Conformal anomaly and consistency of Weyl symmetry

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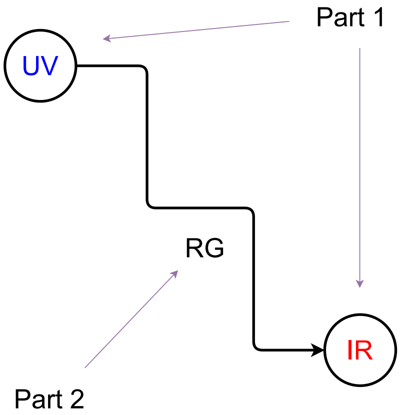
Plan

- Part 1: Integration of the conformal anomaly
- Part 2: Wess-Zumino consistency conditions of the Weyl anomaly

Rather general quantum field theory



Part 1:
Integration of the conformal anomaly

Classical Weyl (conformal) symmetry

Local Weyl rescalings

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

The energy-momentum tensor

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}} \frac{\delta S}{\delta g_{\mu\nu}}$$

Nöther identities of Diff and Weyl symmetries

$$\nabla_\mu T^{\mu\nu} = 0 \qquad T^\mu{}_\mu = 0$$

Quantum Weyl (conformal) symmetry

From the path-integral

$$\mathrm{e}^{-\Gamma} = \int [\mathrm{d}\Phi]\, \mathrm{e}^{-S}$$

The renormalized EMT

$$\langle T^{\mu\nu} \rangle = -\frac{2}{\sqrt{g}} \frac{\delta\Gamma}{\delta g_{\mu\nu}}$$

Conformal anomaly coming from the renormalization

$$\langle T^{\mu}{}_{\mu} \rangle = \text{beta terms} + \text{anomaly}$$

In two dimensions

For zero beta functions $\beta = 0$ the anomaly is

$$\langle T^{\mu}{}_{\mu} \rangle = aR$$

We want to integrate the anomaly, take $g_{\mu\nu} = \mathrm{e}^{2\sigma} \bar{g}_{\mu\nu}$

$$\sqrt{g} R = \sqrt{\bar{g}} (\bar{R} - 2 \bar{\nabla}^2 \sigma)$$

Using $\frac{\delta}{\delta\sigma} \Gamma \sim \langle T \rangle$, find $\Gamma_{\mathrm{ind}} \subset \Gamma$

$$\Gamma_{\mathrm{ind}} = a \int \mathrm{d}^2 x \sqrt{g} \left(\sigma R + \sigma \nabla^2 \sigma \right)$$

On-shell in σ we get Polyakov's

$$\Gamma_{\mathrm{ind}} = \frac{a}{4} \int \mathrm{d}^2 x \sqrt{g} R \frac{1}{-\nabla^2} R$$

In four dimensions

The anomaly is

$$\langle T^{\mu}{}_{\mu}\rangle = bW^2 + a\tilde{E}_4 + a'\Box R$$

Having defined

$$\tilde{E}_4 = E_4 - \frac{2}{3}\Box R = E_4 + \nabla^{\alpha}\left(-\frac{2}{3}\nabla_{\alpha}R\right)$$

The transformations

$$\sqrt{g}\tilde{E}_4 = \sqrt{\bar{g}}\left(\bar{\tilde{E}}_4 + 4\bar{\Delta}_4\sigma\right) \qquad \sqrt{g}W^2 = \sqrt{\bar{g}}\bar{W}^2$$

$$\sqrt{g}\Box R = -\frac{1}{4}\frac{\delta}{\delta\sigma}\int \mathrm{d}^4x\sqrt{g}R^{\mu\nu}R_{\mu\nu}$$

Four dimensional anomaly

We can integrate each term separately

$$\Gamma = \Gamma_{\rm conf}[g] + \frac{a_1'}{12} \int d^4x \sqrt{g}\, R^2 + \int d^4x \sqrt{g} \left(b_1 W^2 + a_1 \tilde{E}_4 \right) \frac{1}{\Delta_4} \tilde{E}_4$$

Applications

- Quantum field theory $\longrightarrow$ C- and A-theorems
- Black holes $\longrightarrow$ corrections to BH entropy
- Cosmology $\longrightarrow$ expanding universe

In general even d

The anomaly is conjectured (Cardy)

$$\langle T^{\mu}{}_{\mu}\rangle = \sum_i b_i \mathcal{W}_i + a\tilde{E}_d + \nabla_\mu \mathcal{J}^\mu$$

Such that

$$\tilde{E}_d = E_d + \nabla_\mu \mathcal{V}^\mu$$

The transformations

$$\sqrt{g}\tilde{E}_d = \sqrt{\bar{g}}\left(\bar{\tilde{E}}_d + d\bar{\Delta}_d\sigma\right) \qquad \sqrt{g}\mathcal{W}_i = \sqrt{\bar{g}}\bar{\mathcal{W}}_i$$

$$\sqrt{g}\nabla_\mu\mathcal{J}^\mu = \frac{\delta}{\delta\sigma}\int \mathrm{d}^4x\sqrt{g}\mathcal{L}_{\rm local}(g, \partial g, \cdots)$$

d-dimensional anomaly

We can integrate each term separately

$$\Gamma = \Gamma_c[g] + \int \mathrm{d}^d x \sqrt{g} \mathcal{L}_{\mathrm{local}} + \int d^d x \sqrt{g} \left(b_i \mathcal{W}_i + a_1 \tilde{E}_d \right) \frac{1}{\Delta_d} \tilde{E}_d$$

Main points

- Existence of $\tilde{E}_d$
- Existence of Δ_d
- Ambiguities in $\mathcal{L}_{\mathrm{local}}$
- Enumeration of $\mathcal{W}_i$

Conformal geometry
and the Fefferman-Graham ambient space

Lightcone embedding in flat space

Move from $\mathbb{R}^d$ to $\mathbb{R}^{d+2}$ on the lightcone

$$Y^A = (Y^\mu, Y^+, Y^-) \qquad \eta_{AB} Y^A Y^B = 0 \qquad Y^A \sim \lambda Y^A$$

Spacetime embedding in the lightcone

$$\begin{aligned} x^\mu &\to Y^A = (Y^\mu, Y^+, Y^-) = Y^+(x^\mu, 1, -x^2) \\ Y^A &\to x^\mu = \frac{Y^\mu}{Y^+} \end{aligned}$$

Embedding Lorentz generates conformal on spacetime

$$(Y'^+)^2 \eta_{\mu\nu} dx'^\mu dx'^\nu = (Y^+)^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

Fefferman-Graham ambient space

Use Cartesian coordinates, $X^2 = 2t^2\rho$, $t = X^+$

$$Y^A \to X^A = (X^\mu, X^{d+1}, X^{d+2}) \stackrel{*}{=} t\Big(x^\mu, \frac{1+2\rho-x^2}{2}, \frac{1-2\rho+x^2}{2}\Big)$$

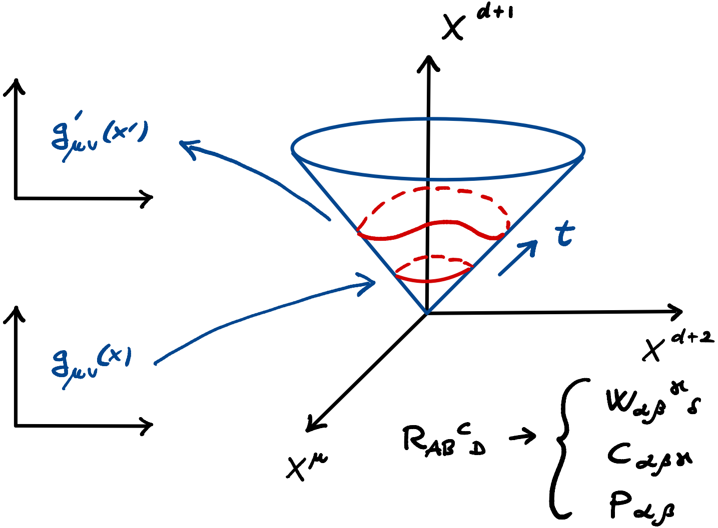
The flat embedding metric

$$\tilde{\eta} = \eta_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 \eta_{\mu\nu} dx^\mu dx^\nu$$

In curved space: FG metric with $R_{AB} = 0$, $\mathcal{L}_{t\partial_t}\tilde{g} = 2\tilde{g}$ and $h_{\mu\nu}(x, \rho = 0) = g_{\mu\nu}$

$$\tilde{g} = \tilde{g}_{AB} dx^A dx^B \stackrel{*}{=} 2\rho dt^2 + 2t dt d\rho + t^2 h_{\mu\nu}(x, \rho) dx^\mu dx^\nu$$

Ambient Space in a nutshell



PBH diffeomorphisms

A diffeomorphism of the ambient

$$\delta_\zeta \tilde{g}_{AB} = \mathcal{L}_\zeta \tilde{g}_{AB} = \zeta^C \partial_C \tilde{g}_{AB} + \tilde{g}_{AC} \partial_B \zeta^C + \tilde{g}_{BC} \partial_A \zeta^C$$

If it preserves the form of the ambient metric

$$\zeta^t = t\sigma(x) \qquad \zeta^\rho = -2\rho\sigma(x) \qquad \zeta^\mu = \xi^\mu(x) + \cdots$$

It generates Diff $\times$ Weyl on spacetime

$$\delta_\zeta h_{\mu\nu}|_{\rho=0} = \delta_\zeta g_{\mu\nu} = \delta_{\sigma,\xi} g_{\mu\nu} = 2\sigma g_{\mu\nu} + \nabla_\mu \xi_\nu + \nabla_\mu \xi_\nu$$

Ricci-flatness determines $h_{\mu\nu}$

Expand in ρ

$$h_{\mu\nu}(x,\rho) = g_{\mu\nu}(x) + \rho {h^{(1)}}_{\mu\nu} + \frac{1}{2}\rho^2 {h^{(2)}}_{\mu\nu} + \cdots$$

The coefficients find obstructions in even d

$$\begin{aligned}
{h^{(1)}}_{\mu\nu} &= 2P_{\mu\nu} = \frac{2}{d-2}\left(R_{\mu\nu} - \frac{R}{2(d-1)} g_{\mu\nu}\right) \\
{h^{(2)}}_{\mu\nu} &= -\frac{2}{d-4} B_{\mu\nu} + 2P_{\mu\sigma}{P^{\sigma}}_{\nu} \\
{h^{(3)}}_{\mu\nu} &= \frac{2}{(d-6)(d-4)} \nabla^2 B_{\mu\nu} + \cdots
\end{aligned}$$

Ambient Laplacian

Scalar Laplacian of the embedding

$$-\Box_{\tilde{g}}\Phi = -\frac{1}{t^2}\Box_h\Phi - \frac{2}{t}\partial_t\partial_\rho\Phi - \frac{1}{2t}\partial_t\Phi - \frac{d-2}{t^2}\partial_\rho\Phi + \frac{\rho}{t^2}{h'_\mu}^\mu\partial_\rho\Phi$$

Consider an embedding scalar field

$$\Phi = t^{\Delta_\varphi}\varphi(x)$$

The projection of the Laplacian gives Yamabe

$$-\Box_{\tilde{g}}(t^{\Delta_\varphi}\varphi(x))|_{\rho=0} = t^{\Delta_\varphi-2}\Big(-\Box_g - \frac{R}{2(d-1)}\Big)\varphi$$

We can construct a family of powers of conformal GJMS Laplacians

$$P_{2n}\varphi(x) \equiv t^{-\frac{2n+d}{2}}(-\Box_{\tilde{g}})^n(t^{\frac{2n-d}{2}}\varphi)|_{\rho=0}$$

Conformal Laplacians

There are derivative and constant parts

$$P_{2n}\varphi(x) = \Delta_{2n} + \frac{d-2n}{2} Q_{2n}$$

Constant part transforms nicely: Q-curvatures in $d = 2n$

$$\sqrt{g} Q_d = \sqrt{\bar{g}} (\bar{Q}_d + \bar{\Delta}_d \sigma)$$

In fact we just found in $d = 2n$

$$\tilde{E}_d = dQ_d + \text{conformal invariants}$$

A physicist proof of Cardy's conjecture

The anomaly is best parametrized

$$\langle T^{\mu}{}_{\mu} \rangle = \sum_i b_i \mathcal{W}_i + a Q_d + \nabla_\mu \mathcal{J}^\mu$$

So that the integration is always possible

$$\Gamma = \Gamma_c[g] + \int \mathrm{d}^d x \sqrt{g} \mathcal{L}_{\text{local}} + \int d^4 x \sqrt{g} \left(b_i \mathcal{W}_i + a_1 Q_d \right) \frac{1}{\Delta_d} Q_d$$

- Ambient curvatures enumerate conformal invariants
- Scaling analysis dictates local anomaly ($\mathcal{J}^\mu$ is like a "virial" current)
- Ambiguities in defining Δ_d come from embedding Riemann in $d \geq 6$

Part 2:
Wess-Zumino consistency conditions of Weyl symmetry

The gauged Weyl group

Introduce an Abelian gauge potential

$$g_{\mu\nu} \to g'_{\mu\nu} = \mathrm{e}^{2\sigma} g_{\mu\nu} \qquad S_\mu \to S'_\mu = S_\mu - \partial_\mu \sigma \qquad \Phi \to \Phi' = \mathrm{e}^{w_\Phi \sigma} \Phi$$

The gauged covariant derivative

$$\hat{\nabla}_\mu \Phi = \nabla_\mu \Phi + L_\mu \cdot \Phi + w_\Phi S_\mu \Phi$$
$$(L_\mu)^\alpha{}_\beta = \frac{1}{2}(S_\beta \delta^\alpha_\mu + S_\mu \delta^\alpha_\beta - S^\alpha g_{\beta\mu})$$

It transforms covariantly under Weyl

$$\hat{\nabla}_\mu \Phi \to \hat{\nabla}'_\mu \Phi' = \mathrm{e}^{w_\Phi \sigma} \hat{\nabla}_\mu \Phi$$

Classical consequences

There is a new dilation current

$$T^{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \qquad\qquad D^{\mu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta S_{\mu}}$$

Gauged Weyl and Diff symmetries imply

$$T^{\mu}{}_{\mu} = \nabla^{\mu} D_{\mu} \qquad\qquad \hat{\nabla}_{\mu} T^{\mu\nu} + D_{\mu} W^{\mu\nu} = 0$$

In flat space $g_{\mu\nu} \to \delta_{\mu\nu}$ and $S_{\mu} \to 0$ imply scale invariance with $J_{\mu} = D_{\mu}$

$$T^{\mu}{}_{\mu} = \partial^{\mu} J_{\mu} \qquad\qquad \partial_{\mu} T^{\mu\nu} = 0$$

Renormalization with local couplings

Suppose $S \supset -\int \mathrm{d}^d x \sqrt{g}\lambda^i(x)\mathcal{O}_i$ and a finite renormalized path-integral

$$\mathrm{e}^{-\Gamma} = \int [\mathrm{d}\Phi]\,\mathrm{e}^{-S}$$

Currents source the expectation values

$$\langle T^{\mu\nu}\rangle = -\frac{2}{\sqrt{g}}\frac{\delta\Gamma}{\delta g_{\mu\nu}} \qquad \langle D^{\mu}\rangle = \frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta S_{\mu}} \qquad \langle\mathcal{O}_i\rangle = -\frac{1}{\sqrt{g}}\frac{\delta\Gamma}{\delta\lambda^i}$$

We expect $[\mathrm{d}\Phi]$ to give an anomaly

$$\langle T^{\mu}{}_{\mu}\rangle = \langle\nabla^{\mu}D_{\mu}\rangle + \text{beta terms} + \text{curvatures}$$

Local rg interpretation

Local scale transformation on the geometrical sources

$$\Delta_\sigma^W = \int \left\{ 2\sigma g_{\mu\nu} \frac{\delta}{\delta g_{\mu\nu}} - \partial_\mu \sigma \frac{\delta}{\delta S_\mu} \right\}$$

Local scale transformations caused by the rg beta functions

$$\Delta_\sigma^\beta = -\int \sigma \beta^i \frac{\delta}{\delta \lambda^i}$$

The anomaly $\langle T^\mu{}_\mu \rangle - \langle \nabla^\mu D_\mu \rangle = \cdots$ becomes

$$\Delta_\sigma^W \Gamma = \Delta_\sigma^\beta \Gamma + A_\sigma \qquad\qquad A_\sigma \supset \{\partial_\mu \lambda^i, R, S_\mu \cdots\}$$

Wess-Zumino consistency

Rewrite

$$\Delta_\sigma \Gamma = (\Delta_\sigma^W - \Delta_\sigma^\beta)\Gamma = A_\sigma$$

For Wess-Zumino's consistency

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = 0$$

Consistency condition for the anomaly

$$(\Delta_\sigma^W - \Delta_\sigma^\beta)A_{\sigma'} - (\sigma \leftrightarrow \sigma') = 0$$

Two dimensions

Most general parametrization of A_σ using $\hat{R} = R - 2\nabla^\mu S_\mu$ in $d = 2$

$$A_\sigma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\Big\{\sigma\frac{\beta_\Phi}{2}\hat{R} - \sigma\frac{\chi_{ij}}{2}\partial_\mu g^i\partial^\mu g^j - \partial_\mu\sigma w_i\partial^\mu g^i + \sigma\beta_\Psi\nabla_\mu S^\mu + \sigma\frac{\beta_2^S}{2}S_\mu S^\mu - \partial_\mu\sigma\beta_3^S S^\mu + \sigma z_i\partial_\mu g^i S^\mu\Big\}$$

Apply Wess-Zumino's

$$[\Delta_\sigma, \Delta_{\sigma'}]\Gamma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\big(\sigma\partial_\mu\sigma' - \sigma'\partial_\mu\sigma\big)\mathcal{Z}^\mu = 0$$

Conditions among tensors and β^i

$$\mathcal{Z}_\mu = \partial_\mu g^i\mathcal{Y}_i + S_\mu\mathcal{X} = 0$$

$$\mathcal{Y}_i = -\partial_i\beta_\Psi + \chi_{ij}\beta^j - \beta^j\partial_j w_i - w^j\partial_i\beta_j + z_i$$

$$\mathcal{X} = \beta_2^S - \beta^i\partial_i\beta_3^S - z_i\beta^i$$

(Ir)reversibility

Define a new charge

$$\tilde{\beta}_\Psi = \beta_\Psi + w_i \beta^i + \beta_3^S$$

Using both $\mathcal{Y}_i = 0$ and $\mathcal{Z} = 0$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{\beta}_\Psi = \beta^i \partial_i \tilde{\beta}_\Psi = \chi_{ij} \beta^i \beta^j + \beta_2^S$$

Reproduce standard local rg taking $\beta_\Psi = \beta_\Phi$, $\beta_2^S = \beta_3^S = 0$. $\tilde{\beta}_\Psi$ becomes Osborn's $\tilde{\beta}_\Phi$
There is a scheme (Zamolodchikov's) in which $\chi_{ij} \to G_{ij} = |x|^4 \langle \mathcal{O}_i(x) \mathcal{O}_j(0) \rangle > 0$

$$\mu \frac{\mathrm{d}}{\mathrm{d}\mu} \tilde{\beta}_\Psi > 0$$

In general: $\beta_2^S \geq 0$?

A weird application: higher derivative scalar

Higher derivative free scalar is a CFT in flat space

$$\mathcal{L} = \frac{1}{2}(\partial^2\varphi)^2$$

Notice that $\langle\varphi(x)\varphi(0)\rangle \sim |x|^2$ for φ primary, in contrast with $(\partial_x^2)^2\langle\varphi(x)\varphi(0)\rangle \sim \delta(x)$

It does not admit a conformal action in $d = 2$ because of the obstruction

$$S_{\rm conf}[\varphi, g] = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\varphi\Delta_4\varphi$$

$$\Delta_4\varphi = (\nabla^2)^2\varphi + 2\nabla^\mu\Big(P_{\mu\nu}\nabla^\nu\varphi + \cdots\Big) - (d-4)\Big(P^{\mu\nu}P_{\mu\nu} + \cdots\Big)\varphi$$

$$P_{\mu\nu} = \frac{1}{d-2}\left\{R_{\mu\nu} - \frac{1}{2(d-1)}Rg_{\mu\nu}\right\}$$

Gauged higher derivative scalar

Assign the weight $w(\varphi) = \frac{4-d}{2} \to 1$

$$S[\varphi, g_{\mu\nu}, S_\mu] = -\frac{1}{2}\int \mathrm{d}^2x\sqrt{g}\varphi(\hat{\nabla}^2)^2\varphi$$

It does exists in $d = 2$

$$(\hat{\nabla}^2)^2\varphi = (\nabla^2)^2\varphi + B^{\mu\nu}\nabla_\mu\partial_\nu\varphi + C^\mu\partial_\nu\varphi + D\varphi$$
$$B_{\mu\nu} = 2g_{\mu\nu}S^\rho S_\rho - 4S_\mu S_\nu + 4\nabla_{(\mu}S_{\nu)}$$

Using heat kernel methods $\beta_2^S = 0$, $\beta_\Phi = \frac{1}{3}$ and $\beta_\Psi = \frac{4}{3}$

$$A_\sigma = \frac{1}{2\pi}\int \mathrm{d}^2x\sqrt{g}\sigma\left\{\frac{R}{6} + \nabla^\mu S_\mu\right\}$$

Conclusions

- FG embedding is the natural framework to study the conformal anomaly
  $\Longrightarrow$ Complete the proof of Cardy's conjecture

- Local RG can be generalized to gauged Weyl symmetry
  $\Longrightarrow$ Extend to $d = 4$ and study "scale vs conformal"

Thank you for listening