \dfrac{dz}{z - z_i}$

Metric Matrix : $\mathbf{C}_{ij} = \langle e_i | h_j \rangle$

$$\mathbf{M} = \begin{pmatrix} \langle \varphi|\psi\rangle & \langle \varphi|h_1\rangle & \langle \varphi|h_2\rangle & \ldots & \langle \varphi|h_\nu\rangle \\ \langle e_1|\psi\rangle & \langle e_1|h_1\rangle & \langle e_1|h_2\rangle & \ldots & \langle e_1|h_\nu\rangle \\ \langle e_2|\psi\rangle & \langle e_2|h_1\rangle & \langle e_2|h_2\rangle & \ldots & \langle e_2|h_\nu\rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle e_\nu|\psi\rangle & \langle e_\nu|h_1\rangle & \langle e_\nu|h_2\rangle & \ldots & \langle e_\nu|h_\nu\rangle \end{pmatrix} \equiv \begin{pmatrix} \langle \varphi|\psi\rangle & \mathbf{A}^\intercal \\ \mathbf{B} & \mathbf{C} \end{pmatrix}$$

$$\det \mathbf{M} = \det \mathbf{C} \left(\langle \varphi|\psi\rangle - \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B} \right) = 0$$

$$\langle \varphi|\psi\rangle = \mathbf{A}^\intercal \, \mathbf{C}^{-1} \, \mathbf{B}$$

$$= \sum_{i,j=1}^{\nu} \langle \varphi|h_j\rangle \, (\mathbf{C}^{-1})_{ji} \, \langle e_i|\psi\rangle$$

Master Decomposition Formula : $\langle \varphi| = \sum_{i,j=1}^{\nu} \langle \varphi|h_j\rangle \, \left(\mathbf{C}^{-1}\right)_{ji} \, \langle e_i|$

Computation of Intersection Number

Fibration Method

Matsumoto (1998)

Goto (2015)

Mizera (2019)   Frellesvig, Gasparotto, Laporta, MKM, Mastrolia, Mattiazzi, Mizera (2019)

Wienzierl (2020)   Frellesvig, Gasparotto, MKM, Mastrolia, Mattiazzi, Mizera (2020)

Caron-Huot, Pokraka (2021)

Secondary Equation

Matsubara-Heo (2019)

Chestnov, Gasparotto, MKM, Mastrolia, Matsubara-Heo, Munch, Takayama (2022)

Multivariate Differential Equation

Matsumoto (1998)

Chestnov, Frellesvig, Gasparotto, MKM, Mastrolia (2022)

Intersection Number Evaluation

$$I = \int_{\mathcal{C}} u\, \varphi = \langle \varphi | \mathcal{C}]$$

$$\langle \varphi_L | \varphi_R \rangle = \frac{1}{2\pi i} \int_X \varphi_L \wedge \varphi_R$$

Uni-variate Intersection Number

$$\langle \varphi_L | \varphi_R \rangle_\omega = \sum_{p \in \mathcal{P}} \mathrm{Res}_{z=p} \Big(\psi_p \, \varphi_R \Big)$$

$$\nabla_{\omega_p} \psi_p = \varphi_{L,p}$$

Multivariate Intersection Number

Recursive Formula :

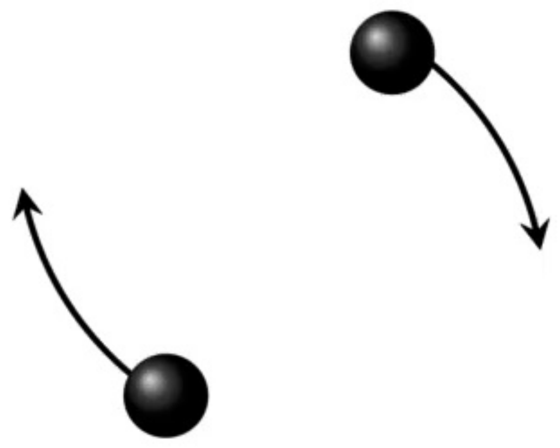
$${}_{\mathbf{n}}\langle \varphi_L^{(\mathbf{n})} | \varphi_R^{(\mathbf{n})} \rangle = -\sum_{p \in \mathcal{P}_n} \mathop{\mathrm{Res}}_{z_n = p} \Big({}_{\mathbf{n-1}}\langle \varphi_L^{(\mathbf{n})} | h_i^{(\mathbf{n-1})} \rangle \, \psi_i^{(n)} \Big)$$

$$\partial_{z_n} \psi_i^{(n)} - \hat{\mathbf{\Omega}}_{ij}^{(n)} \psi_j^{(n)} = \hat{\varphi}_{R,i}^{(n)}$$

Analytical Approximation Methods

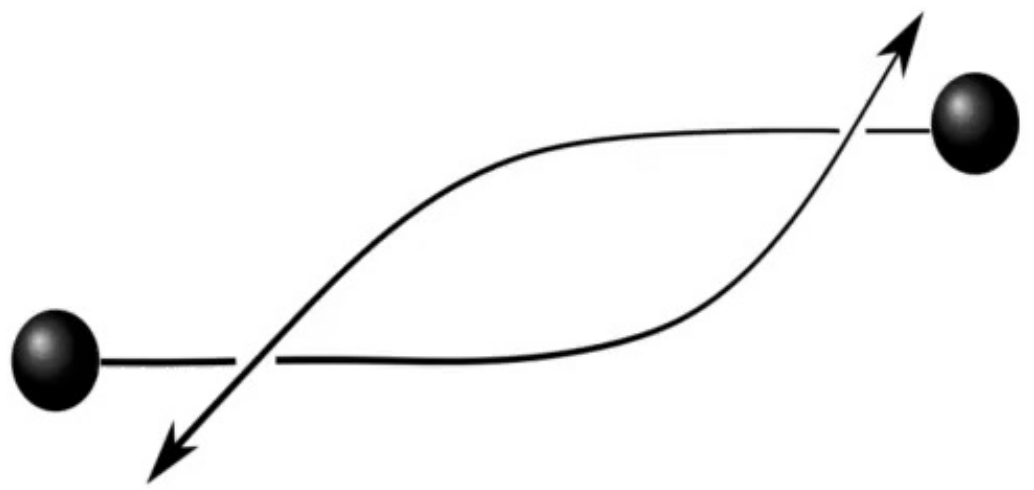
Post-Newtonian (PN)

$$\frac{v^2}{c^2} \sim \frac{GM}{rc^2} \ll 1$$



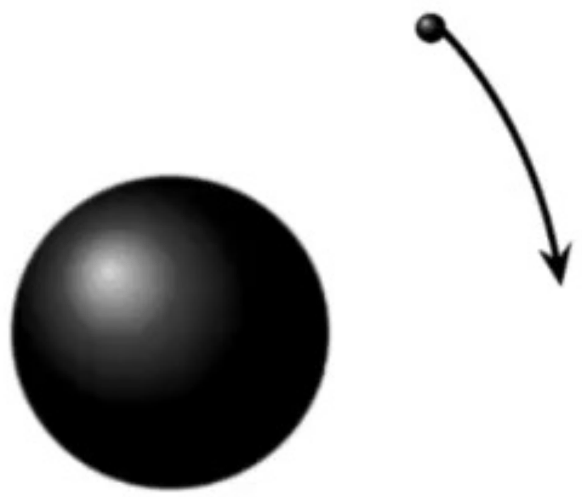
Post-Minkowskian (PM)

$$\frac{GM}{rc^2} \ll 1$$



Self-Force (SF)

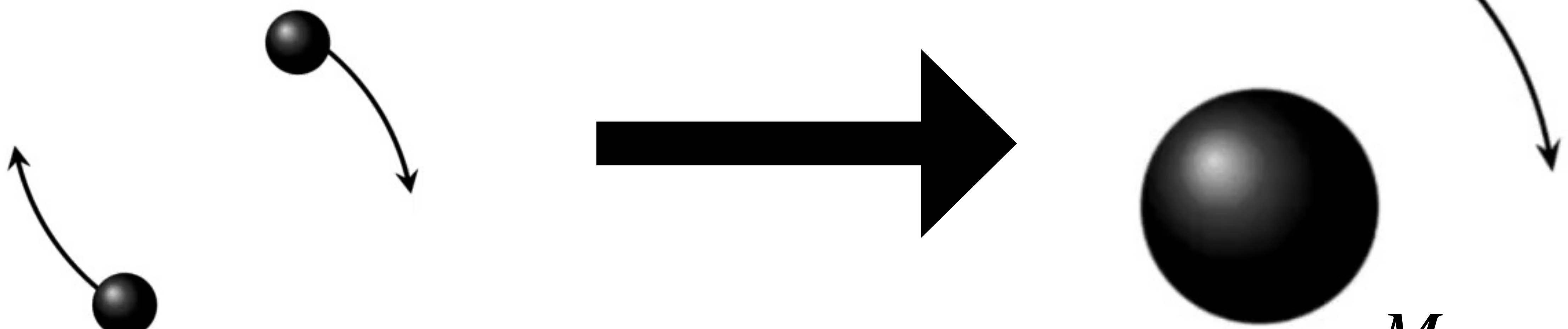
$$\frac{m_1}{m_2} \ll 1$$



Effective One-Body (EOB)

m_2, a_2

$$\mu = \frac{m_1 m_2}{M}$$

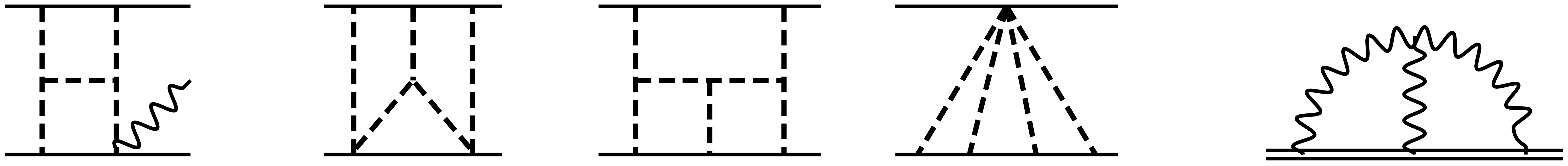


m_1, a_1

$$M = m_1 + m_2$$
$$a = a_1 + a_2$$

Advantage of QFT techniques

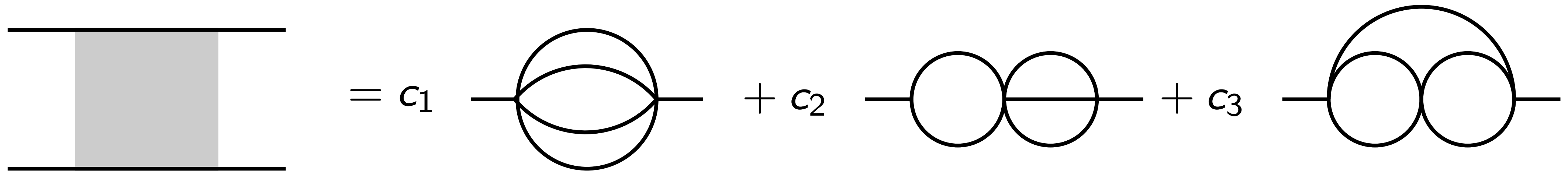
- Use of Feynman diagrams



- Dimensional regularization

Better to handle spurious divergences

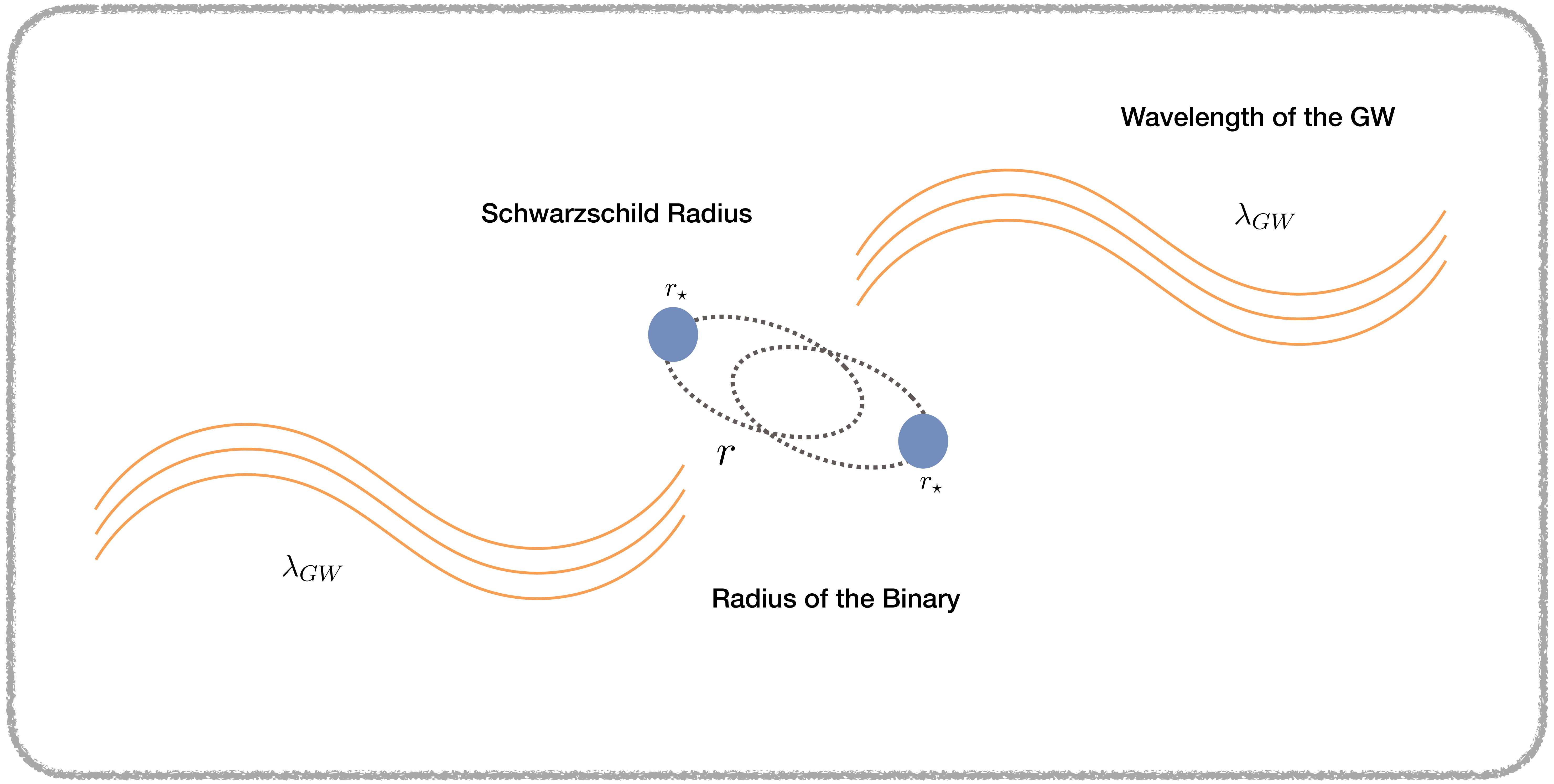
- Multi-loop Techniques



☑ IBP relations

☑ Differential Equations

Post-Newtonian Expansion EFT set up



Equations of Motion

$$\dot{r} = \frac{d\mathcal{H}}{dp_r} \qquad \dot{p}_r = -\frac{d\mathcal{H}}{dr} + \mathcal{F}_r$$

$$\dot{\phi} = \frac{d\mathcal{H}}{dp_\phi} \qquad \dot{p}_\phi = -\frac{d\mathcal{H}}{d\phi} + \mathcal{F}_\phi$$

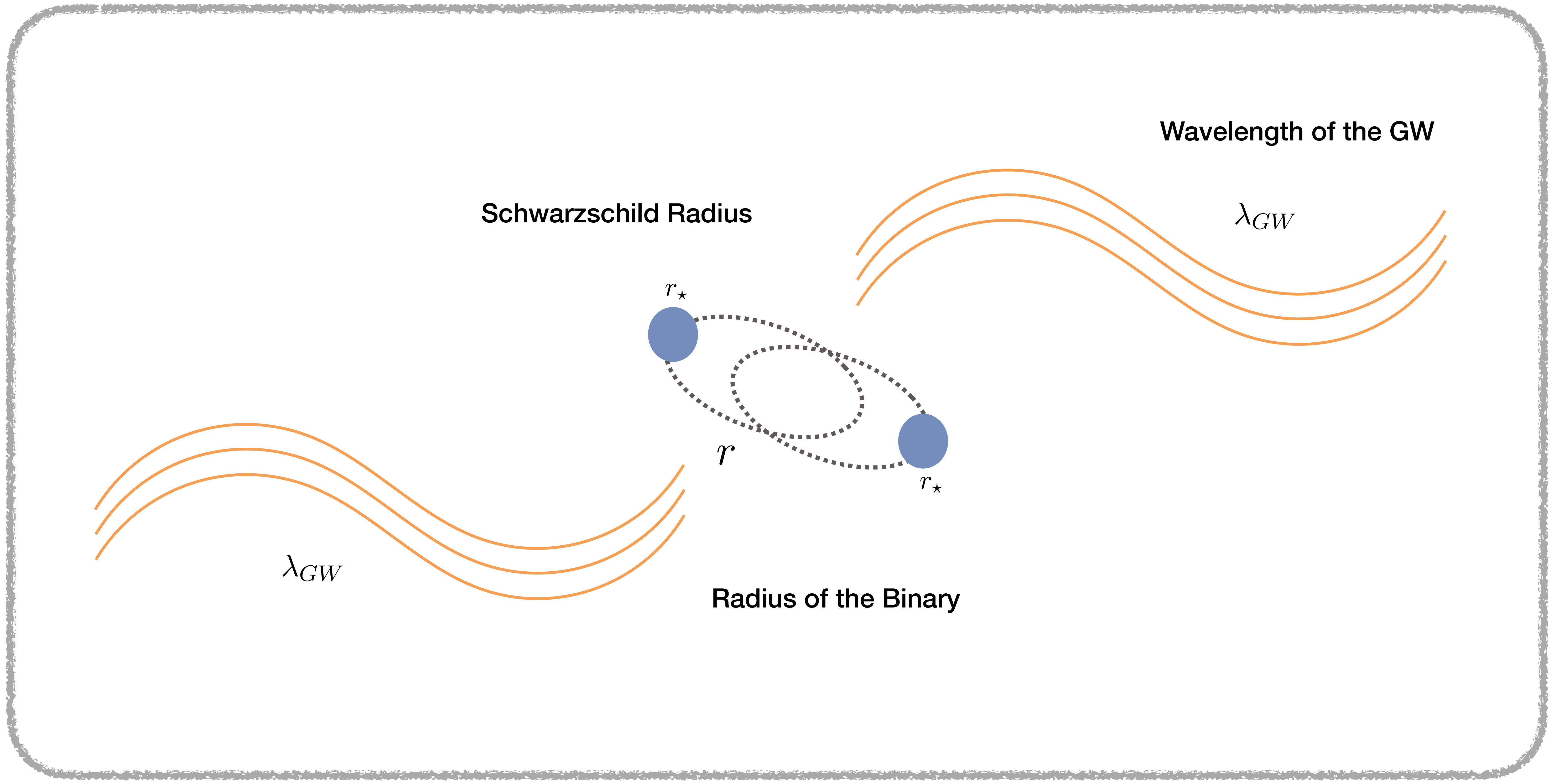
Need:

Hamiltonian $\mathcal{H}$

Radiation Reaction $\mathcal{F}$

Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

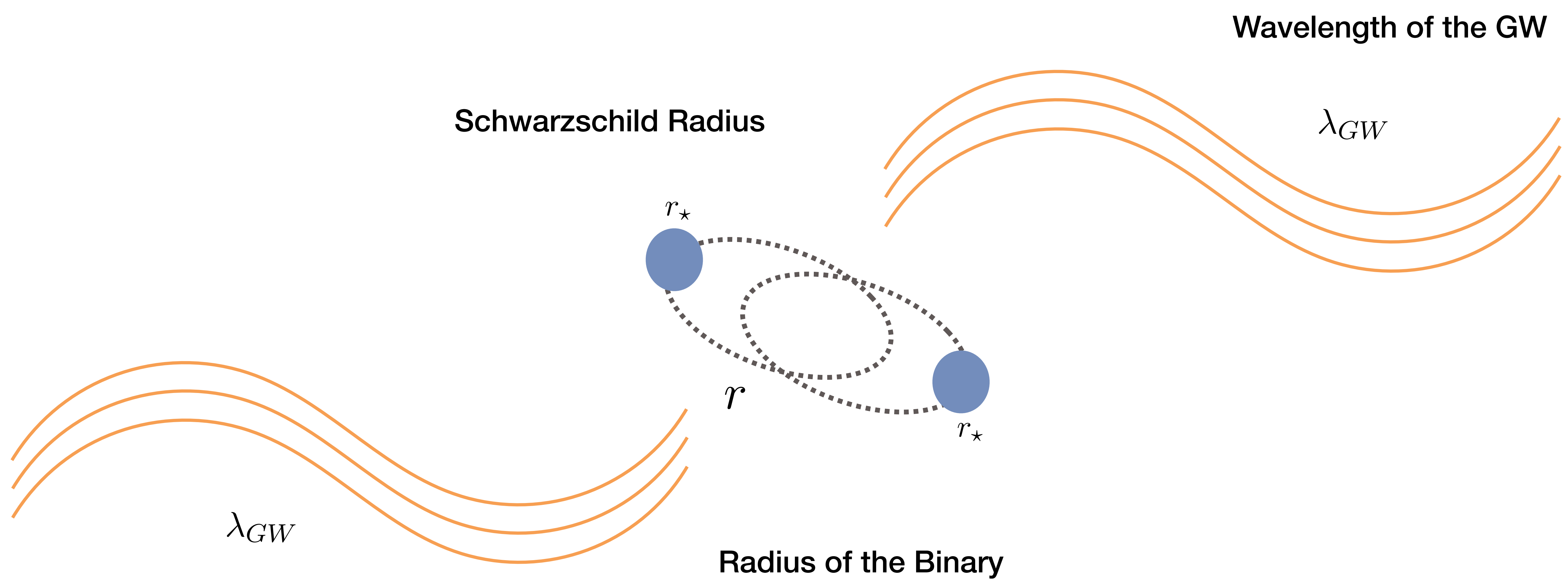
1. One-Particle EFT for Compact Object

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$S_{pp}[g_{\mu\nu}] = -m\int d\sigma\sqrt{u^2}$$

Post-Newtonian Expansion EFT set up

Goldberger, Rothstein



Hierarchy of scales

$$r_\star << r << \lambda_{GW}$$

Tower of EFTs

2. EFT of Composite Particle for Binary

$$S[g_{\mu\nu}] = -\frac{1}{16\pi G}\int d^4x\sqrt{g}R$$

$$g_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + h_{\mu\nu}$$

$$S_{pp}[g_{\mu\nu}, x_K] = \sum_{K=1}^{2} - m_K \int d\sigma \sqrt{u_K^2}$$

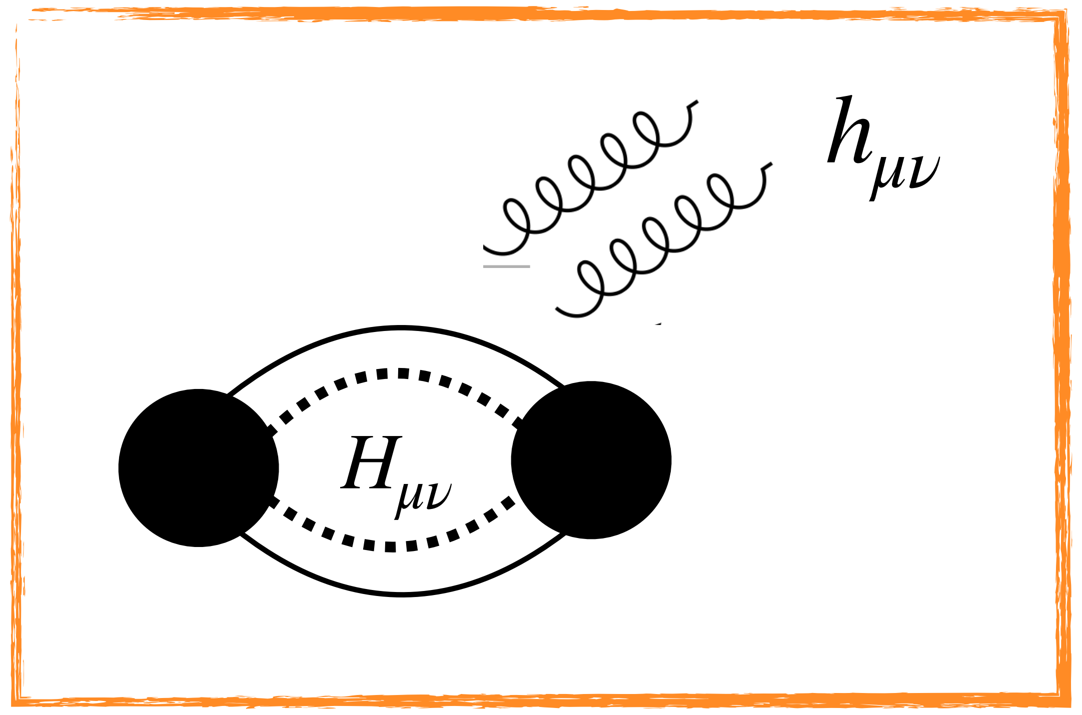
Method of Regions

potential gravitons $H_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, 1/r)$

radiation gravitons $\bar{h}_{\mu\nu}$ with scaling $(k_0, \mathbf{k}) \sim (v/r, v/r)$

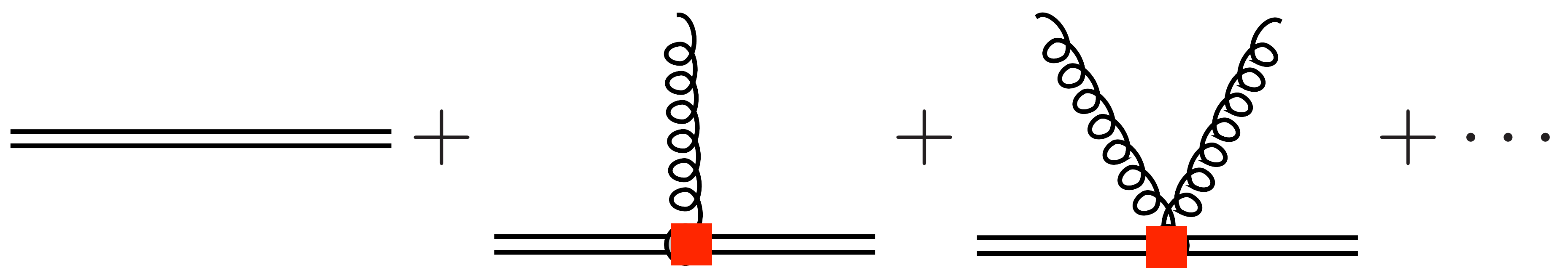
EFT at the orbital scale: Conservative Dynamics

$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \int \mathcal{D}H_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h} + H] + i \sum_{K=1}^{2} S_{pp}[x_K(t), \eta + \bar{h} + H] \right\}$$



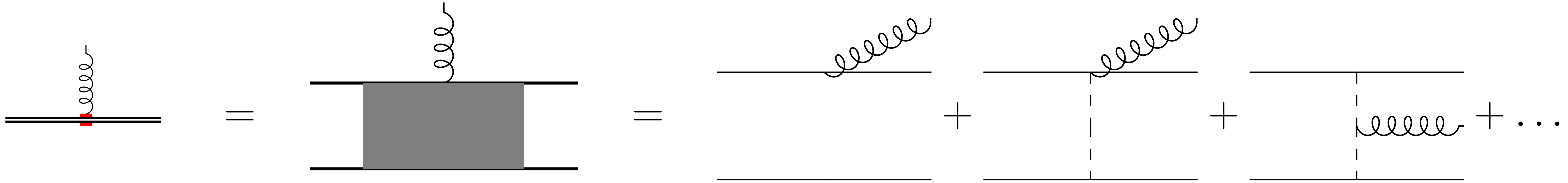
Effective Action for Dynamical Multipoles

$$e^{i S_{eff}[x_K]} = \int \mathcal{D}\bar{h}_{\mu\nu} \, \exp\left\{ iS[\eta + \bar{h}] + \ldots \right\}$$



Conservative Dynamics

$$\int \mathcal{D}H \, \exp\left\{ i\, S\, [\eta + H, h = 0] + i\, S_{pp}\, [x_K, \eta + H, h = 0] \right\} = e^{i\, S_{eff}[h=0, x_K]} = e^{i \int dt\, \mathcal{L}_{eff}}$$



Status of PN Results

PN order	0	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5
no spin	N	1PN		2PN		3PN		4PN		5PN		6PN	
spin-orbit			LO SO		NLO SO		N2LO SO		N3LO SO		N4LO SO		
spin^2				LO S2		NLO S2		N2LO S2		N3LO S2			
spin^3							LO S3		NLO S3		N2LO S3		
spin^4								LO S4		NLO S4			
spin^5											LO S5		NLO S5
spin^6												LO S6	

need up to

- 1PM
- 2PM
- 3PM
- 4PM
- 5PM
- 6PM
- 7PM

credit: Justin Vines

Levi, McLeod, Steinhoff, Teng, Von Hippel,..

Kim, Levi, Yin (2021)

Kim, Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Levi, Yin (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

Brunello, MKM, Mastrolia, Patil (W.I.P)

1PN [Einstein, Infeld, Hoffman '38].

2PN [Ohta *et al.*, '73].

3PN [Jaranowski, Schaefer, '97; Damour, Jaranowski, Schaefer, '97; Blanchet, Faye, '00; Damour, Jaranowski, Schaefer, '01]

4PN [Damour, Jaranowski, Schäfer, Bernard, Blanchet, Bohe, Faye, Marsat, Marchand, Foffa, Sturani, Mastrolia, Sturm, Porto, Rothstein...]

5PN [Foffa, Mastrolia, Sturani, Sturm, Bodabilla, '19; Blümlein, Maier, Marquard, '19; Bini, Damour, Geralico, '19; Blümlein, Maier, Marquard, '19; Almeida, Foffa, Sturani, '22;]

Diagrams for Spinning Binaries

MKM, Mastrolia, Patil, Steinhoff (2022)

MKM, Mastrolia, Patil, Steinhoff (2022)

S^0

Order	Diagrams	Loops	Diagrams
0PN	1	0	1
1PN	4	1	1
		0	3
2PN	21	2	5
		1	10
		0	6
3PN	130	3	8
		2	75
		1	38
		0	9

(a) Non-spinning sector

S^1

Order	Diagrams	Loops	Diagrams
LO	2	0	2
NLO	13	1	8
		0	5
N^2LO	100	2	56
		1	36
		0	8
N^3LO	894	3	288
		2	495
		1	100
		0	11

(b) Spin-orbit sector

S^2

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	7	1	3
		0	4
N^2LO	58	2	27
		1	24
		0	7
N^3LO	553	3	125
		2	342
		1	76
		0	10

(a) Spin1-Spin2 and Spin1^2 (Spin2^2) sector

Order	Diagrams	Loops	Diagrams
LO	1	0	1
NLO	4	1	1
		0	3
N^2LO	25	2	7
		1	12
		0	6
N^3LO	168	3	15
		2	101
		1	43
		0	9

(b) ES^2 sector

Order	Loops	Diagrams
LO	1	1

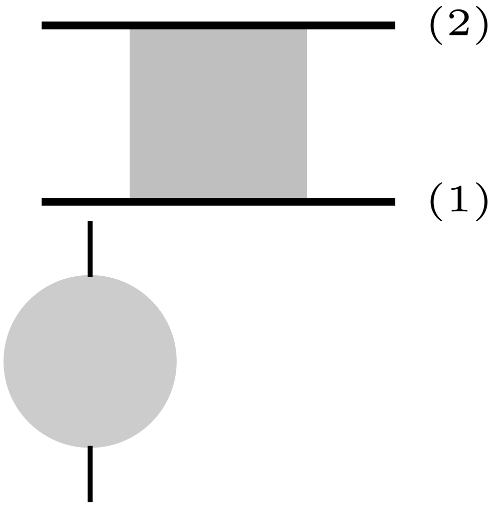
(c) E^2 sector

Order	Loops	Diagrams
LO	1	1

(d) E^2S^2 sector

$$\mathcal{L}(x_a, \dot{x}_a, \ddot{x}_a, \ldots S_a, \dot{S}_a, \ddot{S}_a, \ldots) = \text{- } \mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{ip}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$

$$= \text{- } \mathbf{i} \lim_{d\to 3} \int_{\mathbf{p}} e^{\mathbf{ip}\cdot(\mathbf{x}_{(1)} - \mathbf{x}_{(2)})}$$



- Dimensional Regularization d= 3+ϵ
- IBP Decomposition

Three Loop MIs