HOLOGRAPHIC RENORMALIZED ENTANGLEMENT AND ENTROPIC C-FUNCTION

Mitsutoshi Fujita (University of South China)

Collaborators: Song He, Yuan Sun, Jun Zhang

Based on MF-He-Sun, Phys. Rev. D102, 126019(2020), MF-He-Sun-Zhang JHEP01(2024)079, Chen-Chen-MF-He, work in progress

Motivations of this paper

- QFT with negative energy
  - The IR behavior of $D=4$ $N=4$ SYM on S^1 is $d=3$ pure YM theory
  - The negative Casimir energy of a gauge theory

    ~ The mass of an AdS soliton

Witten `98, Horowitz-Myers `98

- DOF from EE (e.g. the coefficient of the A-type anomaly for a spherical entangling surface in CFT, *Solodkhin `08*)
  - Renormalized EE with a spherical entangling *Liu-Mezei `12*
  - The entropic C function from the EE with the striped subsystem

Nishioka-Takayanagi `06

Motivations of this paper

- What is difference between two entropy? → trace anomaly part
  - Renormalized EE only contains either A-type or B-type anomaly
  - Entropic c function contains both A-type and B-type anomalies

The entanglement entropy (an extension of the thermal entropy)

- System whose total Hilbert space is a direct product:

$$H = H_A \otimes H_B$$

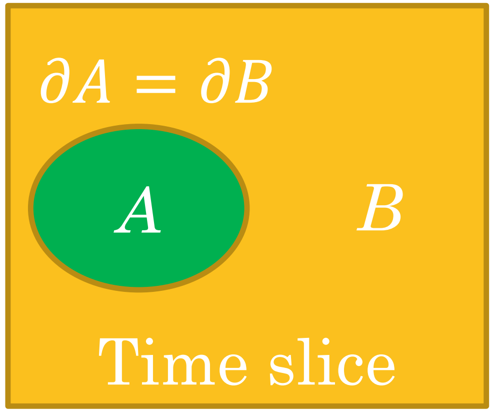
- Definition of the reduced density matrix $\rho_A = Tr_B(\rho)$ taking the trace over H_B

- Entanglement Entropy (EE) defined using the density matrix ρ_A as

$$S_A = -Tr_A(\rho_A log \rho_A)$$

Von Neumann entropy of ρ_A

- In QFT, A and B: often a spatial bipartition of a system



The UV structure of entanglement entropy

- S_{UV} with the entangling surface $S^1 \times S^{d-3}$ vs $S_{UV,0}$ with the surface S^{d-2}
- The UV structure of entanglement entropy is $S_{UV} = \frac{L_\phi}{R} S_{UV,0}$

$$S^A \sim \gamma \frac{Area(\partial A)}{a^{d-1}}$$

$+subleading\ terms$

$$S_{ren} = \frac{1}{R} f_d(R\partial_R) R S_{EE} = L_d(R\partial_R) S_{EE}$$

$$= \begin{cases} \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2)\ldots(R\partial_R - (d-3)) S_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} (R\partial_R + 1)(R\partial_R - 1)\ldots(R\partial_R - (d-3)) S_{EE}, & d = \text{even}, \end{cases}$$

$$f_d(R\partial_R) S_{EE}^{(0)} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3)\ldots(R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2)\ldots(R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{even}. \end{cases}$$

The UV structure of entanglement entropy

- S_{UV} with the entangling surface $S^1 \times S^{d-3}$ vs $S_{UV,0}$ with the surface S^{d-2}
- The UV structure of entanglement entropy is $S_{UV} = \frac{L_\phi}{R} S_{UV,0}$

$$S^A \sim \gamma \frac{Area(\partial A)}{a^{d-1}}$$

$+subleading\ terms$

$$S_{ren} = \frac{1}{R} f_d(R\partial_R) R S_{EE} = L_d(R\partial_R) S_{EE}$$

$$= \begin{cases} \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2)\dots(R\partial_R - (d-3)) S_{EE}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} (R\partial_R + 1)(R\partial_R - 1)\dots(R\partial_R - (d-3)) S_{EE}, & d = \text{even}, \end{cases}$$

$$f_d(R\partial_R) S_{EE}^{(0)} = \begin{cases} \frac{1}{(d-2)!!} (R\partial_R - 1)(R\partial_R - 3)\dots(R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{odd}, \\ \frac{1}{(d-2)!!} R\partial_R (R\partial_R - 2)\dots(R\partial_R - (d-2)) S_{EE}^{(0)}, & d = \text{even}. \end{cases}$$

Equivalent to D-1 dimensional renormalized entanglement entropy on $R^{1,d-2}$

Odd dimensional results

$$S_{ren} = R\partial_R S_{EE} \quad \text{for } d = 3,$$
$$S_{ren} = \frac{1}{3} R\partial_R (R\partial_R - 2) S_{EE} \quad \text{for } d = 5.$$

- Through Kaluza-Klein reduction along S^1, the renormalized entanglement entropy effectively embodies d-1 dimensional one in the low-energy limit
- In systems respecting Lorentz symmetry, the 2-dimensional entropic c-function is both non-negative and monotonically decreasing
- The behavior of the 4d renormalized entanglement entropy displays non-monotonic tendencies

Renormalized entanglement entropy of 4d QFT (even dimensional results)

- The length scale R_1 of the subregion related to rescaling of the metric

$$R_1 \partial_{R_1} S_A = \lim_{n\to 1}(-2\partial_n \int d^{d+1}x g_{\mu\nu}(x) \frac{\delta}{\delta g_{\mu\nu}(x)}[w - nw|_{n=1}])$$
$$\frac{1}{2\pi}\lim_{n\to 1}\partial_n\Big(\langle\int d^{d+1}x\sqrt{g}T_\mu{}^\mu(x)\rangle_{M_n} - n\langle\int d^{d+1}x\sqrt{g}T_\mu{}^\mu(x)\rangle_{M_1}\Big)$$

- The log term of the entanglement entropy $S_{EE} = s\log\left(\frac{\epsilon}{R_1}\right) + ..$

$$s = \frac{a}{180}\int_{\partial\sigma} d^2x\sqrt{h}E_2 + \frac{c}{120}\int_{\partial\sigma} d^2x\sqrt{h}I_2$$

- For cylinder type topology: $S_{ren} = \frac{1}{2}(l\partial_l + 1)(l\partial_l - 1)S_{EE} = s = \frac{cL_\phi}{240l}.$
- For a spherical surface S^2: $S_{ren} = \frac{1}{2}R\partial_R(R\partial_R - 2)S_{EE} = s = \frac{a}{90}$

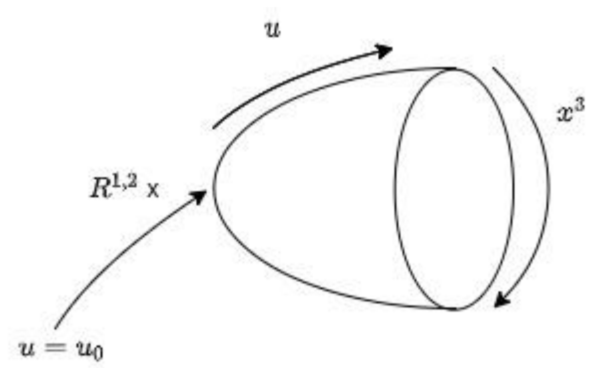
The AdS soliton dual to confining theory

- The double Wick rotation of the AdS black hole

- It corresponds to the ground state of QFT with the anti-periodic boundary condition on fermions

$$ds^2 = \frac{L^2}{u^2}\left(-dt^2 + \frac{du^2}{f(u)} + f(u)d\varphi^2 + \sum dx^i dx^i\right),$$

where $f(u) = 1 - \left(\frac{u}{u_0}\right)^4$



- The mass of the AdS soliton = negative energy

The AdS soliton with a background gauge field

- The metric of the AdS soliton with a background gauge field

$$ds^2_{d+1} = \frac{L^2}{z^2}\left(\frac{dz^2}{f_d(z)} + f_d(z)d\phi^2 - dt^2 + dR^2 + R^2 d\Omega_{d-3}\right) \qquad f_d(z) = 1 - \left(1 + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2}\right)\left(\frac{z}{z_+}\right)^d + \frac{\epsilon_1 z_+^2 a_\phi^2}{\gamma^2}\left(\frac{z}{z_+}\right)^{2(d-1)},$$

a_ϕ a constant gauge field and $\gamma^2 = \frac{(d-1)g_e^2 L^2}{(d-2)\kappa^2}$

$$A_\phi = a_\phi\left(1 - \left(\frac{z}{z_+}\right)^{d-2}\right),$$

- The Kaluza-Klein mass

$$M_0 = \frac{1}{4\pi z_+}\left(d - \frac{\epsilon_1(d-2)z_+^2 a_\phi^2}{\gamma^2}\right) > 0$$

- The solution exists if $a_\phi \leq \frac{2\pi M_0 \gamma}{\sqrt{d(d-2)}} = a_c$

The boundary energy

- $M = \langle T_{00} \rangle V_{d-2} / M_0$

$$M = -\frac{V_{d-2}}{M_0}\frac{L^{d-1}\bar{a}_\phi}{2\kappa^2 z_+^d} \qquad \bar{a}_\phi = 1 - \left(z_+ a_\phi\right)^2$$

- The boundary energy changes the sign when we change Wilson lines a_φ

$$\begin{cases} M < 0 & z_+ a_\phi < 1 \\ M > 0 & z_+ a_\phi > 1. \end{cases}$$

For $a_\varphi = 0$, it realizes Casimir energy of 4d SYM theory.
Casimir energy is different among periodic and anti-periodic b.c.

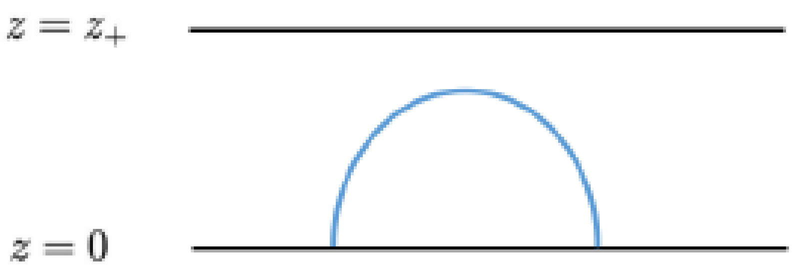
The holographic entanglement entropy

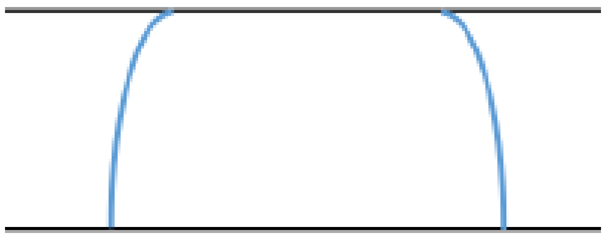
- The boundary region $S^1 \times S^{d-3}$: $\mathrm{R} = l$ and $0 \leq \phi \leq L_\phi$

- The surface action: $$A = \int d^{d-1}x\mathcal{L} = \Omega_{d-3} L_\phi L^{d-1} \int dz \frac{R^{d-3}}{z^{d-1}} \sqrt{1 + f \dot{R}^2}$$

- Boundary conditions:

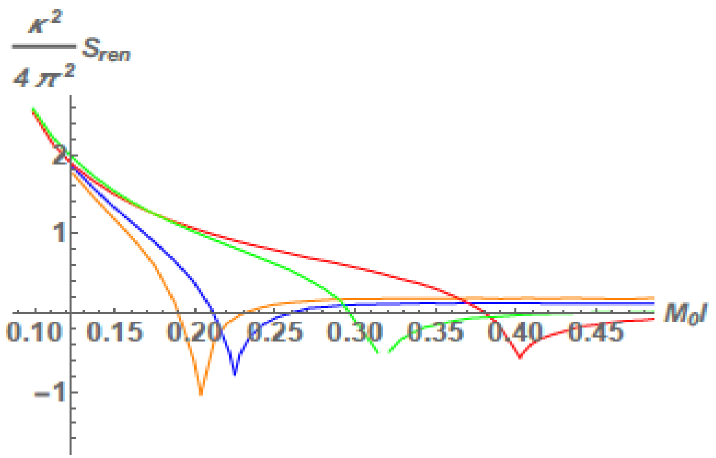
Disk type: $R(z_t) = 0, R'(z_t) = \infty$ A cylinder: $z_t = z_+$



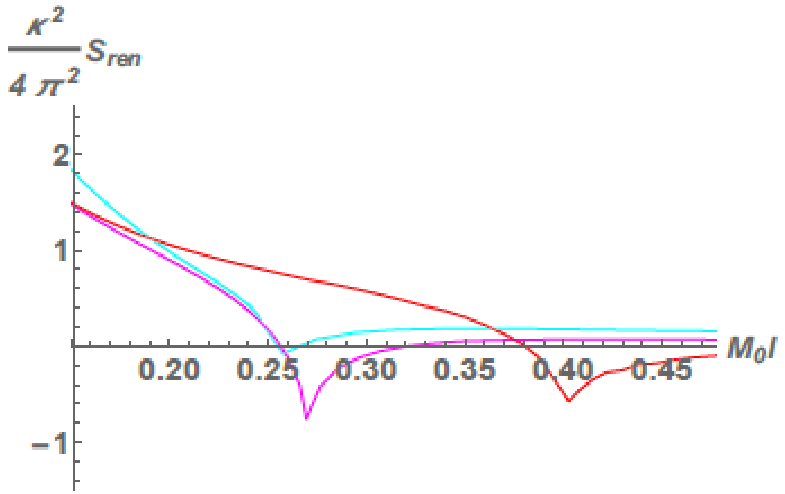


Renormalized EE in *4d*

- REE corresponds to effective DOF from EE with spherical surface
- REE non-monotonically behaves near critical lengths



$a_\phi = \frac{i}{2},\ 0,\ \frac{2}{3},\ \frac{1}{\sqrt{2}}$ from the left to the right

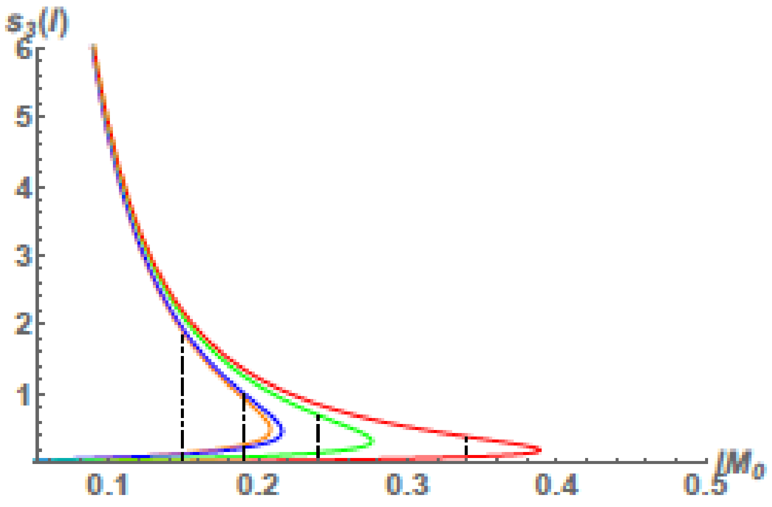


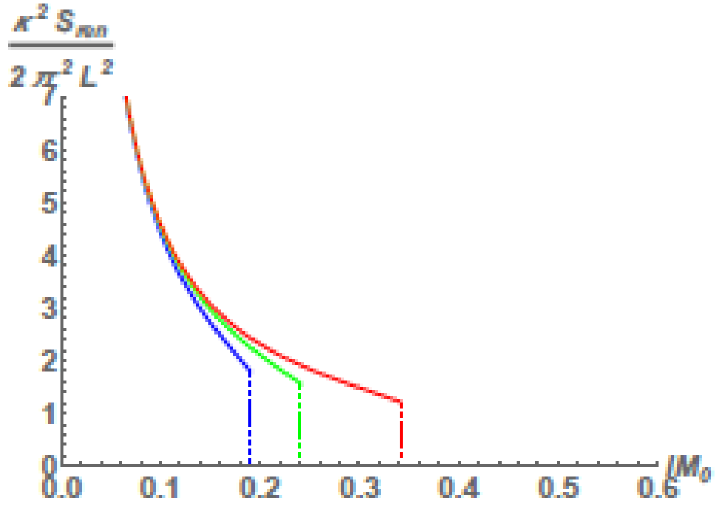
$M_0 = 1/\pi,\ 2/5,\ 3/5$ from the right to the left

- It implies that Wilson lines make particles light
- Massive modes decouple others soon

Renormalized EE in *3d*

- 3d: REE is positive and monotonically decreasing.
  - consistent with the entropic c function in $R^{1,1}$





$a_\phi = i/2,\ 0,\ 1,\ 2/\sqrt{3}$ from the left to the right

  - $s_3 \sim {}^{dS}/_{dl}$. The minimal surface is concave: $S'' < 0$

Discussion

Renormalized EE	Entropic C function
Increase at a region However, $S_{ren}(l \to 0) \geq S_{ren}(l \to \infty)$ (Decrease and positive for d=3)	Increase at a region and positive However, $S_{ren}(l \to 0) \geq S_{ren}(l \to \infty)$
Counting the effective DOF of Wilson lines along the S^1 direction	Counting the effective DOF of Wilson lines along the S^1 direction
Quantum phase transition	Quantum phase transition

Thank you!