QFT at large charge

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This talk will be mostly about new ideas to solve CFT’s.

Typically there is no simplifying limit

In a presence of a global symmetry, however, there can be sub-sectors of the CFT where anomalous dimension and OPE coefficients simplify

Take home message

- **There is a semiclassical technique to study the sectors of the CFT with fixed Noether charge Q. In these sectors the physics is described by a semiclassical configuration and has simple EFT description.**

You can compute correlators of the charged operators, say composite operator ϕ^Q(x). In this talk we will study 2-pt functions but one can go beyond

The example of a symmetry is a global symmetry with the simplest example of a U(1) complex scalar model

Part 1: global symmetry

Consider model with U(1) global symmetry

Badel, Cuomo, Monin, Rattazzi 2019

$$L = \partial_\mu \bar{\phi} \partial^\mu \phi + \frac{\lambda}{4} \left(\bar{\phi}\phi\right)^2$$

The operators $\phi^Q(x)$ and $\bar{\phi}^Q(x)$ carry U(1) charge $+Q(-Q)$

Rescale the field $\phi \to \frac{\phi}{\sqrt{\lambda}}$

$$\Rightarrow \quad L_{new} = \frac{1}{\lambda}\left(\partial_\mu \bar{\phi} \partial^\mu \phi + \frac{1}{4}(\bar{\phi}\phi)^2\right)$$

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle \sim \int D\bar{\phi} D\phi \;\; \phi^Q(x_f)\phi^Q(x_i) e^{-\frac{S}{\lambda}}$$

For **λ**<<1 dominated by the extrema of S

Bring field insertions to the exponent

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle \sim \int D\bar{\phi}D\phi\ e^{-\frac{S_{eff}}{\lambda}}$$

$$S_{eff} = \int d^dx\left[\partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 + \lambda Q(\log\phi(x_f) + \log\phi(x_i)\right]$$

For $\lambda Q \ll 1$ perturbation theory works (expand around) $\phi = 0$

For $\lambda Q \gg 1$ expand around new saddles

λ<<1 so Q >>1 to have new saddles and keep **λ**Q=fixed

$$S_{eff} = \int d^d x \left[\partial\bar{\phi}\partial\phi + \frac{1}{4}(\bar{\phi}\phi)^2 + \lambda Q(\log\phi(x_f) + \log\phi(x_i)\right]$$

E.O.M

$$\partial^2\phi(x) - \frac{1}{2}\phi^2(x)\bar{\phi}(x) = -\frac{\lambda Q}{\bar{\phi}(x_f)}\delta^{(d)}(x - x_f),$$

$$\partial^2\bar{\phi}(x) - \frac{1}{2}\phi(x)\bar{\phi}^2(x) = -\frac{\lambda Q}{\phi(x_i)}\delta^{(d)}(x - x_i).$$

→ $$\partial_\mu j^\mu = Q\delta^{(d)}(x - x_i) - Q\delta^{(d)}(x - x_f)$$

With $j_\mu = \bar{\phi}\partial_\mu\phi - \phi\partial_\mu\bar{\phi}$ Noether current

Field insertions are sources for the Noether current

- E.O.M. can be solved perturbatively but technically challenging

- If we are at the fixed point, however, we can use the power of conformal invariance

In a CFT

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}}$$

Physical critical exponents

$$\Delta_{\phi^Q} \equiv Q\left(\frac{d-2}{2}\right) + \gamma_{\phi^Q}$$

Goal : compute $\Delta_{\phi^Q} \equiv Q\left(\frac{d-2}{2}\right) + \gamma_{\phi^Q}$

We expect scaling dimensions to take the form:

$$\Delta_Q = \sum_{k=-1} \frac{\Delta_k(\lambda_0 Q)}{Q^k}$$

Δ_k is (k+1)-loop correction to the saddle point equation

We will compute Δ_{-1} and Δ_0

In general, we can expand these functions Δ_k's for small and large value of the argument

Semiclassical computation

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

$S(\phi_0) \downarrow \Delta_{-1}$

$\frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) \downarrow \Delta_0$

Semiclassical method

Badel, Cuomo, Monin, Rattazzi 2019

Working in the double scaling limit : $\lambda \to 0, \quad Q \to \infty, \quad \lambda Q = fixed$

- Tune QFT to the (perturbative) fixed point (WF or BZ type)
- Map the theory to the cylinder $\mathbb{R}^d \to \mathbb{R} \times S^{d-1}$
- Exploit operator/state correspondence for the 2-point function to relate anomalous dimension to the energy

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT} = \frac{1}{|x_f - x_i|^{2\Delta_{\phi^Q}}} \qquad E = \Delta_{\phi^Q}/R$$

- To compute this energy, evaluate expectation value of the evolution operator in an arbitrary state with fixed charge Q

Weyl map : $r = Re^{\tau/R}$

$$\mathbb{R}^d \ : \ (r, \Omega_{d-1}) \qquad\qquad \mathbb{R}\times S^{d-1} \ : \ (\tau, \Omega_{d-1})$$

$$ds^2_{cyl} = d\tau^2 + R^2 d\Omega^2_{d-1} = \frac{R^2}{r^2} ds^2_{flat}$$

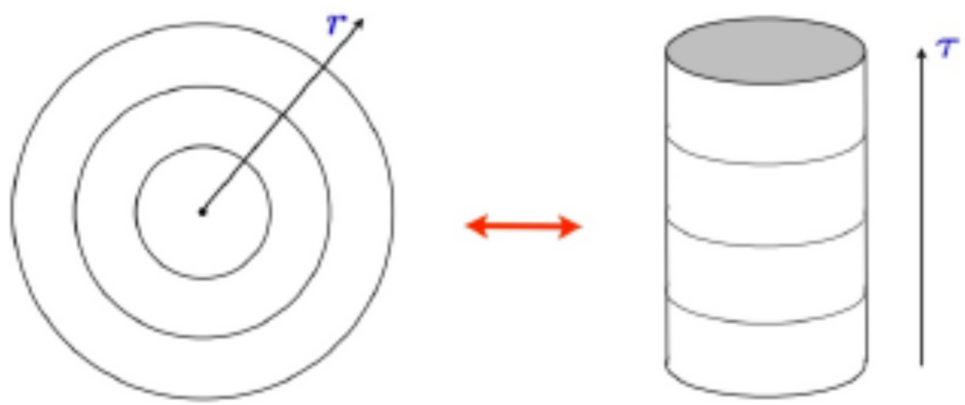
$$\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\rm cyl} = |x_f|^{\Delta_\mathcal{O}}|x_i|^{\Delta_\mathcal{O}}\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\rm flat} \equiv \frac{|x_f|^{\Delta_\mathcal{O}}|x_i|^{\Delta_\mathcal{O}}}{|x_f - x_i|^{2\Delta_\mathcal{O}}}$$

$$\langle \mathcal{O}^\dagger(x_f)\mathcal{O}(x_i)\rangle_{\rm cyl} \overset{\tau_i\to-\infty}{=} e^{-E_\mathcal{O}(\tau_f-\tau_i)}, \qquad E_\mathcal{O} = \Delta_\mathcal{O}/R$$

Weyl map and operator/state correspondence

Working at the WF fixed point we can map the theory to the cylinder.

$$\mathbb{R}^d \to \mathbb{R}\times S^{d-1}, \qquad r = Re^{\tau/R}$$



The eigenvalues of the dilation charge, i.e. the scaling dimensions, become the energy spectrum on the cylinder.

$$\boxed{E_{\phi^Q} = \Delta_{\phi^Q}/R}$$

State-operator correspondence:
States and operators are in 1-to-1 correspondence.

$$\tau_f - \tau_i \equiv T \qquad \langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{cyl} \overset{T\to\infty}{=} Ne^{-E_{\phi^Q}T}$$

- To compute this energy, evaluate expectation value of the evolution operator in an arbitrary state with fixed charge Q

$$\langle Q|e^{-HT}|Q\rangle \overset{T\to\infty}{=} \bar{N}e^{-E_{\phi^Q}T}$$

as long as there is overlap between |Q> and the ground state, the latter will dominate for $T \to \infty$

To study system at fixed charge thermodynamically we have:

$$H \to H + \mu Q$$

μ is chemical potential

Consider model with U(1) global symmetry

$$L = \partial_\mu \bar{\phi} \partial^\mu \phi + \frac{\lambda}{4} \left(\bar{\phi} \phi \right)^2$$

In d=4-ε there is an IR WF fixed point

$$\lambda^* = \frac{3}{10} \epsilon + \cdots$$

Weyl map the theory to the cylinder:

$$S_{cyl} = \int d^d x \sqrt{-g} \left(g_{\mu\nu} \partial^\mu \bar{\phi} \partial^\nu \phi + m^2 \bar{\phi} \phi + \frac{\lambda}{4} (\bar{\phi} \phi)^2 \right)$$

$$m^2 = \left(\tfrac{d-2}{2R} \right)^2$$

stemming from the coupling to Ricci scalar

Classical solution: $$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

$$\phi = \frac{\rho}{\sqrt{2}} e^{i\chi}$$

$$S_{eff} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \Big(\frac{1}{2}(d\rho)^2 + \frac{1}{2}\rho^2 (d\chi)^2 + \frac{m^2}{2}\rho^2 + \frac{1}{16}\rho^4 + \mu^2 f^2 \Big)$$

Stationary solution: $\rho = f$ $\chi = -i\mu\tau$

$$\mu^2 - m^2 = \frac{\lambda}{4} f^2 \qquad \mu f^2 = \frac{Q}{R^{d-1}\Omega_{d-1}}$$

$$m^2 = \left(\tfrac{d-2}{2R}\right)^2$$

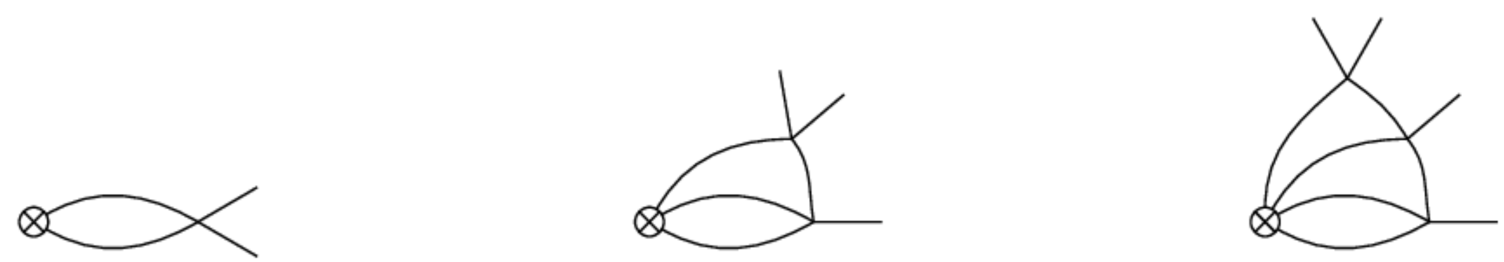
$$S_{eff}R = E_{-1}R = \Delta_{-1}$$

$$4\,\Delta_{-1} = \frac{3^{2/3}\left(x+\sqrt{-3+x^2}\right)^{1/3}}{3^{1/3}+\left(x+\sqrt{-3+x^2}\right)^{2/3}} + \frac{3^{1/3}\left(3^{1/3}+\left(x+\sqrt{-3+x^2}\right)^{2/3}\right)}{\left(x+\sqrt{-3+x^2}\right)^{1/3}}$$

$$x \equiv 6\lambda Q$$

$$\frac{\Delta_{-1}}{\lambda_*} \overset{\lambda Q \ll 1}{=} Q\Big[1+\frac{1}{2}\Big(\frac{\lambda_* Q}{16\pi^2}\Big)-\frac{1}{2}\Big(\frac{\lambda_* Q}{16\pi^2}\Big)^2+\cdots\Big]$$

Resums infinite number of Feynman diagrams



$$\frac{\Delta_{-1}}{\lambda_*} \overset{\lambda Q \gg 1}{=} \frac{8\pi^2}{\lambda^*}\Big[\frac{3}{4}\Big(\frac{\lambda_* Q}{8\pi^2}\Big)^{4/3}+\frac{1}{2}\Big(\frac{\lambda_* Q}{8\pi^2}\Big)^{2/3}+\cdots\Big]$$

Leading quantum correction:

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

$$\rho = f + r(x) \qquad \chi = -i\mu\tau + \frac{\pi(x)}{\sqrt{2}f}$$

$$S^{(2)} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{d-1} \Big(\frac{1}{2}(\partial r)^2 + \frac{1}{2}(\partial \pi)^2 - 2i\mu r \partial_\tau \pi + (\mu^2 - m^2)^2 \Big)$$

One relativistic (Type I) Goldstone boson (the conformal mode=phonon) and one massive state and their excitations

$$\omega_\pm^2(\ell) = J_l^2 + 3\mu^2 - m^2 \pm \sqrt{4J_l^2\mu^2 + (3\mu^2 - m^2)^2}$$

$$J_l^2 = \ell(\ell + d - 2)/R^2$$

Energy= sum of zero point energies

$$\Delta_0 = \frac{R}{2}\sum_{\ell=0}^{\infty} n_\ell\left[\omega_+(\ell) + \omega_-(\ell)\right]$$

The MSbar renormalized result in the limiting cases reads:

$$\lambda Q \ll 1 \quad : \qquad \Delta_0 = -\frac{3\lambda_* Q}{(4\pi)^2} + \frac{\lambda_*^2 Q^2}{2(4\pi)^4} + \cdots$$

$$\lambda Q \gg 1 \;:\; \Delta_0 = [\alpha + \frac{5}{24}\log\left(\frac{\lambda_* Q}{8\pi^2}\right)^{4/3}] + [\beta - \frac{5}{36}\log\left(\frac{\lambda_* Q}{8\pi^2}\right)^{2/3}] + \cdots$$

Regimes

Solve:

$$\mu^2 - m^2 = \frac{\lambda}{4} f^2 \qquad \mu f^2 = \frac{Q}{R^{d-1}\Omega_{d-1}}$$

$$\lambda Q \ll 1 \qquad \mu R = 1 + \frac{\lambda Q}{16\pi^2} + \cdots$$

$$\lambda Q \gg 1 \qquad \mu R = \frac{(\lambda Q)^{1/3}}{2\pi^{2/3}} + \cdots$$

$$\lambda \to 0 \qquad Q \to \infty \qquad \lambda Q = fixed$$

<<1 → Superfluid interacts with light radial mode

>>1 → Radial mode decouples

Large charge expansion, historically

Hellerman, Orlando, Reffert, Watanabe 2015

Started with d=3 $\lambda\phi^4$-model with global U(1) symmetry

EFT for phonon (superfluid phase) in large-Q limit :

$$\Delta_Q = Q^{\frac{d}{d-1}} \left[\alpha_1 + \alpha_2 Q^{\frac{-2}{d-1}} + \alpha_3 Q^{\frac{-4}{d-1}} + \dots\right] + Q^0 \left[\beta_0 + \beta_1 Q^{\frac{-2}{d-1}} + \dots\right] + \mathcal{O}\left(Q^{-\frac{d}{d-1}}\right)$$

- **Comparing to ordinary perturbation theory**

	1-loop	2-loop	3-loop	
Δ_{-1}	$Q^2\lambda_0$	$Q^3\lambda_0^2$	$Q^4\lambda_0^3$	$\cdots$
Δ_0	$Q\lambda_0$	$Q^2\lambda_0^2$	$Q^3\lambda_0^3$	$\cdots$
Δ_1		$Q\lambda_0^2$	$Q^2\lambda_0^3$	$\cdots$
Δ_2			$Q\lambda_0^3$	$\cdots$
$\vdots$				

Reorganizing perturbative expansion

For a well-defined limit need to introduce 't Hooft coupling $\mathcal{A}$

- Large-N_c : Planar limit : $A_c \equiv g^2 N_c = fixed$
- Large-N_f : Bubble diagrams : $A_f \equiv g^2 N_f = fixed$
- Large-charge expansion : $A_Q \equiv gQ = fixed$

Then we have

$$\boxed{observable \sim \sum_{l=loops} g^l P_l(N) = \sum_k \frac{1}{N^k} F_k(\mathcal{A})}$$

$$N = \{N_c, N_f, Q\}$$

Generalisations

- O(N) model (in different dimensions)
- Litim-Sannino (UV BZ fixed point)
- Yukawa interactions
- …..

Part 2: local symmetry

Can we apply these methods to 3d local U(1) model?

$$S=\int d^D x\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+(D_\mu\phi)^\dagger D_\mu\phi+\frac{\lambda(4\pi)^2}{6}(\bar{\phi}\phi)^2\right)$$

$$D_\mu\phi=(\partial_\mu+ieA_\mu)\phi$$

Semiclassical computation on the cylinder can be carried out

BUT

what would be the physical meaning of it in the flat space?

We compute energy $E = \Delta_{\phi^Q}/R$
which is gauge-independent quantity based on

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle^{cylinder}_{CFT}$$

But, in flat space: $\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle^{flat}_{CFT}$

is not gauge-invariant and vanishes due to the Elitzur's theorem (1975)

→ our computation should correspond to gauge-invariant correlator in flat space

But which one? The choice is not unique

Dirac proposal:

Kleinert et al 05

$$G_D = \langle\bar{\phi}(x_f)\exp\left(-i\, e\int d^D x J^\mu(x) A_\mu(x)\right)\phi(x_i)\rangle$$

$$\partial^\mu J_\mu = \delta(x-x_f) - \delta(x-x_i) \qquad \partial^2 J_\mu = 0$$

$$J_\mu(z) = J'_\mu(z-x') - J'_\mu(z-x)$$

$$J'_\mu(z) = -i\int \frac{d^d k}{(2\pi)^d}\frac{k_\mu}{k^2}e^{ik\cdot z} = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}}\partial_\mu\frac{1}{z^{d-2}}$$

$$G_D = \langle\bar{\phi}_{nl}(x_f)\phi_{nl}(x_i)\rangle \qquad \phi_{nl}(x) \equiv e^{-ie\int d^D z J'_\mu(z-x)A^\mu(x)}\phi(x)$$

In Landau gauge $\partial^\mu A_\mu = 0 \implies \phi_{nl}(x) = \phi(x)$

correlators of ϕ(x) in Landau gauge
can be interpreted as that of $\phi_{nl}(x)$

Schwinger proposal:

Wilson line on the shortest path connecting x and x'

$$\langle \bar\phi(x') \exp\Big[-ie \int_x^{x'} dx^\mu A_\mu(x) \Big] \phi(x) \rangle$$

with the external current before "squeezed" into an infinitely thin line along the shortest path connecting x and x'

Schwinger and Dirac correlators lead to different physical results, in particular, different critical exponents Δ_{ϕ^Q}

$\Delta_{\phi^Q}^{perturbative}$ **Landau gauge** $\rightarrow$ $\Delta_{\phi^Q}^{Dirac}$

Gauge-dependent **Gauge-independent**

$\Delta_{\phi^Q}^{perturbative}$ **Traceless gauge** $\rightarrow$ $\Delta_{\phi^Q}^{Schwinger}$

Kleinert et al '03, '05. 1-loop check

Our strategy

- Compute $\Delta_{\phi^Q}^{perturbative}$ via Feynman diagrams in arbitrary linear gauge
- Compare with Δ_{ϕ^Q} from energies $E = \Delta_{\phi^Q}/R$ on the cylinder and look for the match

Doing this we hope to learn to which gauge-invariant correlator in flat space our energies correspond to

Our work

$$E = \Delta_{\phi^Q}/R \quad = \quad \Delta_{\phi^Q}^{Dirac}, \ \Delta_{\phi^Q}^{Schwinger}, \ldots?$$

$$S = \int d^D x \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D_\mu \phi + \frac{\lambda (4\pi)^2}{6} (\bar{\phi}\phi)^2 \right)$$

$$D = 4 - \epsilon$$

- Perturbative WF fixed point at 1-loop reads

$$\lambda^* = \frac{3}{20} \left(19\epsilon \pm i\sqrt{719}\epsilon \right) , \qquad a_g^* = \frac{3}{2}\epsilon \qquad a_g = \frac{e^2}{(4\pi)^2}$$

complex!

- Map to the cylinder

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi + m^2 \bar{\phi}\phi + \frac{\lambda (4\pi)^2}{6} (\bar{\phi}\phi)^2 \right)$$

$m^2 = (D-2)^2/4$ with radius of the cylinder R=1

• State-operator correspondence

$$\langle \bar{\phi}^Q(x_f)\phi^Q(x_i)\rangle_{CFT}^{cylinder}$$

These operators create (annihilate) states with energy

$$E = \Delta_{\phi^Q}/R$$

$$\langle Q|e^{-HT}|Q\rangle = \mathcal{Z}^{-1}\int_{\rho=f}^{\rho=f}\mathcal{D}\rho\mathcal{D}\chi\mathcal{D}A\, e^{-S_{\text{eff}}}$$

$$\phi(x) = \frac{\rho(x)}{\sqrt{2}}e^{i\chi(x)}$$

$$S_{\text{eff}} = \int_{-T/2}^{T/2} d\tau \int d\Omega_{D-1}\Bigg(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(\partial\rho)^2 + \frac{1}{2}\rho^2(\partial\chi)^2$$

$$+ \frac{1}{2}m^2\rho^2 + e\rho^2 A_\mu\partial^\mu\chi + \frac{1}{2}e^2\rho^2 A_\mu A^\mu + \frac{\lambda(4\pi)^2}{24}\rho^4 + \frac{i\,Q}{\Omega_{D-1}}\dot{\chi}\Bigg)$$

- Fixing the charge of the initial and final state to Q

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \dots$$

$S(\phi_0) \rightarrow \Delta_{-1}$

Homogeneous ground state

$$\rho(x) = f, \quad \chi(x) = -i\mu\tau, \quad A_\mu = 0$$

From EOM

$$\mu^3 - \mu = \frac{4}{3}\lambda Q \;, \qquad f^2 = \frac{6}{(4\pi)^2\lambda}\left(\mu^2 - m^2\right)$$

Plugging into S_{eff}.

$$4\,\Delta_{-1} = \frac{3^{2/3}\left(x + \sqrt{-3 + x^2}\right)^{1/3}}{3^{1/3} + \left(x + \sqrt{-3 + x^2}\right)^{2/3}} + \frac{3^{1/3}\left(3^{1/3} + \left(x + \sqrt{-3 + x^2}\right)^{2/3}\right)}{\left(x + \sqrt{-3 + x^2}\right)^{1/3}}$$

The same as in U(1) global case

$$x \equiv 6\lambda Q$$

$$S = S(\phi_0) + \frac{1}{2}(\phi - \phi_0)^2 S''(\phi_0) + \ldots$$

$$\Downarrow$$

$$\Delta_0$$

$$\rho(x) = f + r(x)$$

$$\chi(x) = -i\mu\tau + \frac{\pi(x)}{f}$$

Add gauge-fixing and ghost terms

$$\delta S = \frac{1}{2}\int d^d x \left(G^2 + \mathcal{L}_{\text{ghost}}\right), \qquad G = \frac{1}{\sqrt{\xi}}(\nabla_\mu A^\mu + ef\pi)$$

and expand S_{eff} to quadratic order

$$\mathcal{L}^{(2)}_{\text{eff}} = \frac{1}{2} A_\mu \left(-g^{\mu\nu}\nabla^2 + \mathcal{R}^{\mu\nu} + \left(1 - \frac{1}{\xi}\right)\nabla^\mu\nabla^\nu + (ef)^2 g^{\mu\nu}\right) A_\nu$$

$$+ \frac{1}{2}(\partial_\mu r)^2 - \frac{1}{2}2(m^2 - \mu^2)r^2 + \frac{1}{2}(\partial_\mu \pi)^2 - \frac{1}{2\xi}(ef)^2\pi^2$$

$$- 2i\mu\, r\partial_\tau \pi - 2if\mu r A^0 + ef\left(1 - \frac{1}{\xi}\right) A_\mu \partial^\mu \pi + \bar{c}[-\nabla^2 + (ef)^2]c$$

Spectrum of fluctuations

$$scalars : r, \pi, A_0, h \qquad A_i = B_i + C_i \qquad C^i = \nabla^i h$$

$$vectors : B_i \qquad \nabla_i B^i = 0$$

$$ghosts : c, \bar{c} \qquad -\nabla^2 = -\partial_\tau^2 + \left(-\nabla^2_{S^{D-1}}\right) \quad on \quad \mathbb{R} \times S^{D-1} space$$

$$B_i : \quad \int \frac{d\omega}{2\pi} \sum_\ell n_v(\ell) \det\left(-\partial_\tau^2 + J^2_{\ell(v)} + (D-2) + (ef)^2\right)^{-1/2}$$

$$c, \bar{c} : \quad \int \frac{d\omega}{2\pi} \sum_\ell n_s(\ell) \det\left[-\partial_\tau^2 + J^2_{\ell(s)} + (ef)^2\right]$$

$$scalars : \int \frac{d\omega}{2\pi} \sum_\ell n_s(\ell) \det\left[\mathcal{B}\right]^{-1/2}$$

Scalars

$$\mathcal{B} = \begin{pmatrix} -\omega^2 + J^2_{\ell(s)} + 2(\mu^2 - m^2) & -2i\mu\omega & -2ie\mu f & 0 \\ 2i\mu\omega & -\omega^2 + J^2_{\ell(s)} + \frac{1}{\xi}e^2 f^2 & -ef\left(1-\frac{1}{\xi}\right)\omega & -ief\left(1-\frac{1}{\xi}\right)|J_{\ell(s)}| \\ -2ie\mu f & ef\left(1-\frac{1}{\xi}\right)\omega & -\frac{1}{\xi}\omega^2 + J^2_{\ell(s)} + (ef)^2 & i\left(1-\frac{1}{\xi}\right)\omega|J_{\ell(s)}| \\ 0 & ief\left(1-\frac{1}{\xi}\right)|J_{\ell(s)}| & i\left(1-\frac{1}{\xi}\right)\omega|J_{\ell(s)}| & -\omega^2 + \frac{1}{\xi}J^2_{\ell(s)} + (ef)^2 \end{pmatrix}$$

Determinant factorizes with gauge-independent dispersion relations:

$$\xi \det \mathcal{B} = (\omega^2 + \omega_+^2)(\omega^2 + \omega_-^2)(\omega^2 + \omega_1^2)^2$$

ξ cancels out in the final result due to contribution from Z^{-1}

$$\langle Q|e^{-HT}|Q\rangle = \mathcal{Z}^{-1}\int_{\rho=f}^{\rho=f} \mathcal{D}\rho\mathcal{D}\chi\mathcal{D}A\, e^{-S_{\text{eff}}}$$

$scalars : r, \pi, A_0, h$

$vectors : B_i$

$ghosts : c, \bar{c}$

$$\Delta_0 = \frac{1}{2} \sum_{\ell=\ell_0}^{\infty} d_\ell \omega_i(\ell)$$

Field	d_ℓ	$\omega_i(\ell)$	ℓ_0
B_i	$n_v(\ell)$	$\sqrt{J^2_{\ell(v)} + (D-2) + e^2 f^2}$	1
$h(C_i)$	$n_s(\ell)$	$\sqrt{J^2_{\ell(s)} + e^2 f^2}$	1
$(c, \bar{c})$	$-2n_s(\ell)$	$\sqrt{J^2_{\ell(s)} + e^2 f^2}$	0
A_0	$n_s(\ell)$	$\sqrt{J^2_{\ell(s)} + e^2 f^2}$	0
ϕ	$n_s(\ell)$	$\sqrt{J^2_{\ell(s)} + 3\mu^2 - m^2 + \frac{1}{2}e^2 f^2 \pm \sqrt{\left(3\mu^2 - m^2 - \frac{1}{2}e^2 f^2\right)^2 + 4J^2_{\ell(s)}\mu^2}}$	0

The MSbar renormalized result reads

$$\Delta_0 = \frac{1}{16}\left(-15\mu^4 - 6\mu^2 + 8\sqrt{6\mu^2-2}+5\right) \qquad a_g \equiv \frac{e^2}{16\pi^2}$$

$$+\frac{1}{2}\sum_{\ell=1}\sigma(\ell) - \frac{3a_g}{8\lambda}(\mu^2-1)\left(\frac{3a_g}{\lambda}(7\mu^2+5)-9\mu^2+5\right)$$

$$\sigma(\ell) = \frac{9a_g}{2\lambda\ell}\left(\mu^2-1\right)\left[\left(\frac{3a_g}{\lambda}-1\right)\left(\mu^2-1\right)-2\ell(\ell+1)\right]$$

$$+\frac{5}{4\ell}\left(\mu^2-1\right)^2 - 2(\ell+1)(2\ell(\ell+2)+\mu^2),$$

subtraction terms

$$+(\ell+1)^2\left[\omega_+^*(\ell)+\omega_-^*(\ell)\right]+2\ell(\ell+2)\omega^*(\ell)$$

$$(\omega_\pm^*)^2 = \frac{3a_g}{\lambda}\left(\mu^2-1\right)+3\mu^2+\ell(\ell+2)-1$$

$$\pm\sqrt{\left(\frac{3a_g}{\lambda}\left(\mu^2-1\right)-3\mu^2+1\right)^2+4\ell(\ell+2)\mu^2}$$

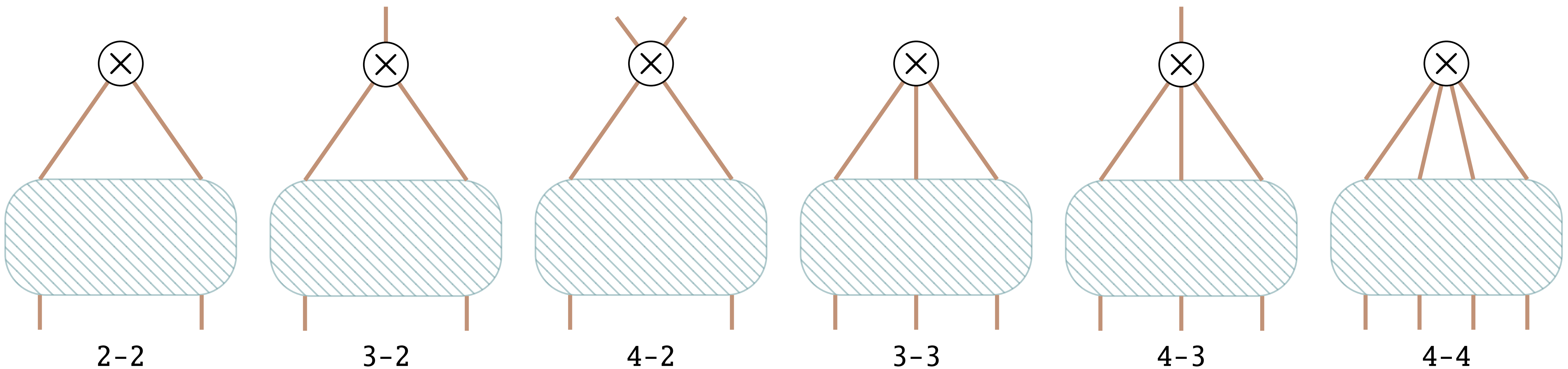
scalars

$$(\omega^*)^2 = \frac{6a_g}{\lambda}\left(\mu^2-1\right)+\ell(\ell+2)+1$$

vectors

Explicit 3-loop gauge-dependent result for ϕ^Q

We compute 3-loop AD for ϕ^Q for fixed Q=2,3,4 in $D = 4 - \epsilon$



and ``fit'' all coefficients C_{kl} in

$$\gamma_Q(\lambda, a_g, \xi) = \sum_{l=1}^{3} \gamma_Q^{(l-\text{loop})}(\lambda, a_g, \xi) \ , \quad \gamma_Q^{(l-\text{loop})} \equiv \sum_{k=0}^{l} C_{kl} Q^{l+1-k}$$

Explicit 3-loop gauge-dependent result for ϕ^Q

$$\gamma_Q^{(1)}(\lambda, a_g, \xi) = \underbrace{\frac{\lambda}{3}Q^2}_{\text{leading}} - \underbrace{Q\left(3a_g + \frac{\lambda}{3}\right)}_{\text{sub-leading}} + a_g Q^2 \xi$$

$$\gamma_Q^{(2)}(\lambda, a_g) = \underbrace{-\frac{2\lambda^2}{9}Q^3}_{\text{leading}} + \underbrace{\left(a_g^2 - \frac{4a_g\lambda}{3} + \frac{2\lambda^2}{9}\right)Q^2}_{\text{sub-leading}} + \left(\frac{7a_g^2}{3} + \frac{4a_g\lambda}{3} + \frac{\lambda^2}{9}\right)Q$$

$$\gamma_Q^{(3)}(\lambda, a_g) = \underbrace{\frac{8\lambda^3}{27}Q^4}_{\text{leading}} + \underbrace{Q^3\left[\frac{4a_g\lambda^2}{3}(3-2\zeta_3) - \frac{8a_g^2\lambda}{3}(1+3\zeta_3) + 4a_g^3(9\zeta_3-1) + \frac{2\lambda^3}{27}(16\zeta_3-17)\right]}_{\text{sub-leading}}$$
$$+ Q^2\left[\frac{29a_g^2\lambda}{6} + a_g^3(95-108\zeta_3) + \frac{\lambda^3}{18}(57-64\zeta_3) - \frac{4a_g\lambda^2}{9}(31-30\zeta_3)\right] + Q\left[\frac{13a_g^2\lambda}{6} + \frac{2a_g\lambda^2}{9}(49-48\zeta_3) - \frac{2\lambda^3}{27}(31-32\zeta_3) - a_g^3\left(\frac{3251}{54} - 72\zeta_3\right)\right]$$

In Landau gauge we find perfect agreement for the leading and subleading terms with large-Q results!

$$E = \Delta_{\phi^Q}/R \overset{\text{Our work}}{=} \Delta_{\phi^Q}^{Dirac}$$

Can we understand deeper why?

1. When gauge-fixing parameter considered as running parameter, Landau gauge emerges as a FP of the RG since

$$\beta_\xi = -\gamma_A \xi$$

2. Schwinger correlator does not lead to long-range order and decays to zero while Dirac correlator does

Frohlich et al'81

Kennedy et al'85

3. Correlators of ϕ(x) in Landau gauge can be interpreted as that of $\phi_{nl}(x)$

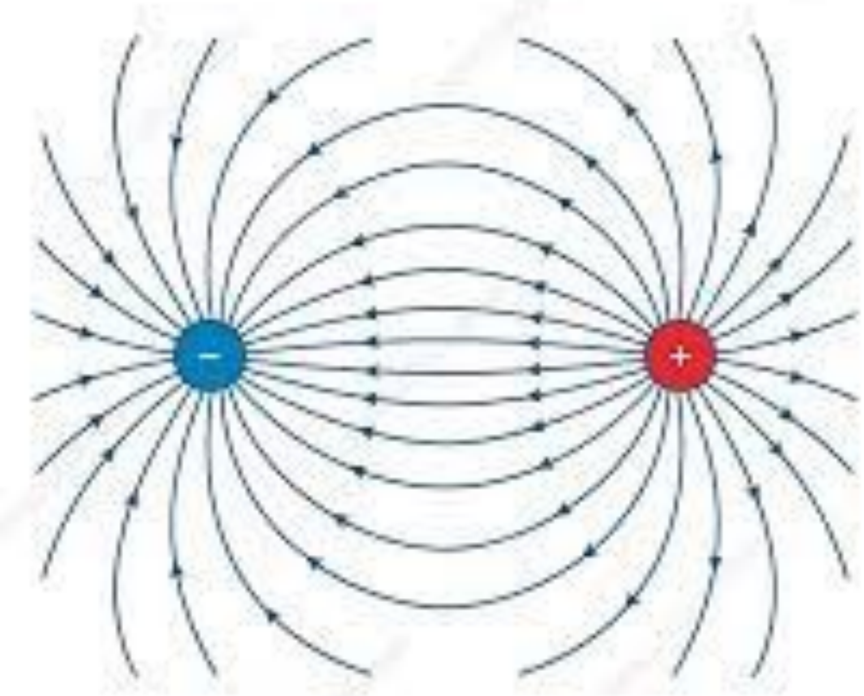
The field insertions act as sources for the current

$$\partial_\mu j^\mu = n\delta^{(d)}(x - x_i) - n\delta^{(d)}(x - x_f)$$

This is precisely the constraint for the external current in Dirac's proposal (for n=1)

$$\partial^\mu J_\mu = \delta(x - x_f) - \delta(x - x_i) \qquad J_\mu(z) = J'_\mu(z - x') - J'_\mu(z - x)$$

$$J'_\mu(z) = -i \int \frac{d^d k}{(2\pi)^d} \frac{k_\mu}{k^2} e^{ik\cdot z} = -\frac{\Gamma(d/2-1)}{4\pi^{d/2}} \partial_\mu \frac{1}{z^{d-2}}$$



In fact, original Dirac's proposal (1955)

$$\phi_{Dirac}(\vec{r}) \equiv e^{-i\int d^3r' \vec{E}_{cl}(\vec{r}'-\vec{r})\cdot\vec{A}(\vec{r}')} \phi(\vec{r})$$

$$\nabla \cdot \vec{E}_{cl} = \delta(\vec{r})$$

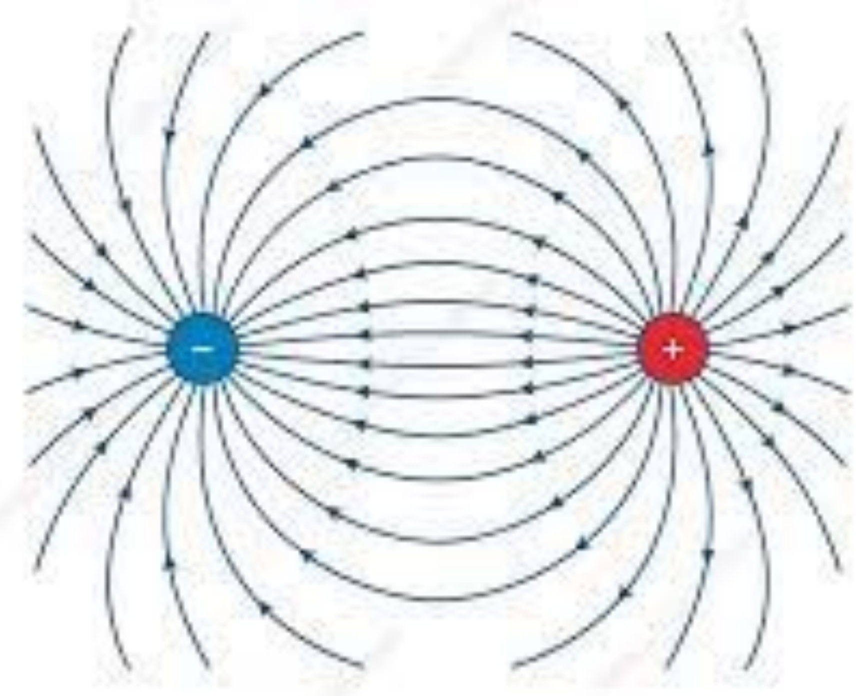
classical electric field corresponding to a point charge at the origin

Covariant generalisation

$$\phi_{nl}(x) \equiv e^{-ie\int d^D z J'_\mu(z-x)A^\mu(x)}\phi(x)$$

Physical meaning of $\phi_{nl}(x)$: creation operator of a charged particle dressed with a coherent state of photons describing its Coulomb field.

These are the lowest-lying operators with charge Q corresponding to the energies we have computed



Conclusions on Part 2

- We showed that the large-charge expansion can be applied also to gauge theories where the relevant gauge-invariant observables are in general non-local

- We demonstrated that the non-local operators $\phi_{nl}(x)$ are the lowest-lying operators with charge Q well-defined at criticality. In particular, this signals that $\phi_{nl}(x)$ is the relevant order parameter for long-range order in superconductors and it is automatically selected by the large charge approach

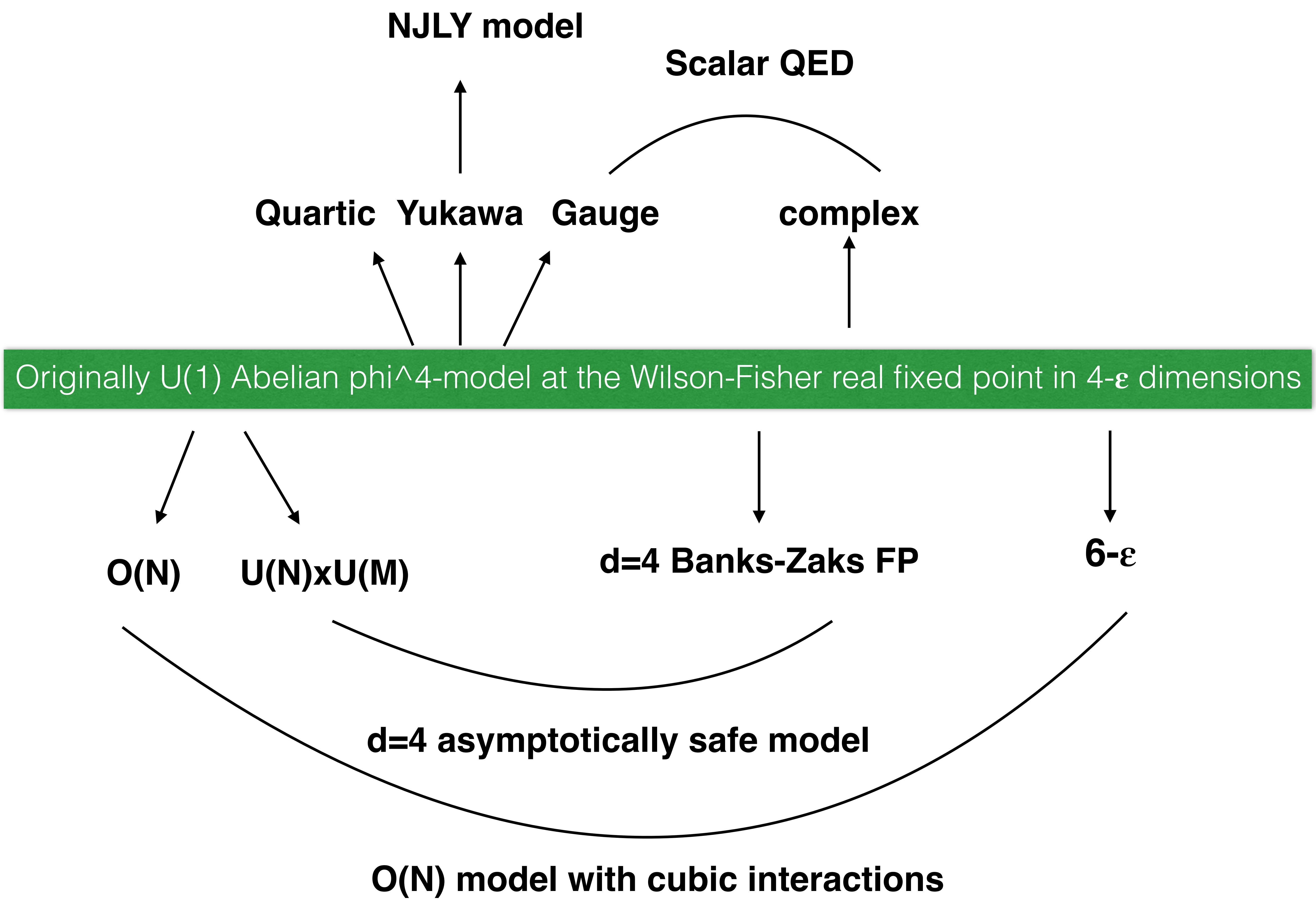
- We computed Δ_{ϕ^Q} to the next-to-leading order in the large-charge expansion and all orders in the loop expansion. Moreover, we explicitly calculate the full three-loop Δ_{ϕ^Q} in perturbation theory and find perfect agreement with our semiclassical result.

Other directions/aspects

- We can add Yukawa and non-Abelian gauge interactions
- Large order behaviour of the series (resurgence)
- Higher correlation functions, OPE coefficients,….
- Condensed matter applications (superconductors, superfluids,..)
- Inhomogeneous ground state (operators with spin/derivatives)
- Test dualities between different CFTs in their charged sectors
- Global charged corresponding to generalised global symmetry?
- ….

Thank you!

Extending the method



Identify the operator

We want the smallest dimension operator carrying a total charge $\bar{Q}$

1. Derivatives increase the scaling dimension $\Longrightarrow$ we consider operator without derivatives.
2. The latter belong to the fully symmetric $O(N)$ space $\Longrightarrow$ m-index traceless symmetric tensors, $T^{(m)}_{(i_1 \ldots i_m)} \phi^{2p}$. They have charge m and classical dimension $m + 2p \Longrightarrow p = 0$.
3. **Thus our operator is the $\bar{Q}$-index traceless symmetric tensor with classical dimension $\bar{Q}$.** It can be represented as a $\bar{Q}$-boxes Young tableau with one row.

$$\mathcal{O}_{\bar{Q}} = \underbrace{\Box\Box\Box\Box\ldots\Box}_{Q}$$

$\Delta_{\bar{Q}}$ define a set of **crossover (critical) exponent** which measures the stability of the system (e.g. critical magnets) against anisotropic perturbations (e.g. crystal structure).