The spectrum of Asymptotic Safety from 2+ε gravity

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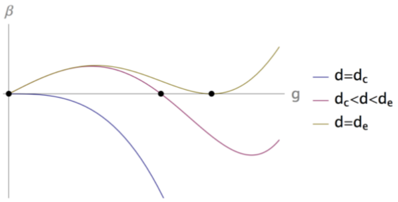
Motivation I

- Original formulation of asymptotic safety [**Weinberg, 1979**] based on results in gravity close to $d = 2$



Credit: H. Gies

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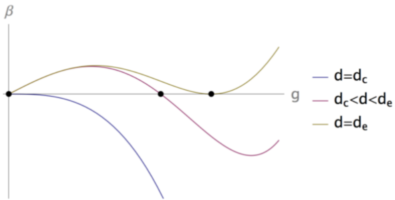

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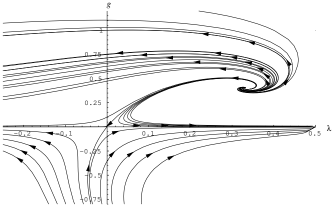

credit: Reuter & Saueressig, 2001

• First application of non-perturbative RG flow to test the conjecture in $d = 4$ [**Reuter, 1998**]

• Non-perturmative computations based on Functional renormalization group [**Wetterich, 1993; Morris, 1993**]

Motivations: II

Functional Renormalization Group cons:

- Theory space is infinitely dimensional $\to$ need of truncation
- Strong regulator and scheme dependence
- Gauge dependent results

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Essential Renormalization Group •

[Weinberg, 1979; Benedetti, 2011; Anselmi, 2013; Baldazzi, Zinati, Falls; 2021]

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Revise perturbative methods

- Revise perturbative computations in $d = 2 + \varepsilon$
- Does the analytic continuation $d = 2 \rightarrow 4$ interpolate higher derivative gravity?

[Stelle, 1976; Fradkin & Tseyling, 1982; Avramidi & Barvinsky, 1985...]

How to identify universality classes?

• Identify the symmetries of the system

• Identify a critical dimension d_c

• At d_c the perturbative expansion of the theory is controlled by a set of marginal operators.

• Require possible analytic continuation in some dimensional range

For a metric field content one has:

$$\mathcal{O} \sim \mathcal{R}^n \qquad d_c = 2n\,.$$

- $d_c = 2$ Kawai-Ninomiya universality class **Kawai, Ninomiya, Aida, Kitazawa, Nishimura, Tsuchiya, ..., 1993-1997**
- $d_c = 4$ Stelle universality class **Stelle, 1976**
- $d_c = 6$ Cubic gravity **Knorr, 2021**

Divergences in dimensional regularization

Consider a massive scalar ϕ^4-theory in 4-dimensions

$$\mu^{4-d}\int \frac{\mathrm{d}^d k}{k^2+m^2} = \mu^{4-d}(\pi m^2)^{d/2}\Gamma\left(1-\frac{d}{2}\right)$$

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Expanding in $\varepsilon = 4-d$ one finds the following mapping between cut-off divergences and ε-poles

$$\log\Lambda \leftrightarrow \frac{1}{\varepsilon}$$

$$\Lambda^2 \leftrightarrow 0$$

$$\Lambda^4 \leftrightarrow 0$$

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$$\log\Lambda \leftrightarrow \frac{1}{\varepsilon^L}$$

$$\Lambda^2 \leftrightarrow \frac{\mu^2}{\varepsilon^{(L)}}$$

$$\Lambda^4 \leftrightarrow \frac{\mu^4}{\bar{\varepsilon}^{(L)}} \quad \text{(vacuum diagrams)}$$

where $\varepsilon^{(L)} = 4 - \frac{2}{L} - d$ and $\bar{\varepsilon}^{(L)} = 4 - \frac{4}{L} - d$. **[Al-sarhi, Jack, Jones; '90-'91]**

The theory can be made finite for **all** dimensions $d < 4$.

2d Gravity: an old puzzle

Linear splitting
(ex. **Jack and Jones, 1990**)

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$$

$$S_E[g] = \int \mathrm{d}^d x \sqrt{g} \Big\{ g_0 - g_1 R \Big\}$$

central charge = 19

$$\beta_G = \varepsilon G - \frac{19}{24\pi} G^2$$

$$\boxed{\langle T^\mu_\mu \rangle \overset{\varepsilon \to 0}{=} \beta_G = ?}$$

Exponential splitting
(ex. **Aida, Kitazawa, Kawai, Ninomiya, 1994**)

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} \varphi^{\frac{4}{d-2}} = \bar{g}_{\mu\rho} \left(e^h\right)^\rho_{\ \nu} \varphi^{\frac{4}{d-2}}$$

$$S_D[\varphi, \tilde{g}] = -\, g_1 \int \mathrm{d}^d x \sqrt{\tilde{g}} \Big\{ \varphi^2 \tilde{R} + \frac{1}{\xi_c} \tilde{g}^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \Big\} + g_0 \int \mathrm{d}^d x \sqrt{\tilde{g}} \varphi^{\frac{2d}{d-2}}$$

$\xi_c = \frac{d-2}{4(d-1)} \Rightarrow$ central charge = 25

$$\beta_G = \varepsilon G - \frac{25}{24\pi} G^2$$

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Einstein-Hilbert gravity

Consider the Einstein-Hilbert action

$$S_{EH}[g] = \int d^d x \sqrt{g}\ (g_0 - g_1 R)$$

Diffeomorphism-invariance plays the role of a gauge symmetry: $\delta_\zeta g_{\mu\nu} = \nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu$
We look at the parametrization dependence

$$g_{\mu\nu} = \overline{g}_{\mu\nu} + h_{\mu\nu} + \frac{\lambda}{2} h_{\mu\rho} \overline{g}^{\rho\theta} h_{\theta\nu} + \mathcal{O}(h^3)$$

and gauge dependence of the 1-loop divergences: $\delta_\zeta h_{\mu\nu} = \bar{g}_{\mu\rho} \bar{\nabla}_\nu \zeta^\rho + \bar{g}_{\nu\rho} \bar{\nabla}_\mu \zeta^\rho + o(h)$

$$\begin{aligned} S_{gf}[h;\bar{g}] =& \frac{1}{2} \int d^d x \sqrt{\bar{g}}\ \bar{g}^{\mu\nu} F_\mu F_\nu\,, \\ F_\mu =& \bar{\nabla}_\rho h^\rho_\mu - \frac{1+\delta\xi}{2} \bar{\nabla}_\mu h^\rho_\rho\,, \\ S_{gh}[h,c,\bar{c};\bar{g}] =& \int d^d x \sqrt{\bar{g}}\ \ \bar{c}^\mu \delta_\zeta F_\mu |_{\zeta\to c} \end{aligned}$$

Dimensions as symmetry parameter

In $d = 2$ we have a discontinuity in the number of the degrees of freedom

$$\frac{\delta^2}{\delta h_{\mu\nu}\delta h_{\alpha\beta}}(S_{EH} + S_{gf}) = -K^{\mu\nu\alpha\beta}\nabla^2 + E^{\mu\nu\alpha\beta}\,,$$

$$K^{-1}_{\mu\nu\alpha\beta} = \frac{1}{2}\left(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - \frac{1}{d-2}g_{\mu\nu}g_{\alpha\beta}\right)$$

The kinematic pole enters the structure of subdivergences in the loop expansion

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Modified dimensional regularization

$$\int \mathrm{d}^2 x = \mu^{-\varepsilon}\int \mathrm{d}^d x \qquad\qquad g^{\mu}_{\mu} = d \neq 2 - \varepsilon$$

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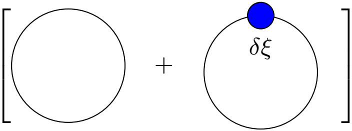
d in $\mathrm{Diff}(\mathcal{M}_d)$ has a similar role to N in $SU(N)$ gauge theories

On-shell divergences

$$\Gamma[0;\bar{g}] = \frac{1}{2}$$



The 1-loop divergences in $d = 2 - \varepsilon$ can be arranged in the following way

$$\Gamma^{1-loop}_{\text{div}}[0;\bar{g}] = \frac{\mu^{-\varepsilon}}{\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\left\{A\overline{R} + J_{\mu\nu}\left(\overline{G}^{\mu\nu} + \frac{g_0}{2g_1}\bar{g}^{\mu\nu}\right)\right\},$$

$$A = \frac{36 + 3d - d^2}{48\pi}, \qquad J_{\mu\nu} = \frac{\bar{g}_{\mu\nu}}{4\pi}\left\{\frac{d^2 - d - 4}{2(d-2)}\lambda - \delta\xi\left(2 + \frac{2\lambda}{d-2}\right) - d - 1\right\}$$

And the consequent β-function for $G_N = \frac{1}{g_1}$ is
[Falls, 2015; R.M. et al., 2021; Bastianelli et al., 2022]

$$\beta_G = \varepsilon G - \frac{36 + 3d - d^2}{48\pi}G^2 \quad \rightarrow \quad \beta_G = -\frac{19}{24\pi}G^2$$

$$G^* = -\frac{48\pi(d-2)}{d^2 - 3d - 36} \quad \text{for} \quad d \lesssim 7.6$$

The dilaton realization

We can repeat the same analysis for unimodular-dilaton action and postulate

$$\Gamma_{\text{div}}[0,0;\bar{g},\bar{\varphi}] = \frac{\mu^{-\varepsilon}}{\varepsilon}\int \mathrm{d}^d x\sqrt{\bar{g}}\left\{B\bar{R} + Je\right\}$$

e being the e.o.m. for $\bar{\varphi}$. However

$$\begin{aligned} B = &-\frac{11d^4 - 44d^3 - 78d^2 + 180d - 72}{96\pi d(d-1)} \\ &- \frac{(d-2)(18d^5 - 35d^4 - 132d^3 + 152d^2 + 48d - 48)}{192\pi d^3(d-1)}\delta\beta\,, \\ J = &-\frac{3d^3 - 6d^2 - 12d + 16}{8\pi d} - \frac{(d-2)(3d^2-4)(2d^2+d-2)}{16\pi d^3}\delta\beta \end{aligned}$$

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- Not suitable for continuation away from $d = 2$.
- As $d \to 2$ the volume operator $\lambda(\varphi)$ has a trivial scaling
- The topological charge has a non-trivial running proportional to $(d-2)$
- As $d \to 2$ we recover the known result $\beta_G = -\frac{25}{24\pi}G^2$
- Choosing the scheme where e.o.m. are solved for the volume operator we recover $\beta_G = -\frac{19}{24\pi}G^2$

Deformation of Einstein-Hilbert theory

Include higher derivative operators as composite:

$$S_{\rm hd}[g] = \int \mathrm{d}^d x \sqrt{g} \Big\{ \alpha_1 R^2_{\mu\nu\rho\sigma} + \alpha_2 R^2 + I_{2,\mu\nu} E[g]^{\mu\nu} \Big\} \equiv -\overrightarrow{\alpha} \cdot \mathcal{R}_2$$

$$I_{2,\mu\nu} = \alpha_3 R g_{\mu\nu} + \alpha_4 R_{\mu\nu} + \alpha_5 \Lambda g_{\mu\nu} \, .$$

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Divergences of effective action for the composite operator

$$\Gamma_\infty[0,\alpha;\bar{g}] = -\overrightarrow{\alpha}\left[\frac{1}{2}\mathrm{Tr}\frac{\delta^2\mathcal{R}_2}{\delta h_{\mu\nu}\delta h_{\rho\sigma}} G_{\rho\sigma\mu\nu} - \frac{g_1}{\varepsilon} J_{\mu\nu}\frac{\delta\mathcal{R}_2}{\delta h_{\mu\nu}}\right]_{h=0}$$

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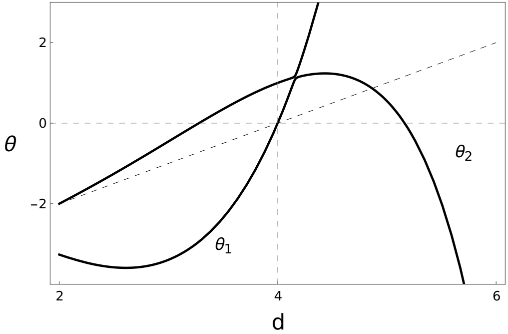
Reduce the system to only two operators with the aid of e.o.m.s

$$\begin{pmatrix}\tilde{\beta}_1\\ \tilde{\beta}_2\end{pmatrix} = \begin{pmatrix}\hat{\gamma}_{11} & \hat{\gamma}_{12}\\ \hat{\gamma}_{21} & \hat{\gamma}_{22}\end{pmatrix}\begin{pmatrix}\tilde{\alpha}_1\\ \tilde{\alpha}_2\end{pmatrix}$$

with eigenvalues $\{-\theta_1, -\theta_2\}$
In $d = 4$

$$v_1 = (2,1)\,, \quad \theta_1 = 0$$
$$v_2 = (0,1)\,, \quad \theta_2 = 1$$



Special cases: $d = 2, 3$

In $d = 2$ one has the following identities

$$R_{\mu\nu\rho\sigma} = \frac{R}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \,, \quad R_{\mu\nu} = \frac{R}{2}g_{\mu\nu}$$

because of which α_1 and α_2 are not independent. Therefore

$$\beta_2^{d=2} = (\beta_1 + \beta_2)_{\alpha_1=0}^{d->2}$$

We find the classical scaling for α_2: $\theta = -2$.

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In $d = 3$

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = 4R_{\mu\nu}R^{\mu\nu} - R^2$$

and we have to consider

$$\beta_2^{d=3} = (-\beta_1 + \beta_2)^{d->3}_{\alpha_1=0}$$

with critical exponent $\theta = -\frac{49}{108}$.

Conclusions and Outlook

Conclusions:

- Non trivial flow for G_N from poles in $d \to 2$
- finite size conformal window for the UV-fixed point
- Dilaton realization not suitable for analytic continuation $d \to 4$
- One relevant operator of order $\sim \mathcal{R}^2$

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Outlook

- Test gauge dependence of composite operators
- Inspect $\sim \mathcal{R}^3$ operators
- Two-loop computations of Einstein-Hilbert gravity in $d = 2 + \varepsilon$ to prove renormalizability

Thank you!

Wave function renormalization

Given an action $S[g]$ for a field g_x over a background $\bar{g}_x$ we have the functionals

$$Z[j;\bar{g}] = e^{W[j;\bar{g}]} = \int \mathcal{D}h\ \exp\left\{-S[\bar{g}+h] + j_x h_x\right\}\ ,$$

$$\Gamma[H;\bar{g}] = j_x[H]H_x - W[j[H]] \quad \text{with} \quad H_x \equiv \frac{\delta W[j]}{\delta J_x} = \langle h_x \rangle_c$$

One loop approximation close to some critical dimension $d = d_c - \varepsilon$

$$\frac{1}{2}\mathrm{Tr}\log \bar{S}'' = -\frac{1}{\varepsilon}\mathcal{J}_x \bar{S}'_x + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

$$\Gamma[H;\bar{g}] = \bar{S} + \bar{S}'_x\left(H_x - \frac{1}{\varepsilon}\mathcal{J}_x\right) + \frac{1}{2}H_x\bar{S}''_{xy}H_y + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

However $j_x[H] = 0 \Leftrightarrow H_x = -G_{xy}\bar{S}'_y \overset{\text{o.s.}}{=} 0$

$$\Gamma[0;\bar{g}] = \bar{S} - \bar{S}'_x\frac{1}{\varepsilon}\mathcal{J}_x + \frac{1}{\varepsilon}\Delta\Gamma_\infty + \Delta\Gamma_f$$

Field redefinition

Change variable $\tilde{h}_x = h_x - \frac{1}{\varepsilon}\mathcal{J}_x$ and add the counterterm

$$Z_R[j;\bar{g}] = e^{\frac{1}{\varepsilon}\Delta\Gamma_\infty} \int \mathcal{D}\tilde{h}\ \exp\left\{-S\left[\bar{g} + \tilde{h} + \frac{1}{\varepsilon}\mathcal{J}\right] + j_x\tilde{h}_x\right\}$$

We can read the renormalized equations of motion from the relation between j_x and $\tilde{H}_x$

$$j_x[\tilde{H}] = \bar{S}''_{xy}\tilde{H}_y + \bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_x$$

leading to

$$\Gamma_R[\tilde{H};\bar{g}] = \bar{S} + \Delta\Gamma_f + \tilde{H}_x\left(\bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_y\right) + \frac{1}{2}\tilde{H}_x\bar{S}''_{xy}\tilde{H}_y + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{S}''_{xy}\mathcal{J}_y$$

$$\Downarrow$$

$$\Gamma_R[0;\bar{g}] = \bar{S} + \Delta\Gamma_f + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{S}''_{xy}\mathcal{J}_y$$

Composite operators

We are interested in computing the expectation value of an operator $O[g]$

$$\langle O[g] \rangle \equiv \int \mathcal{D}h \; O[\bar{g}+h] \exp\left\{-S[\bar{g}+h] + j_x h_x\right\}$$

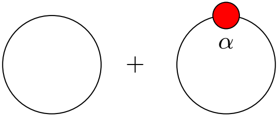
$$= \frac{\partial}{\partial\alpha}\bigg|_{\alpha=0} \int \mathcal{D}h \;\; \exp\left\{-S[\bar{g}+h] + \alpha O[\bar{g}+h] + j_x h_x\right\} \equiv \frac{\partial}{\partial\alpha}\bigg|_{\alpha=0} \exp\{W[j,\alpha;\bar{g}]\}$$

define

$$\Gamma[H,\alpha;\bar{g}] = H_x j_x - W[j,\alpha;\bar{g}] \quad \Rightarrow \quad \langle O[\bar{g}+h] \rangle = -\frac{\partial}{\partial\alpha}\bigg|_{\alpha=0} \Gamma[j^{-1}[0],\alpha,\bar{g}]$$

Effectively the action is shifted $A[g] = S[g] - \alpha O[g]$

$$\frac{1}{2}\mathrm{Tr}\log A'' = \frac{1}{2}\mathrm{Tr}\log\left(\bar{S}'' - \alpha\bar{O}''\right) = -\frac{1}{\varepsilon}\mathcal{J}_x \bar{S}'_x + \frac{1}{\varepsilon}\Delta\Gamma^O_\infty + \Delta\Gamma^O_f \,,$$

$\Gamma[H,\alpha;\bar{g}] =$ 

Composite operators and wave function renormalization

$$j_x[H] = \left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y + \bar{S}'_x - \alpha\bar{O}'_x \,,$$

leading to

$$\begin{aligned}\Gamma[H,\alpha;\bar{g}] =& \bar{S} - \alpha\bar{O} + (H_x - \frac{1}{\varepsilon}\mathcal{J}_x)\bar{S}'_x - \alpha H_x\bar{O}'_x + \frac{1}{2}H_x\left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y \\ &+ \frac{1}{\varepsilon}\Delta\Gamma^O_\infty + \Delta\Gamma_f\end{aligned}$$

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$$\Gamma[H,\alpha;\bar{g}] = \bar{S} - \alpha\bar{O} + (H_x - \frac{1}{\varepsilon}\mathcal{J}_x)\bar{S}'_x - \alpha H_x\bar{O}'_x + \frac{1}{2}H_x\left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right)H_y + \frac{1}{\varepsilon}\Delta\Gamma^{O}_{\infty} + \Delta\Gamma_f$$

- $H[0] = j^{-1}[0] = 0 \quad \Leftrightarrow \quad \bar{S}'_x = \alpha\bar{O}'_x$

  Then $-\frac{1}{\varepsilon}\mathcal{J}_x\bar{S}' \to -\frac{\alpha}{\varepsilon}\mathcal{J}_x\bar{O}'$ new counterterm to include

Composite operators and wave function renormalization

$$j_x[H] = \left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y + \bar{S}'_x - \alpha\bar{O}'_x \,,$$

leading to

$$\Gamma[H,\alpha;\bar{g}] = \bar{S} - \alpha\bar{O} + (H_x - \frac{1}{\varepsilon}\mathcal{J}_x)\bar{S}'_x - \alpha H_x\bar{O}'_x + \frac{1}{2}H_x\left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right)H_y + \frac{1}{\varepsilon}\Delta\Gamma^{O}_{\infty} + \Delta\Gamma_f$$

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  Then $-\frac{1}{\varepsilon}\mathcal{J}_x\bar{S}' \to -\frac{\alpha}{\varepsilon}\mathcal{J}_x\bar{O}'$ new counterterm to include
- Shift H_x as before to take care of $\mathcal{J}_x$

  Then $-\alpha H_x\bar{O}'_x \to -\alpha\tilde{H}_x\bar{O}'_x - \frac{\alpha}{\varepsilon}\mathcal{J}_x\bar{O}'$ new counterterm

Composite operators and wave function renormalization

$$j_x[H] = \left(\bar{S}''_{xy} - \alpha\bar{O}''_{xy}\right) H_y + \bar{S}'_x - \alpha\bar{O}'_x \,,$$

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- Shift H_x as before to take care of $\mathcal{J}_x$

  Then $-\alpha H_x\bar{O}'_x \to -\alpha\tilde{H}_x\bar{O}'_x - \frac{\alpha}{\varepsilon}\mathcal{J}_x\bar{O}'$ new counterterm

$$\frac{\langle O\rangle}{\langle 1\rangle} = -\frac{\partial}{\partial\alpha}\Gamma_R[0,\alpha;\bar{g}]\Big|_{\alpha=0} = \bar{O} + \frac{1}{2\varepsilon^2}\mathcal{J}_x\bar{O}''_{xy}\mathcal{J}_y - \frac{\partial}{\partial\alpha}\Delta\Gamma^O_f\Big|_{\alpha=0}$$

$$\bar{S}'_x + \frac{1}{\varepsilon}\bar{S}''_{xy}\mathcal{J}_x = 0$$