UV properties of some Higher Derivative models

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UV Complete Quantum Field Theories for Particle Physics

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Outline :

- Higher Derivative (HD) Scalar Theories and phase structure.

  HD → New scaling dimensions of fields.

  Appearance of new fixed points : Lifshitz points (Anisotropic and Isotropic).

- Anisotropic model with 6 space derivatives and 2 time derivatives.

  Number of time derivatives =2 → Unitary models.

  HD in space → Renormalizable model. (Lorentz symmetry violated).

  Direct extension to fermionic and gauge theories.

- Isotropic model with 4 derivatives (both in space and time).

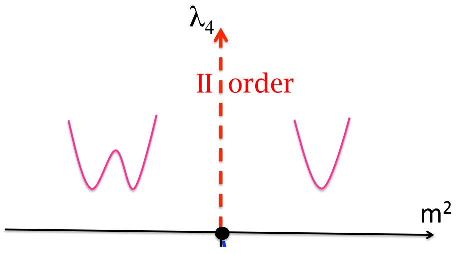
  Symmetry among coordinates recovered. Unitarity is lost.

  Affinity of HD vector model O(2) in d=4 with standard O(2) model in d=2.

  Appearance of a topological phase and of a line of UV attractive FPs.

Standard kinetic sector $\partial\varphi\partial\varphi$ → minima of the potential

Critical point

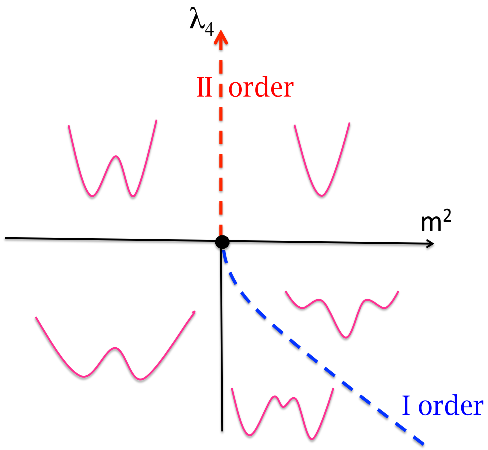


$V_{eff} = m^2 \varphi^2 + \lambda_4 \varphi^4$

Phase transition from
interplay of two parameters

$\lambda_4 > 0 \quad m^2$

Tricritical point



$V_{eff} = m^2 \varphi^2 + \lambda_4 \varphi^4 + \lambda_6 \varphi^6$

Phase transition from
interplay of three parameters

$\lambda_6 > 0 \quad \lambda_4 \quad m^2$

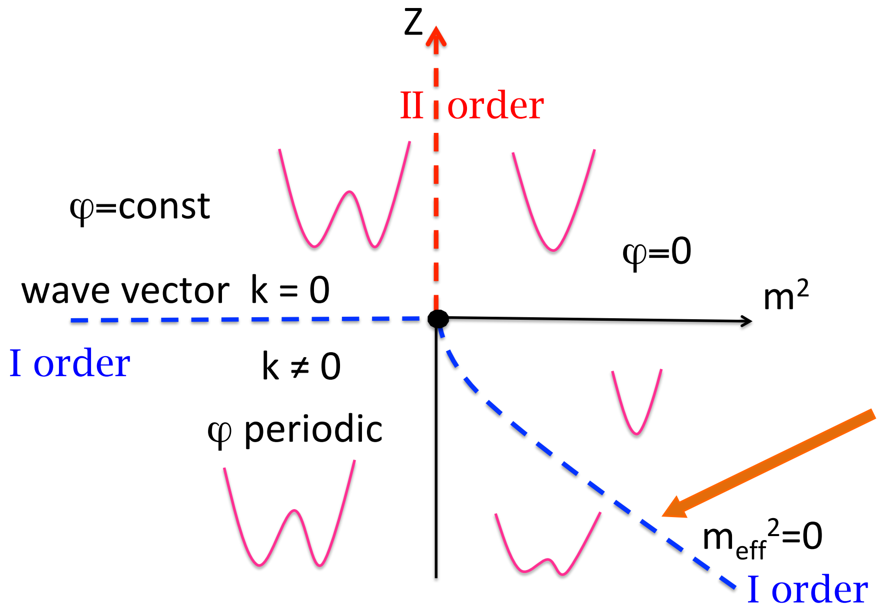
Improved kinetic sector with Higher Derivatives $W\,(|\partial^2\varphi|)^2 + Z\,(|\partial\varphi|)^2$

With W > 0, and potential $V_{eff} = m^2\,|\varphi|^2 + \lambda_4\,|\varphi|^4$ →

interplay of three parameters λ_4 m^2 Z

If $Z < 0$ → new type of ground state : non-constant, modulated φ

Ground state	Minimize derivative sector	Change in Effective Potential
$\varphi = A \exp[i\,\mathbf{k_0}\cdot\mathbf{x}]$	$\mathbf{k}_0^2 = -\dfrac{Z}{2W}$	$m^2 \to m_{eff}^2 = m^2 - \dfrac{Z^2}{4W}$



New type of tricritical point

Tricritical Lifshitz Point

Singularities at m_{eff} =0 forbid II order transition

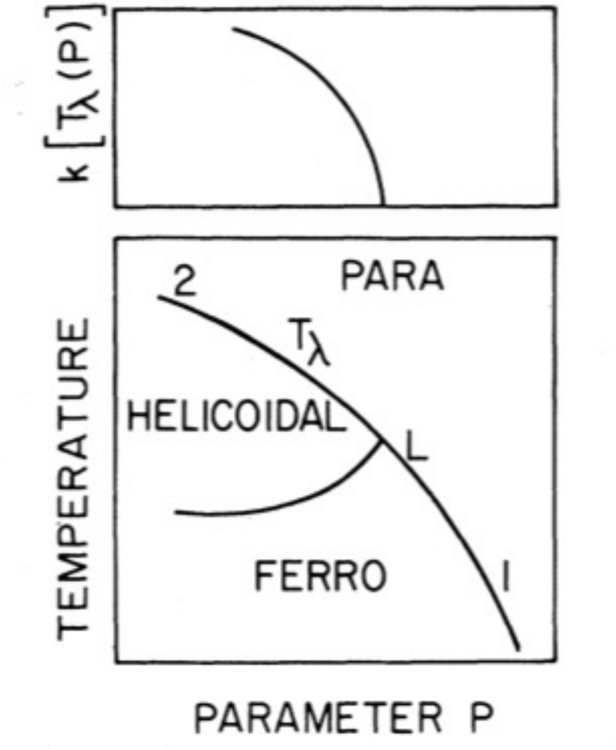
Brazovskii, Zh.Eksp.Teor.Fiz., 175(1975)

Coexistence of three phases at a TLP :

a modulated phase $\varphi = A \exp[\, i\, \mathbf{k}\bullet\mathbf{x}\,]$,

an ordered phase with φ =const $\neq 0$,

a disordered phase with $\varphi = 0$.



Anisotropic LP find application in Condensed Matter

(magnetic systems, liquid crystal, polymer mixtures)

R. Hornreich, M. Luban, S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.

R. Hornreich, J. Magnetism and Magnetic Materials 15 (1980) 387.

H.Diehl, Acta Phys. Slovaca 52 (2002) 271

Besides phase diagrams - related to ground state structure – of condensed matter systems, the general properties of the LP can open new scenarios on the UV nature of field theories especially because of the higher derivative property of smoothening the UV divergences.

E.g. Horava-Lifshitz gravity

P. Horava, Phys. Rev. D79, (2009) 084008.

LP from modified scaling :

Example of Euclidean anisotropic case with 2 and 4 derivatives

d-m "transverse" and m "parallel" space coordinates

$$\Gamma[\phi] = \int \mathrm{d}^{d-m}x_\perp \, \mathrm{d}^m x_\parallel \left\{ W_\parallel \, (\partial_\parallel^2 \phi)^2 + W_\perp \, (\partial_\perp^2 \phi)^2 + \frac{Z_\parallel}{2} (\partial_\parallel \phi)^2 + \frac{Z_\perp}{2} (\partial_\perp \phi)^2 + V(\phi) \right\}$$

If $Z_\parallel = 0$, $Z_\perp = 1$ → regular scaling dimension of transverse coordinates BUT

modified scaling of parallel coor. set by the 4-derivative term

↓

2 correlation lengths and anomalous dimensions: η_2 , η_4 associated to $Z_\perp$ and $W_\parallel$

$$\Gamma^{(2)}(\mathbf{k}_\parallel = 0, \mathbf{k}_\perp \to 0) \sim \mathrm{k}_\perp^{2-\eta_2} \qquad \Gamma^{(2)}(\mathbf{k}_\parallel \to 0, \mathbf{k}_\perp = 0) \sim \mathrm{k}_\parallel^{4-\eta_4}$$

and appearance of anisotropy parameter : θ $\qquad \mathrm{k}_\parallel = \mathrm{k}_\perp^\theta \qquad \theta = \frac{2-\eta_2}{4-\eta_4}$

Modified scaling admits stationary LPs.

R. Hornreich, M. Luban, S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.

Unitary models must have only 2 time-derivatives

but no constraint on the number of space derivatives is required

R.P.Woodard, Lect. Notes Phys. 720 (2007) 403.
D.Anselmi, M. Halat, Phys. Rev. D 76 (2007) 125011.

A suitable model (Minkowski metric) with polynomial potential is

$$S = \int d^3x\, dt \left(\frac{1}{2} (\partial_t \phi)^2 - \frac{\widehat{a}_z}{2} (\partial_i^z \phi)^2 - \frac{\widehat{a}_{z-1}}{2} (\partial_i^{z-1} \phi)^2 - \dots - \frac{\widehat{a}_1}{2} (\partial_i \phi)^2 - V \right)$$

$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!} \phi^n$$

Under scale transformations

$$x^i \to b\, x^i$$
$$t \to b^z\, t$$

with $b > 1$

The main scaling dimensions are $[x^i]_s = -1 \qquad [t]_s = -z \qquad [\widehat{a}_z]_s = -\eta$

and zero anomalous dimension of time is assumed.

Derivative operators with lower number of derivatives are relevant

Lifshitz Points and number of dimensions.

GENERAL CASE in d dim. with d_s space-like dim. and $d-d_s$ time-like dim.

$$[\phi]_s = [z(d-d_s-2)+d_s+\eta]/2$$
$$[\widehat{g}_n]_s = \left(1-\frac{n}{2}\right)[\,z(d-d_s)+d_s\,]+nz$$

Field scaling dimension

Coupling scaling dimension

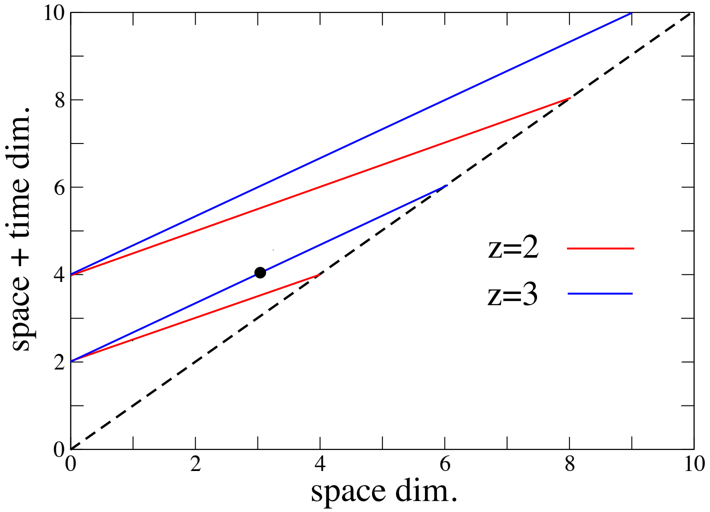
Upper and lower critical dimension established from vanishing of quartic coupling scaling dim. or of field scaling dim. respectively

$$[\widehat{g}_4]_s = 0$$
$$[\phi]_s = 0$$

Non-trivial LP admitted between red lines (z = 2) or blue lines (z = 3), by approximating $\eta=0$

Outside, only Gaussian-like LP is expected

$$S_{FP} = \int d^3x\,dt\left(\frac{1}{2}(\partial_t\phi)^2 - \frac{\widehat{a}_z}{2}(\partial_i^z\phi)^2\right)$$



The phenomenologically interesting case with d = 3+1 and d_s = 1 sits on the lower blue line

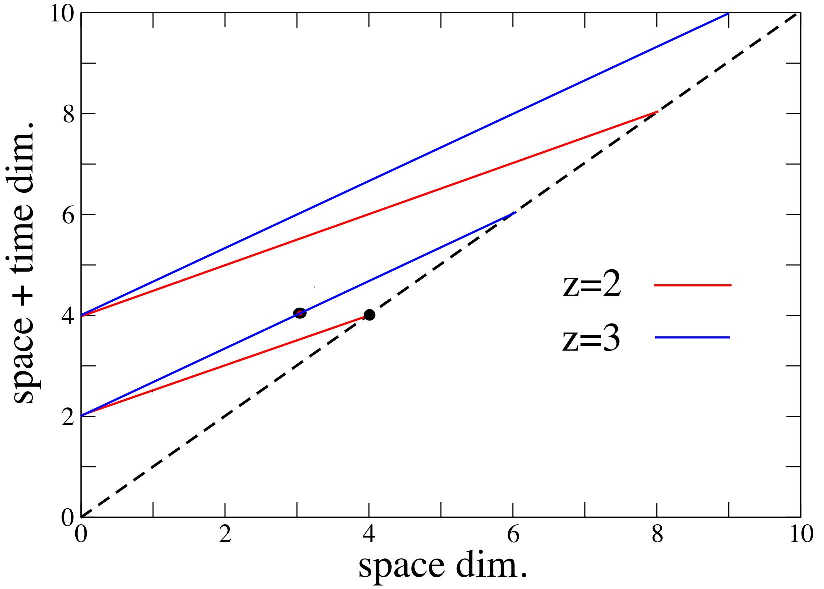
The larger z, the stronger the UV suppression.

The case z = 3 with only Gaussian-like LP should present a more favorable UV picture.

Isotropic case $d = d_s$

Restored symmetry among coordinates.

Only higher derivative terms (restriction of the plane to the dashed line).



Even in this case, $d = d_s = 4$ sits on the lower red line i.e. $z = 2$

but novel properties similar to the standard theory in d=2 show up.

Anisotropic case d = 3+1 and d_s = 3 and z=3

$$[\phi]_s = 0$$

zero scaling dimension

$$S = \int d^3x\, dt \left[\frac{1}{2}\left(\partial_t\phi\right)^2 - \frac{\widehat{a}_3}{2}\left(\partial^2\partial_i\phi\partial^2\partial_i\phi\right) - \frac{\widehat{a}_2}{2}\left(\partial^2\phi\partial^2\phi\right) - \frac{\widehat{a}_1}{2}\left(\partial_i\phi\partial_i\phi\right) - V(\phi)\right]$$

$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!}\phi^n$$

and, in principle, non-polynomial $\overline{n} \to \infty$

The canonical dimension of S requires the introduction of a reference mass scale M that re-defines all couplings

$$\widehat{a}_s = \frac{a_s}{M^{2(s-1)}}\Bigg|_{s=1,2,3}$$

$$\widehat{g}_n = \frac{g_n}{M^{n-4}}\Bigg|_{n=2,3,4,\ldots.}$$

This model violates Lorentz symmetry and the dispersion relation is

$$E^2 = \vec{k}^2\left[a_1 + a_2\left(\frac{k}{M}\right)^2 + a_3\left(\frac{k}{M}\right)^4\right] + g_2\, M^2$$

M separates two energy ranges :

$k >> M$ with manifest Lorentz violation

$k << M$ with suppression of the anomalous terms.

Pattern of divergences above the scale M

The propagator with HD suppressing factors

$$\frac{1}{\left[k_0^2 - A^2 + i\epsilon\right]}$$

with

$$A = \sqrt{\widehat{a}_3\,\vec{k}^{\,6} + \widehat{a}_2\,\vec{k}^{\,4} + \widehat{a}_1\,\vec{k}^{\,2} + \widehat{g}_2}$$

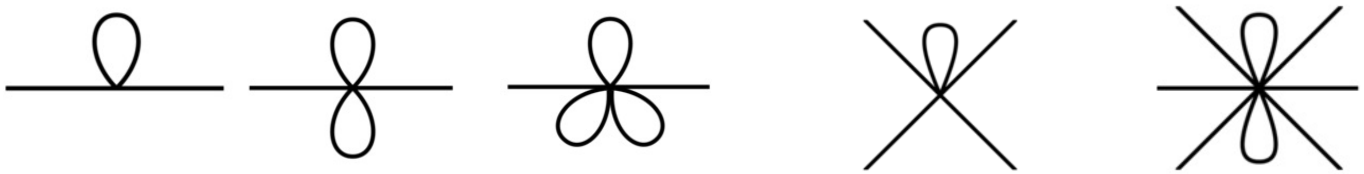
The corresponding primitive divergence depends only on the number of vertices

$$D_\Lambda = \left[3L - 2z\,(I - L) - z\,L\right]\big|_{z=3} = 6(L - I) = 6\left(1 - \sum_n V_n\right)$$

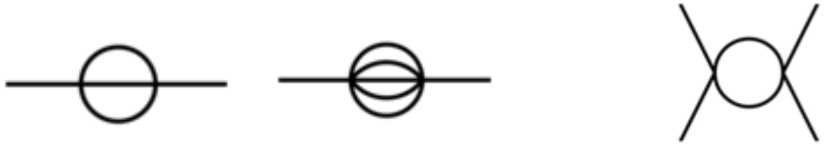
Only the tadpole diverges !

Use a three-momentum cut-off Λ (Lorentz violating)

$$\widehat{I}_1 = \int \frac{d^3k\,dk_0}{(2\pi)^4}\,\frac{i}{\left(k_0^2 - \widehat{a}_3\vec{k}^{\,6} - \widehat{a}_2\,\vec{k}^{\,4} - \widehat{a}_1\vec{k}^{\,2} - \widehat{g}_2 + i\epsilon\right)} = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\,\frac{1}{\sqrt{\widehat{a}_3\vec{k}^{\,6} + \widehat{a}_2\,\vec{k}^{\,4} + \widehat{a}_1\vec{k}^{\,2} + \widehat{g}_2}}$$



Divergent diagrams



UV Finite! → $\eta = 0$

Close form of the Action

No UV divergent correction that depends on the external momenta is generated by tadpoles

$$(p_i\, p_i')^s\ \mathcal{W}_{m,s} = (p_i\, p_i')^s \left[\left(\frac{\partial^2}{\partial p_j\, \partial p_j'} \right)^s \Gamma^{(m+2)}(p, p', 0, \ldots, 0) \right]_{p=p'=0}$$

UV finite →

→ no counterterm is needed of the form $w_{m,s}\, \phi^m \left(\partial_i^s \phi \partial_i^s \phi \right)$

Since $[\phi]_s = 0$, terms like these are acceptable as they are renormalizable.

But the absence of radiative corrections of that kind implies that the bare action

$$S = \int d^3x\, dt \left[\frac{1}{2} (\partial_t \phi)^2 - \frac{\hat{a}_3}{2} \left(\partial^2 \partial_i \phi \partial^2 \partial_i \phi \right) - \frac{\hat{a}_2}{2} \left(\partial^2 \phi \partial^2 \phi \right) - \frac{\hat{a}_1}{2} \left(\partial_i \phi \partial_i \phi \right) - V(\phi) \right]$$

with $w_{m,s} = 0$ is consistently renormalizable

Inclusion of terms proportional to $w_{m,s}$ produce logarithmically divergent corrections to a_1.

This yields a difference in the speed of light a_1 of different fields

Experimentally forbidden –> Fine tuning on $w_{m,s}$ required.

R. Iengo, J.G Russo, M.Serone, , JHEP 11 (2009) 020.
J. Alexandre, Int.J. Mod.Phys. A 26 (2011), 4523.

Resummation of diagrams

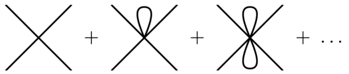
DZ., Eur. Phys. J. C 82 (2022) 341.

The potential

$$V = \sum_{n=2}^{\overline{n}} \frac{\widehat{g}_n}{n!}\phi^n \qquad \widehat{g}_n = \frac{g_n}{M^{n-4}}\Bigg|_{n=2,3,4,....}$$

The tadpole 

$$\widehat{I}_1(\Lambda, M) = M^2\, I_1(\Lambda, M) = \frac{M^2}{(2\pi)^2}\ \ln\left(\frac{\Lambda}{M}\right) + O\left(\frac{M^4}{\Lambda^2}\right)$$

The sequence of divergent diagrams is the same for each n-point vertex 

and the same truncated exponential series is obtained for each vertex

$$g_{2R} = g_2 + g_4\left(\frac{I_1}{2}\right) + \frac{g_6}{2!}\left(\frac{I_1}{2}\right)^2 + \frac{g_8}{3!}\left(\frac{I_1}{2}\right)^3 + ... + \frac{g_{\overline{n}}}{(\overline{n}/2-1)!}\left(\frac{I_1}{2}\right)^{(\overline{n}/2-1)}$$

$$g_{4R} = g_4 + g_6\left(\frac{I_1}{2}\right) + \frac{g_8}{2!}\left(\frac{I_1}{2}\right)^2 + \frac{g_{10}}{3!}\left(\frac{I_1}{2}\right)^3 + ... + \frac{g_{\overline{n}}}{(\overline{n}/2-2)!}\left(\frac{I_1}{2}\right)^{(\overline{n}/2-2)}$$

UV regime :

1) Non-polynomial simple case: $\overline{n} \to \infty$ and $g_n = g > 0,\ \forall n$ ➡ $g_R = g\ e^{I_1/2}$ $\quad \lim_{\Lambda\to\infty} g = 0$

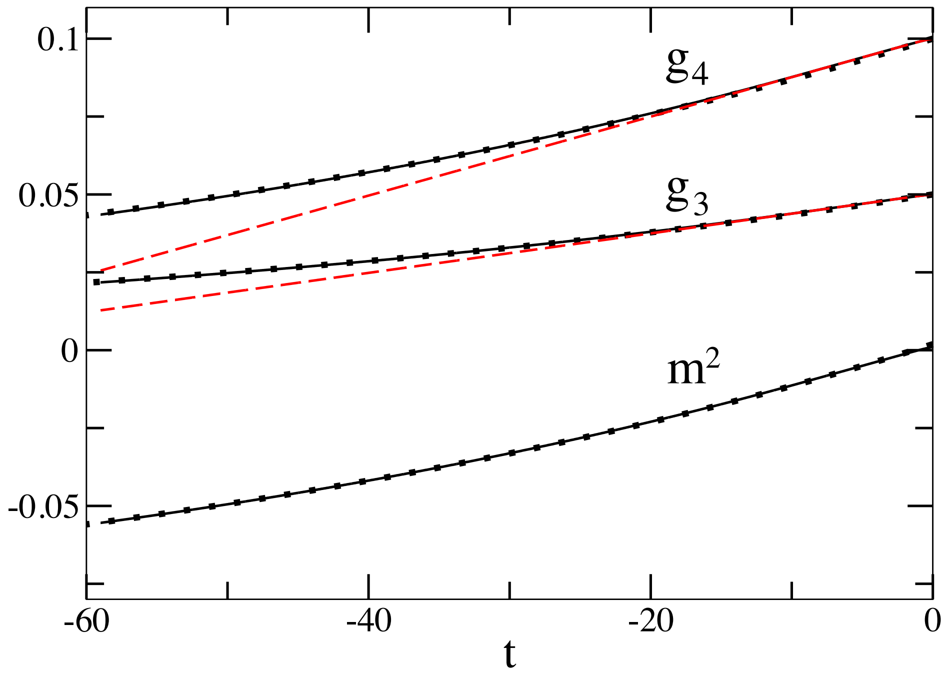
2) Non-polynomial, different couplings: $\overline{n} \to \infty,\ g_n > 0,\ \forall n,\ g = Sup\{g_n\}$ ➡ $g_{nR} \in \Re$ $\quad \lim_{\Lambda\to\infty} g_n = 0$

3) Polynomial case: $\overline{n}$ finite ➡ g_n grows as a power of the logarithm ${I_1}^n$

Cases 1), 2) show asymptotic freedom. Not in case 3).

UV regime of couplings

$g_4,\ g_3,\ m^2 = g_2$ plotted vs. $t = \ln(M/\mu)$ with boundaries at *t=0*



Black solid lines : case 1) and AF → $g_j(t) = g_j(t=0)\ \mathrm{e}^{\frac{t}{8\pi^2}}$

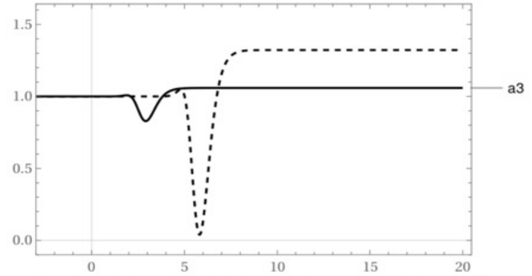
Black dotted lines : case 3) with $\overline{n} = 22$ Red dashed lines : case 3) with $\overline{n} = 6$

Different boundary for the mass $g_2(t)$ and $g(t) = g_4(t) = g_6(t) = g_8(t) = \dots$

so that in case 1) the square mass has a finite UV limit $m^2(t) = m^2(0) + g(0)\left(e^{\frac{t}{8\pi^2}} - 1\right)$

Crossover and IR regime

Boundaries at *t*= - 3 : $m^2 = 0.007$ (solid), $m^2 \simeq -0.004$ (dashed)



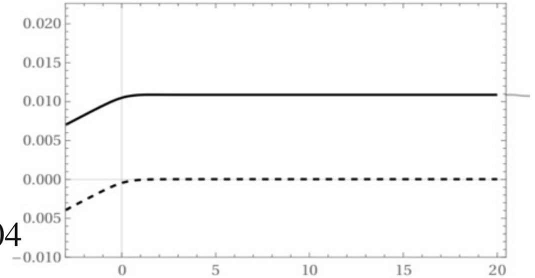
Finite correction for a_3 a_1 and a_2 get extremely smaller corrections

Bump related to the combination of couplings in the propagator around *t=0*

$$\frac{1}{\sqrt{\widehat{a}_3\vec{k}^6 + \widehat{a}_2\vec{k}^4 + \widehat{a}_1\vec{k}^2 + \widehat{g}_2}}$$

$m^2 = 0.007$

$m^2 \simeq -0.004$

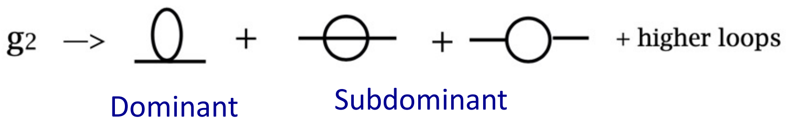


$m^2 \simeq 10^{-2}$

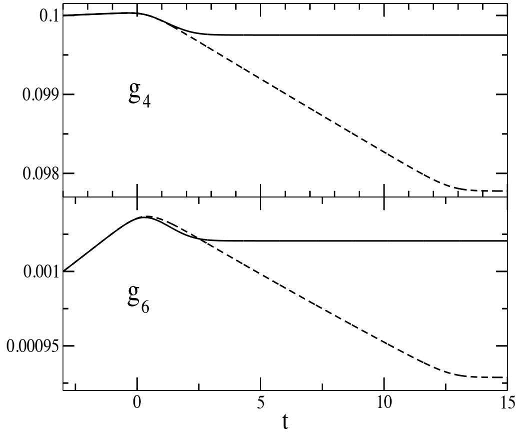
$m^2 \simeq 10^{-12}$

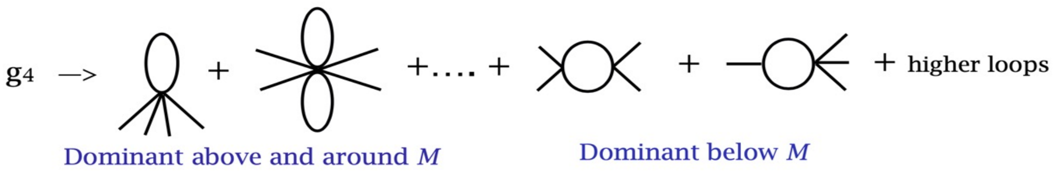
IR square masses normalized to *M²*.

Change of slope due to the change of behavior of diagrams in the two regimes



IR mass values determined by fine tuning in UV.





Standard IR scenario can be recovered.

Regardless of the UV limit of m^2, a quadratic growth of m^2 is unavoidable in the region around and below M^2 .

Size of the scale M dividing the two regimes

Observational indications on the size of M

Bounds on M from time of flight difference of high and low energy photons from GRB. Bounds depend on the form of the dispersion relation and the modified speed of light.

$$c_g(k) = 1 + \lambda \left(\frac{k}{M} \right)^{\alpha}$$

$$M \simeq 10^{17}\ GeV \quad \alpha = 1$$

$$M \simeq 10^{10}\ GeV \quad \alpha = 2$$

J. Ellis, R.Konoplich, et al. Phys. Rev. D 99 (2019) 083009.
B. Chen, Q.G: Huang, Phys. Lett. B 683, (2010) 108.
J. Ellis, N. Mavromatos, et al. Astron.Astrophys.402 (2003) 409.

Our specific case has the dispersion relation

$$E^2 = \vec{k}^2 \left[a_1 + a_2 \left(\frac{k}{M} \right)^2 + a_3 \left(\frac{k}{M} \right)^4 \right] + m^2 \quad \longrightarrow \quad M \simeq 10^{10}\ GeV$$

Such large mass scale brings back the naturalness problem.

Scalar & fermion fields

E. Rizza, DZ., Mod. Phys. Lett. A 37 (2022) 2 250203

Generalization of the HD sector and Yukawa interaction

$$S_{ferm} = \int d^3x\, dt\, \bar{\psi} \left[i\gamma^0 \partial_0 - \left(a_1 + a_3 \frac{\partial_j \partial_j}{M^2} \right) (i\gamma^i \partial_i) - m_f \right] \psi$$

$$[\psi]_s = \frac{3}{2}$$

$$S = S_{scal} + S_{ferm} - \int d^3x\, dt\, V_Y$$

$$V_Y = \sum_i^{\overline{n}} y_n \bar{\psi}\, \psi \frac{\phi^n}{M^{n-1}}$$

$$\mathcal{G}_n (\bar{\psi}\psi)^n$$

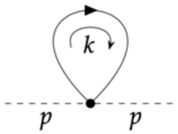
2n-fermion interaction is renormalizable only for n=2
Can be reduced to Yukawa int. through Hubbard –Stratonovich transf.

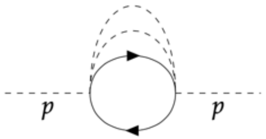
Divergent diagrams with Yukawa vertices

$$D_\Lambda = -3V - \frac{3}{2} E_f + 6$$

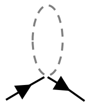
Only logarithmic divergences

$$E_f = 0 \qquad V = 1 \qquad V = 2$$





$$E_f = 2 : \quad V = 1$$



Log. diverg.

Finite

Fermion induced corrections in the scalar sector

Simplest explicative example

$\overline{n} \to \infty, \; g_n = g, \; y_n = y, \; \forall n$

except $g_2 = m^2$

$I_1 = (1/4\pi^2)\log(\mu/M)$

$E = \exp(I_1/2)$

Diagram resummation

2-point function $m^2(M^2) = m^2 - g + gE - 4\left(\frac{m_f}{M}\right) yEI_1 - 2(yE^2I_1)^2$

4-point function $g(M^2) = gE - 4\left(\frac{m_f}{M}\right) yEI_1 - (yEI_1)^2\,(12 + 16E^2)$

Made finite by renormalization conditions $g^{-1} \propto E \qquad y^{-1} \propto E^2 I_1$

Both coupling are asymptotically free

The scalar mass in the UV limit $\mu >> M$

$m^2(M^2) - m^2(\mu^2) = g(M^2) + 14(yE^2I_1)^2 > 0$

No boson-fermion cancellation if $g(M^2) > 0$

For dimensional variables

$\widehat{m}^2(M^2) - \widehat{m}^2(\mu^2) \simeq O(M^2\, g(M^2))$

Comparable to corrections generated in the IR region below M .

Inclusion of Gauge fields

E. Rizza, DZ., Mod. Phys. Lett. A 37 (2022) 2 250203

Gauge invariant form

$$S_{YM} = \int d^3x\,dt\,\left[\frac{1}{2}\mathrm{Tr}(E_i E_i) - \frac{1}{2}\mathrm{Tr}\left(D_h D_j F_{ik}\right)^2\right]$$

P. Horava, Phys. Lett. B 694, (2011), 172.
W. Chao, Commun. Theor.Phys. 65 (2016), 743.
J. Alexandre, Int.J. Mod.Phys. A 26 (2011), 4523

It contains terms like $A_i A_i \left(\partial^2 A\, \partial^2 A\right)$

These class of terms are those rejected in our model as they are responsible of generating model dependent corrections to the speed of light which is experimentally forbidden.

R. Iengo, J.G Russo, M.Serone, , JHEP 11 (2009) 020.
J. Alexandre, Int.J. Mod.Phys. A 26 (2011), 4523.

To overcome the inconvenience use a different scheme

Modified covariant derivative

$$\partial'_\mu = \left\{\partial_0,\ \left(a_1 + a_3\frac{\partial_j\partial_j}{M^2}\right)\partial_i\right\} \qquad D'_\mu = \partial'_\mu - iqA_\mu \qquad F'_{\mu\nu} = \partial'_\mu A_\nu - \partial'_\nu A_\mu$$

$$S_{SQED} = \int d^3x\,dt\left(D'_\mu\phi\left(D'^\mu\phi\right)^* - V(\phi) - \frac{1}{4}F'_{\mu\nu}F'^{\mu\nu}\right)$$

Gauge invariance violated although non-invariant terms are suppressed as $(E/M)^2$

Gauge and Lorentz symm. are observable only at energies E<< M

HD in anisotropic models

Anisotropic (z=3) HD smoothen the UV behavior preserving unitarity
and producing in some cases asymptotically free interactions.

Price to pay : breaking of Lorentz and Gauge symm. at extremely large energies,
Symmetries recoverable as emergent properties below the crossover scale *M*.

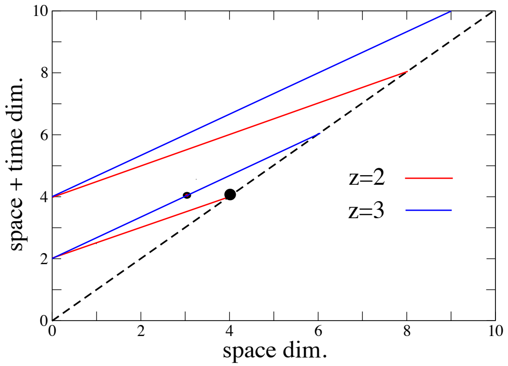
Observations constrain *M* at very high values. Therefore naturalness of
the mass scales and fine tuning are unavoidable also in this approach.

HD in isotropic models

R. Hornreich, M. Luban, S. Shtrikman, Phys. Rev. Lett. 35 (1975) 1678.
A. Erzan, G. Stell, Phys. Rev. B 16 (1977) 4146.
H. Diehl, M. Shpot, Journal of Physics. A 35 (2002) 6249.

$$S = \int \mathrm{d}^d x_{\parallel} \left[\frac{W_{\parallel}}{2} (\partial_{\parallel}^2 \phi)^2 + \frac{Z_{\parallel}}{2} (\partial_{\parallel} \phi)^2 + V \right]$$

$Z_{\parallel}$ = 0 at tree level enforces the Lifshitz scaling.



No transverse components, only parallel coordinates.

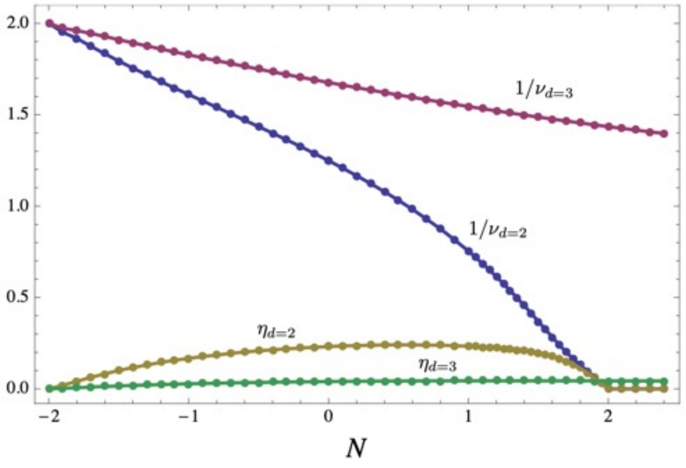
Only one anomalous dimension.

Isotropy is recovered.

Non trivial Lifshitz point structure.

O(N) models at lower critical dimension (d=2)

A.Codello, N. Defenu, G. D'Odorico, Phys.Rev.D 91 (2015) 10, 105003

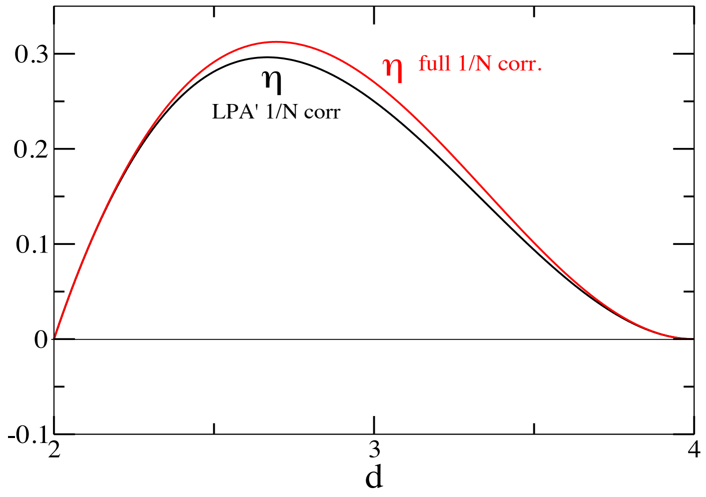


Models with no Higher Derivatives are constrained by Coleman-Mermin-Wagner theorem in d=2:

SSB of a CONTINUOUS SYMMETRY cannot be realized.
VEV = 0 because of Goldstone boson fluctuations.

This is reflected in the values of η and $1/\nu$:
both vanish for N=2 and N>2.

Analytical computation of η in the large N limit at order O(1/N) (symmetric phase)



$$\eta = \frac{4\epsilon}{(4-\epsilon)\pi} \, \frac{\sin(\pi\epsilon/2)\,\Gamma(2-\epsilon)}{\Gamma(1-\epsilon/2)\,\Gamma(2-\epsilon/2)}$$

(red)

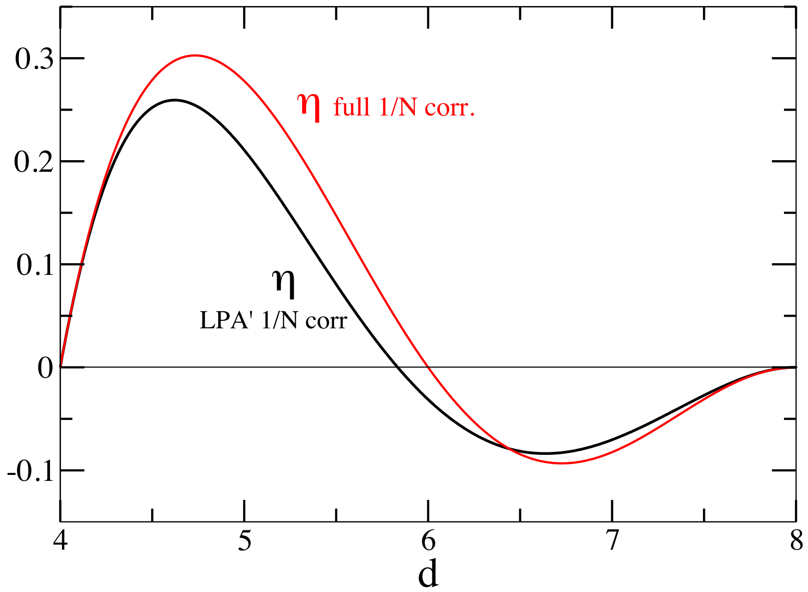
RG computation in LPA' (black)

$$d = 2 + \delta \qquad \eta = \delta$$

$\eta = 0$ in d=2 at order O(1/N)

Does it repeat in Higher Derivative models with z=2 in d=4 ?

Only indications in the large N limit at O(1/N)



$$\eta_N = \frac{(8-d)}{d\,(d+2)}\,\frac{3\cdot 2^{d-2}}{\pi^{3/2}}\,\frac{\Gamma\left(\frac{d-3}{2}\right)}{\Gamma\left(\frac{d}{2}\right)}\,\sin\frac{\pi\, d}{2}$$

(red)

R. Hornreich, M. Luban, S. Shtrikman, Phys. Lett. A 55 (1975) 269.
S.S. Gubser, C. Jepsen, S. Parikh, B. Trundy, JHEP 1711, (2017) 107.

RG computation in LPA' (black)

DZ, Phys Lett B 773 (2017) 213.

$\eta = 0$ in d=4 at order O(1/N)

$$\eta_N = \delta \qquad d = 4 + \delta$$

In line with the conjecture that d=4 is the lower critical dimension of O(N) HD models

Note also the change of sign at d=6 and $\eta < 0$ when approaching d=8.

$\eta < 0$ for d < 8 also observed in N=1 HD models.

A. Bonanno, DZ, Nucl. Phys. B 893 (2015) 891.
M. Safari, GP Vacca, Phys. Rev D 97 (2018) 041701.

Topological effects in O(2) model in d=2

$$S = \int d^2x \left[\frac{Z}{2} \vec{\nabla}\phi\, \vec{\nabla}\phi^* + V(\phi\phi^*) \right]$$

$$\phi(\vec{r}) = \sqrt{\rho(\vec{r})}\, e^{i\theta(\vec{r})}$$

By neglecting the dynamics of the ρ field at energies below its mass → Only angular component is left

$$S_{KT} = \int d^2x \left[\frac{K}{2} \vec{\nabla}\theta\, \vec{\nabla}\theta \right]$$

with STIFFNESS : $K = Z\rho$

2D TOPOLOGICAL EXCITATIONS (VORTICES or COULOMB CHARGES) play crucial role $\vec{\nabla} \times \vec{\nabla}\theta(\vec{r}) = 2\pi n\, \delta(\vec{r} - \vec{r}_0)$

They are parametrized by solution of Laplace equation $\vec{\nabla}\theta(\vec{r}) = -\vec{\nabla} \times \hat{z}\, \psi(\vec{r})$

$$\Delta\psi(\vec{r}) = 2\pi n\, \delta(\vec{r} - \vec{r}_0) \Rightarrow \psi(\vec{r}) = \ln(|\vec{r} - \vec{r}_0)|)/R$$

Logarithmic interaction between opposite charges $n_{\mathbf{r}} = 1$ $n_{\mathbf{r}'} = -1$ $H_I^{(2)} = 2\,\pi\, K \ln|\mathbf{r}' - \mathbf{r}|$

Formal mapping onto sine-Gordon model

$$S_{sg} = \int d^2x \left[\frac{w}{2} \vec{\nabla}\varphi\, \vec{\nabla}\varphi^* + g(1 - \cos(\varphi)) \right]$$

with stiffness and fugacity, respectively

$$\begin{cases} K \sim w^{-1} \\ g \sim e^{-\epsilon_S} \end{cases}$$

Topological effects in O(2) model in d=2 (cont.)

Both Coulomb gas and sine-Gordon model show :

1) a line of fixed points at g=0 ;

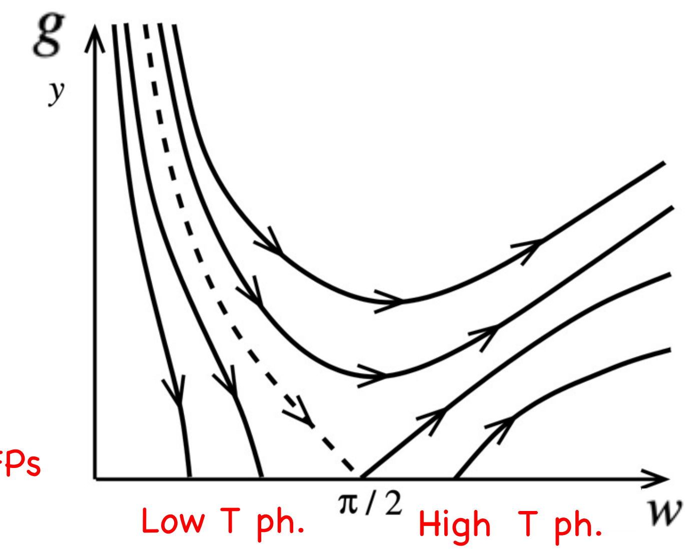
2) a transition from a strongly interacting phase (K>> 1 or low T phase) where pairs of charges with opposite sign are tightly bound to a weakly interacting (K<< 1 or high T) phase with unbound charges ;

3) The universal transition point at g=0 corresponds to a discontinuity of the renormalized stiffness from $K_c = 2/\pi$ to $K=0$;

4) The anomalous dimension at the transition jumps from η_c= 1/4 to $\eta = 0$;

5) Mermin-Wagner theorem respected as $\rho = 0$ *(no ssb)* but $K=Z\rho$ finite as $Z = k^{-\eta}$ diverges in IR.

IR stable FPs



IR unstable FPs

Low T ph. High T ph.

Preliminary analysis of U(1) model in d=4

Which N in O(N) could reproduce topological solutions ?

Adapt one approach used for the d=2 Kosterlitz-Thouless phase

P. Jakubczyk, W. Metzner, Phys. Rev. B 95, (2017) 085113.

$$\Gamma_k[\phi] = \int d^4\mathbf{r}\left\{\frac{u_k}{8}\left(|\phi|^2-\alpha_k^2\right)^2+\frac{W_k^A}{2}\left[\partial^2\phi\,\partial^2\phi^*\right]+\frac{W_k^B}{8}\left[\partial^2|\phi|^2\right]^2+\frac{Z_k^A}{2}\left[\partial\phi\,\partial\phi^*\right]+\frac{Z_k^B}{8}\left[\partial|\phi|^2\right]^2\right\}$$

$$\phi(\mathbf{r}) = \alpha + \sigma(\mathbf{r}) + i\pi(\mathbf{r})$$

with the parametrization

$$W_\sigma = W^A + W^B\alpha^2 \quad ; \quad Z_\sigma = Z^A + Z^B\alpha^2$$
$$W_\pi = W^A \quad ; \quad Z_\pi = Z^A$$
$$m_\sigma^2 = u\alpha^2$$

→

$$\Gamma_\sigma^{(2)}(q) = W_\sigma q^4 + Z_\sigma q^2 + m_\sigma^2\,,$$
$$\Gamma_\pi^{(2)}(q) = W_\pi q^4 + Z_\pi q^2\,.$$

FRG flow reduced to the set of ODEs for the 6 parameters α , m_σ , W_π , W_σ , Z_π , Z_σ

Unsatisfactory indications on KT transition in this approximation.

M.Grater, C. Wetterich, Phys. Rev.Lett.75 (1995) 378.
G.v. Gersdorff, C. Wetterich, Phys. Rev. B 64 (2001) 054513.
P. Jakubczyk, N. Dupuis, B. Delamotte, Phys. Rev. E 90, (2014) 062105.

Further approximated by neglecting fluctuations of the longitudinal field : m_σ , W_σ , Z_σ FIXED

Three RG eqs. in terms of the rescaled variables [$R = W_\pi k^4$]

$$a^2 = k^{-\eta}\alpha^2, \quad w_\pi = k^\eta W_\pi\,, \quad z_\pi = k^{\eta-2} Z_\pi$$

Preliminary analysis of U(1) model in d=4 (cont.)

DZ. Phys.Rev. D 98 (2018)085005

$$\partial_t \mathrm{a}^2 - \eta \mathrm{a}^2 = (\eta - 4)\, w_\pi \int_{\tilde{q}} \widetilde{G}_\pi^2$$

$$\partial_t w_\pi + \eta w_\pi = \frac{(4-\eta)\, w_\pi^2}{\mathrm{a}^2} \int_{\tilde{q}} \widetilde{G}_\pi^2$$

$$\partial_t z_\pi + (\eta - 2)\, z_\pi = \frac{(4-\eta)\, w_\pi}{\mathrm{a}^2} \int_{\tilde{q}} (z_\pi + 3 w_\pi \tilde{q}^2)\, \widetilde{G}_\pi^2$$

With propagator $\widetilde{G}_\pi = \left[w_\pi (\tilde{q}^4 + 1) + z_\pi \tilde{q}^2 \right]^{-1}$

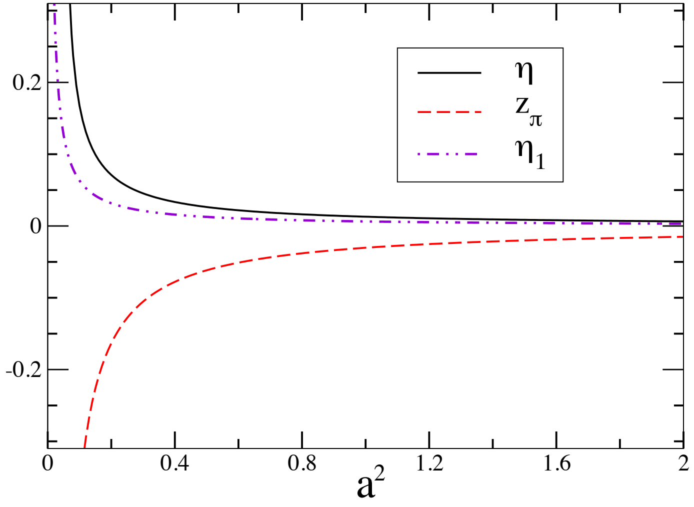
Stationary FP solutions for a, η, z_π

with normalization $w_\pi = 1$

2 eqs. identical (ONLY U(1) i.e. O(2) MODEL)

→ 1 unconstrained parameter → line of FPs

Free parameter: $J = W_\pi \alpha^2$ Scaled param. → $J = \mathrm{a}^2$



$Z_\pi < 0$ and $\eta > 0$: FP line down to $\mathrm{a}^2 = 0.065$

Eigenvalue spectrum from linearized eqs. around FPs at large J indicates 2 relevant (z_π and a) and 1 irrelevant (w_π) eigenvect.

$$\lambda_1 = 2 - \frac{24}{\mathrm{a}^2} \int_{\tilde{q}} \tilde{q}^4 \left[\tilde{q}^4 + z_\pi \tilde{q}^2 + 1 \right]^{-3}$$

$$\lambda_2 = \eta$$

$$\lambda_3 = -\eta$$

This analysis finds KT-like features in d=4 for O(2) symmetry only.

Are there topological excitations even in d=4 ?

N. Defenu, A. Trombettoni and D. Z. Nucl. Phys. B 964 (2021) 115295.

Start from a complex scalar field (or O(2)) model :

$$S[\phi] = \int d^4\mathbf{r} \left[\frac{W}{2} \partial^2\phi\, \partial^2\phi^* + \frac{Z}{2} \partial\phi\, \partial\phi^* + V(\phi\,\phi^*) \right]$$

$$\phi(\mathbf{r}) = \sqrt{\rho(\mathbf{r})}\; e^{i\vartheta(\mathbf{r})}$$

After selecting Z=0, focus on the angular degree of freedom

$$H[\vartheta(\mathbf{r})] = \frac{\mathcal{K}}{2} \int d^4\mathbf{r}\, [\Delta\vartheta(\mathbf{r})\, \Delta\vartheta(\mathbf{r})]$$

Use the Laplace equation to build the Green F. of the square Laplacian in the equation of motion

d=4

$$\Delta\ (\mathbf{r}-\mathbf{r}')^{-2} = -(2\pi)^2 \delta^4(\mathbf{r}-\mathbf{r}')$$

d=2

$$\Delta \ln|\mathbf{r}-\mathbf{X}| = 2\pi\, \delta^2(\mathbf{r}-\mathbf{X})$$

$$\Delta^2_{\mathbf{r}}\, \mathcal{G}(\mathbf{r}-\mathbf{r}') = (2\pi)^2 \delta^4(\mathbf{r}-\mathbf{r}') \quad \Longrightarrow \quad \Delta_{\mathbf{r}}\, \mathcal{G}(\mathbf{r}-\mathbf{r}') = \frac{-1}{(\mathbf{r}-\mathbf{r}')^2}$$

The solution is :

$$\mathcal{G}(\mathbf{r}-\mathbf{r}') = \int \frac{d^4\mathbf{r}''}{(2\pi)^2} \frac{1}{(\mathbf{r}-\mathbf{r}'')^2} \frac{1}{(\mathbf{r}''-\mathbf{r}')^2} = \frac{1}{4} \ln \frac{R^2}{(\mathbf{r}-\mathbf{r}')^2}$$

Logarithmic function as for the Laplacian GF in d=2

By generalizing to $n_i \in Z$ charges

$$\mathcal{G}^C(\mathbf{r}) = \sum_i n_i\, \mathcal{G}(\mathbf{r}-\mathbf{r_i})$$

the interaction strength among charges is logarithmic

$$H[\mathcal{G}^C] = \frac{\mathcal{K}}{2} \int d^4\mathbf{r} \sum_{i,j} n_i n_j\, [\Delta\mathcal{G}(\mathbf{r}-\mathbf{r_i})][\Delta\mathcal{G}(\mathbf{r}-\mathbf{r_j})]$$

Topological configurations

$$H[\vartheta(\mathbf{r})] = \frac{\mathcal{K}}{2} \int d^4\mathbf{r} \, [\Delta\vartheta(\mathbf{r}) \, \Delta\vartheta(\mathbf{r})]$$

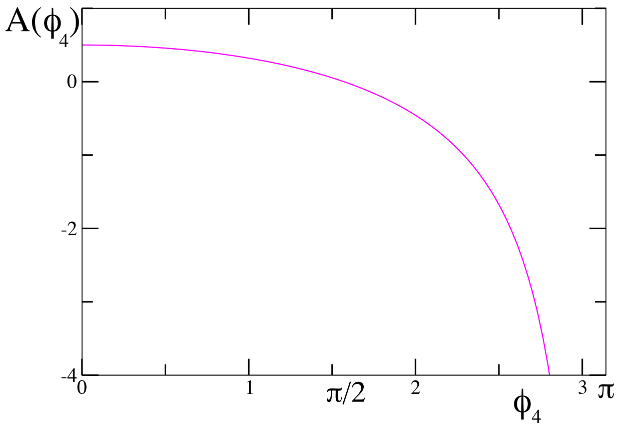
$\vartheta(\mathbf{r})$ is the phase of a complex field and is defined in a Euclidean 4-dimensional space.

Use d=4 spherical coordinates

$$\begin{aligned} x_1 &= \mathrm{r}\sin(\phi_4)\sin(\phi_3)\sin(\phi_2) \\ x_2 &= \mathrm{r}\sin(\phi_4)\sin(\phi_3)\cos(\phi_2) \\ x_3 &= \mathrm{r}\sin(\phi_4)\cos(\phi_3) \\ x_4 &= \mathrm{r}\cos(\phi_4) \end{aligned}$$

$$\mathrm{r} = \sqrt{x_i \, x_i} \qquad \phi_4 = \mathrm{ArcCos}\left(\frac{x_4}{\mathrm{r}}\right)$$

$$\Delta = \frac{\partial_{\mathrm{r}}[\mathrm{r}^3 \partial_{\mathrm{r}}]}{\mathrm{r}^3} + \frac{\partial^2_{\phi_2}}{\mathrm{r}^2 \sin^2(\phi_4)\sin^2(\phi_3)} + \frac{\partial_{\phi_3}[\sin(\phi_3)\partial_{\phi_3}]}{\mathrm{r}^2\sin^2(\phi_4)\sin(\phi_3)} + \frac{\partial_{\phi_4}[\sin^2(\phi_4)\partial_{\phi_4}]}{\mathrm{r}^2\sin^2(\phi_4)}$$



Configuration **A(r)**, associated to a singularity in **r = 0**, depends on ϕ_4 only

$$A(\mathbf{r}) = \frac{1}{2}\,\phi_4 \,\frac{\cos(\phi_4)}{\sin(\phi_4)}$$

With the exception of **r = 0**, **A(r)** differentiable

$$\Delta A(\mathbf{r}) = -\frac{1}{\mathrm{r}^2} \Longrightarrow \Delta^2 A(\mathbf{r}) = (2\pi)^2 \delta^4(\mathbf{r})$$

Same effect of the Green F. and **A(r)** when replaced in **H**.

A(r) is a solution of the equation of motion.

Obvious generalization to a generic configuration with singularity located in **r'**

$$\Delta^2_{\mathbf{r}} A_{\mathbf{r}'}(\mathbf{r}) = (2\pi)^2 \delta^4(\mathbf{r} - \mathbf{r}')$$

Topological configurations (cont.)

$$H[\vartheta(\mathbf{r})] = \frac{\mathcal{K}}{2} \int d^4\mathbf{r}\, [\Delta\vartheta(\mathbf{r})\, \Delta\vartheta(\mathbf{r})]$$

$$A(\mathbf{r}) = \frac{1}{2}\, \phi_4\, \frac{\cos(\phi_4)}{\sin(\phi_4)}$$

$$\Delta A(\mathbf{r}) = -\frac{1}{\mathrm{r}^2} \qquad \Delta^2 A(\mathbf{r}) = (2\pi)^2 \delta^4(\mathbf{r})$$

Full analogy with the vortex configurations in d=2

Argument on Free energy used in d=2 by Kosterlitz-Thouless can be reproduced in d=4

DZ, Int. J. Geom. Methods Mod. Phys. 17 (2020) 2050053.

$$H[A_{\mathbf{r}'}] = \mathcal{K}\pi^2 \ln\left(R/r_0\right)$$

Energy of a single charge

$$S[A_{\mathbf{r}'}] = \ln\left(R^4/r_0^4\right)$$

Configurational entropy of a single charge

R = Large distance cutoff
r_0 = Short distance cutoff

Vanishing of the FREE ENERGY F = H - T S = 0 signals a transition from paired charge-anticharge phase to a phase of isolated charges . The critical coupling (in units of temperature)

$$\mathcal{K}_c = \frac{4}{\pi^2}$$

Also in d=2 the logarithmic dependence is recovered. The critical KT coupling is $\mathcal{K}_c = 2/\pi$

Mapping onto HD Sine-Gordon model

Same procedure of the d=2 case

Start from the Hamiltonian

$$H[\mathcal{G}^C] = \frac{\mathcal{K}}{2}\int d^4\mathbf{r} \sum_{i,j} n_i n_j \; [\Delta\mathcal{G}(\mathbf{r}-\mathbf{r_i})][\Delta\mathcal{G}(\mathbf{r}-\mathbf{r_j})]$$

Define a scalar field to account for all charges

$$n(\mathbf{r}) = \sum_i n_i \, \delta^4(\mathbf{r}-\mathbf{r_i})$$

Use the property

$$\Delta^2_{\mathbf{r}}\,\mathcal{G}(\mathbf{r}-\mathbf{r}') = (2\pi)^2\delta^4(\mathbf{r}-\mathbf{r}')$$

$$H_I = \frac{\mathcal{K}}{2}(2\pi)^2 \int d^4\mathbf{r}_1 \int d^4\mathbf{r}_2 \; n(\mathbf{r}_1)\,\mathcal{G}(\mathbf{r}_1-\mathbf{r}_2)\,n(\mathbf{r}_2)$$

with a short distance cutoff on $\mathbf{r}_1-\mathbf{r}_2$ and an additional selfenergy term ϵ_S

The partition function $\mathcal{Z} = \sum_{conf} \exp\left[-H_I - \sum_i n_i^2 \epsilon_S\right]$

With the help of an auxiliary field $\exp[-H_I] = \mathcal{N}\int D\phi \, \exp\left[-\frac{1}{2\,(2\pi)^4\mathcal{K}}\int d^4\mathbf{r}\,\Delta\phi(\mathbf{r})\,\Delta\phi(\mathbf{r}) + i\int d^4\mathbf{r}\,(\phi(\mathbf{r})\,n(\mathbf{r}))\right]$

by defining the two couplings $y = \mathrm{e}^{-\epsilon_S}$ $\qquad (2\pi)^4\mathcal{K} = w^{-1}$

for configurations with zero or one positive or one negative charge , $\mathcal{Z}$ is

$$\mathcal{Z} = \mathcal{N}\int D\phi\,\exp\left[-\frac{1}{2\,(2\pi)^4\mathcal{K}}\int d^4\mathbf{r}\,\Delta\phi(\mathbf{r})\,\Delta\phi(\mathbf{r})\right]\left\{1 + \int d^4\mathbf{r}_s\,\mathrm{e}^{-\epsilon_S + i\phi(\mathbf{r}_s)} + \int d^4\mathbf{r}_s\,\mathrm{e}^{-\epsilon_S - i\phi(\mathbf{r}_s)}\right\}$$

$$= \mathcal{N}\int D\phi\,\exp\left[-\frac{w}{2}\int d^4\mathbf{r}\,\Delta\phi(\mathbf{r})\,\Delta\phi(\mathbf{r}) + 2\,y\int d^4\mathbf{r}\,\cos(\phi(\mathbf{r}))\right]$$

Sine- Gordon Model in d=4 and with HD term

Renormalization Group analysis in d=4

RG analysis of the couplings y and $\mathcal{K}$ in the KT framework is equivalently replaced by the RG analysis of y and w of SG model (in d=2).

J.V. Jose´, L.P. Kadanoff, S. Kirkpatrick, D.R. Nelson, Phys. Rev. B 16 (1977) 1217.

S. Nagy, I. Nandori, J. Polonyi, K. Sailer, Phys. Rev. Lett. 102 (24) (2009) 241603.
N.O.Defenu, A.Trombettoni, I.Nandori, T.Enss, Phys. Rev. B 96 (2017), 174505.

Average effective action

$$\Gamma_k[\varphi] = \int \left\{ \frac{w_k}{2} (\Delta\varphi)^2 + g_k (1 - \cos\varphi) \right\} d^4x$$

Define

$$V_k(\varphi) = g_k(1 - \cos\varphi) \qquad g_k = k^4 \tilde{g}_k$$

$$R_k(q) = k^4 \qquad G(q) = \frac{1}{w_k q^4 + V_k''(\varphi) + R_k(q)}$$

and get the potential FRG flow

$$\partial_t V_k(\varphi) = \int \frac{d^d q}{(2\pi)^d} G(q) \partial_t R_k(q)$$

$$t = -\log(k/\Lambda)$$

Use projectors to select the flow eqs. for the two couplings

$$\begin{cases} g_k = \frac{-1}{\pi} \int_{-\pi}^{\pi} [V_k(\varphi)] \cos(\varphi) d\varphi \\ \partial_t w_k = \lim_{p \to 0} \int_{-\pi}^{\pi} \frac{d\varphi}{2\pi} \int \frac{d^d q}{(2\pi)^d} \partial_t R_t(q) G(q)^2 \, V_k'''(\varphi)^2 \frac{d^4}{dp^4} G(p+q) \end{cases}$$

The FRG flow eqs. of the couplings are

$$\begin{cases} (4 - \partial_t)\tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k} \left(1 - \sqrt{1 - \tilde{g}_k^2}\right) \\ \partial_t w_k = -\frac{9}{160\pi^2} \frac{\tilde{g}_k^2}{(1 - \tilde{g}_k^2)^{\frac{3}{2}}} \end{cases}$$

N. Defenu, A. Trombettoni and D. Z.
Nucl. Phys. B 964 (2021) 115295.

RG analysis (cont.)

Flow equations of the two couplings

$$(4-\partial_t)\tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k}\left(1-\sqrt{1-\tilde{g}_k^2}\right) \qquad \partial_t w_k = -\frac{9}{160\pi^2}\frac{\tilde{g}_k^2}{(1-\tilde{g}_k^2)^{\frac{3}{2}}}$$

$\tilde{g}_k = 0$ is a line of FPs

$\partial_t \tilde{g}_k$ changes sign at $w_k^{-1} = 64\pi^2$ → equal to the F=0 critical value $\mathcal{K}_c = \frac{4}{\pi^2}$

according to $(2\pi)^4\mathcal{K} = w^{-1}$

Anomalous dimension from long distance decay of correlator: $C(\mathbf{r}) = \langle e^{i\vartheta(\mathbf{r})}\, e^{i\vartheta(\mathbf{0})}\rangle \propto r^{-\eta}$

At $\tilde{g}_k = 0$ gaussian structure $C(\mathbf{r}) = \exp\left(-\frac{1}{2}\langle(\vartheta(\mathbf{r})-\vartheta(0))^2\rangle\right)$ → $\eta = (8\pi^2\mathcal{K})^{-1}$ → $\eta_c = \frac{1}{32}$

Same RG analysis in d=2 : renowned properties of SG

$$\begin{cases} (2-\partial_t)\tilde{g}_k = \dfrac{\tilde{g}_k}{4\pi w_k} \\ \partial_t w_k = \dfrac{\tilde{g}_k^2}{4\pi} \end{cases}$$

Line of FPs at $\tilde{g}_k = 0$

Change of sign of $\partial_t \tilde{g}_k$ at $w_{KT} = 8\pi$; $\mathcal{K}_{KT} = 2/\pi$; $\eta_{KT} = 1/4$

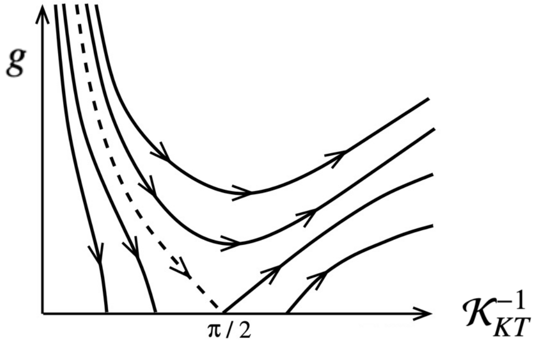
d=2

$$(2-\partial_t)\tilde{g}_k = \frac{\tilde{g}_k}{4\pi w_k}$$

$$\partial_t w_k = \frac{\tilde{g}_k^2}{4\pi}$$

positive in d=2.

Determines the KT phase transition



d=4

$$(4-\partial_t)\tilde{g}_k = \frac{1}{8\pi^2 w_k \tilde{g}_k}\left(1-\sqrt{1-\tilde{g}_k^2}\right)$$

$$\partial_t w_k = -\frac{9}{160\pi^2}\frac{\tilde{g}_k^2}{(1-\tilde{g}_k^2)^{\frac{3}{2}}}$$

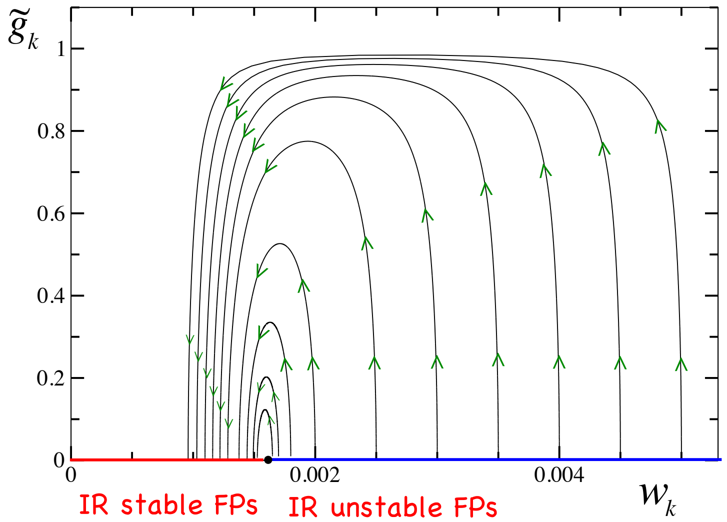
negative in d=4.

In the diagram all poins are attracted toward the red line.

This corresponds to charge-anticharge pairing.

The line $\tilde{g}_k = 0$ is unmodified, but

the ph. tr. disappears in this diagram.



The phase separatrix is the plane itself.

$$\Gamma_k[\varphi] = \int \left\{\frac{w_k}{2}(\Delta\varphi)^2 + g_k(1-\cos\varphi)\right\} d^4x$$

Turning on Z_k opens a new relevant direction and a novel IR limit (new phase).

$$S = \int \mathrm{d}^d x_\parallel \left[\frac{W_\parallel}{2}(\partial_\parallel^2\phi)^2 + \frac{Z_\parallel}{2}(\partial_\parallel\phi)^2 + V\right]$$

Phase with paired charges occurs at IR FPs (red line) if $z_k = 0$ with finite renormalized inverse stiffness $w_k < 1/(64\pi^2)$

Same mechanism as in Kosterlitz- Thouless. $w_k^{-1} = \mathcal{K} = k^{-\eta}\rho$

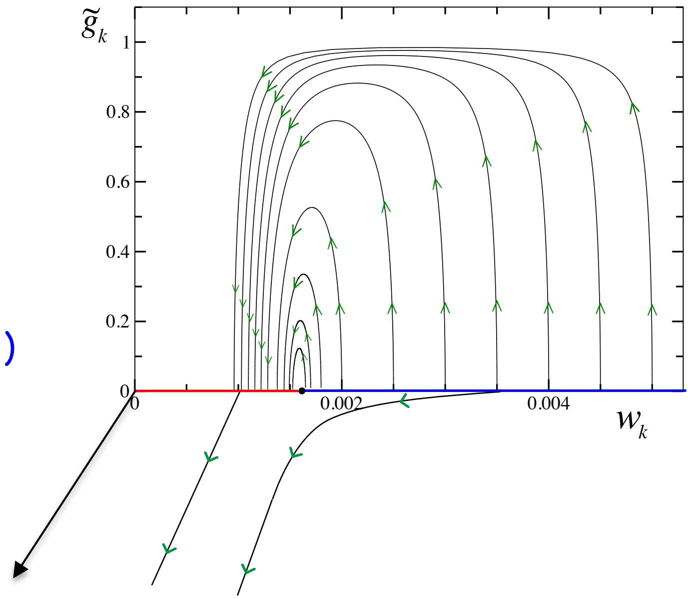
Phase Transition show up when $z_k \neq 0$

Similarity with 'branched polymer phase' of the conformal factor effective action of QG,

Antoniadis, P. O. Mazur, E. Mottola,, *Physics Letters B 394 (1997) 49–56.*
S. Catterall, E. Mottola, *Physics Letters B 467 (1999) 29–39.*

and also with second order ph. tr. observed in causal dynamical triangulation.

J. Ambjørn, S. Jordan, J. Jurkiewicz, R. Loll, Phys. Rev. Lett. 107 (2011) 211303; Phys. Rev. D 85 (2012) 124044. J. Ambjørn et al Eur. Phys. J. C 77 (2017) 152.



Red line of FPs has one irrelevant (g) and one relevant direction (z_k) .

The blue line of FPs ensures a safe UV limit where

$w_{k=\infty}$ = finite $\quad \eta = 0 \quad \rho \neq 0$

Flow towards a new basin of attraction and new IR phase.

Possibly flows toward the standard 4D Gaussian FP (2-derivative).

D. Buccio, R.Percacci JHEP 10 (2022) 113

Conclusions

Anisotropic (z=3) HD models

1. Models in d=3+1 are renormalizable and in some cases show asymptotically free interactions.

2. Action restricted by phenomenological constraints.
   Still, above scale M, theory affected by Lorenz and Gauge violating effects.

3. Quantum corrections to the scalar square mass are O(gM^2) :
   No progress with respect to a standard effective theory with UV cutoff = M.

Isotropic (z=2) HD non-unitary models

1. O(2) models in d=4 resemble in many aspects the d=2 KT phase transition.
   Mapping on Coulomb gas and on Sine-Gordon model realized.

2. But, sign flip in one RG equation brings substantial changes.
   Only at $z_k = 0$ the phase with paired charges is observed.

3. A line of UV stable FPs guarantees the non-perturbative renormalizability of these models.
   Non-vanishing z_k should lead to standard IR picture dominated by Gaussian FP with 2 derivatives.

Backup slides

Functional RG analysis of the ILP for one field: N = 1

A. Bonanno, DZ,
Nucl. Phys. B893 (2015) 891.

Fixed point equations from Proper Time RG

$$k\partial_k v - d\, v + D_\phi\, x\partial_x v = \int \frac{d^d p}{(2\pi)^d}\, e^{\left(-\frac{a_0}{2w}\right)}$$

$$k\partial_k w + \eta w + D_\phi\, x\partial_x w = -\int \frac{d^d p}{(2\pi)^d}\, e^{\left(-\frac{a_0}{2w}\right)}\, K_w$$

$$k\partial_k z - (2-\eta) z + D_\phi\, x\partial_x z = -\int \frac{d^d p}{(2\pi)^d}\, e^{\left(-\frac{a_0}{2w}\right)}\, K_z$$

in terms of rescaled variables ($k = k_{||}$)

$$\phi = k^{D_\phi}\, x \qquad D_\phi = \frac{d-4+\eta}{2}$$

$$V(k,\phi) = k^d\, v(k,x)$$

$$Z(k,\phi) = k^{2-\eta}\, z(k,x)$$

$$W(k,\phi) = k^{-\eta}\, w(k,x)$$

$$a_0 = \partial_x^2 v + z\, p^2 + 2w\, p^4$$

with K_w and K_z depending on v, w, z

D. Litim, DZ, Phys RevD 83 (2011) 085009

Local Potential Approx. : V only (Z = 0, W = 1, $\eta = 0$)

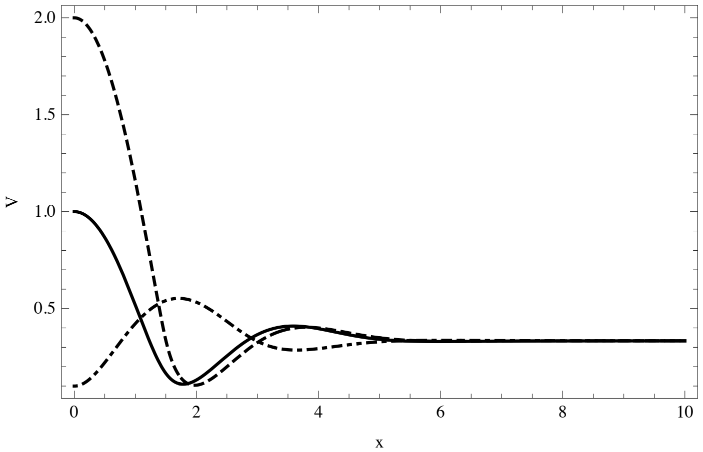
LP solution found for 4 < d < 8

When d ≤ 4 : $D_\phi \leq 0$

No meaningful LP solution.

Dimensional V is flat at k=0

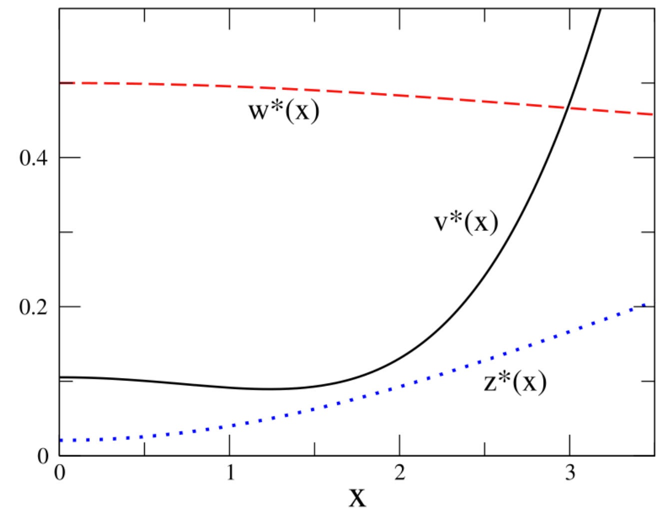
as for the standard scaling with d < 2



N =1 case. Including v, z, w

LP solution observed for d < 8

Failure in numerical resolution for d < 5.4 possibly due to mixing with multicritical solutions appearing at lower d.



Full solution for η (red) and v″ (blue)

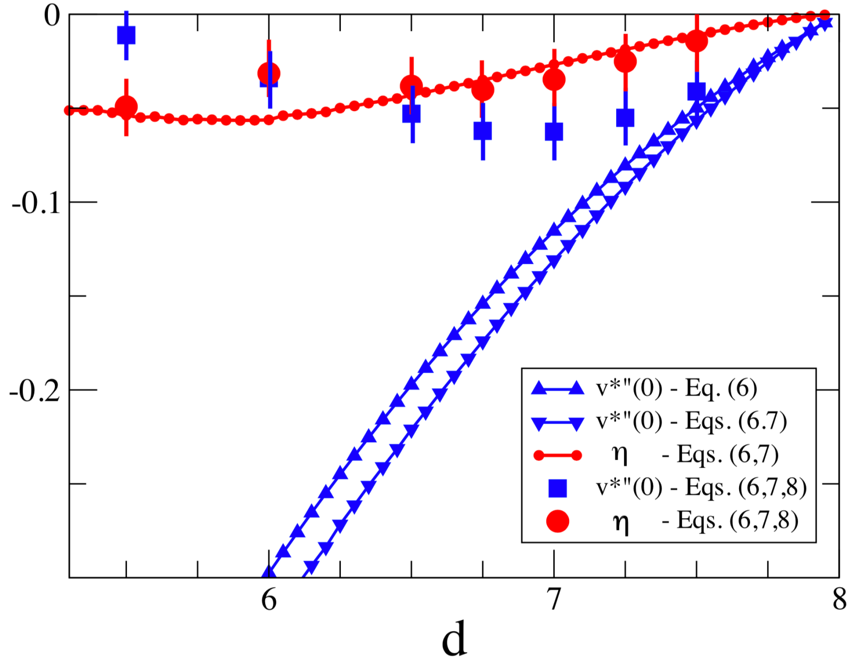
with 5.4 < d < 8

Negative anomalous dimension.

$\eta < 0$ suggests $d_l > 4$.

Sign of η confirmed via different approach

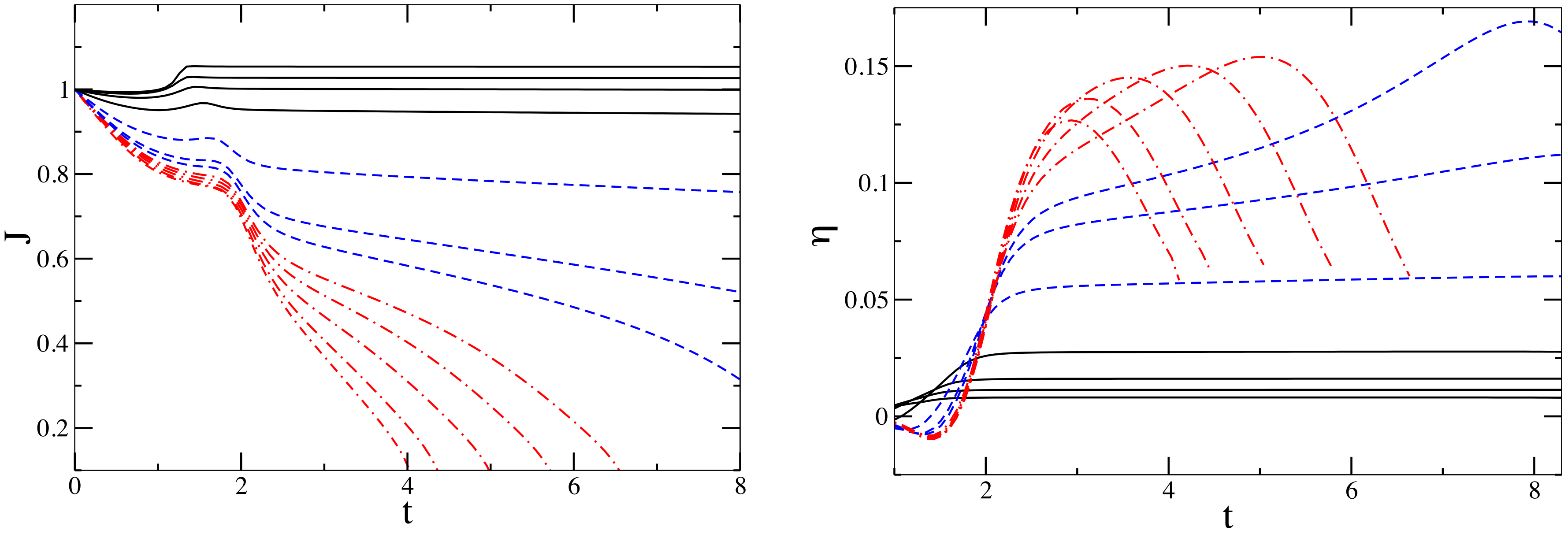
M.Safari, GP Vacca, Phys. Rev D 97 (2018) 041701.



U(1) theory including longitudinal component σ

Resolution of the flow eqs. for the parameters α , m_σ , W_π ,W_σ , Z_π, Z_σ

with various initial values of α^2 ($1.5 - 0.2\ K_{UV}$)



Decreasing a^2 (K_{UV}), 3 regimes : 1) const. (LP) ; 2) linear deviations ; 3) J = 0 at small t

Apparently: regime 1) → line of fixed points in the large a^2 (low T) region

No true LP observed : even the constant regime breaks down at very large t .

Same picture as for the BKT in the U(1) theory in d=2

(in Jakubczyk, Metzner, Phys.Rev. B95,(2017)085113 the effect is ascribed to the limited approximation)

U(1) theory in polar coordinates

The most general action to $O(\partial^4)$ and ϕ^4

$$\Gamma_k^E[\phi] = \int d^4\mathbf{r}\Big\{V_k(\rho) + (a_1 + a_2\rho)\,(\partial\phi\partial\phi^*) + a_3\left[\phi^*\overleftrightarrow{\partial}\phi\right]^2 + (b_1 + b_2\rho)\,\left[\partial^2\phi\,\partial^2\phi^*\right] + b_3\left[\phi^*\overleftrightarrow{\partial^2}\phi\right]^2$$
$$+b_4\,\partial\phi\,\partial\phi^*\left[\phi^*\overleftrightarrow{\partial^2}\phi\right] + b_5\left[(\partial\phi\partial\phi)\left(\phi^*\partial^2\phi^*\right) + (\partial\phi^*\partial\phi^*)\left(\phi\partial^2\phi\right)\right] + b_6\left[\partial\phi\partial\phi^*\right]^2 + b_7\left[(\partial\phi\partial\phi)\,(\partial\phi^*\partial\phi^*)\right]\Big\}$$

After substitution and by neglecting fluctuations of ρ :

$$\Gamma_k^{IR}[\phi] = \int d^4\mathbf{r}\Big\{\left(a_1\rho_0 + a_2\rho_0^2\right)(\partial\vartheta\,\partial\vartheta) + \left(b_1\rho_0 + b_2\rho_0^2\right)\left(\partial^2\vartheta\,\partial^2\vartheta\right)$$
$$+\left(\frac{b_1}{\rho_0} + b_2 + 4b_3 - 2b_4 + b_5 + b_6 + b_7\right)\rho_0^2\,(\partial\vartheta\,\partial\vartheta)^2\Big\}$$

A suitable combination of the parameters at the ILP cancels quartic powers of ϑ

U(1) theory in polar coordinates

With the effective action quadratic in ϑ the field correlator is:

$$\langle \phi(\mathbf{r}) \phi^*(\mathbf{0}) \rangle = \left\langle \sqrt{\rho(\mathbf{r})\rho(\mathbf{0})} \; e^{i[\vartheta(\mathbf{r}) - \vartheta(\mathbf{0})]} \right\rangle = \rho_0 \exp\left[-\frac{1}{2} \left\langle [\vartheta(\mathbf{r}) - \vartheta(\mathbf{0})]^2 \right\rangle \right]$$

and

$$\left\langle [\vartheta(\mathbf{r}) - \vartheta(\mathbf{0})]^2 \right\rangle = \int \frac{d^4\mathbf{q}}{(2\pi)^4} \frac{|e^{i\mathbf{qr}} - 1|^2}{[2\rho_0 (b_1 q^4 + a_1 q^2)]} = 2\,\eta_1 \int_0^{\Lambda r} \mathrm{d}x \frac{x^3 [1 - (2/x) J_1(x)]}{[x^4 + (a_1 r^2 / b_1) x^2]} \approx 2\,\eta_1 \; \ln(\Lambda r)$$

where $\eta_1 = \dfrac{1}{16\pi^2 \rho_0 b_1}$ Therefore $\rightarrow$ $\langle \phi(\mathbf{r}) \phi^*(\mathbf{0}) \rangle \propto \dfrac{\rho_0}{r^{\eta_1}}$

The algebraic long-distance decay is obtained

IF the action is quadratic in the phase.