KUNGL. VETENSKAPS-AKADEMIEN

THE ROYAL SWEDISH ACADEMY OF SCIENCES

Buns, doughnuts, pretzels, physics, and math….

© Kungl. Vetenskapsakademien

Nobelpriset i fysik 2016

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

With one half to

Photo: American Physical Society

David J. Thouless
University of Washington
Seattle, WA, USA

and the other half to

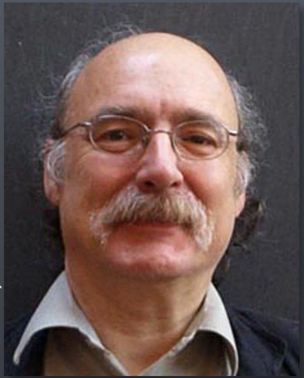


Photo: IOP Physics World

F. Duncan M. Haldane
Princeton University
NJ, USA

Photo: Brown University

J. Michael Kosterlitz
Brown University
Providence, RI, USA

“ for theoretical discoveries of topological phase transitions and topological phases of matter ”

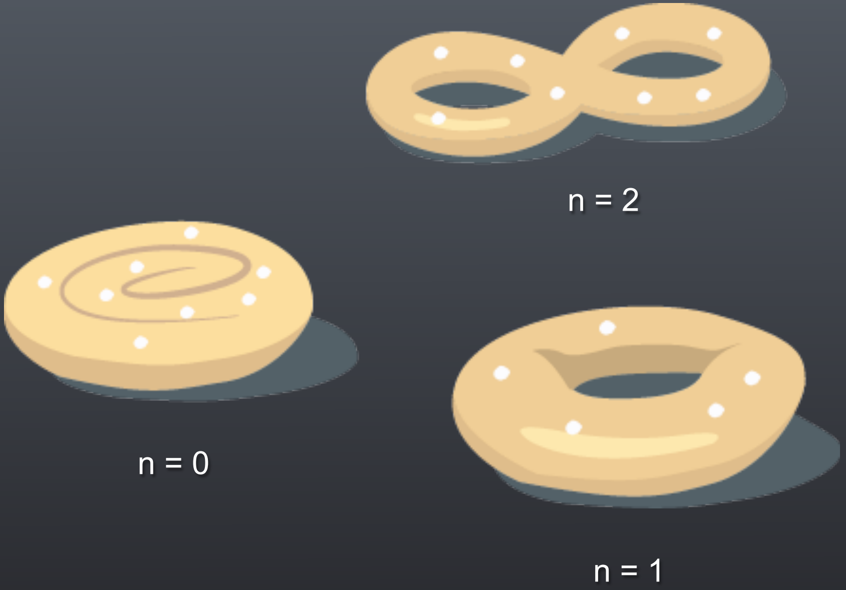
© Kungl. Vetenskapsakademien

Topology – what is it?

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Topologogy is a field of mathematics that describes properties that are stable and only change in integer steps: 1, 2, 3…

The number of holes is a topological invariant that is always an integer, but never anything in between.



© Kungl. Vetenskapsakademien

Topology – the physics connection

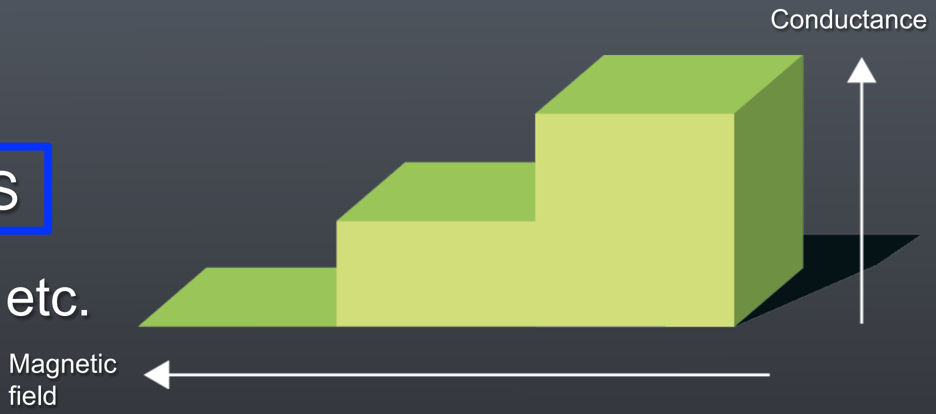
KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Electrons in a thin cold layer and in a strong magnetic field lead to

the Quantum Hall Effect (NP 1985)

$$\boxed{\text{Conductance} = n \times 3.874\,045\,8673\ \text{S}}$$

regardless of impurities, temperature, etc.



© Kungl. Vetenskapsakademien

Topology – the physics connection

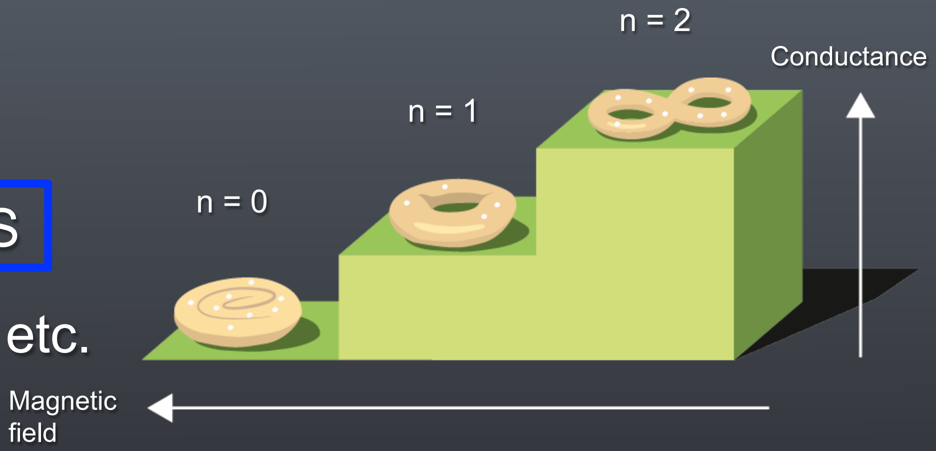
KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

Electrons in a thin cold layer and in a strong magnetic field lead to

the Quantum Hall Effect (NP 1985)

$\boxed{\text{Conductance} = n \times 3.874\ 045\ 8673\ \text{S}}$

regardless of impurities, temperature, etc.



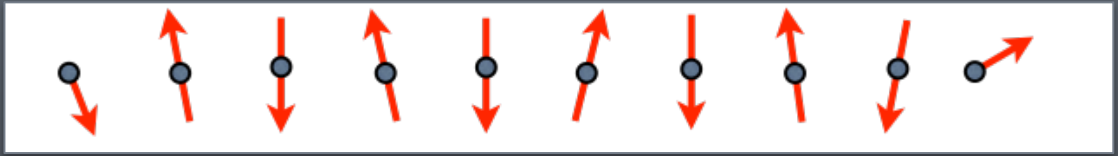
Thouless, et al., 1982: n is a topological invariant

Haldane, 1988: magnetic field is not necessary

© Kungl. Vetenskapsakademien

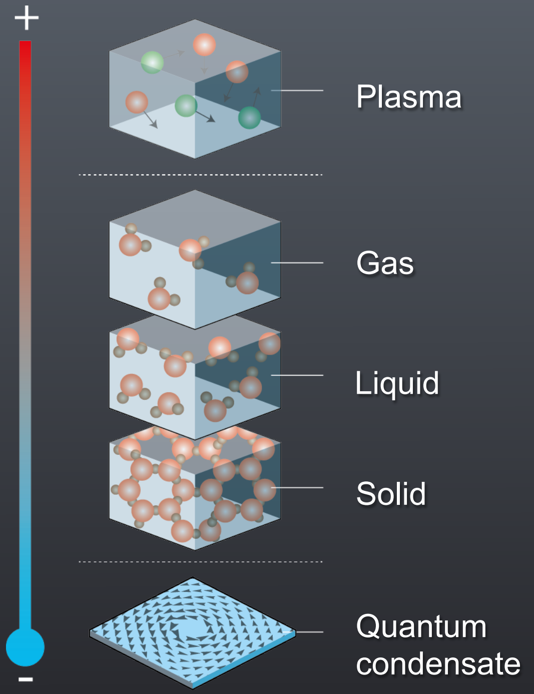
Exotic topological phases of matter

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES



Haldane, 1983: chains of atomic magnets can form a topological phase

The hunt is now on for other topological phases of matter in 1, 2 and 3 dimensions.



© Kungl. Vetenskapsakademien

A topological phase transition

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

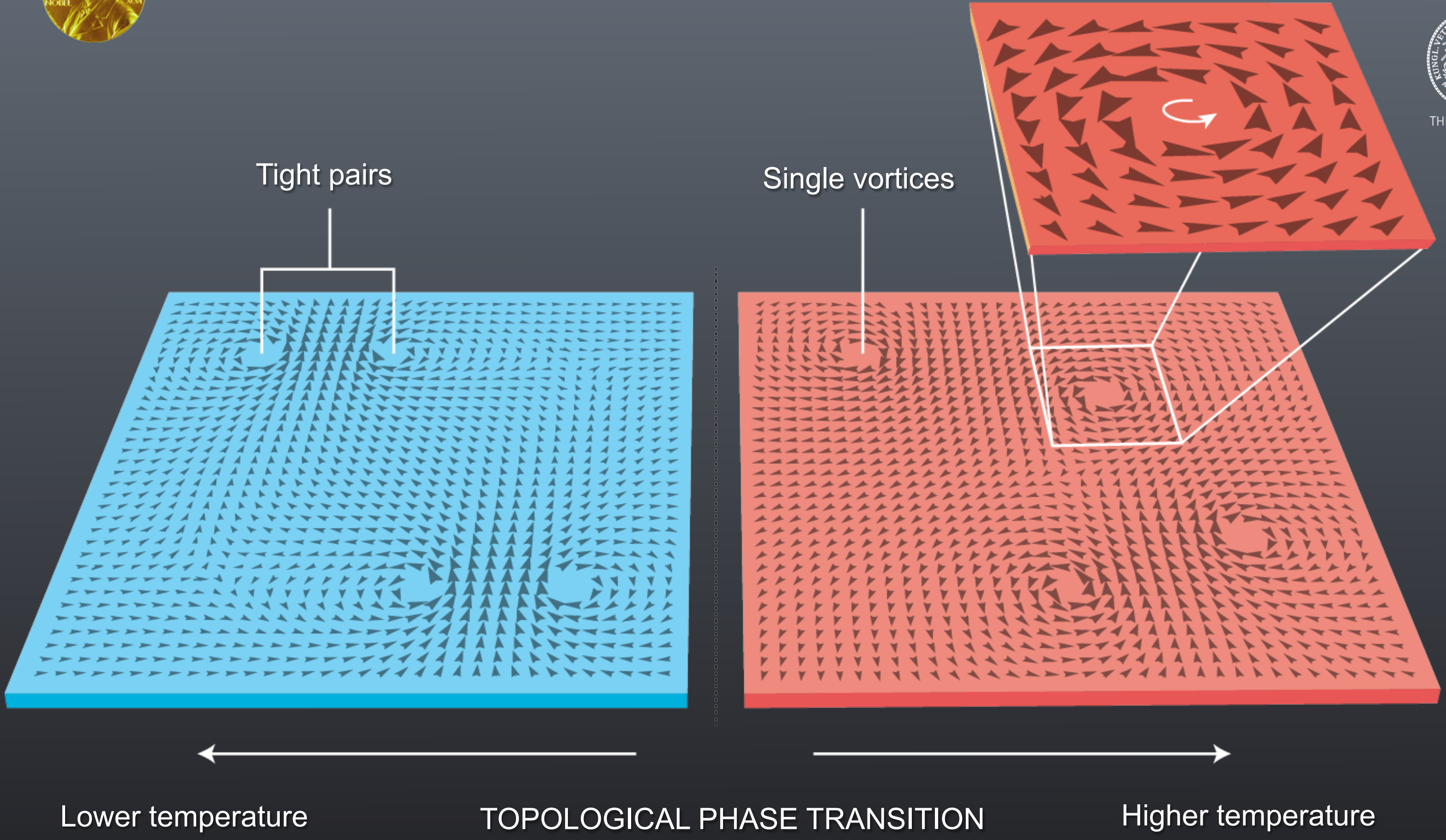
When temperature changes, one phase can turn into another – a phase transition.

Until the beginning of the 1970's it was believed that phase transitions could not occur in thin films of liquid helium.

© Kungl. Vetenskapsakademien


The Nobel Prize in Physics 2016
A LARGE VORTEX
and many small ones….
Firenze 20170328

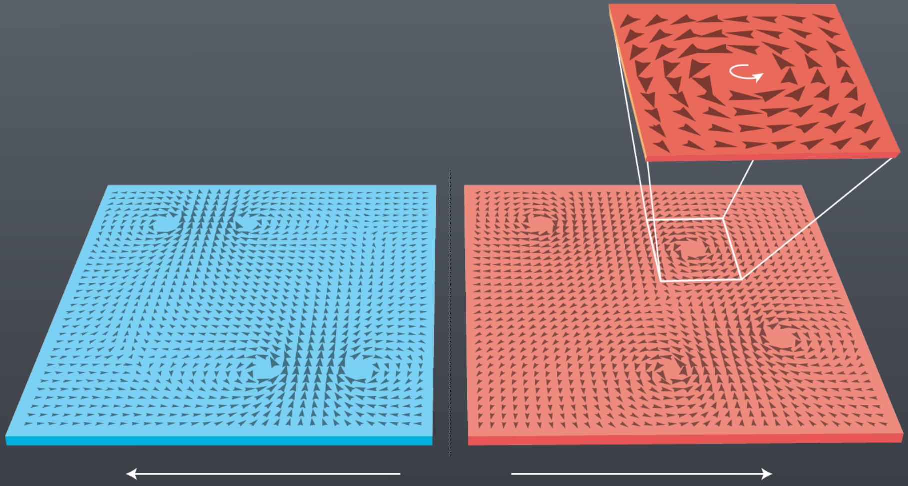
KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES



© Kungl. Vetenskapsakademien

A topological phase transition

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES



Kosterlitz & Thouless found a topological phase transition driven by vortices.

Kosterlitz later worked out a detailed mathematical theory.

The KT transition applies to many very different systems.

© Kungl. Vetenskapsakademien

Potential future benefits

KUNGL. VETENSKAPS-AKADEMIEN
THE ROYAL SWEDISH ACADEMY OF SCIENCES

This prize is for theoretical discoveries. They combined beautiful mathematics and profound insights into physics, achieving unexpected results that have been confirmed experimentally.

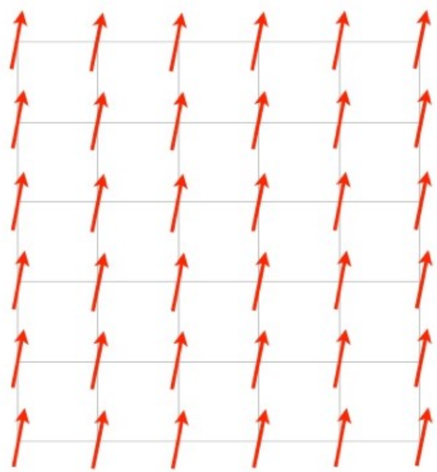
It has inspired a huge amount of international research. Scientists are hoping for practical applications in new electronics, new materials and even components for future quantum computers.

© Kungl. Vetenskapsakademien

Phases and Phase transitions

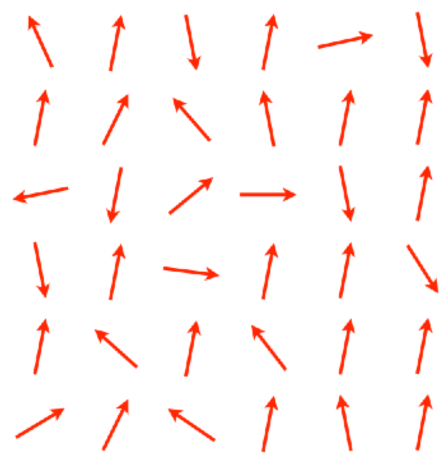
Ferromagnetism:

Order parameter: $\vec{m}_i = \mu \vec{S}_i$



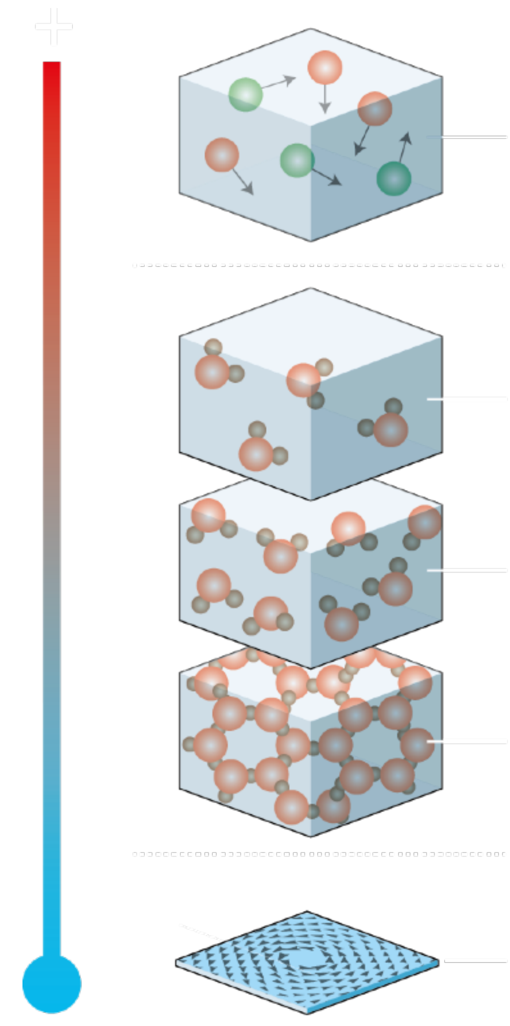
Ordered low temperature phase

$$\langle \vec{m}_i \rangle \neq 0$$



Disordered high temperature phase

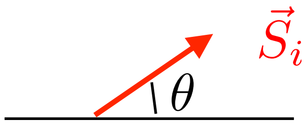
$$\langle \vec{m}_i \rangle = 0$$



The 2d XY - model

Describes: **"easy plane" magnets, superfluid films, superconducting films, melting of two-dimensional crystals, …..**

Two-dimensional spins: $\vec{S}_i = S(\cos\theta_i, \sin\theta_i)$



$$H_{XY} = -J\sum_{\langle ij\rangle} S_i \cdot S_j = -J\sum_{\langle ij\rangle}\cos(\theta_i - \theta_j) \qquad H_{XY} = \frac{J}{2}\int d^2r\,(\vec{\nabla}\theta(\vec{r}))^2$$

- nearest neighbours

Three-dimensional result:

$$\lim_{r\to\infty}\langle e^{i(\theta(\vec{r})-\theta(\vec{0}))}\rangle = \begin{cases} c_1 & T < T_c \\ c_2\, e^{-r/\xi} & T > T_c \end{cases}$$

Precisely at the critical temperature, T_c, the correlation function falls as a power signalling scale invariance and critical behavior.

Franz Wegner:

Spin-ordering in a planar classical Heisenberg model. Zeitschrift für Physik, 206(5):465–470, 1967.

Thermal fluctuations prevent phase transitions in the two-dimensional XY-model.

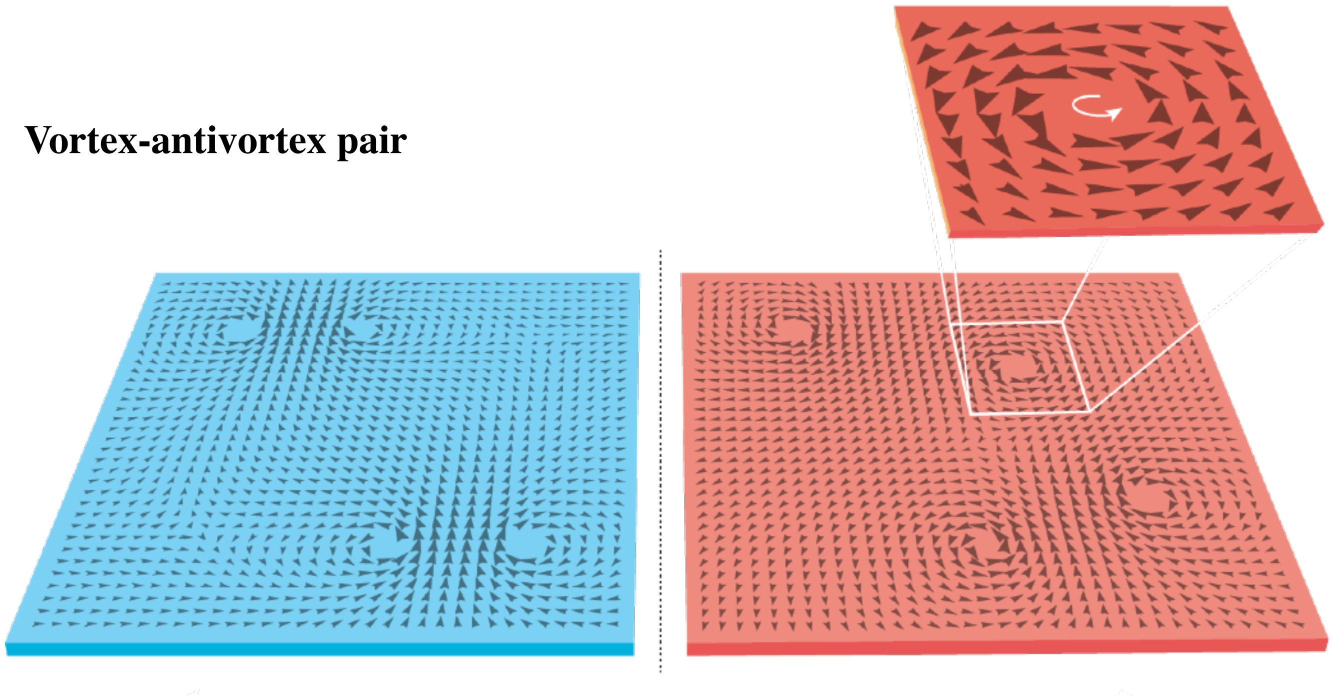
(A variant of the "Mermin-Wegner" theorem)

But a naive calculation shows that the fluctuating spin waves give rise to a power law behaviour in the correlation function for arbitrary T ! It is hard to believe that the system is critical at high temperature…

Kosterlitz och Thouless identified a completely new type of "topological phase transition" that does not correspond to any spontaneous breaking of a symmetry, and thus does not fit into the usual Landau classification of phase transitions.

The Kosterlitz-Thouless transition

Vortex-antivortex pair



Below critical T_{KT} Above critical T_{KT} **Free vortices and anti-vortices**

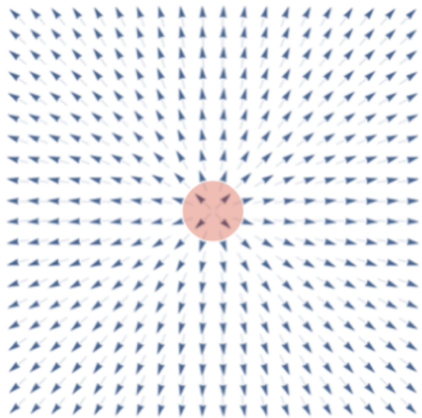
- **JM Kosterlitz and DJ Thouless,**
  Ordering, metastability and phase transitions in two-dimensional systems,
  Journal of Physics C: Solid State Physics, **6,** 1181, 1973.

- **JM Kosterlitz,** *The critical properties of the two-dimensional XY model,*
  Journal of Physics C: Solid State Physics, **7**, 1046, 1974.

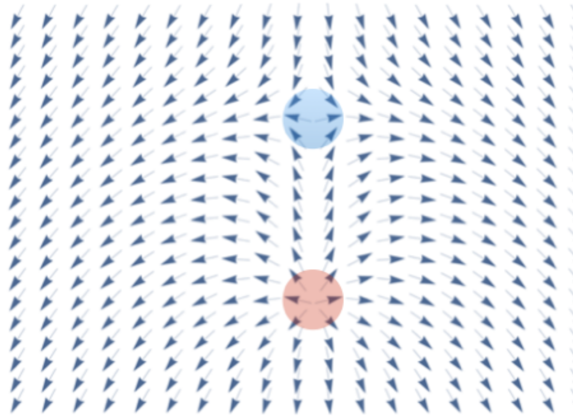
Kosterlitz' and Thouless' thermodynamical argument for a phase transition driven by "vortex liberation"

The direction of the arrows shows θ

single vortex



vortex - antivortex pair



$$E_v = J\pi \ln \frac{L}{a}$$

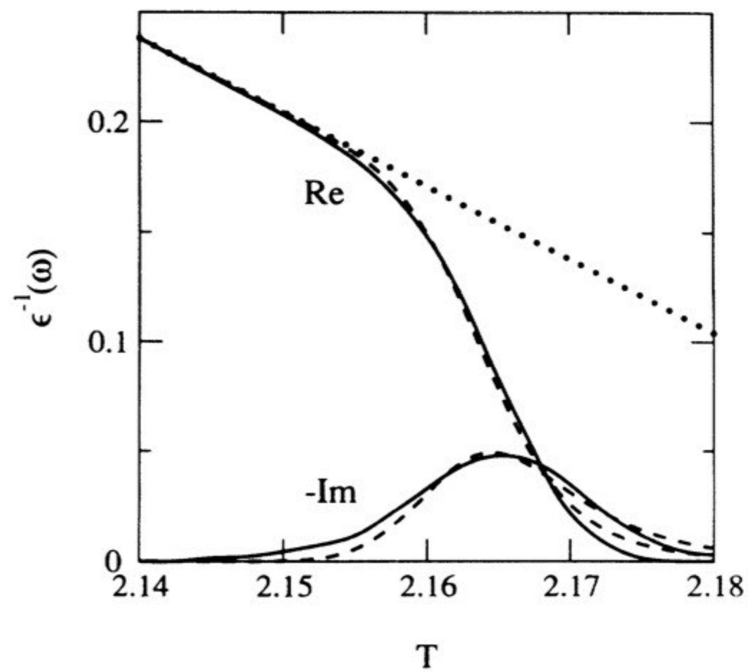
$$E_{pair} = J2\pi \ln \frac{r}{a}$$

Free energy for a single vortex

$$F = E - TS = J\pi \ln\left(\frac{L}{a}\right) - T\ln\left(\frac{L^2}{a^2}\right)$$

The entropy balances the energy for: $T_{KT} = \dfrac{J\pi}{2}$

One of many experiments where the KT-transition has been observed



Experimental evidence of the vortex-unbinding at the KT-transition can be found in e.g thin superconducting films. The figure shows the measurements of the ac-conductance in such a case . A narrow temperature range reveals a drop in the superfluid density ~ Re(ε(ω)) and a peak in the disssipation ~ Im(ε(ω)) caused by vortex-unbinding around the transition temperature. The data (full curves) are adapted from measurements by Jeanneret et al (PRB 89) [Wallin PRB89]

Something to remember:

From:

Early Work on Defect Driven Phase Transitions

J. Michael Kosterlitz, David J. Thouless

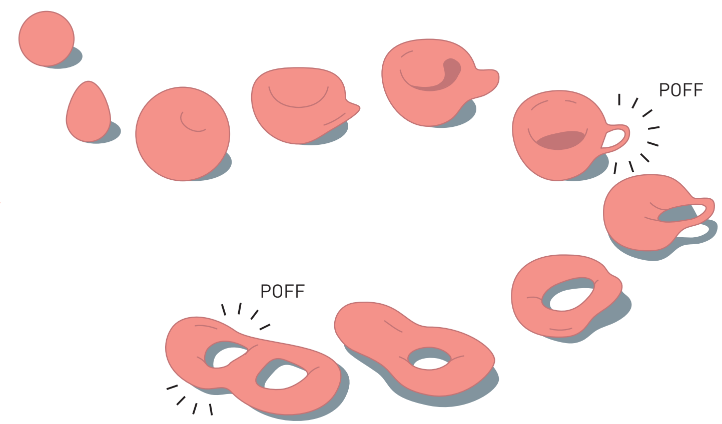
"There seem to have been nearly 2200 papers published since 1972 that mention our work in the title or abstract, only 3 of which were published in the first five years."

(This appears to have been written perhaps around year 2000; the article is in the collection of papers ”40 years of BKT transition”)

So, work hard, and be critical about your own work, but do not all the time measure yourself by what other people think.

Topological phases of matter

- Topologically ordered states
- Symmetry protected topological states
- Topological band theory



The number of holes is a topological invariant that can only be an integer 0, 1, 2, …
Never anything in between.

Using algebraic topology, one can express the topological invariants as integrals over manifolds.

The Quantum Hall effect

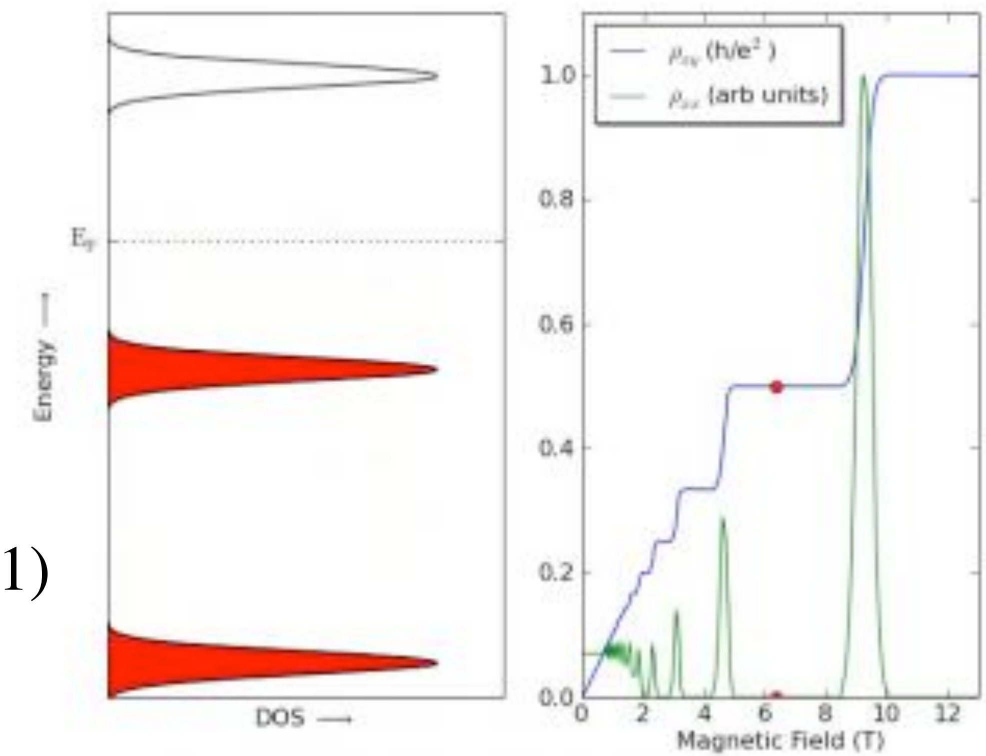
(von Klitzing, 80, NP 85)

- **2d elektron gas**
- **Clean samples, low temperature, high magnetic fields**
- **Quantized Hall conductance on the plateaux**
- **Longitudinal conductance = 0 on the plateaux**

Similar to a band insulator, but

Why is the quantization so exact??

- **Gauge invariance** (Laughlin -81)
- **Topology** (Thouless et.al. -82)



Thouless: The topology explains !

Using linear response theory, the Hall conductance can be related to the single particle wave functions in the filled bands:

$$\sigma_H = \frac{e^2}{2\pi h} \sum_n \int_{BZ} d^2k \, \mathcal{B}(\vec{k}, n)$$

DJ Thouless, M. Kohmoto, M.P Nightingale, and M. Den Nijs. PRL, **49**, 405, 1982.

$$\mathcal{B}(\vec{k}, n) = \partial_{k_x} \mathcal{A}_y(\vec{k}, n) - \partial_{k_y} \mathcal{A}_x(\vec{k}, n)$$

Berry field

$$\mathcal{A}_j(\vec{k}, n) = i \langle u_{\vec{k},n} | \partial_{k_j} | u_{\vec{k},n} \rangle$$

Berry potential

The Brillouinzone is a closed surface, so the integral over a magnetic field must, according to Dirac, be quantized!

$$\frac{1}{2\pi} \int_{Bz} \mathcal{B}(\vec{k}, n) = C_1(n)$$

is the first Chern number, which is a topological invariant

Anomalous QHE - the Chern insulator

Q: Can you have a Hall effect without a magnetic field?

A: Yes! The anomalous Hall effect. Early observations, theory much later (Karplus & Luttinger -54)

Q: Can you have a quantized anomalous Hall effect?

A: Yes! If the band have non-zero Chern number.

F.D.M. **Haldane**, PRL, **61**, 2015, 1988.

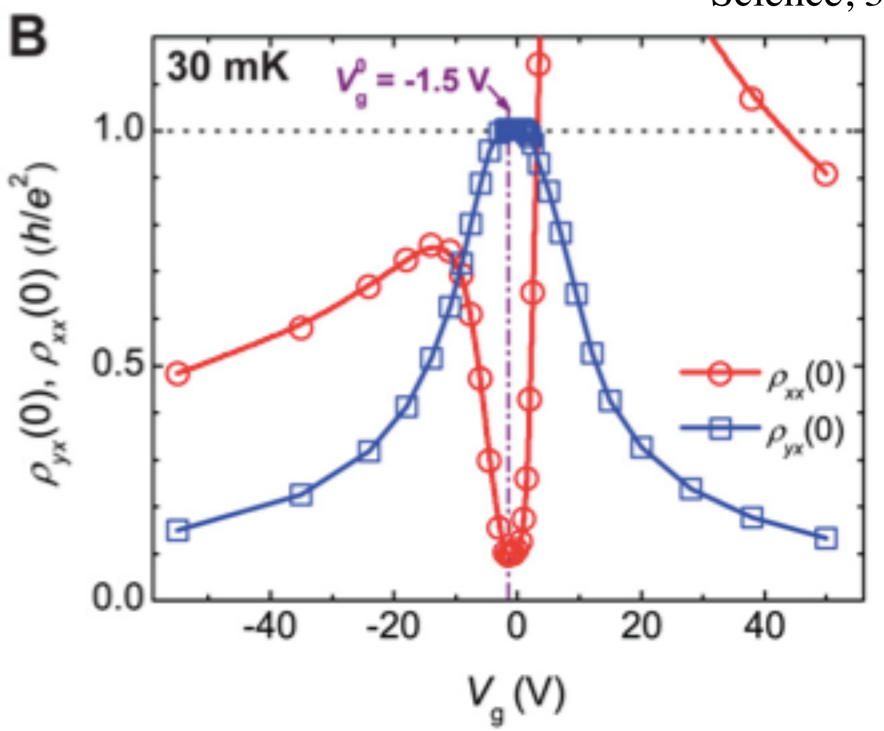
- **AH and QAH only when time reversal invariance is broken**
- **Experimental observations 2013 -**
- **Exact quantization of Hall conductance**

The topological argument is basically the same as the one for the quantum Hall effect given by Thouless et.al.
The conductance again equals the Chern number!

Experimental Observation of the Quantum Anomalous Hall Effect in a Magnetic Topological Insulator

Cui-Zu Chang,[1,2*] Jinsong Zhang,[1*] Xiao Feng,[1,2*] Jie Shen,[2*] Zuocheng Zhang,[1] Minghua Guo,[1] Kang Li,[2] Yunbo Ou,[2]
Pang Wei,[2] Li-Li Wang,[2] Zhong-Qing Ji,[2] Yang Feng,[1] Shuaihua Ji,[1]
Xi Chen,[1] Jinfeng Jia,[1] Xi Dai,[2] Zhong Fang,[2] Shou-Cheng Zhang,[3] Ke He,[2†] Yayu Wang,[1†] Li Lu,[2] Xu-Cun Ma,[2] Qi-Kun Xue[1,2†]

Science, 340(6129):167–170, 2013.



"……we report the observation of the quantum anomalous Hall (QAH) effect in thin films of Cr-doped $(Bi,Sb)_2Te_3$, a magnetic topological insulator. At zero magnetic field, the gate-tuned anomalous Hall resistance reaches the predicted quantized value of h/e^2, ….. "

Spin chains and the Haldane gap

The antiferromagnetic Heisenberg chain:

$$\vec{S} = (S_x, S_y, S_z) \qquad H_{\text{Heis}} = -J \sum_{\langle ij \rangle} S_i \cdot S_j \qquad J < 0$$

It was known since the 60’s that the spin halv version of this chain was gapless, and this was tacitly assumed to be true for all spins

Q: What is the order parameter for an antiferromagnet?
A: “Staggered magnetization!

Configuration of spins

Staggered spins: $\vec{\mathcal{S}}_i = (-1)^i \vec{S}_i$

Corresponding staggered spins

In 1983, Haldane derived a large *S* theory:

F.D.M. **Haldane**,
Physics Letters A, **93**, 1983.

$$S_{NLS} = \frac{1}{2g} \int dt dx \left(\frac{1}{v} (\partial_t \vec{m})^2 - v (\partial_x \vec{m})^2 \right) \qquad \vec{m} = \frac{\vec{\mathcal{S}}_i}{|\vec{\mathcal{S}}|}$$

which might look massless, but

Polyakov had shown that this O(3) sigma model actually has a gap generated by quantum fluctuations. Like in QCD the beta function is negative, so the coupling constant, *g*, becomes large in the infrared.

Now it looks like *all* spin chains are gapped??

Haldane: extra topological piece,

F.D.M. Haldane,
PRL **50**, 2015, 1988.

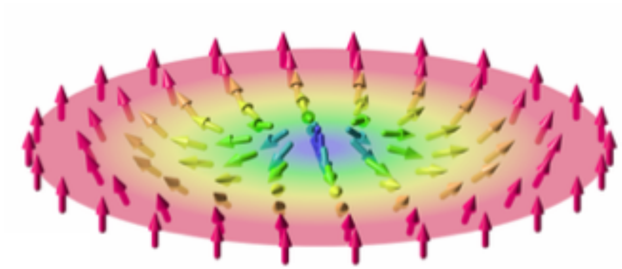
$$S_{top} = i\frac{\theta}{4\pi}\int d^2x\ \vec{m}\cdot(\partial_1\vec{m}\times\partial_2\vec{m}) = \theta\,\mathcal{W} \quad , \qquad \theta = 2\pi S$$

The winding number, $\mathcal{W}$, is a topological invariant !

In the path integral, configurations with winding numbers get a phase:

$$e^{i2\pi S} = (-1)^{\mathcal{W}}$$

which for half-integer spins give different signs!

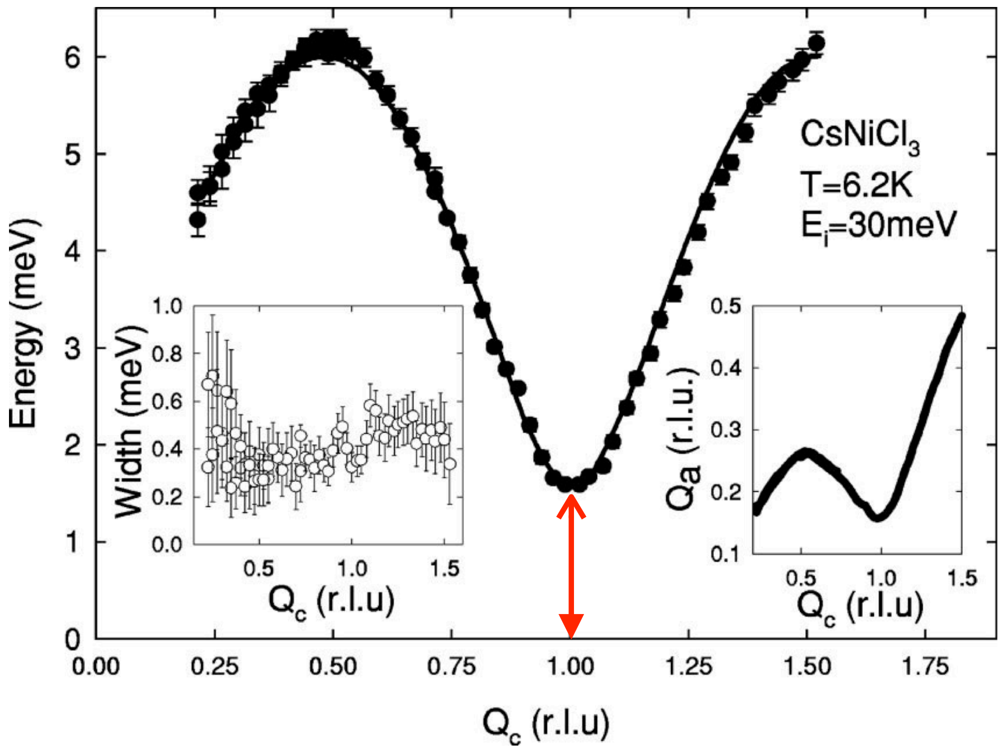


$\mathcal{W} = 1$

Experiment

PHYSICAL REVIEW B **66**, 024407 (2002)

Properties of Haldane excitations and multiparticle states in the antiferromagnetic spin-1 chain compound $CsNiCl_3$



The curve shows the energy for a spin excitation as a function of the wave vector close to the Néel value, $Q = \pi$; The Haldane gap is marked by the red arrow.

Outlook I

Quantum simulations, and artificial states of matter

Haldane 1988: **"While the particular model presented here, is unlikely to be directly physically realizable, it indicates ..."**

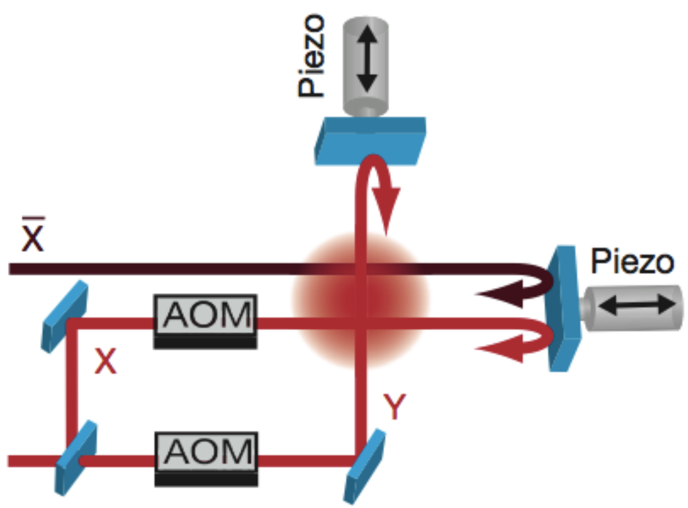
LETTER

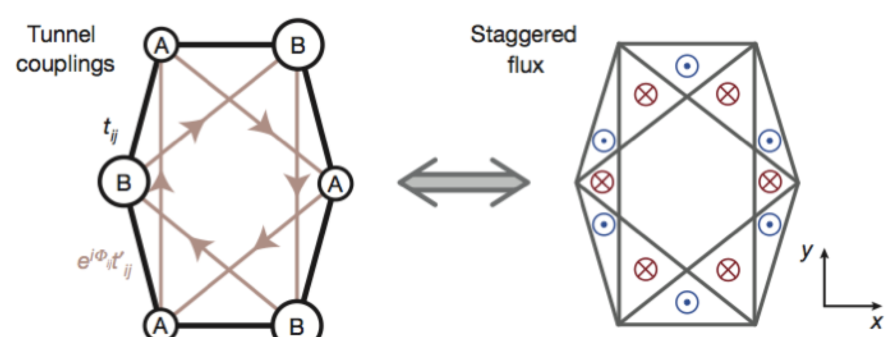
Nature, **515**, 237, 2014

doi:10.1038/nature13915

Experimental realization of the topological Haldane model with ultracold fermions

Gregor Jotzu[1], Michael Messer[1], Rémi Desbuquois[1], Martin Lebrat[1], Thomas Uehlinger[1], Daniel Greif[1] & Tilman Esslinger[1]

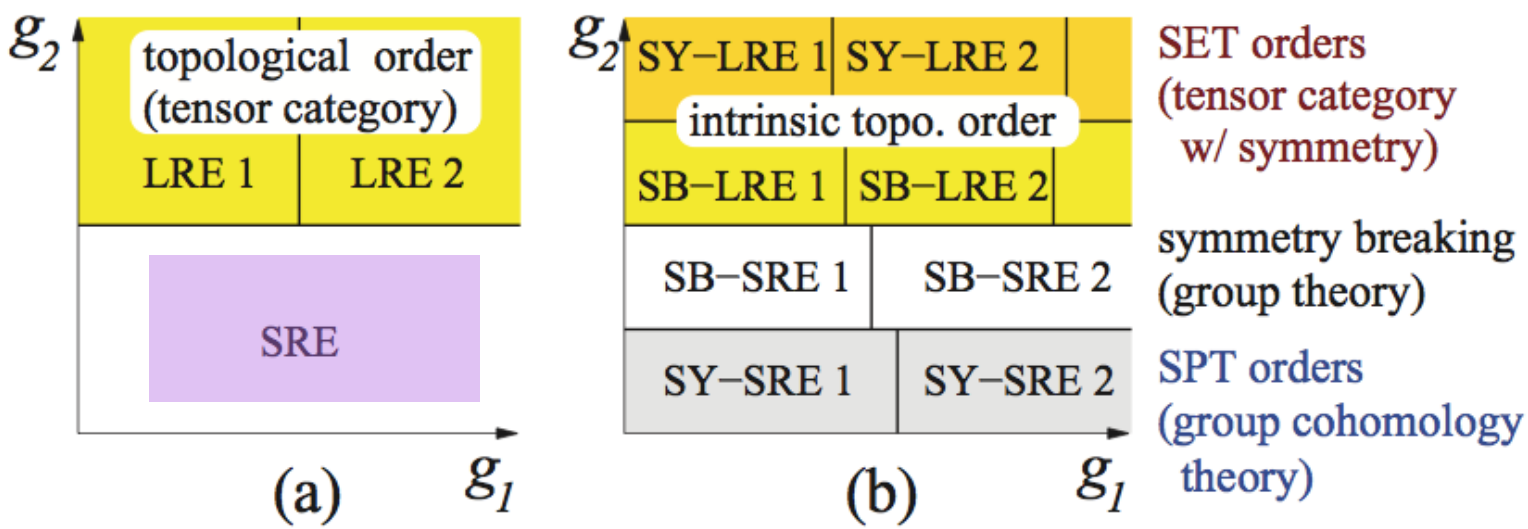




Outlook II Classification of states of matter

Gapped states

Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen
PHYSICAL REVIEW B **87**, 155114 (2013)



- **Symmetry breaking patterns**
- **Entanglement patterns**
- **10-fold way of free fermion states**
- **Topological band theory**

Also gapless states such as Weyl sem-metals

Topological band theory

	Experiment	Teori
Q. Hall effect	von Klitzing -80	Laughlin -81 Thouless *et.al* -82
Chern-insulator	Zhang *et.al*. -08 Chang *et.al* -13	Haldane -88
2d Top.insulator/ spinhalleffect	Bernevig *et.al*. -06 Zhang *et.al*. -07	Kane, Mele -05
3d top. insulator	Fu, Kane - 2007 Hasan *et.al*. 2008	Fu, Kane, Mele - 07 Moore, Balents - 07 Roy - 09

Blue indicates that theory predated the experiment.