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Spin-orbit part of the EOB Hamiltonian at the third subleading PN order

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Recap: the Hamiltonian of a spinning particle

Particle → mass μ and spin S_*

Kerr black hole → mass M and spin S

$$\hat{H}^K \equiv \frac{H^K}{\mu} = \sqrt{A^K \underbrace{\left[1 + p_\varphi^2 \left(u_c^K\right)^2 + \frac{A^K}{D^K} p_r^2\right]}_{\equiv\, Q^K}} + \left(G_S^K \hat{a} + G_{S_*}^K \tilde{a}_*\right) p_\varphi$$

$$\hat{a} \equiv S/M^2, \quad \tilde{a}_* \equiv S_*/\mu M$$

$$u_c^K \equiv 1/r_c^K, \quad \left(r_c^K\right)^2 \equiv r^2 + \hat{a}^2 \left(1 + \frac{2}{r}\right)$$

Orbital and SS part

$$A^K = \left(1 - 2u_c^K\right) \frac{1 + 2u_c^K}{1 + 2u}$$

$$D^K = \frac{(u_c^K)^2}{u^2}$$

SO part

$$G_S^K = 2\,u\,(u_c^K)^2$$

$$G_{S_*}^K = (u_c^K)^2 \left\{ \frac{\sqrt{A^K}}{\sqrt{Q^K}} \left[1 - \frac{(u_c^K)'\sqrt{A^K}}{\left(u_c^K\right)^2 \sqrt{D^K}}\right] + \frac{\left(A^K\right)'}{2u_c^K \left(1 + \sqrt{Q^K}\right) \sqrt{D^K}} \right\}$$

Recap: the Hamiltonian of TEOBResumS

Spin-aligned binary system → masses m_i and spins S_i, $\quad i = 1{,}2$

$$\hat{H}_{\rm EOB} \equiv \frac{1}{\nu}\sqrt{1 + 2\nu\left(\hat{H}_{\rm eff} - 1\right)}$$

$$\hat{H}_{\rm eff} \equiv \sqrt{A\left[1 + p_\varphi^2 u_c^2 + 2\nu(4-3\nu)p_{r_*}^4 u_c^2\right] + p_{r_*}^2} + \left(G_S \hat{S} + G_{S_*}\hat{S}_*\right)p_\varphi$$

$$\hat{S} \equiv \frac{S_1 + S_2}{M^2}, \quad \hat{S}_* \equiv \frac{1}{M^2}\left(\frac{m_2}{m_1}S_1 + \frac{m_1}{m_2}S_2\right), \quad \tilde{a}_0 \equiv \hat{S} + \hat{S}_*$$

$$M = m_1 + m_2, \quad \mu = m_1 m_2/M, \quad \nu \equiv \mu/M$$

$$u_c \equiv 1/r_c, \quad r_c^2 \equiv r^2 + \tilde{a}_0^2\left(1 + \frac{2}{r}\right) + \frac{\delta a^2}{r}$$

δa^2 → NLO spin-spin term

$$p_{r_*} \equiv A/D^{1/2}\, p_r$$

Orbital and SS part

$$A = P_5^1\left[A_{\rm orb}^{\rm 5PN}\right](u_c)\,\frac{1+2u_c}{1+2u}$$

$$D = P_3^0\left[D_{\rm orb}^{\rm 3PN}\right](u_c)\,\frac{u_c^2}{u^2}$$

SO part

$$G_S = G_S^0\,\hat{G}_S, \quad G_S^0 \equiv 2uu_c^2, \quad \hat{G}_S = \frac{1}{T_{\rm PN}\left[\left(G_S/G_S^0\right)^{-1}\right]}$$

$$G_{S_*} = G_{S_*}^0\hat{G}_{S_*}, \quad G_{S_*}^0 \equiv \frac{3}{2}u_c^3, \quad \hat{G}_{S_*} = \frac{1}{T_{\rm PN}\left[\left(G_{S_*}/G_{S_*}^0\right)^{-1}\right]}$$

Recap: more on the SO term in TEOBResumS

$$\boxed{G_S = G_S^0 \; \hat{G}_S} \qquad \boxed{G_{S_*} = G_{S_*}^0 \hat{G}_{S_*}}$$

Prefactors:

$G_S^0 \equiv 2uu_c^2 = G_S^K(u_c^K \to u_c)$ however $G_{S_*}^0 \equiv \frac{3}{2}u_c^3, \quad G_{S_*}^K = \frac{3}{2}u_c^K + \mathcal{O}(1/c^2)$

We would like $G_{S_*}^0 \equiv G_{S_*}^K(u_c^K \to u_c, A^K \to A, D^K \to D)$ but it is impossible in the "usual" DJS spin gauge

Inverse-resummed PN residuals:

$$\hat{G}_S = \frac{1}{T_{\rm PN}\left[\left(G_S/G_S^0\right)^{-1}\right]}, \qquad \hat{G}_{S_*} = \frac{1}{T_{\rm PN}\left[\left(G_{S_*}/G_{S_*}^0\right)^{-1}\right]}$$

They are both built from PN-expanded results at $\mathrm{N^2LO}$ but we have now enough analytical information to compute the $\mathrm{N^3LO}$

We have therefore two possible directions of improvement:

- **Change the spin gauge in which the PN expressions of G_S and are G_{S_*} obtained**
- **Include the $\mathrm{N^3LO}$**

Computation outline

Generic ansatz for the N^{3}LO-accurate expressions of (G_S, G_{S_*})

↓

4.5PN effective Hamiltonian with the ansatz in the SO part

↓

Effective ansatz-dependent scattering angle $\chi_{\rm eff}$

↓

Equivalence of the scattering angle, $\chi_{\rm eff} = \chi_{\rm PM}$

↓

N^{3}LO-accurate expressions of (G_S, G_{S_*}) in gauge-unfixed form

↓

Gauge fixing, such that $G^0_{S_*} \equiv G^K_{S_*}(u^K_c \to u_c, A^K \to A, D^K \to D)$

↓

N^{3}LO-accurate expressions of (G_S, G_{S_*}) in the spin gauge we want

Source of dynamical information

4PM scattering angle expanded through the third subleading PN order:

[Antonelli et al. 2020]

$$
\begin{aligned}
\frac{\chi_{\mathrm{PM}}}{\Gamma} = {} & \left(\frac{GM}{b\sqrt{\varepsilon}}\right) 2\frac{1+2\varepsilon}{\sqrt{\varepsilon}} + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^2 \frac{3\pi}{4}(4+5\varepsilon) \\
& - \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^3 \frac{1}{\sqrt{\varepsilon}} \left[2\frac{1-12\varepsilon-72\varepsilon^2-64\varepsilon^3}{3\varepsilon} + \nu\left(\frac{8+94\varepsilon+313\varepsilon^2}{12} + \mathcal{O}(\varepsilon^3)\right)\right] \\
& + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^4 \pi \left[\frac{105}{64}(16+48\varepsilon+33\varepsilon^2) + \nu\left(-\frac{15}{2} + \left(\frac{123}{128}\pi^2 - \frac{557}{8}\right)\varepsilon + \mathcal{O}(\varepsilon^2)\right)\right] \\
& - \left(\frac{a_b}{b\sqrt{\varepsilon}}\right) \Bigg\{ \left(\frac{GM}{b\sqrt{\varepsilon}}\right) 4\gamma\sqrt{\varepsilon} + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^2 2\pi\gamma\,(2+5\varepsilon) \\
& + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^3 \frac{\gamma}{\sqrt{\varepsilon}} \left[12(1+12\varepsilon+16\varepsilon^2) - \nu\left(10\varepsilon + \frac{97}{2}\varepsilon^2 + \frac{177}{4}\varepsilon^3 + \mathcal{O}(\varepsilon^4)\right)\right] \\
& + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^4 \pi\gamma \left[\frac{21}{2}(8+36\varepsilon+33\varepsilon^2) - \nu\left(\frac{21}{2} + \frac{495}{4}\varepsilon + \left(\frac{17423}{48} - \frac{241\pi^2}{128}\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)\right] \Bigg\} \\
& - \left(\frac{a_t}{b\sqrt{\varepsilon}}\right) \Bigg\{ \left(\frac{GM}{b\sqrt{\varepsilon}}\right) 4\gamma\sqrt{\varepsilon} + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^2 \frac{3\pi}{2}\gamma\,(2+5\varepsilon) \\
& + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^3 \frac{\gamma}{\sqrt{\varepsilon}} \left[8(1+12\varepsilon+16\varepsilon^2) - \nu\left(10\varepsilon + \frac{97}{2}\varepsilon^2 + \frac{177}{4}\varepsilon^3 + \mathcal{O}(\varepsilon^4)\right)\right] \\
& + \left(\frac{GM}{b\sqrt{\varepsilon}}\right)^4 \pi\gamma \left[\frac{105}{16}(8+36\varepsilon+33\varepsilon^2) - \nu\left(9 + \frac{219}{2}\varepsilon + \left(\frac{2759}{8} - \frac{123}{32}\pi^2\right)\varepsilon^2 + \mathcal{O}(\varepsilon^3)\right)\right] \Bigg\}
\end{aligned}
$$

$\hat{S}\,\chi_{\mathrm{PM}}^{\mathrm{SO,S}}$

$\hat{S}_*\,\chi_{\mathrm{PM}}^{\mathrm{SO,S_*}}$

$$a_b = \hat{S}, \qquad a_t = \hat{S}_*$$

$$\gamma \equiv \hat{H}_{\mathrm{eff}} \qquad \varepsilon \equiv \hat{H}_{\mathrm{eff}}^2 - 1 \sim 1/c^2 \qquad \Gamma \equiv \sqrt{1 + 2\nu\left(\hat{H}_{\mathrm{eff}} - 1\right)} \qquad \frac{GM}{b\sqrt{\varepsilon}} \equiv \frac{1}{p_\varphi}$$

Starting ansatz and corresponding Hamiltonian

We consider the **general ansatz**: (p is the dimensionless total momentum, $p^2 = p_r^2 + p_\varphi^2 u^2$)

$$G_S^{\rm gen} = u^3 \Big[2 + \frac{1}{c^2} \left(p^2\, g_1^{\rm NLO} + p_r^2\, g_2^{\rm NLO} + u\, g_3^{\rm NLO} \right) + \frac{1}{c^4} \Big(p^4\, g_1^{\rm N^2LO} + p^2 p_r^2\, g_2^{\rm N^2LO} + p^2 u\, g_3^{\rm N^2LO} + p_r^4\, g_4^{\rm N^2LO}$$
$$+ p_r^2 u\, g_5^{\rm N^2LO} + u^2\, g_6^{\rm N^2LO} + \Big) + \frac{1}{c^6} \Big(p^6\, g_1^{\rm N^3LO} + p^4 p_r^2\, g_2^{\rm N^3LO} + p^4 u\, g_3^{\rm N^3LO} + p^2 p_r^4\, g_4^{\rm N^3LO} + p^2 p_r^2 u\, g_5^{\rm N^3LO}$$
$$+ p^2 u^2\, g_6^{\rm N^3LO} + p_r^6\, g_7^{\rm N^3LO} + p_r^4 u\, g_8^{\rm N^3LO} + p^2 u^2\, g_9^{\rm N^3LO} + u^3\, g_{10}^{\rm N^3LO} \Big) \Big] ,$$

$G_{S_*}^{\rm gen}$ has the same structures, with $2 \to 3/2$ and $g_n^{{\rm N}^m{\rm LO}} \to g_{*n}^{{\rm N}^m{\rm LO}}$

Corresponding 4.5PN-accurate effective Hamiltonian:

$$\hat{H}_{\rm eff}^{\rm gen} \equiv \sqrt{A_{\rm 4PN} \left(1 + p_\varphi^2 u^2 + \frac{A_{\rm 4PN}}{D_{\rm 4PN}} p_r^2 + Q_{\rm 4PN} \right)} + \left(G_S^{\rm gen} \hat{S} + G_{S_*}^{\rm gen} \hat{S}_* \right) p_\varphi$$

$$A_{\rm 4PN} = 1 - 2u + 2\nu u^3 + \nu \left(\frac{94}{3} - \frac{41\pi^2}{32} \right) u^4 + \left(a_{5,c} + \frac{64\nu}{5} \log(u) \right) u^5,$$

$$D_{\rm 4PN} = 1 - 6\nu u^2 + 2\nu(-26 + 3\nu) u^3 + \left(d_{4,c} - \frac{592\nu}{15} \log(u) \right) u^4,$$

$$Q_{\rm 4PN} = p_r^4 \left(q_{43}\, u^3 + 2\nu(4 - 3\nu) u^2 \right) + q_{62}\, u^2 p_r^6$$

The effective scattering angle

Inverse of $r_{\min}$ i.e. the largest real root of $p_r = 0$

$$\chi_{\text{eff}}(\varepsilon, p_\varphi, g_n^{\text{N}^m\text{LO}}, g_{*n}^{\text{N}^m\text{LO}}) \equiv -\pi - 2\int_0^{u_{\max}} \frac{du}{u^2}\frac{\partial}{\partial p_\varphi} p_r(\varepsilon, p_\varphi, u, g_n^{\text{N}^m\text{LO}}, g_{*n}^{\text{N}^m\text{LO}})$$

Obtained by inverting perturbatively $\varepsilon = (\hat{H}_{\text{eff}}^{\text{gen}})^2 - 1$

To properly solve the integral:

- Change of integration variable $u \rightarrow \frac{\sqrt{\varepsilon}}{p_\varphi}x, \qquad u_{\max} \rightarrow x_{\max} = 1 + \mathcal{O}(G/c^2)$
- From “canonical” to “covariant” angular momentum

$$p_\varphi \rightarrow \frac{p_\varphi}{\Gamma} + \frac{\Gamma - 1}{2\nu c^2}\left(\hat{S} + \hat{S}_* - \frac{\hat{S} - \hat{S}_*}{\Gamma}\right), \qquad \Gamma = \sqrt{1 + 2\nu\left(\sqrt{1+\varepsilon} - 1\right)}$$

- Expansion of the integrand in $\underbrace{1/p_\varphi \text{ up to } 1/p_\varphi^4}_{\text{PM expansion}}$ and in $\underbrace{1/c \text{ up to } 1/c^8}_{\text{PN expansion}}$
- Hadamard partie finie $\int_0^{1+\mathcal{O}(G/c^2)} dx \rightarrow \text{Pf}\int_0^1 dx$ [Damour-Schafer, 1988]

The effective scattering angle

$\chi_{\text{eff}}(\varepsilon, p_\varphi, g_n^{\text{N}^m\text{LO}}, g_{*n}^{\text{N}^m\text{LO}})$ is then given by a series of integrals of the type:

$$\text{Pf}\int_0^1 dx\,(1-x^2)^{-1/2-n}\,x^m \quad \to \quad \text{Pf of an Euler Beta function } \left(x \to \sqrt{y}\right)$$

We can evaluate them by **analytical continuation**:

$$\text{Pf}\int_0^1 dx\,(1-x^2)^{-1/2-n}\,x^m = \lim_{z\to 0}\int_0^1 dx\,(1-x^2)^{-1/2+z-n}\,x^m$$

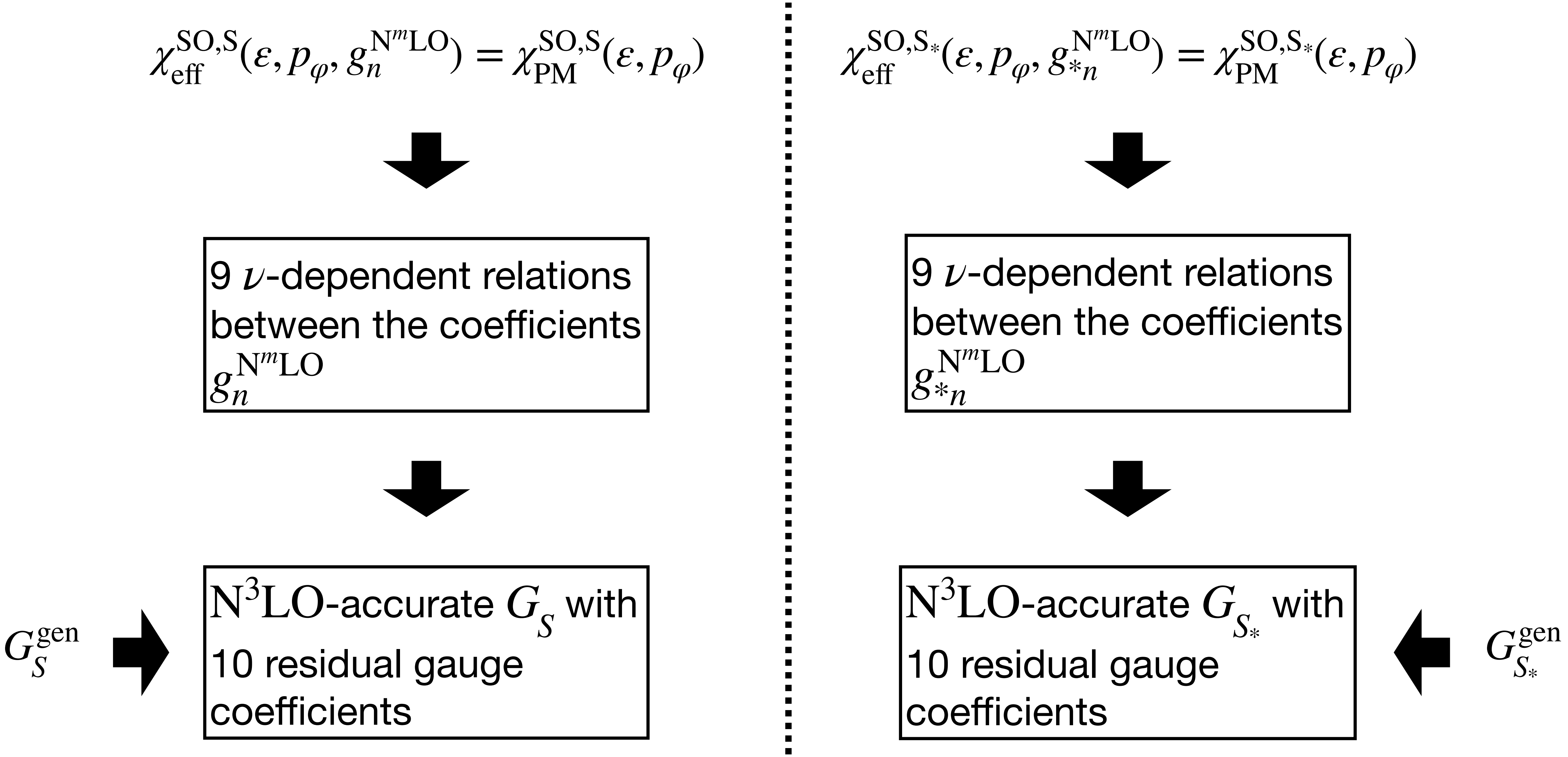
The resulting χ_{eff} has a three-component structure like χ_{PM} :

$$\chi_{\text{eff}}(\varepsilon, p_\varphi, g_n^{\text{N}^m\text{LO}}, g_{*n}^{\text{N}^m\text{LO}}) = \chi_{\text{eff}}^{\text{orb}}(\varepsilon, p_\varphi) + \hat{S}\,\chi_{\text{eff}}^{\text{SO,S}}(\varepsilon, p_\varphi, g_n^{\text{N}^m\text{LO}}) + \hat{S}_*\,\chi_{\text{eff}}^{\text{SO,S}_*}(\varepsilon, p_\varphi, g_{*n}^{\text{N}^m\text{LO}})$$

Moreover $\chi_{\text{eff}}^{\text{orb}}$ coincide with the non-spinning part of χ_{PM}

$\mathrm{N^3LO}$-accurate (G_S, G_{S_*}) in gauge-unfixed form

We can now impose the following equivalences:



$$\chi_{\rm eff}^{\rm SO,S}(\varepsilon, p_\varphi, g_n^{\mathrm{N}^m\mathrm{LO}}) = \chi_{\rm PM}^{\rm SO,S}(\varepsilon, p_\varphi)$$

9 ν-dependent relations between the coefficients $g_n^{\mathrm{N}^m\mathrm{LO}}$

$G_S^{\rm gen}$

$\mathrm{N^3LO}$-accurate G_S with 10 residual gauge coefficients

$$\chi_{\rm eff}^{\rm SO,S_*}(\varepsilon, p_\varphi, g_{*n}^{\mathrm{N}^m\mathrm{LO}}) = \chi_{\rm PM}^{\rm SO,S_*}(\varepsilon, p_\varphi)$$

9 ν-dependent relations between the coefficients $g_{*n}^{\mathrm{N}^m\mathrm{LO}}$

$G_{S_*}^{\rm gen}$

$\mathrm{N^3LO}$-accurate G_{S_*} with 10 residual gauge coefficients

Coefficient conditions for G_S

NLO

$$g_1^{\mathrm{NLO}} \to -\frac{g_2^{\mathrm{NLO}}}{3} - \frac{9\nu}{8}, \qquad g_3^{\mathrm{NLO}} \to \frac{g_2^{\mathrm{NLO}}}{3} + \frac{\nu}{2},$$

N^{2}LO

$$\begin{aligned}
g_1^{\mathrm{N^2LO}} &\to -\frac{g_2^{\mathrm{N^2LO}}}{3} - \frac{g_4^{\mathrm{N^2LO}}}{5} + \frac{7\nu^2}{8} + \frac{\nu}{8}, \\
g_3^{\mathrm{N^2LO}} &\to \frac{g_2^{\mathrm{N^2LO}}}{2} + \frac{g_2^{\mathrm{NLO}}}{4} + \frac{9g_4^{\mathrm{N^2LO}}}{20} - \frac{g_5^{\mathrm{N^2LO}}}{4} - \frac{5\nu^2}{4} - \frac{33\nu}{16}, \\
g_6^{\mathrm{N^2LO}} &\to -\frac{g_2^{\mathrm{N^2LO}}}{6} + \frac{3g_2^{\mathrm{NLO}}}{4} - \frac{g_4^{\mathrm{N^2LO}}}{4} + \frac{g_5^{\mathrm{N^2LO}}}{4} + \frac{\nu^2}{4} - \frac{119\nu}{16},
\end{aligned}$$

N^{3}LO

$$\begin{aligned}
g_1^{\mathrm{N^3LO}} \to{}& -\frac{g_2^{\mathrm{N^3LO}}}{3} - \frac{g_4^{\mathrm{N^3LO}}}{5} - \frac{g_7^{\mathrm{N^3LO}}}{7} - \frac{95\nu^3}{128} - \frac{9\nu^2}{32} + \frac{\nu}{128}, \\
g_3^{\mathrm{N^3LO}} \to{}& \frac{g_2^{\mathrm{N^2LO}}}{3} + \frac{2g_2^{\mathrm{N^3LO}}}{3} + \frac{3g_4^{\mathrm{N^2LO}}}{8} + \frac{3g_4^{\mathrm{N^3LO}}}{5} - \frac{g_5^{\mathrm{N^3LO}}}{4} + \frac{29g_7^{\mathrm{N^3LO}}}{56} - \frac{g_8^{\mathrm{N^3LO}}}{8} + \frac{31\nu^3}{16} + \frac{21\nu^2}{8} + \frac{\nu}{2}, \\
g_6^{\mathrm{N^3LO}} \to{}& \frac{2g_2^{\mathrm{N^2LO}}}{3} - \frac{7g_2^{\mathrm{N^3LO}}}{15} + \frac{2g_2^{\mathrm{NLO}}\nu}{5} + \frac{17g_2^{\mathrm{NLO}}}{30} + \frac{63g_4^{\mathrm{N^2LO}}}{200} - \frac{3g_4^{\mathrm{N^3LO}}}{5} + \frac{g_5^{\mathrm{N^2LO}}}{10} + \frac{7g_5^{\mathrm{N^3LO}}}{20} - \frac{5g_7^{\mathrm{N^3LO}}}{8} \\
& + \frac{11g_8^{\mathrm{N^3LO}}}{40} - \frac{g_9^{\mathrm{N^3LO}}}{5} - \frac{11\nu^3}{8} + \frac{431\nu^2}{40} - \frac{1231\nu}{80}, \\
g_{10}^{\mathrm{N^3LO}} \to{}& -\frac{g_2^{\mathrm{N^2LO}}}{2} + \frac{2g_2^{\mathrm{N^3LO}}}{15} - \frac{7g_2^{\mathrm{NLO}}\nu}{5} + \frac{101g_2^{\mathrm{NLO}}}{60} - \frac{21g_4^{\mathrm{N^2LO}}}{25} + \frac{g_4^{\mathrm{N^3LO}}}{5} + \frac{13g_5^{\mathrm{N^2LO}}}{20} - \frac{g_5^{\mathrm{N^3LO}}}{10} + \frac{g_7^{\mathrm{N^3LO}}}{4} \\
& - \frac{3g_8^{\mathrm{N^3LO}}}{20} + \frac{g_9^{\mathrm{N^3LO}}}{5} + \frac{\nu^3}{8} - \frac{123\nu^2}{20} + \frac{241\pi^2\nu}{192} - \frac{28331\nu}{720}
\end{aligned}$$

Coefficient conditions for G_{S_*}

NLO

$$g_{*1}^{\mathrm{NLO}} \to -\frac{g_{*2}^{\mathrm{NLO}}}{3} - \frac{3\nu}{4} - \frac{5}{8}, \qquad g_{*3}^{\mathrm{NLO}} \to \frac{g_{*2}^{\mathrm{NLO}}}{3} - \frac{1}{2},$$

N^{2}LO

$$g_{*1}^{\mathrm{N^2LO}} \to -\frac{g_{*2}^{\mathrm{N^2LO}}}{3} - \frac{g_{*4}^{\mathrm{N^2LO}}}{5} + \frac{9\nu^2}{16} + \frac{\nu}{2} + \frac{7}{16},$$

$$g_{*3}^{\mathrm{N^2LO}} \to \frac{g_{*2}^{\mathrm{N^2LO}}}{2} + \frac{g_{*2}^{\mathrm{NLO}}}{4} + \frac{9g_{*4}^{\mathrm{N^2LO}}}{20} - \frac{g_{*5}^{\mathrm{N^2LO}}}{4} - \frac{3\nu^2}{8} - \frac{9\nu}{8} + \frac{9}{16},$$

$$g_{*6}^{\mathrm{N^2LO}} \to -\frac{g_{*2}^{\mathrm{N^2LO}}}{6} + \frac{3g_{*2}^{\mathrm{NLO}}}{4} - \frac{g_{*4}^{\mathrm{N^2LO}}}{4} + \frac{g_{*5}^{\mathrm{N^2LO}}}{4} - \frac{3\nu^2}{8} - \frac{55\nu}{8} - \frac{13}{16},$$

N^{3}LO

$$g_{*1}^{\mathrm{N^3LO}} \to -\frac{g_{*2}^{\mathrm{N^3LO}}}{3} - \frac{g_{*4}^{\mathrm{N^3LO}}}{5} - \frac{g_{*7}^{\mathrm{N^3LO}}}{7} - \frac{15\nu^3}{32} - \frac{33\nu^2}{64} - \frac{25\nu}{64} - \frac{45}{128},$$

$$g_{*3}^{\mathrm{N^3LO}} \to \frac{g_{*2}^{\mathrm{N^2LO}}}{3} + \frac{2g_{*2}^{\mathrm{N^3LO}}}{3} + \frac{3g_{*4}^{\mathrm{N^2LO}}}{8} + \frac{3g_{*4}^{\mathrm{N^3LO}}}{5} - \frac{g_{*5}^{\mathrm{N^3LO}}}{4} + \frac{29g_{*7}^{\mathrm{N^3LO}}}{56} - \frac{g_{*8}^{\mathrm{N^3LO}}}{8} + \frac{3\nu^3}{4} + \frac{3\nu^2}{2} + \frac{3\nu}{8} - \frac{5}{8},$$

$$g_{*6}^{\mathrm{N^3LO}} \to \frac{2g_{*2}^{\mathrm{N^2LO}}}{3} - \frac{7g_{*2}^{\mathrm{N^3LO}}}{15} + \frac{2g_{*2}^{\mathrm{NLO}}\nu}{5} + \frac{17g_{*2}^{\mathrm{NLO}}}{30} + \frac{63g_{*4}^{\mathrm{N^2LO}}}{200} - \frac{3g_{*4}^{\mathrm{N^3LO}}}{5} + \frac{g_{*5}^{\mathrm{N^2LO}}}{10} + \frac{7g_{*5}^{\mathrm{N^3LO}}}{20} - \frac{5g_{*7}^{\mathrm{N^3LO}}}{8}$$
$$+ \frac{11g_{*8}^{\mathrm{N^3LO}}}{40} - \frac{g_{*9}^{\mathrm{N^3LO}}}{5} + \frac{3\nu^3}{8} + \frac{213\nu^2}{20} - \frac{21\nu}{4} + \frac{51}{80},$$

$$g_{*10}^{\mathrm{N^3LO}} \to -\frac{g_{*2}^{\mathrm{N^2LO}}}{2} + \frac{2g_{*2}^{\mathrm{N^3LO}}}{15} - \frac{7g_{*2}^{\mathrm{NLO}}\nu}{5} + \frac{101g_{*2}^{\mathrm{NLO}}}{60} - \frac{21g_{*4}^{\mathrm{N^2LO}}}{25} + \frac{g_{*4}^{\mathrm{N^3LO}}}{5} + \frac{13g_{*5}^{\mathrm{N^2LO}}}{20} - \frac{g_{*5}^{\mathrm{N^3LO}}}{10} + \frac{g_{*7}^{\mathrm{N^3LO}}}{4}$$
$$- \frac{3g_{*8}^{\mathrm{N^3LO}}}{20} + \frac{g_{*9}^{\mathrm{N^3LO}}}{5} - \frac{3\nu^3}{4} - \frac{201\nu^2}{40} + \frac{41\pi^2\nu}{32} - \frac{701\nu}{24} - \frac{121}{80}$$

Fixing the spin gauge: DJS gauge

(G_S, G_{S_*}) in the **DJS gauge** are obtained by setting to zero each term proportional to p_φ in the corresponding gauge-unfixed expressions

All the remaining coefficients are fixed and we find:

$$\begin{aligned}\frac{G_S^{\mathrm{DJS}}}{u^3} &= 2 - \frac{1}{c^2}\left(\frac{27\nu}{8}p_r^2 + \frac{5\nu}{8}u\right) + \frac{1}{c^4}\left[\left(\frac{5\nu}{16} + \frac{35\nu^2}{16}\right)p_r^4 + \left(-\frac{21\nu}{4} + \frac{23\nu^2}{16}\right)p_r^2 u + \left(-\frac{51\nu}{8} - \frac{\nu^2}{16}\right)u^2\right] \\ &+ \frac{1}{c^6}\left\{\left(\frac{7\nu}{128} - \frac{63\nu^2}{32} - \frac{665\nu^3}{128}\right)p_r^6 + \left(\frac{781\nu}{128} + \frac{831\nu^2}{32} - \frac{771\nu^3}{128}\right)p_r^4 u + \left(-\frac{5283\nu}{64} + \frac{1557\nu^2}{16} + \frac{69\nu^3}{64}\right)p_r^2 u^2\right. \\ &\left. + \left[2\left(-\frac{80399}{2304} + \frac{241\pi^2}{384}\right)\nu + \frac{379\nu^2}{32} - \frac{7\nu^3}{128}\right]u^3\right\},\end{aligned}$$

$$\begin{aligned}\frac{G_{S_*}^{\mathrm{DJS}}}{u^3} &= 3/2 - \frac{1}{c^2}\left[\left(\frac{15}{8} + \frac{9\nu}{4}\right)p_r^2 + \left(\frac{9}{8} + \frac{3\nu}{4}\right)u\right] + \frac{1}{c^4}\left[\left(\frac{35}{16} + \frac{5\nu}{2} + \frac{45\nu^2}{16}\right)p_r^4 + \left(\frac{69}{16} - \frac{9\nu}{4} + \frac{57\nu^2}{16}\right)p_r^2 u\right. \\ &\left. + \left(-\frac{27}{16} - \frac{39\nu}{4} - \frac{3\nu^2}{16}\right)u^2\right] + \frac{1}{c^6}\left\{\left(-\frac{315}{128} - \frac{175\nu}{64} - \frac{231\nu^2}{64} - \frac{105\nu^3}{32}\right)p_r^6\right. \\ &+ \left(-\frac{1105}{128} - \frac{53\nu}{64} + \frac{351\nu^2}{64} - \frac{243\nu^3}{32}\right)p_r^4 u + \left(-\frac{45}{64} - \frac{837\nu}{32} + \frac{2361\nu^2}{32} + \frac{27\nu^3}{16}\right)p_r^2 u^2 \\ &\left. + \left[-\frac{405}{128} + \left(-\frac{7627}{192} + \frac{41\pi^2}{32}\right)\nu + \frac{711\nu^2}{64} - \frac{3\nu^3}{32}\right]u^3\right\},\end{aligned}$$

Corresponds to the result of [Antonelli et al. 2020]

Fixing the spin gauge: SP gauge

We now want to compute (G_S, G_{S_*}) in the a spin gauge where also G_{S_*} reduces to its spinning particle analog in the limit $\nu \to 0$, at each PN order

This condition turns out to be equivalent to removing all the ν-dependent terms proportional to p_r and defines a new spin gauge, which we dub **SP gauge**

In this gauge we find:

$$\frac{G_S^{\rm SP}}{u^3} = 2 + \frac{1}{c^2}\left(-\frac{9\nu}{8}p^2 + \frac{\nu}{2}u\right) + \frac{1}{c^4}\left[\left(\frac{\nu}{8} + \frac{7\nu^2}{8}\right)p^4 + \left(-\frac{33\nu}{16} - \frac{5\nu^2}{4}\right)p^2 u + \left(-\frac{119\nu}{16} + \frac{\nu^2}{4}\right)u^2\right]$$
$$+ \frac{1}{c^6}\left\{\left(\frac{\nu}{128} - \frac{9\nu^2}{32} - \frac{95\nu^3}{128}\right)p^6 + \left(\frac{\nu}{2} + \frac{21\nu^2}{8} + \frac{31\nu^3}{16}\right)p^4 u + \left(-\frac{1231\nu}{80} + \frac{431\nu^2}{40} - \frac{11\nu^3}{8}\right)p^2 u^2\right.$$
$$\left. + \left(-\frac{28331\nu}{720} - \frac{123\nu^2}{20} + \frac{\nu^3}{8} + \frac{241\nu\pi^2}{192}\right)u^3\right\},$$

$$\frac{G_{S_*}^{\rm SP}}{u^3} = 3/2 - \frac{1}{c^2}\left[\left(\frac{5}{8} + \frac{3\nu}{4}\right)p^2 + \frac{u}{2}\right] + \frac{1}{c^4}\left[\left(\frac{7}{16} + \frac{\nu}{2} + \frac{9\nu^2}{16}\right)p^4 + \frac{5p_r^2 u}{4} + \left(\frac{1}{4} - \frac{9\nu}{8} - \frac{3\nu^2}{8}\right)p^2 u\right.$$
$$\left. + \left(-\frac{1}{2} - \frac{55\nu}{8} - \frac{3\nu^2}{8}\right)u^2\right] + \frac{1}{c^6}\left\{\left(-\frac{45}{128} - \frac{25\nu}{64} - \frac{33\nu^2}{64} - \frac{15\nu^3}{32}\right)p^6 + \left(-\frac{3}{16} + \frac{3\nu}{8} + \frac{3\nu^2}{2} + \frac{3\nu^3}{4}\right)p^4 u\right.$$
$$\left. - \frac{7}{4}p^2 p_r^2 u + \left(\frac{1}{4} - \frac{21\nu}{4} + \frac{213\nu^2}{20} + \frac{3\nu^3}{8}\right)p^2 u^2 - \frac{p_r^2 u^2}{2} - \left[\frac{5}{8} + \nu\left(\frac{701}{24} - \frac{41\pi^2}{32}\right) + \frac{201\nu^2}{40} + \frac{3\nu^3}{4}\right]u^3\right\},$$

Checking the computation: binding energy

We restrict to **circular orbits**:

$$E_b(x) \equiv \frac{1}{\nu}\left[\sqrt{1 + 2\nu\left(\hat{H}^{\rm circ}_{\rm eff}(x) - 1\right)} - 1\right], \qquad x \equiv \left(\frac{GM\Omega}{c^2}\right)^{2/3}$$

$\hat{H}^{\rm circ}_{\rm eff}(x)$ is obtained from $\hat{H}_{\rm eff}(u, p_r, p_\varphi)$ by taking the limit $p_r \to 0$ and replacing p_φ and u with their circular expansion in terms of x

Repeating this computation with (G_S, G_{S_*}) in DJS gauge, SP gauge, and even gauge-unfixed form, the SO part of E_b is the same and reads:

$$\begin{aligned} E_b = & -x^{5/2}\left(\frac{4}{3}\hat{S} + \hat{S}_*\right) + x^{7/2}\left[\left(-4 + \frac{31\nu}{18}\right)\hat{S} + \left(-\frac{3}{2} + \frac{5\nu}{3}\right)\hat{S}_*\right] + x^{9/2}\left[\left(-\frac{27}{2} + \frac{211\nu}{8} - \frac{7\nu^2}{12}\right)\hat{S}\right. \\ & \left. + \left(-\frac{27}{8} + \frac{39\nu}{2} - \frac{5\nu^2}{8}\right)\hat{S}_*\right] + x^{11/2}\left[\left(-45 + \frac{19679\nu}{144} + \frac{29\pi^2\nu}{24} - \frac{1979\nu^2}{36} - \frac{265\nu^3}{3888}\right)\hat{S}\right. \\ & \left. + \left(-\frac{135}{16} + \frac{565\nu}{8} - \frac{1109\nu^2}{24} - \frac{25\nu^3}{324}\right)\hat{S}_*\right] \end{aligned}$$

Corresponds to the result of [Antonelli et al. 2020]

Checking the computation: periastron advance

Fractional advance of the periastron per radial period in the **quasi-circular limit**:

$$\frac{\Delta\Phi}{2\pi} = K - 1, \qquad K \equiv \frac{\Omega_\varphi}{\Omega_r}\bigg|_{p_r\to 0} = \left(\frac{\partial^2 \hat{H}_{\text{eff}}}{\partial r^2}\frac{\partial^2 \hat{H}_{\text{eff}}}{\partial p_r^2}\right)^{-1}\frac{\partial \hat{H}_{\text{eff}}}{\partial p_\varphi}\Bigg|_{p_r\to 0}$$

Again, expressing p_φ and u in x we find the same result regardless of the spin gauge used for (G_S, G_{S_*}):

Gives back the 3PN non-spinning result of [Le Tiec et al. 2011]

$$\begin{aligned}
K = {} & \boxed{1 + 3x + x^2\Big(\frac{27}{2} - 7\nu\Big) + x^3\Big[\frac{135}{2} + \Big(-\frac{649}{4} + \frac{123\pi^2}{32}\Big)\nu + 7\nu^2\Big]} + x^4\Big[\frac{2835}{8} + \nu\Big(-\frac{275941}{360} - \frac{2512\gamma}{15} \\
& + \frac{48007\pi^2}{3072} - \frac{592\log 2}{15} - \frac{1458\log 3}{5} - \frac{1256\log x}{15}\Big) + \Big(\frac{5861}{12} - \frac{451\pi^2}{32}\Big)\nu^2 - \frac{98\nu^3}{27}\Big] \\
& + x^{3/2}\Big(-4\hat{S} - 3\hat{S}_*\Big) + x^{5/2}\Big[\Big(-34 + \frac{17\nu}{2}\Big)\hat{S} + \Big(-18 + \frac{15\nu}{2}\Big)\hat{S}_*\Big] + x^{7/2}\Big[\Big(-252 + \frac{5317\nu}{24} - \frac{22\nu^2}{3}\Big)\hat{S} \\
& + \Big(-\frac{243}{2} + \frac{1313\nu}{8} - 7\nu^2\Big)\hat{S}_*\Big] + x^{9/2}\Big\{\Big[-1755 + \Big(\frac{504173}{144} - \frac{3655\pi^2}{96}\Big)\nu - \frac{4419\nu^2}{8} + 3\nu^3\Big]\hat{S} \\
& + \Big[-810 + \Big(\frac{111401}{48} - \frac{533\pi^2}{16}\Big)\nu - \frac{3661\nu^2}{8} + 3\nu^3\Big]\hat{S}_*\Big\}
\end{aligned}$$

Factorization and final results

$$\boxed{G_S = G_S^0\ \hat{G}_S}$$

$$\boxed{G_{S_*} = G_{S_*}^0 \hat{G}_{S_*}}$$

In the **SP gauge** the prefactors are:

$$G_S^0 = 2uu_c^2, \qquad G_{S_*}^0 = u_c^2\left\{\frac{\sqrt{A}}{\sqrt{Q}}\left[1 - \frac{u_c{}'\sqrt{A}}{u_c^2\sqrt{D}}\right] + \frac{(A)'}{2u_c\left(1+\sqrt{Q}\right)\sqrt{D}}\right\}$$

$$Q = 1 + p_\varphi^2 u_c^2 + \frac{A}{D}p_r^2$$

Resulting inverse-resummed PN residuals:

$$\begin{aligned}\hat{G}_S^{-1} = 1 &+ \left(\frac{9\nu}{16}p^2 - \frac{\nu}{4}u_c\right) + \frac{1}{c^4}\left[\left(-\frac{\nu}{16} - \frac{31\nu^2}{256}\right)p^4 + \left(\frac{33\nu}{32} + \frac{11\nu^2}{32}\right)p^2u_c + \left(\frac{119\nu}{32} - \frac{\nu^2}{16}\right)u_c^2\right] \\ &+ \frac{1}{c^6}\left\{\left(-\frac{\nu}{256} + \frac{9\nu^2}{128} + \frac{233\nu^3}{4096}\right)p^6 + \left(-\frac{\nu}{4} - \frac{31\nu^2}{256} - \frac{291\nu^3}{1024}\right)p^4u_c + \left(\frac{1231\nu}{160} - \frac{2201\nu^2}{1280} + \frac{87\nu^3}{256}\right)p^2u_c^2\right. \\ &+ \left.\left[\nu\left(\frac{28331}{1440} - \frac{241\pi^2}{384}\right) + \frac{389\nu^2}{320} - \frac{\nu^3}{64}\right]u_c^3\right\},\end{aligned}$$

$$\begin{aligned}\hat{G}_{S_*}^{-1} = 1 &+ \left(\frac{\nu}{2}p^2 - 2\nu u_c\right) + \frac{1}{c^4}\left[\left(-\frac{\nu}{8} - \frac{\nu^2}{8}\right)p^4 + \left(\frac{13\nu}{12} - \frac{3\nu^2}{4}\right)p^2u_c + \left(-\frac{121\nu}{12} + \frac{9\nu^2}{4}\right)u_c^2\right] \\ &+ \frac{1}{c^6}\left\{\left(\frac{\nu}{16} + \frac{\nu^2}{16} + \frac{\nu^3}{16}\right)p^6 + \left(-\frac{13\nu}{144} + \frac{11\nu^2}{48}\right)p^4u_c - \frac{5\nu}{12}p^2p_r^2u_c + \left(\frac{1031\nu}{144} - \frac{933\nu^2}{80} + \frac{\nu^3}{2}\right)p^2u_c^2 - \frac{17\nu}{6}p_r^2u_c^2\right. \\ &+ \left.\left[\nu^2\left(\frac{398}{5} - \frac{41\pi^2}{16}\right) + \nu\left(\frac{4328}{135} - \frac{1184\gamma}{45} + \frac{25729\pi^2}{4608} + \frac{6496\log 2}{45} - \frac{972\log 3}{5} - \frac{592\log u_c}{45}\right)\right]u_c^3\right\}\end{aligned}$$

BONUS SLIDES

$\chi_{\text{eff}}^{\text{SO,S}}(\varepsilon, p_\varphi, g_n^{\text{N}^m\text{LO}})$

χSoS

invj $\equiv 1/\text{p}_\varphi$

$$\text{invj}\,\Big(-4\sqrt{\epsilon}+\epsilon^{3/2}\Big(-2\,\text{g1NLO}-\frac{2\,\text{g2NLO}}{3}-\frac{9\,\nu}{4}\Big)+\epsilon^{5/2}\Big(-2\,\text{g1N2LO}-\frac{2\,\text{g2N2LO}}{3}-\frac{2\,\text{g4N2LO}}{5}+\Big(\frac{1}{4}-\text{g1NLO}-\frac{\text{g2NLO}}{3}\Big)\nu+\frac{5\,\nu^2}{8}\Big)+$$

$$\epsilon^{7/2}\Big(-2\,\text{g1N3LO}-\frac{2\,\text{g2N3LO}}{3}-\frac{2\,\text{g4N3LO}}{5}-\frac{2\,\text{g7N3LO}}{7}+\Big(\frac{1}{64}-\text{g1N2LO}+\frac{\text{g1NLO}}{4}-\frac{\text{g2N2LO}}{3}+\frac{\text{g2NLO}}{12}-\frac{\text{g4N2LO}}{5}\Big)\nu+\Big(-\frac{5}{32}+\frac{\text{g1NLO}}{4}+\frac{\text{g2NLO}}{12}\Big)\nu^2-\frac{21\,\nu^3}{64}\Big)\Big)+$$

$$\text{invj}^2\,\Big(-4\,\pi+\epsilon\,(-10\,\pi-4\,\text{g1NLO}\,\pi-\text{g2NLO}\,\pi-\text{g3NLO}\,\pi-4\,\pi\,\nu)+$$

$$\epsilon^2\Big(-6\,\text{g1N2LO}\,\pi-\frac{15\,\text{g1NLO}\,\pi}{2}-\frac{3\,\text{g2N2LO}\,\pi}{2}-\frac{9\,\text{g2NLO}\,\pi}{4}-\text{g3N2LO}\,\pi-\frac{3\,\text{g4N2LO}\,\pi}{4}-\frac{\text{g5N2LO}\,\pi}{4}+\Big(-\frac{1}{4}\,(39\,\pi)-4\,\text{g1NLO}\,\pi-\text{g2NLO}\,\pi-\text{g3NLO}\,\pi\Big)\nu\Big)+$$

$$\epsilon^3\Big(-10\,\text{g1N2LO}\,\pi-8\,\text{g1N3LO}\,\pi-3\,\text{g2N2LO}\,\pi-2\,\text{g2N3LO}\,\pi-\text{g3N3LO}\,\pi-\frac{13\,\text{g4N2LO}\,\pi}{8}-\text{g4N3LO}\,\pi-\frac{\text{g5N3LO}\,\pi}{4}-\frac{5\,\text{g7N3LO}\,\pi}{8}-$$

$$\frac{\text{g8N3LO}\,\pi}{8}+\Big(\frac{29\,\pi}{16}-6\,\text{g1N2LO}\,\pi-\frac{13\,\text{g1NLO}\,\pi}{2}-\frac{3\,\text{g2N2LO}\,\pi}{2}-2\,\text{g2NLO}\,\pi-\text{g3N2LO}\,\pi+\frac{\text{g3NLO}\,\pi}{4}-\frac{3\,\text{g4N2LO}\,\pi}{4}-\frac{\text{g5N2LO}\,\pi}{4}\Big)\nu+\frac{3\,\pi\,\nu^2}{8}\Big)\Big)+$$

$$\text{invj}^3\,\Big(-\frac{12}{\sqrt{\epsilon}}+\sqrt{\epsilon}\,\Big(-144-30\,\text{g1NLO}-6\,\text{g2NLO}-12\,\text{g3NLO}-\frac{71\,\nu}{4}\Big)+\epsilon^{3/2}\Big(-192-70\,\text{g1N2LO}-168\,\text{g1NLO}-14\,\text{g2N2LO}-40\,\text{g2NLO}-$$

$$20\,\text{g3N2LO}-24\,\text{g3NLO}-6\,\text{g4N2LO}-4\,\text{g5N2LO}-4\,\text{g6N2LO}+\Big(-\frac{763}{4}-45\,\text{g1NLO}-9\,\text{g2NLO}-18\,\text{g3NLO}\Big)\nu-\frac{35\,\nu^2}{8}\Big)+$$

$$\epsilon^{5/2}\Big(-\frac{1}{5}\,(1512\,\text{g1N2LO})-126\,\text{g1N3LO}-176\,\text{g1NLO}-72\,\text{g2N2LO}-\frac{126\,\text{g2N3LO}}{5}-48\,\text{g2NLO}-\frac{168\,\text{g3N2LO}}{5}-28\,\text{g3N3LO}-\frac{168\,\text{g4N2LO}}{5}-\frac{54\,\text{g4N3LO}}{5}-$$

$$8\,\text{g5N2LO}-\frac{28\,\text{g5N3LO}}{5}-4\,\text{g6N3LO}-6\,\text{g7N3LO}-\frac{12\,\text{g8N3LO}}{5}-\frac{4\,\text{g9N3LO}}{5}+\Big(-\frac{74\,181}{320}-105\,\text{g1N2LO}-\frac{4479\,\text{g1NLO}}{20}-21\,\text{g2N2LO}-$$

$$\frac{1011\,\text{g2NLO}}{20}-30\,\text{g3N2LO}-\frac{63\,\text{g3NLO}}{2}-9\,\text{g4N2LO}-6\,\text{g5N2LO}-6\,\text{g6N2LO}\Big)\nu+\Big(-\frac{4129}{160}-\frac{45\,\text{g1NLO}}{4}-\frac{9\,\text{g2NLO}}{4}-\frac{9\,\text{g3NLO}}{2}\Big)\nu^2+\frac{45\,\nu^3}{64}\Big)\Big)+$$

$$\text{invj}^4\,\Big(-84\,\pi-12\,\text{g1NLO}\,\pi-2\,\text{g2NLO}\,\pi-6\,\text{g3NLO}\,\pi+\epsilon\,(-378\,\pi-48\,\text{g1N2LO}\,\pi-168\,\text{g1NLO}\,\pi-8\,\text{g2N2LO}\,\pi-33\,\text{g2NLO}\,\pi-18\,\text{g3N2LO}\,\pi-$$

$$42\,\text{g3NLO}\,\pi-3\,\text{g4N2LO}\,\pi-3\,\text{g5N2LO}\,\pi-6\,\text{g6N2LO}\,\pi+(-120\,\pi-24\,\text{g1NLO}\,\pi-4\,\text{g2NLO}\,\pi-12\,\text{g3NLO}\,\pi)\,\nu)+$$

$$\epsilon^2\Big(-\frac{1}{2}\,(693\,\pi)-\frac{3\,\text{g10N3LO}\,\pi}{2}-420\,\text{g1N2LO}\,\pi-120\,\text{g1N3LO}\,\pi-\frac{957\,\text{g1NLO}\,\pi}{2}-\frac{165\,\text{g2N2LO}\,\pi}{2}-20\,\text{g2N3LO}\,\pi-\frac{429\,\text{g2NLO}\,\pi}{4}-84\,\text{g3N2LO}\,\pi-36\,\text{g3N3LO}\,\pi-$$

$$\frac{189\,\text{g3NLO}\,\pi}{4}-\frac{135\,\text{g4N2LO}\,\pi}{4}-\frac{15\,\text{g4N3LO}\,\pi}{2}-\frac{33\,\text{g5N2LO}\,\pi}{2}-6\,\text{g5N3LO}\,\pi-\frac{21\,\text{g6N2LO}\,\pi}{2}-9\,\text{g6N3LO}\,\pi-\frac{15\,\text{g7N3LO}\,\pi}{4}-\frac{9\,\text{g8N3LO}\,\pi}{4}-\frac{3\,\text{g9N3LO}\,\pi}{2}+$$

$$\Big(-\frac{1}{8}\,(4233\,\pi)-96\,\text{g1N2LO}\,\pi-285\,\text{g1NLO}\,\pi-16\,\text{g2N2LO}\,\pi-50\,\text{g2NLO}\,\pi-36\,\text{g3N2LO}\,\pi-\frac{153\,\text{g3NLO}\,\pi}{2}-6\,\text{g4N2LO}\,\pi-6\,\text{g5N2LO}\,\pi-12\,\text{g6N2LO}\,\pi\Big)\nu+$$

$$\Big(-\frac{1}{4}\,(81\,\pi)-12\,\text{g1NLO}\,\pi-2\,\text{g2NLO}\,\pi-6\,\text{g3NLO}\,\pi\Big)\nu^2\Big)\Big)$$

$\chi_{\text{eff}}^{\text{SO},S_*}(\varepsilon, p_\varphi, g_{*n}^{\text{N}^m\text{LO}})$

χSoSstar

invj ≡ $1/\text{p}_\varphi$

$$\text{invj}\,\Big(-4\sqrt{\epsilon}+\epsilon^{3/2}\Big(-\frac{5}{4}-2\,\text{gStar1NLO}-\frac{2\,\text{gStar2NLO}}{3}-\frac{3\,\nu}{2}\Big)+$$

$$\epsilon^{5/2}\Big(\frac{7}{8}-2\,\text{gStar1N2LO}-\frac{2\,\text{gStar2N2LO}}{3}-\frac{2\,\text{gStar4N2LO}}{5}+\Big(\frac{3}{8}-\text{gStar1NLO}-\frac{\text{gStar2NLO}}{3}\Big)\nu+\frac{3\,\nu^2}{8}\Big)+\epsilon^{7/2}\Big(-\frac{45}{64}-2\,\text{gStar1N3LO}-\frac{2\,\text{gStar2N3LO}}{3}-\frac{2\,\text{gStar4N3LO}}{5}-$$

$$\frac{2\,\text{gStar7N3LO}}{7}+\Big(-\frac{3}{16}-\text{gStar1N2LO}+\frac{\text{gStar1NLO}}{4}-\frac{\text{gStar2N2LO}}{3}+\frac{\text{gStar2NLO}}{12}-\frac{\text{gStar4N2LO}}{5}\Big)\nu+\Big(-\frac{3}{16}+\frac{\text{gStar1NLO}}{4}+\frac{\text{gStar2NLO}}{12}\Big)\nu^2-\frac{3\,\nu^3}{16}\Big)\Big)+$$

$$\text{invj}^2\,\Big(-3\,\pi+\epsilon\,\Big(-\frac{1}{2}\,(21\,\pi)-4\,\text{gStar1NLO}\,\pi-\text{gStar2NLO}\,\pi-\text{gStar3NLO}\,\pi-3\,\pi\,\nu\Big)+\epsilon^2\,\Big(-\frac{1}{2}\,(3\,\pi)-6\,\text{gStar1N2LO}\,\pi-\frac{15\,\text{gStar1NLO}\,\pi}{2}-\frac{3\,\text{gStar2N2LO}\,\pi}{2}-$$

$$\frac{9\,\text{gStar2NLO}\,\pi}{4}-\text{gStar3N2LO}\,\pi-\frac{3\,\text{gStar4N2LO}\,\pi}{4}-\frac{\text{gStar5N2LO}\,\pi}{4}+\Big(-\frac{1}{4}\,(27\,\pi)-4\,\text{gStar1NLO}\,\pi-\text{gStar2NLO}\,\pi-\text{gStar3NLO}\,\pi\Big)\nu\Big)+$$

$$\epsilon^3\,\Big(\frac{15\,\pi}{16}-10\,\text{gStar1N2LO}\,\pi-8\,\text{gStar1N3LO}\,\pi-3\,\text{gStar2N2LO}\,\pi-2\,\text{gStar2N3LO}\,\pi-\text{gStar3N3LO}\,\pi-\frac{13\,\text{gStar4N2LO}\,\pi}{8}-\text{gStar4N3LO}\,\pi-\frac{\text{gStar5N3LO}\,\pi}{4}-\frac{5\,\text{gStar7N3LO}\,\pi}{8}-$$

$$\frac{\text{gStar8N3LO}\,\pi}{8}+\Big(\frac{3\,\pi}{2}-6\,\text{gStar1N2LO}\,\pi-\frac{13\,\text{gStar1NLO}\,\pi}{2}-\frac{3\,\text{gStar2N2LO}\,\pi}{2}-2\,\text{gStar2NLO}\,\pi-\text{gStar3N2LO}\,\pi+\frac{\text{gStar3NLO}\,\pi}{4}-\frac{3\,\text{gStar4N2LO}\,\pi}{4}-\frac{\text{gStar5N2LO}\,\pi}{4}\Big)\nu\Big)\Big)$$

$$\text{invj}^3\,\Big(-\frac{8}{\sqrt{\epsilon}}+\sqrt{\epsilon}\,\Big(-\frac{483}{4}-30\,\text{gStar1NLO}-6\,\text{gStar2NLO}-12\,\text{gStar3NLO}-\frac{25\,\nu}{2}\Big)+\epsilon^{3/2}\,\Big(-\frac{1651}{8}-70\,\text{gStar1N2LO}-168\,\text{gStar1NLO}-14\,\text{gStar2N2LO}-$$

$$40\,\text{gStar2NLO}-20\,\text{gStar3N2LO}-24\,\text{gStar3NLO}-6\,\text{gStar4N2LO}-4\,\text{gStar5N2LO}-4\,\text{gStar6N2LO}+\Big(-\frac{1037}{8}-45\,\text{gStar1NLO}-9\,\text{gStar2NLO}-18\,\text{gStar3NLO}\Big)\nu-\frac{27\,\nu^2}{8}\Big)+$$

$$\epsilon^{5/2}\,\Big(-\frac{1155}{64}-\frac{1512\,\text{gStar1N2LO}}{5}-126\,\text{gStar1N3LO}-176\,\text{gStar1NLO}-72\,\text{gStar2N2LO}-\frac{126\,\text{gStar2N3LO}}{5}-48\,\text{gStar2NLO}-\frac{168\,\text{gStar3N2LO}}{5}-$$

$$28\,\text{gStar3N3LO}-\frac{168\,\text{gStar4N2LO}}{5}-\frac{54\,\text{gStar4N3LO}}{5}-8\,\text{gStar5N2LO}-\frac{28\,\text{gStar5N3LO}}{5}-4\,\text{gStar6N3LO}-6\,\text{gStar7N3LO}-\frac{12\,\text{gStar8N3LO}}{5}-\frac{4\,\text{gStar9N3LO}}{5}+$$

$$\Big(-\frac{2637}{20}-105\,\text{gStar1N2LO}-\frac{4479\,\text{gStar1NLO}}{20}-21\,\text{gStar2N2LO}-\frac{1011\,\text{gStar2NLO}}{20}-30\,\text{gStar3N2LO}-\frac{63\,\text{gStar3NLO}}{2}-9\,\text{gStar4N2LO}-6\,\text{gStar5N2LO}-6\,\text{gStar6N2LO}\Big)$$

$$\nu+\Big(-\frac{1809}{80}-\frac{45\,\text{gStar1NLO}}{4}-\frac{9\,\text{gStar2NLO}}{4}-\frac{9\,\text{gStar3NLO}}{2}\Big)\nu^2+\frac{9\,\nu^3}{16}\Big)\Big)+$$

$$\text{invj}^4\,\Big(-63\,\pi-12\,\text{gStar1NLO}\,\pi-2\,\text{gStar2NLO}\,\pi-6\,\text{gStar3NLO}\,\pi+\epsilon\,(-336\,\pi-48\,\text{gStar1N2LO}\,\pi-168\,\text{gStar1NLO}\,\pi-8\,\text{gStar2N2LO}\,\pi-33\,\text{gStar2NLO}\,\pi-$$

$$18\,\text{gStar3N2LO}\,\pi-42\,\text{gStar3NLO}\,\pi-3\,\text{gStar4N2LO}\,\pi-3\,\text{gStar5N2LO}\,\pi-6\,\text{gStar6N2LO}\,\pi+(-75\,\pi-24\,\text{gStar1NLO}\,\pi-4\,\text{gStar2NLO}\,\pi-12\,\text{gStar3NLO}\,\pi)\,\nu)+$$

$$\epsilon^2\,\Big(-378\,\pi-\frac{3\,\text{gStar10N3LO}\,\pi}{2}-420\,\text{gStar1N2LO}\,\pi-120\,\text{gStar1N3LO}\,\pi-\frac{957\,\text{gStar1NLO}\,\pi}{2}-\frac{165\,\text{gStar2N2LO}\,\pi}{2}-20\,\text{gStar2N3LO}\,\pi-\frac{429\,\text{gStar2NLO}\,\pi}{4}-84\,\text{gStar3N2LO}\,\pi-$$

$$36\,\text{gStar3N3LO}\,\pi-\frac{189\,\text{gStar3NLO}\,\pi}{4}-\frac{135\,\text{gStar4N2LO}\,\pi}{4}-\frac{15\,\text{gStar4N3LO}\,\pi}{2}-\frac{33\,\text{gStar5N2LO}\,\pi}{2}-6\,\text{gStar5N3LO}\,\pi-\frac{21\,\text{gStar6N2LO}\,\pi}{2}-9\,\text{gStar6N3LO}\,\pi-$$

$$\frac{15\,\text{gStar7N3LO}\,\pi}{4}-\frac{9\,\text{gStar8N3LO}\,\pi}{4}-\frac{3\,\text{gStar9N3LO}\,\pi}{2}+\Big(-259\,\pi-96\,\text{gStar1N2LO}\,\pi-285\,\text{gStar1NLO}\,\pi-16\,\text{gStar2N2LO}\,\pi-50\,\text{gStar2NLO}\,\pi-36\,\text{gStar3N2LO}\,\pi-$$

$$\frac{153\,\text{gStar3NLO}\,\pi}{2}-6\,\text{gStar4N2LO}\,\pi-6\,\text{gStar5N2LO}\,\pi-12\,\text{gStar6N2LO}\,\pi-\frac{123\,\pi^3}{64}\Big)\nu+(-18\,\pi-12\,\text{gStar1NLO}\,\pi-2\,\text{gStar2NLO}\,\pi-6\,\text{gStar3NLO}\,\pi)\,\nu^2\Big)\Big)$$