Next-to-leading power corrections to event shape variables

[arXiv:2306.17601][1]

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QCD@work-2024 : International workshop on QCD theory and experiment
Trani, Italy

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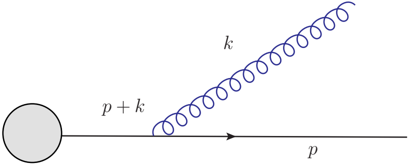
Content

Introduction

Infrared Divergence and eikonal approximation



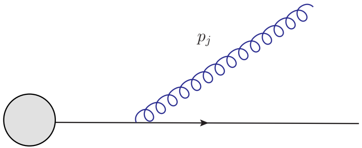
$$\mathcal{M}_\mu = \mathcal{M}_0\, i g_s\, \bar{u}(p)\gamma_\mu \left(\frac{\not{p} + \not{k} + m}{(p+k)^2 - m^2} \right) T^a$$

$$\propto \frac{1}{2p_0\, k_0 (1 - \cos\theta)}$$

- Emitted radiation have vanishing momenta ($k_0 \to 0$)
- Collinear to the emitting particle ($\theta \to 0$)
- Infrared singularities $\cdots$ now what? $\longrightarrow$ KLN Theorem

Infrared safe observables

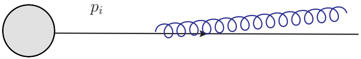
An observable $\mathcal{X}$ is Infrared safe if



➡ Soft emission

$$\mathcal{X}(p_1, ...p_i, p_j, p_k, ...p_n) = \mathcal{X}(p_1, ...p_i, p_k, ...p_n)$$
$$p_j \to 0$$

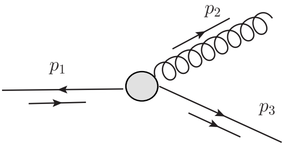
➡ Collinear splitting



$$\mathcal{X}(p_1, ...p_i, ...p_n) = \mathcal{X}(p_1, ...zp_i, (1-z)p_i, ...p_n)$$

Event Shape Variables

Different Events and different *Shapes*



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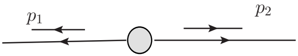

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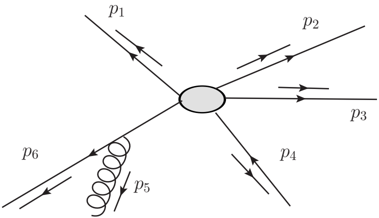
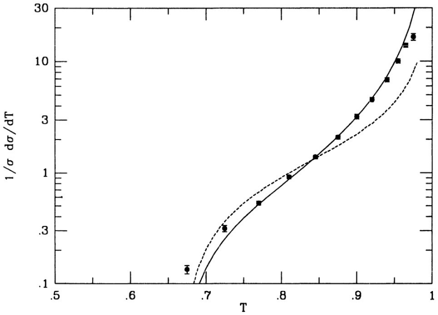


C

Event shape and α_s

- Thrust distribution at [LEP, DELPHI collab.]



- Prospects for strong coupling measurement at hadron colliders using soft-drop jet mass [JHEP04(2023)087]
- Fitting the Strong Coupling Constant with Soft-Drop Thrust [JHEP11(2019)179]

Event shape variables

Thrust	1977 [E. Farhi]
Spherocity	1977 [H. Georgi and M. Machacek]
C-parameter	1978 [G. C. Fox, S. Wolfram ...]
Jet mass	1981 [L. Clavelli et. al.]
Jet broadening	1992 [S. Catani et. al.]
Angularities	2003 [C.F. Berger et. al.]

Thrust and C-parameter

- Thrust :

$$T = \max_{\mathbf{n}} \frac{\sum_i |\mathbf{p_i} \cdot \mathbf{n}|}{\sum_i |\mathbf{p_i}|} = \max\{x_1, x_2, x_3\}$$

- C-parameter :

$$C = 3 - \frac{3}{2} \sum_{i,j} \frac{\left(p^{(i)} \cdot p^{(j)}\right)^2}{\left(p^{(i)} \cdot q\right)\left(p^{(j)} \cdot q\right)}$$

$$c = \frac{C}{6} = \frac{(1 - x_1)(1 - x_2)(1 - x_3)}{x_1 \; x_2 \; x_3}$$

Next-to-leading power terms

Next-to-leading Power terms

Distribution in variable ξ

$$\frac{d\sigma}{d\xi} \propto \sum_{n=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \sum_{m=0}^{2n-1} \left[\overbrace{c_{nm}^{(-1)} \left(\frac{\log^m \xi}{\xi}\right)_+ + c_n \delta(\xi)}^{\text{LP terms}} + \underbrace{c_{nm}^{(0)} \log^m \xi}_{\text{NLP terms}} + \dots \right]$$

Thrust distribution

$$\frac{1}{\sigma_0}\frac{d\sigma}{d\tau} = \frac{2\alpha_s}{3\pi}\left(\overbrace{\frac{-3 - 4\log\tau}{\tau}}^{\text{LP terms}} - \underbrace{2 + 2\log\tau}_{\text{NLP terms}} + \dots \right)$$

Arrangement of leading terms

$$\frac{d\sigma}{d\zeta} \propto \sum_{n=0}^{\infty}\left(\frac{\alpha_s}{\pi}\right)^n \left[\sum_{m=0}^{2n-1} c_{nm}^{\mathsf{LP}}\left(\frac{\log^m \zeta}{\zeta}\right)_+ + c_n^{(\delta)}\delta(\zeta) + \sum_{m=0}^{2n-1} c_{nm}^{\mathsf{NLP}} \log^m \zeta + \ldots\right].$$

For $n = 1$, the above expression reads as

$$\left[\left\{c_{(1,0)}^{\mathsf{LP}} + c_{(1,1)}^{\mathsf{LP}} \log \zeta\right\}\frac{1}{\zeta} + \left\{c_{(1,0)}^{\mathsf{NLP}} + c_{(1,1)}^{\mathsf{NLP}} \log \zeta\right\}\zeta^0 + \left\{c_{(1,0)}^{\mathsf{NNLP}} + c_{(1,1)}^{\mathsf{NNLP}} \log \zeta\right\}\zeta^1 + \cdots\right].$$

Event shape distribution

Event Shape distribution

$$\frac{d\sigma}{dX} = \frac{1}{2s} \int |\mathcal{M}|^2 \overbrace{\delta(X - f(x_1, x_2, x_3))}^{\text{new condition}} d\Phi$$

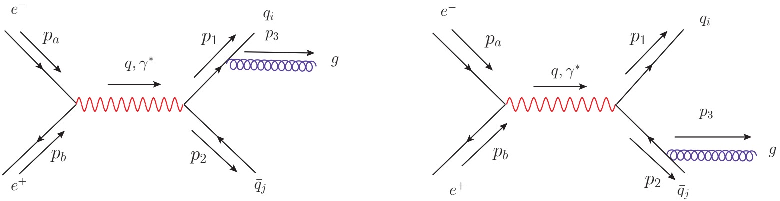
Thrust distribution for $e^+ + e^- \to q + \bar{q} + g$

$$\frac{1}{\sigma_0} \frac{d\sigma}{dT} = \frac{2\alpha_s}{3\pi} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \right] \delta[T - \max(x_1, x_2, x_3)]$$

The formalism of shifted kinematics

➡ Shifted kinematic approximation[2]

$$\overline{\sum}|\mathcal{M}_{\text{shift}}|^2 = g_s^2 C_F \underbrace{\frac{2p_1 \cdot p_2}{(p_1 \cdot k)(p_2 \cdot k)}}_{\text{Eikonal}} |\mathcal{M}_0(p_1 - \delta p_1, p_2 - \delta p_2)|^2$$



➡ Focuses on contribution from next-to-soft gluon emissions

[2] Del Duca et. al. (2017)

The Matrix Elements

- Exact approach

$$\overline{\sum}|\mathcal{M}_{\text{exact}}|^2 = 8(e^2 e_q)^2 N_c g_s^2 C_F \frac{1}{3Q^2} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)}$$

- Shifted kinematics

$$\overline{\sum}|\mathcal{M}_{\text{shift}}|^2 = 8(e^2 e_q)^2 N_c g_s^2 C_F \frac{1}{3Q^2} \frac{2x_1 + 2x_2 - 2}{(1-x_1)(1-x_2)}$$

- Soft quark approximation

$$\overline{\sum}|\mathcal{M}_{\text{rem}}|^2 = 8(e^2 e_q)^2 N_c g_s^2 C_F \frac{1}{3Q^2} \left(\frac{1-x_1}{1-x_2} + \frac{1-x_2}{1-x_1}\right)$$

Thrust distribution

Exact approach

$$\frac{2\alpha_s}{3\pi}\left(\frac{-3-4\log\tau}{\tau}-2+2\log\tau\right)$$

Shifted kinematics

$$\frac{2\alpha_s}{3\pi}\left(\frac{-4-4\log\tau}{\tau}+4+4\log\tau\right)$$

Soft quark

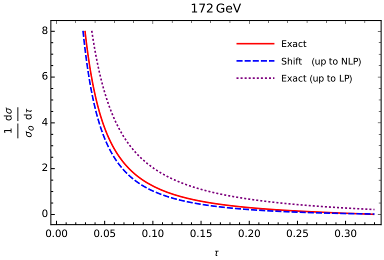
$$\frac{2\alpha_s}{3\pi}\left(\frac{1}{\tau}-6+2\log\tau\right)$$

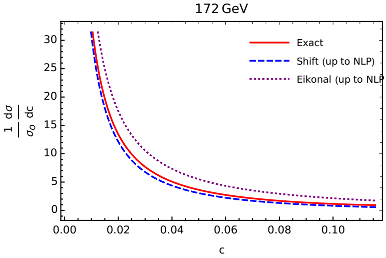
c-parameter distribution

Exact approach	$$\frac{2\alpha_s}{3\pi}\left(\frac{-3-4\log c}{c}-3+4\log c\right)$$
Shifted kinematics	$$\frac{2\alpha_s}{3\pi}\left(\frac{-4-4\log c}{\tau}+4+8\log c\right)$$
Soft quark	$$\frac{2\alpha_s}{3\pi}\left(\frac{1}{c}-7-4\log c\right)$$

Plots for thrust and c-parameter distribution

A comparison graph between Eikonal approximation and Shifted formalism





The shifted formalism provides a better approximation!

NLP corrections to Spherocity distribution at NLO in QCD

[Shubham Mishra]

[arXiv: 2403.16449] [hep-ph]
[Under review with PRD]

Spherocity

The expression for spherocity is

$$s_p = \min \left(\frac{\sum_i |\mathbf{p_i} \times \hat{\mathbf{n}}|}{\sum_i |\mathbf{p_i}|} \right)^2 ,$$

in terms of EF variables

$$s(x_1, x_2, x_3) = \frac{\pi^2}{16} \times s_p = \left[\frac{(1 - x_1)(1 - x_3)(1 - x_3)}{\max(x_1^2, x_2^2, x_3^2)} \right] .$$

Spherocity Distribution

$$\frac{d\sigma}{ds} = \frac{1}{2s_0} \int d\Phi_3 \left[\overline{\sum} |\mathcal{M}(x_1, x_2, x_3)|^2 \ \delta\big(s - s(x_1, x_2, x_3)\big) \right],$$

substituing these we get

$$\frac{1}{\sigma_0(s_0)} \frac{d\sigma}{ds} = \frac{2\alpha_s}{3\pi} \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \times \left[\frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \ \delta\left(s - \frac{(1-x_1)(1-x_3)(1-x_3)}{\max(x_1^2, x_2^2, x_3^2)} \right) \right],$$

Thus, depending upon which x_i is largest, above integral breaks into three separate integrals.

Result from region-I

$$\frac{1}{\sigma_0(s_0)}\frac{d\sigma}{ds}\bigg|_{\mathrm{I}} = \frac{2\alpha_s}{3\pi}\left[\frac{-2\left(4s^2+1\right)\log 4s+2s\log 4s}{s}-\frac{(s(7s-10)+6)\sqrt{\frac{4}{s-1}+5}}{2s}\right.$$
$$\left.+\frac{4(s(4s-1)+1)\log\left(\sqrt{1-s}+\sqrt{1-5s}\right)}{s}\right].$$

This expression under the dijet limit takes the form

$$\frac{1}{\sigma_0(s_0)}\frac{d\sigma}{ds}\bigg|_{(\mathrm{NLO+NNLP})_I} = \frac{2\alpha_s}{3\pi}\left[\frac{-3-2\log s}{s}+5+2\log s+\mathcal{O}(s)\right],$$

Result from region-III

NLO results from region-I

$$\frac{1}{\sigma_0(s_0)}\frac{d\sigma}{ds}\bigg|_{\rm III} = \frac{2\alpha_s}{3\pi}\left[\frac{8\tanh^{-1}\left(\sqrt{\frac{1-5s}{4s^2-5s+1}}\right)}{\sqrt{1-4s}s} + \log\left(\frac{256s^8}{\left(\sqrt{5s^2-6s+1}-3s+1\right)^8}\right)\right.$$
$$+\frac{\left(s^2+1\right)\log\left(\frac{16s^4}{\left(\sqrt{5s^2-6s+1}-3s+1\right)^4}\right)}{s}$$
$$\left.+\frac{(-s-3)\sqrt{(s-1)(5s-1)}}{s}\right].$$

Expanding the above expression around $s=0$, we get

$$\frac{1}{\sigma_0(s_0)}\frac{d\sigma}{ds}\bigg|_{\rm III} = \frac{2\alpha_s}{3\pi}\left[\frac{-3-4\log s}{s} - 8\log s + \mathcal{O}(s)\right].$$

Final form of spherocity distribution

The first result of spherocity distribution at NLO

$$\frac{1}{\sigma_0(s_0)}\frac{d\sigma}{ds}\bigg|_{\text{NLO+NNLP}} = \frac{2\alpha_s}{3\pi}\left[\frac{-9-8\log s}{s}+10-4\log s+\mathcal{O}(s)\right].$$

Summary and Outlook

- Shifted kinematics together with soft quark approximation captures LL and NLL upto NLP accuracy.
- Spherocity distribution has been computed for the first time.
- Furthet steps towards the resummation NLP terms
- Application of shifted kinematics to other event shapes such as, Angularities and Jet broadening


OBRIGADO
gracias
どうも
hvala
DANK U
takk
MERSI
merci
obrigado
danke schön
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PALDIES
muchas gracias
ありがとう
TEŞEKKÜR EDERIM
MOLTE GRAZIE
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muchas gracias
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NA GODE

Struggles with Elliptic integrals

Elliptic integrals appearing in c-parameter distribution and their quandaries

- Final integrand

$$\frac{1}{\sigma_0(s)}\frac{\mathrm{d}\sigma}{\mathrm{dc}}\bigg|_{\mathsf{NLO}} = \frac{2\alpha_s}{3\pi}\int_{y_1}^{y_2} dy\,\frac{2(1-y)\big(y\left(c(y-2)^2+(y-3)y+4\right)-2\big)}{c(cy+y-1)\sqrt{y(cy+y-1)\left(c(y-2)^2+(y-1)y\right)}}\,.$$

- The limits

$$y_1 = \frac{1+4c-\sqrt{1-8c}}{2(1+c)}\,, \qquad y_2 = \frac{1+4c+\sqrt{1-8c}}{2(1+c)}\,.$$

- Final form

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{dc}\bigg|_{\mathsf{NLO}} = \frac{2\alpha_s}{3\pi}\Big(e(c)\ \mathrm{E}[m_1(c)] + p(c)\ \Pi[n_1(c), m_1(c)] + k(c)\ \mathrm{K}[m_1(c)]\Big)\,,$$

Transformation of elliptic integrals

The three kinds of incomplete elliptic integrals are

$$F[\phi, m] = \int_0^{\sin\phi} \frac{dt}{\sqrt{(1-t^2)(1-mt^2)}}\,,$$

$$E[\phi, m] = \int_0^{\sin\phi} dt\sqrt{\frac{1-mt^2}{1-t^2}}\,,$$

$$\Pi[n, \phi, m] = \int_0^{\sin\phi} \frac{dt}{(1-nt^2)\sqrt{(1-t^2)(1-mt^2)}}\,.$$

ϕ, m and n are called the amplitude, parameter and characteristic respectively

Transformation rules

If the amplitude $\phi = \frac{\pi}{2}$

$$\begin{aligned} F[\phi, m] &= K[m]\,, \\ E[\phi, m] &= E[m]\,, \\ \Pi[n, \phi, m] &= \Pi[n, m]\,. \end{aligned}$$

The incomplete elliptic integral reduces to complete elliptic integral!

Amplitude of elliptic integrals

Non reducible	$\phi_1\Big\vert_{y=y_2}$ & $\phi_2\Big\vert_{y=y_1}$
	$\phi_1\Big\vert_{y=y_2} = \phi_2\Big\vert_{y=y_1} = 0$
Reducible	$\phi_1\Big\vert_{y=y_1} \& \phi_2\Big\vert_{y=y_2}$

Trouble with transformations

Elliptic integral and their amplitudes

$$\phi_{1,2}(c,y) = \left(\frac{-1+\sqrt{1-8c}-4c\pm 8c/y}{2\sqrt{1-8c}}\right)^{1/2} .$$

- Non-reducible for $\phi_1\Big|_{y=y_2}$ & $\phi_2\Big|_{y=y_1}$
  - $\phi_1\Big|_{y=y_2} = \phi_2\Big|_{y=y_1} = 0$
- Reducible for $\phi_1\Big|_{y=y_1}$ & $\phi_2\Big|_{y=y_2}$
  - $\phi_1\Big|_{y=y_1} = \phi_2\Big|_{y=y_2} = \frac{\pi}{2}$

Fixing the trouble with non-reducible elliptic integrals

Use an off-shell parameter[3]

$$E\left[\sin^{-1}\left(\frac{\left(-4c+\sqrt{1-8c}-1\right)(c+1)e}{\sqrt{1-8c}\left(2c(e+2)+\sqrt{1-8c}+2e+1\right)}\right)^{1/2},\ m_1(c)\right]$$

Argument $(\phi) \to 0$ as $e \to 0$

$$E[\phi_1(c,e), m_1(c)] = \sqrt{\frac{\left(-4c+\sqrt{1-8c}-1\right)(c+1)}{\sqrt{1-8c}\left(4c+\sqrt{1-8c}+1\right)}}\sqrt{e} + \mathcal{O}(e^{3/2})$$

[3] $y_{(1,2)} \to y_{(1,2)} + e$

The final form

$$\left.\frac{1}{\sigma_0(s)}\frac{d\sigma}{dc}\right|_{\text{NLO}} = \frac{2\alpha_s}{3\pi}\Big(e(c)\ \mathrm{E}[m_1(c)] + p(c)\ \Pi[n_1(c), m_1(c)] + k(c)\ \mathrm{K}[m_1(c)]\Big),$$

$$\left.\frac{1}{\sigma_0(s)}\frac{d\sigma}{dc}\right|_{\text{NLO}} = \frac{2\alpha_s}{3\pi}\left(\frac{-3-4\log c}{c} - 3 + 4\log c\right).$$

- Modify the limits, by introducing an off-shell parameter.
- Categorize the elliptic integrals into reducible and non-reducible type
- Expand the non-reducible integral around $e = 0$

Back up slides

Breakdown of thrust distribution from region-I

- Upper limit contributions

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau}\bigg|_{\text{I,u}} = \frac{2\alpha_s}{3\pi}\left(\frac{-2\log\tau}{\tau} + 2 + 2\log\tau - \frac{\tau}{2} + \tau\log\tau + \mathcal{O}\left(\tau^2\right)\right),$$

- Lower limit contributions

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau}\bigg|_{\text{I,l}} = \frac{2\alpha_s}{3\pi}\left(\frac{3}{2\tau} + 2 - 2\tau + \mathcal{O}(\tau^2)\right).$$

- LLs at LP and NLP from soft and next-to-soft gluon emissions

Breakdown of thrust distribution from region-III

➡ Upper limit contributions

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau}\bigg|_{\mathsf{I,u}} = \frac{2\alpha_s}{3\pi}\left(-2-\log\tau+(1-\log\tau)\tau+\mathcal{O}\left(\tau^2\right)\right),$$

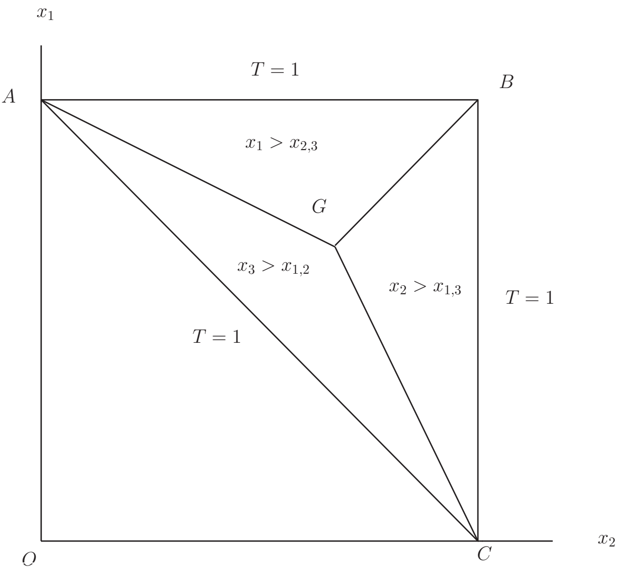
➡ Lower limit contributions

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau}\bigg|_{\mathsf{I,l}} = \frac{2\alpha_s}{3\pi}\left(\log\tau+(-1+\log\tau)\tau+\mathcal{O}(\tau^2)\right).$$

➡ LL at NLP from soft (anti-) quark emissions[4]

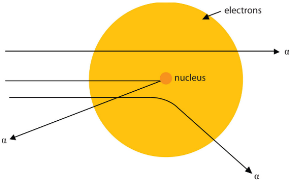
[4] JHEP03(2020)106

Phase space plot

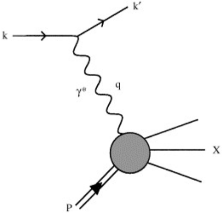


The Gold Foil Experiment and Deep inelastic scattering

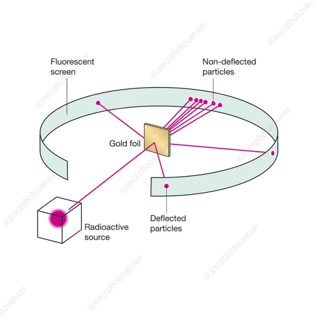
- Rutherford scattering

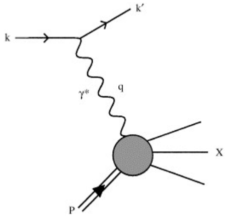


- Deep inelastic scattering



The


Fluorescent screen
Non-deflected particles
Gold foil
Radioactive source
Deflected particles


k
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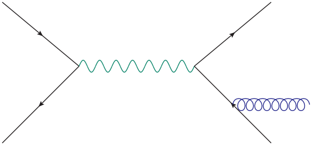
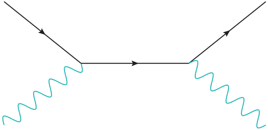
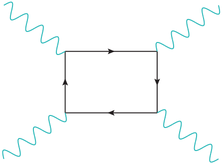
Scattering in QFT

- Computation of physical observables







- $$\sigma = \frac{1}{2s} \int d\Phi \; |\mathcal{M}|^2 \quad , \quad \frac{d\sigma}{d\tau} \quad , \quad \tau$$

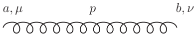
QCD - The Theory for Strong Interaction

- Interaction between quarks and gluons
- Non-abelian gauge theory with gauge group $SU(3)$
- Asymptotically free theory

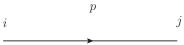
$$\mathcal{L}_{\text{QCD}} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi - \frac{1}{4}(F^a_{\mu\nu})^2 + g\bar{\psi}\gamma^\mu T^a \psi A^a_\mu + \mathcal{L}_{\text{GF}} + \mathcal{L}_{\text{ghost}}$$

- $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$
- $[T^a, T^b] = i f^{abc} T^c$
- $\mathcal{L}_{\text{GF}} = \frac{-1}{2\xi}(\partial^\mu A^a_\mu)(\partial^\nu A^a_\nu)$
- $\mathcal{L}_{\text{ghost}} = -\bar{c}^a \left(\partial^\mu (\delta^{ab}\partial_\mu + g_s f_{abc} A^c_\mu)\right) c^b$

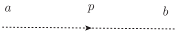
Feynman Rules for QCD



$$\delta_{ab}\frac{-ig_{\mu\nu}+(1-\alpha)\frac{p_\mu p_\nu}{p^2}}{p^2+i\epsilon}$$

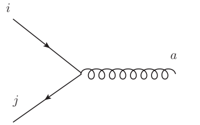


$$\delta_{ij}\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

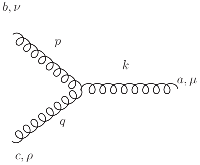


$$\frac{i\delta_{ab}}{p^2+i\epsilon}$$

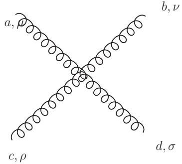
Feynman Rules for QCD



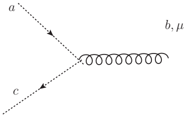
$$ig\gamma^\mu T^a_{ij}$$



$$gf^{abc}[g^{\mu\nu}(k-p)^\rho + CP]$$



$$-ig^2[f^{abc}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + CP]$$



$$-gf^{abc}p^\mu$$

Infrared singularities now what?

KLN Theorem

Singularities from loop integrations will cancel with the singularities from phase space integrations

leaving behind large logs! .... Resummation

$$\delta p_1^\alpha = -\frac{1}{2}\left(\frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\alpha - \frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\alpha + k^\alpha\right),$$

$$\delta p_2^\alpha = -\frac{1}{2}\left(\frac{p_1 \cdot k}{p_1 \cdot p_2} p_2^\alpha - \frac{p_2 \cdot k}{p_1 \cdot p_2} p_1^\alpha + k^\alpha\right).$$

Angularity Distribution and the Mystery of NLP logs

➡ FKS variables[5]

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau} = \frac{2\alpha_s}{3\pi}\left(\frac{-3-4\log\tau}{\tau} + 4 - 2(1+\log\tau)\tau + \mathcal{O}(\tau^2)\right)$$

➡ Berger's variable[6]

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau} = -\frac{\left(q^2(q+6)-4\right)\tau^3}{16q} + \left(\tau - \frac{\tau^3}{2}\right)\log(q) - \frac{\left(\tau^2+4\right)\log(q+1)}{\tau} + q\tau + \frac{-4q-3}{(q+1)^2\tau}$$

[5] Frixione et. al. (95)
[6] Berger et. al. (04)

Angularity distribution contd.

- Energy fractions

$$\frac{1}{\sigma_0(s)}\frac{d\sigma}{d\tau} = \frac{2\alpha_s}{3\pi}\left(\frac{-3-4\log\tau}{\tau} + \frac{5}{2} + \frac{1}{8}(-5+8\log\tau)\tau + \mathcal{O}(\tau^2)\right)$$

- No NLP logs for Angularity
- A pattern is noticed, logs are present at odd powers only!