Hydrodynamics of Onsager's Vortex Flow

Fractional quantum Hall effect

P. Wiegmann,

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- Turbulent Chiral Flows in 2D

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- Rotating superfluid (say, He^4)

- Turbulent Chiral Flows in 2D
- Rotating superfluid (say, He^4)
- Electronic liquid in a strong magnetic field (Quantum Hall Effect)

2D INCOMPRESSIBLE FLOWS CONSIST OF VORTICES

Say, on a sphere, there are no flow at all unless there are vortices or boundaries

How important that vortices are quantized?

Two step quantization:

- Classical fluid: Circulation of each vortex is $\Gamma = n \times \hbar$

$$n \to \infty, \quad \hbar \to 0, \quad \Gamma = \text{fixed}$$

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Two step quantization:

- Classical fluid: Circulation of each vortex is $\Gamma = n \times \hbar$

$$n \to \infty, \quad \hbar \to 0, \quad \Gamma = \text{fixed}$$

- Quantum vortices $n = 2, 3....$

Hydrodynamics of the vortex fluid (2D)

- Vortices can be viewed as constituencies of a secondary fluid

the vortex fluid

or vortex matter

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- Slow motion: drift of vortices

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- **What is hydrodynamics of quantum vortex fluid?**
- classical fluid subject of the paper with Alexander Abanov

OUTLINE

- Hydrodynamics of vortex flow: fluid of topological constitences

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- Exotic effects of quantization

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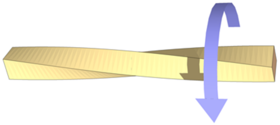
- Hydrodynamics of vortex flow: fluid of topological constitences
- Exotic effects of quantization
- Example: Effect of torsion



ONSAGER: NUOVO CIMENTO PAPER, 1949

- The idea to treat vortices as a macroscopical system goes back to Onsager

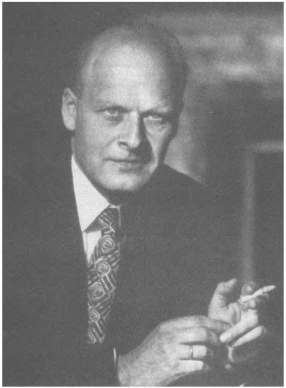
SUPPLEMENTO AL VOLUME VI, SERIE IX DEL NUOVO CIMENTO N. 2, 1949

XIII.

Statistical Hydrodynamics. (*)

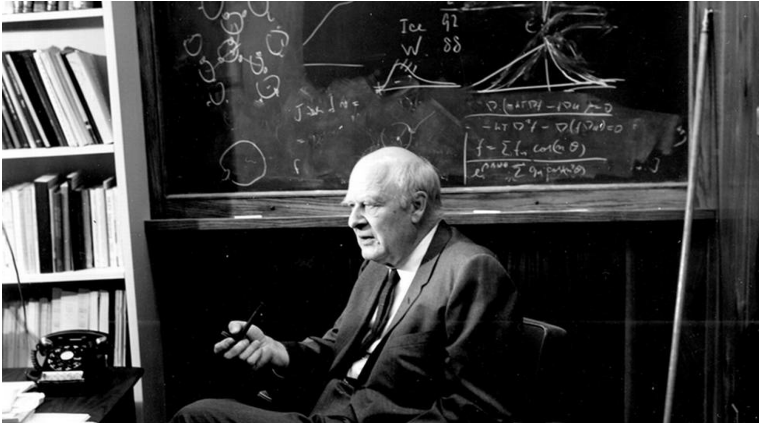
L. ONSAGER
New Haven, Conn.

It is a familiar fact of hydrodynamics, than when the « Reynolds number » exceeds a certain critical value, which depends on the type of flow, no steady flow is stable. The unsteady flow which occurs under these conditions calls for statistical analysis; but early attempts in this direction encountered formidable difficulties. Within the last few years, however, the most important remaining questions concerning the stability of laminar flow were settled by C. C. LIN [1], and a promising start towards a quantitative theory of turbulence was achieved by KOLMOGOROFF [2]. For good measure, KOLMOGOROFF's main result was rediscovered at least twice [3], [4]. The theories involved deal with the mechanism of turbulent dissipation. We shall return to this subject; it seems logical to discuss first a different, new application of statistics to hydrodynamics.



Onsager's vortex matter

✓ The idea to treat vortices as a macroscopical system goes back to Onsager



- Excellent survey of Onsager's archive by G. L. Eyink, K. R. Sreenivasan,

Physica **80A** (1975) 217–233 © *North-Holland Publishing Co.*

VORTICES IN He II, CURRENT ALGEBRAS AND QUANTUM KNOTS

M. RASETTI and T. REGGE

Institute for Advanced Study, Princeton, New Jersey 08540, USA

Received 21 February 1975

A canonical quantization scheme is developed for vortices in superfluid He II, using Dirac's technique for constrained hamiltonian systems. Quantization introduces in the theory in natural way the structure of the infinite Lie algebra of incompressible flows. We argue that all the topological invariants of the vortex, considered as a knot, can be regarded as observables of the system. Finally unitary representations of measure preserving flows on R^3 and current algebra are discussed.

Clustering: Chiral flow

- Sign-like vortices of turbulent flows tend to cluster

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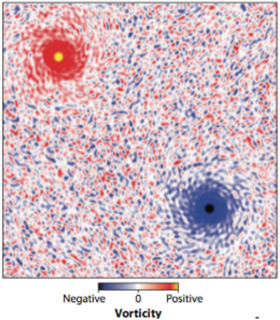
- 

"The formation of large, isolated vortices is an extremely common, yet spectacular phenomenon in unsteady flow . " (Onsager)

Red Spot: the longest observed vortex: after Galileo 1610

1945 Pauling Note, 1949 Publication in Nuovo Cimento:

The little vortices who wanted to play

Once upon a time there were n vortices of strengths $K_1, \dots, K_n$ in a dimensional frictionless incompressible fluid. They were enclosed by a bounda could play ring-around-the-rosy otherwise. The rule of that game was 1)

$$K_i \, dx_i/dt = -\partial W/\partial y_i \quad ; \quad K_i \, dy_i/dt = \partial W/\partial x_i$$

where $-\rho W(x_1, y_1, \dots, x_n, y_n)$ equals the energy apart from an additive (which is infinite on account of the self-energies). The function W is some like this:

$$W = \frac{1}{2\pi} \sum_{i,j}' K_i K_j \log(r_{ij}) + \text{(potential of image forces)}$$

and the image forces are finite except near the boundary, --- Now the vortice very playful like I said and they liked to distribute themselves in completel fashion but they could not do that because they had too much energy. You see not like molecules which have more room in momentum-space the more energy the The vortices had only a finite configuration-space. So when they had more e the average over that space, they could not play quite the way they wanted t

“The little vortices who wanted to play"

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<u>The little vortices we want to play</u>

Once upon a time there were n vortices of strength $\Gamma_1, \ldots, \Gamma_n$ in a two dimensional frictionless fluid. They were enclosed by a boundary and could play ring-around-the rosy otherwise. The rule of that game was

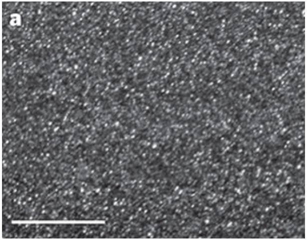
$$\Gamma_i dx_i/dt = -\partial H/\partial y_i; \quad \Gamma_i dy_i/dt = +\partial H/\partial x_i$$

where H equals the energy apart from additive infinite self-energies. It is something like this

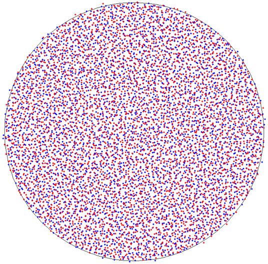
$$H = \sum_{i>j} \Gamma_i \Gamma_j \log r_{ij}$$

Now vortices are very playful like I said and they liked to distribute themselves in a random fashion, but they could not do that because they had <u>too much energy</u>. The vortices had only a finite configuration-space, so they can not play the way they want to.

You can figure out that there is no way to take care about much energy unless you let the same sign to get closer together. And now you know how the little vortices arranged - they just pushed the bigger vortices together until the big vortex has all the energy the little one do not want, and then the little ones played ring-around-the rosy until you could not tell which was where.



Quantized vortices in superfluid He^4
(after Bewley et all 2006)



Monet-Carloimulation of 10^3 electrons of
$\nu = \frac{1}{n}$ filled QH state

Feynman wave function for superfuid

$$\Psi \sim \prod_{i>j}^{N} (z_i - z_j)^{\frac{\Gamma}{\hbar}} e^{-\frac{\Omega}{2}\sum_i |z_i|^2}$$

Laughlin FQH wave function

$$\Psi \sim \prod_{i\neq j}^{N} (z_i - z_j)^{n} e^{-\frac{B}{4}\sum_i |z_i|^2}$$

$$\text{Identification :} \quad \text{filling fraction} = \frac{\Gamma}{\hbar}, \quad \text{magnetic field} = \frac{1}{2}\ \text{frequency of rotation}$$

Euler Equation of rotating fluid

$$\dot{\mathrm{u}} + (\mathrm{u}\cdot\nabla)\,\mathrm{u} = 2\,\Omega\times\mathrm{u} - \nabla p, \qquad \nabla\cdot\mathrm{u} = 0$$

How does the wave function

$$\Psi \sim \prod_{i>j}^{N}(z_i - z_j)^{\frac{\Gamma}{\hbar}}\, e^{-\frac{\Omega}{2}\sum_i |z_i|^2}$$

follow from the Euler equation?

within short three slides away

Euler Equation

$$\dot{u} + \overbrace{(u \cdot \nabla)\, u}^{\text{advection}} = 2\,\Omega \times u - \nabla p, \qquad \nabla \cdot u = 0$$

is a consequence only of Galilean invariance. It has the same form in terms of quantum operators

The problem is to evaluate advection term

$$\langle \mathbf{u} \cdot \nabla \mathbf{u} \rangle - \langle \mathbf{u} \rangle \cdot \nabla \langle \mathbf{u} \rangle = ?$$

$$\lim_{\epsilon \to 0} \mathbf{u}(x+\epsilon)\, \mathbf{u}(x)$$

short distance cut-off depends on the flow

$$\epsilon = \text{function of } \mathbf{u}$$

Digression: 2D Euler Hydrodynamics of incompressible fluid

In units of mass

Euler Equation	$D_t \mathbf{u} = -\nabla p,$
Material Derivative	$D_t \equiv (\partial_t + \mathbf{u} \cdot \nabla)$
Incompressibility	$\nabla \cdot \mathbf{u} = 0,$
Vorticity	$\omega = \nabla \times \mathbf{u}$

Vorticity is transported along the velocity field: the material derivative of the vorticity in that flow vanishes:

$$\text{Helmholtz Equation :} \qquad D_t \omega \equiv \dot{\omega} + \mathbf{u} \cdot \nabla \omega = 0.$$

Kirchhoff Equations

$$\frac{D\omega}{Dt} \equiv \dot{\omega} + \mathbf{u} \cdot \nabla \omega = 0.$$

Helmholtz (and later Kirchhoff)

$$\mathrm{u}(z,t) = u_x - \mathrm{i}u_y = -\mathrm{i}\Omega\bar{z} + \mathrm{i}\sum_{i=1}^{N} \frac{\Gamma}{z - z_i(t)}$$

Kirchhoff equations: $\mathrm{i}\,\dot{\bar{z}}_i = \Omega\bar{z}_i - \sum_{i \neq j}^{N} \frac{\Gamma}{z_i(t) - z_j(t)}$

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Hamiltonian structure and quantization of Kirchhoff equations

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- Hamiltonian (in units of mass) $\quad \mathscr{H} = \Omega \sum_i |z_i|^2 - \Gamma \sum_{j \neq i} \log |z_i - z_j|$
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- Quantum Kirchhoff equations:

$$\dot{\bar{z}}_i = p_i,$$
$$i p_i = \hbar\partial_{z_i} + \Omega \bar{z}_i - \sum_{j\neq i} \frac{\Gamma}{z_i - z_j}$$

Identification with Quantum Hall states and superfluid

- Quantum Kirchhoff equations: $\dot{z}_i = p_i, \qquad \mathrm{i}p_i = \hbar\partial_{z_i} + \Omega\bar{z}_i - \sum_{j\neq i}\frac{\Gamma}{z_i - z_j}$
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- Electrons in FQH states are identified with vortices in rotating superfluid
- Identification with FQH state: $\Omega = eB/(2c), \quad$ fraction $\nu = \beta^{-1} = \hbar/\Gamma$
- Identification with superfluid: $\beta = \text{integer} = 2\Gamma/\hbar$: if $\Gamma = \hbar$, then $\beta = 2$

Hydrodynamics of vortex flow

- Kirchhoff Hamiltonian $\mathscr{H} = \Omega \sum_i |z_i|^2 - \Gamma \sum_{j \neq i} \log |z_i - z_j|$

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- Kirchhoff Hamiltonian $\mathscr{H} = \Omega \sum_i |z_i|^2 - \Gamma \sum_{j \neq i} \log |z_i - z_j|$
- Express in terms of density of vortices/electrons $\rho = \frac{1}{2\pi\Gamma} \times [\text{vorticity}]$
- The (density of the) Hamiltonian (PW 2013, Can et al 2014)

$$\mathscr{H} = \underbrace{\frac{1}{2}\left(: \mathrm{u}^2 : - \pi\Gamma\rho \log\rho \right)}_{\text{classical part}} + \underbrace{\underbrace{\hbar \left(\rho \log \rho - \frac{1}{96\pi} (\nabla \log \rho)^2 \right)}_{\text{quantum correction}}}_{\text{Entropy=effect of the gravitational anomaly}}$$

- The energy of the vortex flow diverges as $\log|z_i - z_j|$ at merging points. Regularization by a typical distance between vortices $\ell \sim 1/\sqrt{\rho}$

$$\log|z_i - z_j| \to \log \ell = \log(1/\sqrt{\rho})$$

$$-\sum_{i\neq j} \log|z_i - z_j| \to -\int \rho(z) \log|z - z'|\rho(z')\, dV dV' + \int \rho \log \ell \, dV =$$
$$\frac{1}{2\pi\Gamma}\int \left(: \mathrm{u}^2 : -\pi\Gamma\rho \log \rho\right) dV$$

- Integration over paths of Lagrangian particles (vortices)

$$\int \mathscr{D}z_1(t)\dots\mathscr{D}z_N(t)$$

- Instead integration over "*collective modes*"

$$a_{-n} = \frac{1}{n}\sum_{i=1}^{N} z_i^n, \quad \varphi(z) = \sum_{n>0} a_{-n} z^n$$

Large N : $$\mathscr{D}z_1(t)\dots\mathscr{D}z_N(t) = e^{-S[\rho]}\prod_n \mathscr{D}a_n(t) = e^{-S[\rho]}\mathscr{D}\varphi$$

QUANTUM PART

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- $$S[\rho] = \int \left(\rho\log\rho - \frac{1}{96\pi}(\nabla\log\rho)^2\right) dVdt$$

quantum correction=effect of the gravitational anomaly

$$\mathscr{H} = \frac{1}{2}\int \left(: \mathrm{u}^2 : -\pi\Gamma\rho\log\rho\right) dV + \hbar\int\left(\rho\log\rho - \frac{1}{96\pi}(\nabla\log\rho)^2\right) dV$$

Stress

- Once the energy is known, we can find the momentum flux tensor

$$\Pi_{ij} = \mathrm{u}_i \mathrm{u}_j + p\delta_{ij} + \frac{\Gamma}{2\pi} T_{ij}$$

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- Anomalous stress in complex coordinates (Abanov& PW 2014, Can et al, 2014-15)

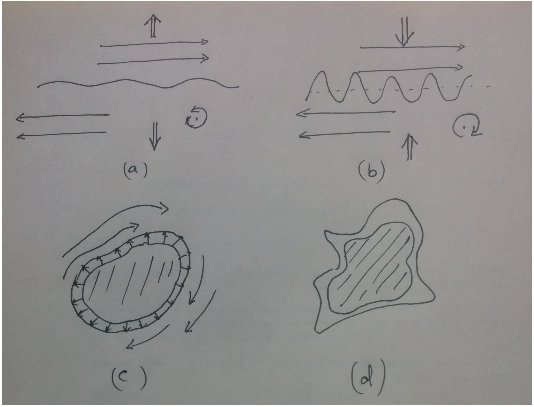
$$\text{shear} = T_{zz} = -\underbrace{\frac{1}{8\pi}\partial_z \mathrm{u}_z}_{\text{odd-viscosity}} + \frac{\hbar}{48\pi}\underbrace{\left(-\frac{1}{2}(\partial_z \log\rho)^2 + \partial_z^2 \log\rho\right)}_{\text{Schwarzian}[\log\rho]=\text{effect of grav. anomaly}}$$

$$\text{compression} = T_{z\bar{z}} = -\frac{\Gamma}{2}(\rho - \bar{\rho}) + \underbrace{\frac{\hbar}{48}\Delta\log\rho}_{\text{trace anomaly}}$$

Trace anomaly yields the intrinsic angular momentum the flow

$$L = \frac{2}{\pi} T_{z\bar{z}}$$

Effects of shear and compression



Force acts normal to shear: (a) repelling; (b) compressing - Kelvin-like instability of the interface; (c) shear causes compression/expansion; (d) fingering instability (PW 2001)

Coupling with Geometry

- Once the stress is known, we can determine effects of geometry

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- Vortices are pushed to higher curvature (Klevtsov, Can et al 2014)

$$\text{vorticity} = \rho - \bar{\rho} = \frac{1}{8\pi}(R - \bar{R}) - \frac{1}{176\pi^2\bar{\rho}}(3 - \nu)\Delta R + \dots$$

$$\nu = \frac{\hbar}{\Gamma}$$

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- Fluid spins (Klevtsov& PW 2016)

$$\text{angular momentum} = \mathrm{L} = \frac{\hbar}{2\pi}\frac{c_H}{24}R, \qquad c_H = 1 - 3\,\nu^{-1}$$

Cone is a flat surface except a flux of curvature inserted through the apex.

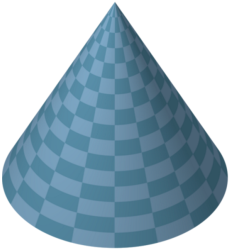
Ho do vortices feel the curvature if they are not there? Vortices know about metric!

Singular geometry : T. Can et al, 2016,

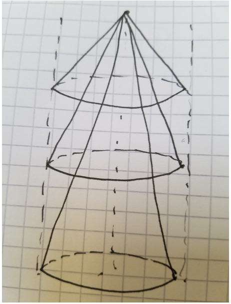
$$\text{circulation} = \frac{1}{2} \times [\text{deficit angle}]$$

$$\text{angular momentum} = \frac{c_H}{12}\frac{(2-\alpha)\alpha}{1-\alpha}$$

α – deficit angle, $\quad c_H = 1 - 3\nu^{-1}$



What happen if the cone angle slowly changes?

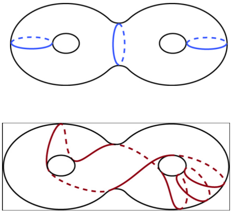


Geometric transport

$$\text{current} = \frac{1}{2} \times [\text{rate of change of } \alpha]$$

$$\text{torque} = \hbar \times [\text{rate of change of } \Delta_\alpha]$$

$$\Delta_\alpha = \frac{c_H}{12} \frac{(2-\alpha)\alpha}{1-\alpha}$$



Geometric singularity

$\tau_1 \to \tau_1 + 1$

transferred vorticity $= \frac{\Gamma}{2}$

angular momentum $= \hbar \frac{c_H}{24}$

Electric charge is transferred in units of $\boxed{e}$

Conjecture: a quantum of angular momentum is transferred in units of

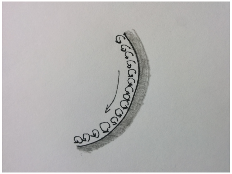
$$\boxed{\frac{\hbar}{24}}$$

Boundary : Forces exert by the fluid to a boundary

Force $\boxed{T_{nn} = \frac{1}{4\nu}\kappa}$ boundary curvature

Edge double layer (Can, Forrester, PW 2013)

$$\boxed{\rho_\nu \approx \rho_{\nu=1} + \frac{1-\nu}{4\pi}\nabla_n \delta_{\text{boundary}}}$$



Boundary vortex layer

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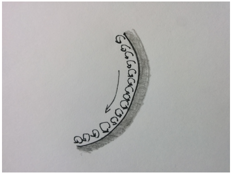

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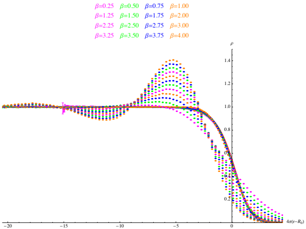
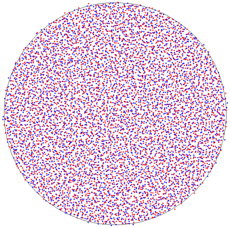


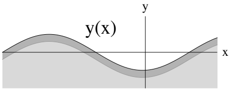
Overshoot (after A. Shytov)



Do you see a boundary layer?

Edge waves: Benjamin-Ono equation PW 2011

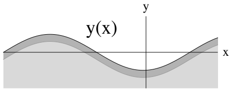
- Motion of the boundary $\dot{\rho} + \nabla_x T_{xy}[\rho] = 0$



Boundary wave

EDGE WAVES: BENJAMIN-ONO EQUATION PW 2011

- Motion of the boundary $\dot{\rho} + \nabla_x T_{xy}[\rho] = 0$



Boundary wave

- (Quantum) Benjamin-Ono equation:

Waves on interface of a sharp change of density (*Benjamin 1968*)

$$\boxed{\dot{\rho} - \Gamma \nabla_x \left(\frac{\Gamma}{2} \rho^2 + \frac{\hbar}{4\pi} \cdot \nabla_x \rho^H \right) = 0}$$

$$\rho^H = \frac{1}{\pi} \int \frac{\rho(x') - \rho(x)}{x' - x} dx'$$

Fractionally charged solitons on the edge

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- Two branches of solitons: *holes/particles* propagating to the left/right;

$$\text{Charge: } q = \int \rho dx = -\nu,\ 1$$

- Topological nature of constituencies of the vortex fluid affect the hydrodynamics

Hydrodynamics of topological fluids

- Topological nature of constituencies of the vortex fluid affect the hydrodynamics
- Hydrodynamics reveales quantization through non-linear soliton motion