# Brownian motion

## Gypsum cristals in a closterium moniliferum

Movie

source: IWF Wissen und Medien gGmbH, 1949

# The ring of Brownian motion: Past - Presence and Future Trends


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D-86135 Augsburg

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# Why you should **not do** Brownian motion

- You know nothing about the subject

- Many very good people worked on it

  (Einstein, Langevin, Smoluchowski, Ornstein, Uhlenbeck, Wiener, Onsager, Stratonovich, …)

- You don't have your own pet theory yet

# Why you should **do** Brownian motion

- You know nothing about the subject

- Many very good people worked on it

- You still can do your own pet theory

# Robert Brown (1773-1858)

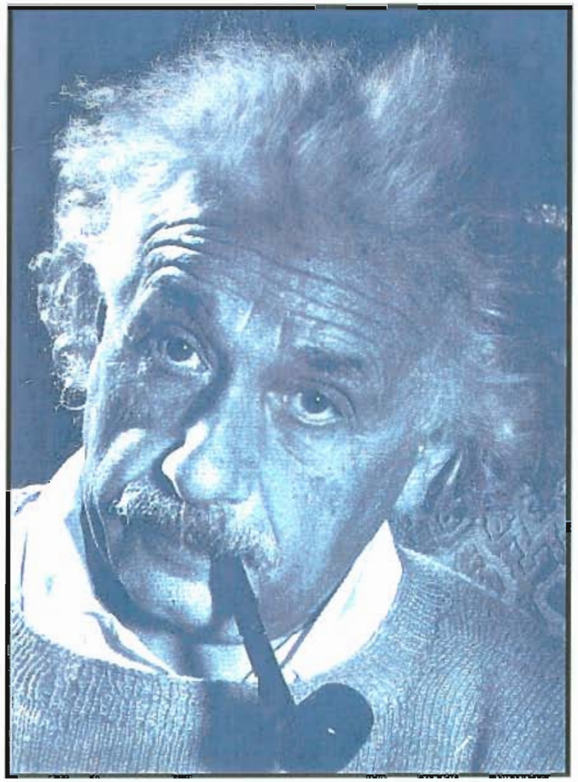
Source: www.anbg.gov.au

Source: permission kindly granted by Prof. Brian J. Ford
http://www.brianjford.com/wbbrowna.htm

1827 – irregular motion of granules of pollen in liquids

- Brown, Phil. Mag. **4**, 161 (1928)
- Deutsch: *Did Robert Brown observe Brownian Motion: probably not*, Sci. Am. **256**, 20 (1991)
- Ford: “*Brownian movement in clarkia pollen: a reprise of the first observations*”, The Microscope **39**, 161 (1991)

**Robert Brown**



**Albert Einstein**

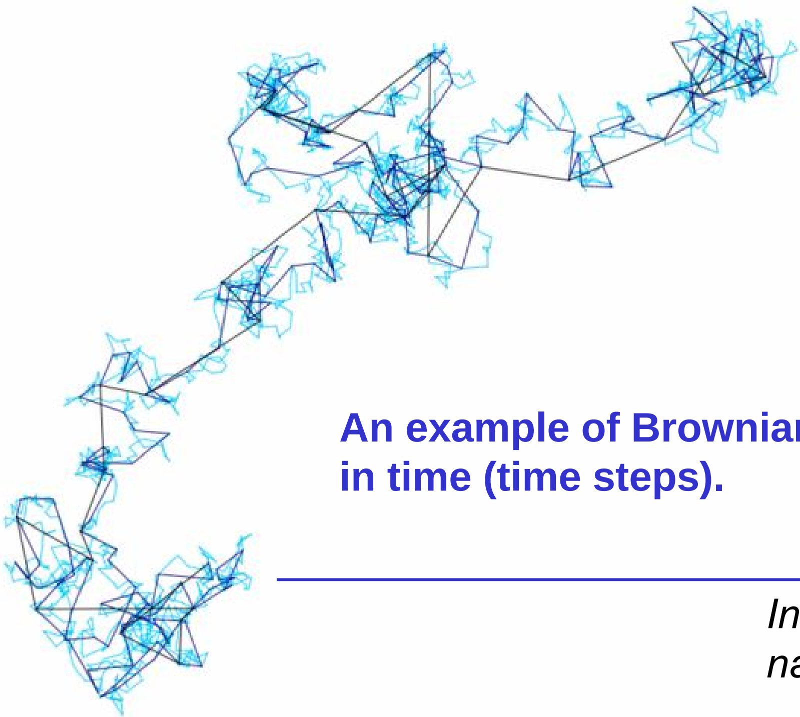
**Marian Smoluchowski**

# Brownian motion

**Robert Brown (1773-1858)**

In 1827, the botanist Robert Brown published a study *”A brief account of microscopical observations on the particles contained in the pollen of plants...”,* where we reported his observations of irregular, jittery motion of small (clay) particles in pollen grains.

He repeated the same experiment with particles of dust, showing that the motion could not be due to the pollen particles being alive.



Although several people worked on this phenomenon over the years, a proper physical explanation of it had to wait for almost 80 years.

**An example of Brownian motion of a particle, recorded for three different resolutions in time (time steps).**

---

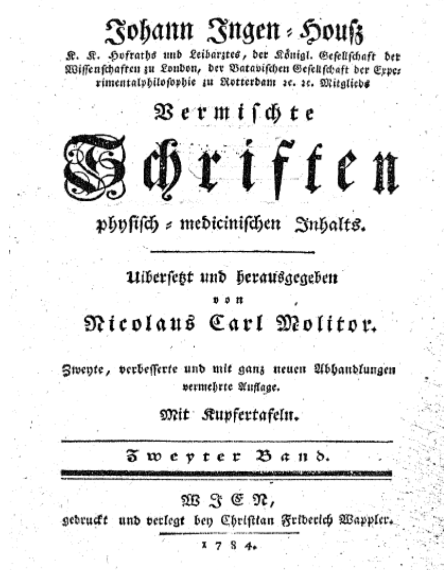
*Incidentally, Robert Brown was also the first to note the ubiquitous nature of a part of eukaryotic cells which he named the ”cell nucleus”.*

permission kindly granted by Prof. Brian J. Ford
http://www.brianjford.com/wbbrowna.htm

**Jan Ingen-Housz (1730-1799)** **William Sutherland (1859-1911)**

# Jan Ingen-Housz (1730-1799)

Source: www.americanchemistry.com



Johann Ingen-Housz

K. K. Hofraths und Leibarztes, der Königl. Gesellschaft der Wissenschaften zu London, der Batavischen Gesellschaft der Experimentalphilosophie zu Rotterdam rc. rc. Mitglieds

Vermischte

Schriften

physisch-medicinischen Inhalts.

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Zweyter Band.

Wien,

gedruckt und verlegt bey Christian Friderich Wappler.

1784.

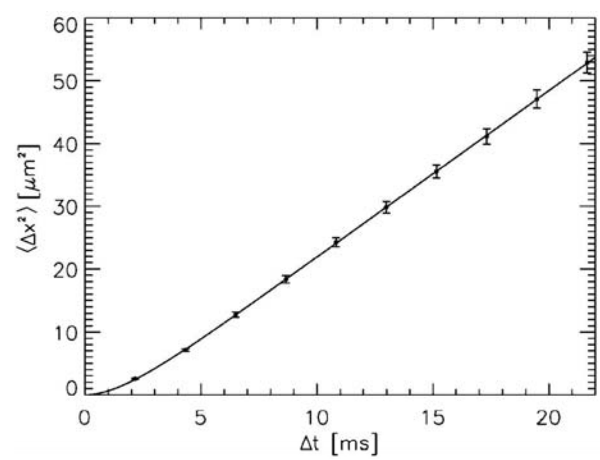
To see clearly how one can deceive one's mind on this point if one is not careful, one has only to place a drop of alcohol in the focal point of a microscope and introduce a little finely ground charcoal therein, and one will see these corpuscules in a confused, continuous and violent motion, as if they were animalcules which move rapidly around.

# Measurement of the Translational and Rotational Brownian Motion of Individual Particles in a Rarefied Gas

Jürgen Blum,* Stefan Bruns, Daniel Rademacher, Annika Voss, Björn Willenberg, and Maya Krause
*Institut für Geophysik und Extraterrestrische Physik, Technische Universität Braunschweig, Mendelssohnstraße 3, 38114 Braunschweig, Germany*
(Received 2 August 2006; published 4 December 2006)

We measured the free Brownian motion of individual spherical and the Brownian rotation of individual nonspherical micrometer-sized particles in rarefied gas. Measurements were done with high spatial and temporal resolution under microgravity conditions in the Bremen drop tower so that the transition from diffusive to ballistic motion could be resolved. We find that the translational and rotational diffusion can be described by the relation given by Uhlenbeck and Ornstein [Phys. Rev. **36**, 823 (1930)]. Measurements of rotational Brownian motion can be used for the determination of the moments of inertia of small particles.

$$\langle \Delta x^2 \rangle = 2D\Delta t\left(1 - \frac{\tau_f}{\Delta t} + \frac{\tau_f}{\Delta t}\exp\left[-\frac{\Delta t}{\tau_f}\right]\right)$$



# Theory of Brownian motion

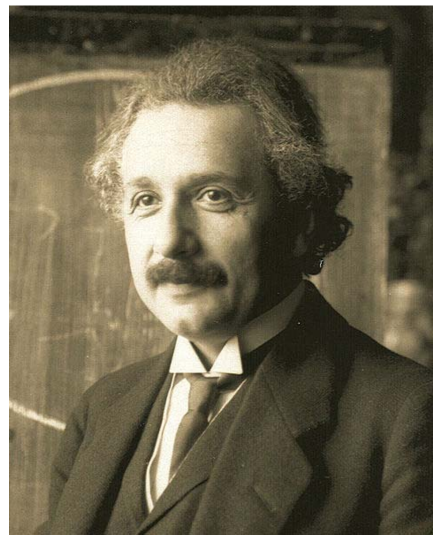
W. Sutherland (1858-1911)

Source: www.theage.com.au

$$D = \frac{RT}{6\pi\eta a C}$$

Phil. Mag. **9**, 781 (1905)

A. Einstein (1879-1955)



Source: wikipedia.org

$$\langle x^2(t)\rangle = 2Dt$$

$$D = \frac{RT}{N}\frac{1}{6\pi k P}$$

Ann. Phys. **17**, 549 (1905)

M. Smoluchowski (1872-1917)

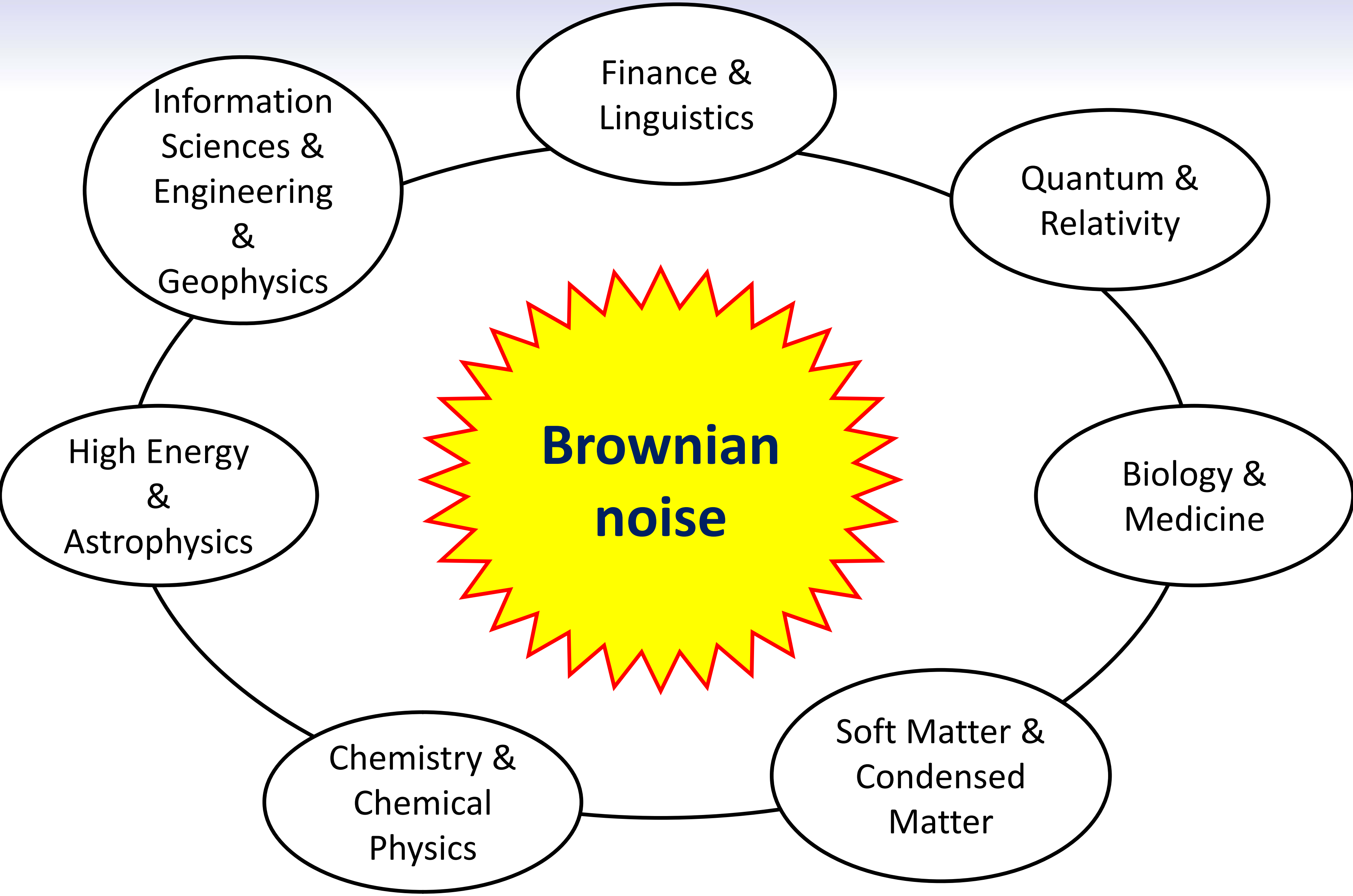
Source: wikipedia.org

$$D = \frac{32}{243}\frac{mc^2}{\pi\mu R}$$

Ann. Phys. **21**, 756 (1906)

http://www.physik.uni-augsburg.de/theo1/hanggi/

- Curriculum Vitae
- Feature articles
- Research areas
- Books & Reviews
- List of publications in chronological order
- List of publications in reverse chronological order
- Das Universum bremst nicht
- 100 years of Brownian motion: historical items and surveys
- Thermodynamics and statistical physics of small systems
- Palermo Lecture Series, November 2016 "Classical and Quantum Thermodynamics and Statistical Physics"


Brownian noise
Finance & Linguistics
Quantum & Relativity
Biology & Medicine
Soft Matter & Condensed Matter
Chemistry & Chemical Physics
High Energy & Astrophysics
Information Sciences & Engineering & Geophysics

# Quantum-Mechanics
# = Brownian Motion ?
# = Stochastic Mechanics ?

(E. Nelson; 1966, 1986)

$$p(x,t) = |\Psi(x,t)|^2$$

Schrödinger-
gleichung

$$\Psi(x,t) = |\Psi(x,t)|\, e^{iS(x,t)}$$

$$\dot{p}(x,t) = -\frac{\hbar}{m}\nabla\left[(\nabla \ln|\Psi(x,t)| + \nabla S(x,t))\, p(x,t)\right] + \frac{\hbar}{2m}\nabla^2 p(x,t)$$

$$\mathbf{f}_1 \geq 0\,,\ \mathbf{f}_2 \geq 0\ :\quad \frac{1}{2}\langle \mathbf{f}_1(t_1)\mathbf{f}_2(t_2) + \mathbf{f}_2(t_2)\mathbf{f}_1(t_1)\rangle \geq 0$$

QM: **NO !**

H. Grabert, P.H., P. Talkner, Phys. Rev. A **19**, 2440 (1979)

# Diffusion: space-time only

## classical diffusion (Markovian)

$$p(t, x|t_0, x_0) = \left[\frac{1}{4\pi\, \mathcal{D}(t-t_0)}\right]^{1/2} \exp\left[-\frac{(x-x_0)^2}{4\mathcal{D}(t-t_0)}\right].$$

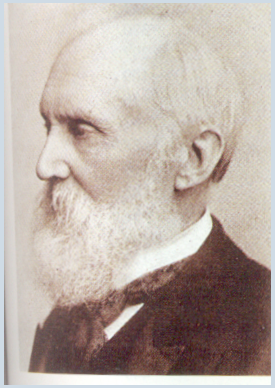
## telegraph equation (non-Markovian)

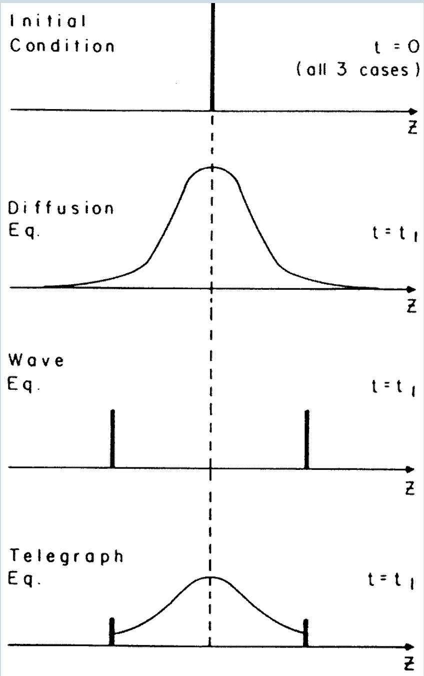
$$\tau_v \frac{\partial^2}{\partial t^2}\varrho + \frac{\partial}{\partial t}\varrho = \mathcal{D}\,\nabla^2 \varrho,$$

## alternative approach

$$\boxed{p(\bar{x}|\bar{x}_0) \propto \int_{a_-(\bar{x}|\bar{x}_0)}^{a_+(\bar{x}|\bar{x}_0)} \mathrm{d}a\ \exp\left(-\frac{a}{2\mathcal{D}}\right)}$$

*PRD **75**:043001 (2007)*






*J Masoliver & G H Weiss*
*Eur J Phys **17**:190 (1996)*

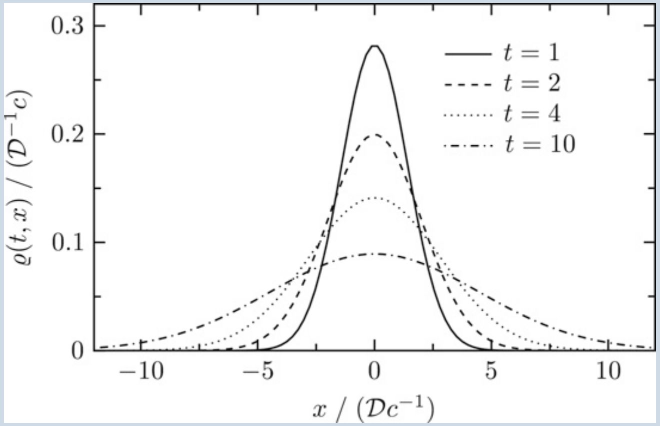
# Diffusion propagator

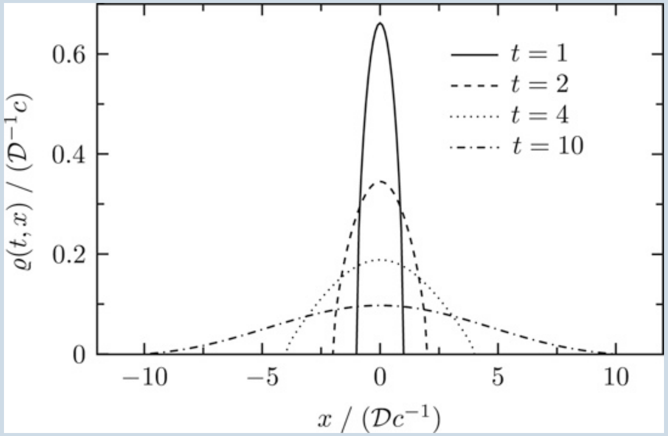
JD et al, PRD **75**:043001 (2007)

$$a(\bar{x}|\bar{x}_0) = \frac{1}{2}\int_{t_0}^{t} \mathrm{d}t' \; v(t')^2,$$

$$a(\bar{x}|\bar{x}_0) = -\int_{t_0}^{t} \mathrm{d}t' \; \left[1 - v(t')^2\right]^{1/2}$$

$$\boxed{p(\bar{x}|\bar{x}_0) \propto \int_{a_-(\bar{x}|\bar{x}_0)}^{a_+(\bar{x}|\bar{x}_0)} \mathrm{d}a \; \exp\left(-\frac{a}{2\mathcal{D}}\right)}$$



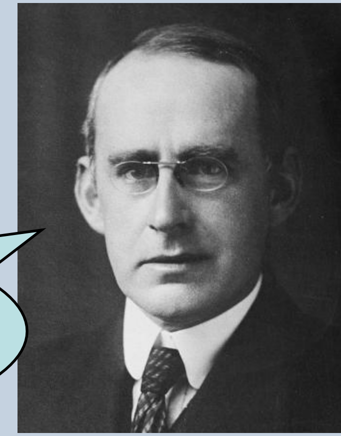

$$a_+ = +\infty$$

$$a_-(\bar{x}|\bar{x}_0) = \frac{(x - x_0)^2}{2(t - t_0)}$$

$$a_+ = 0$$

$$a_-(\bar{x}|\bar{x}_0) = -\left[(t - t_0)^2 - (x - x_0)^2\right]^{1/2}$$

# "Temperature" problem in RTD?



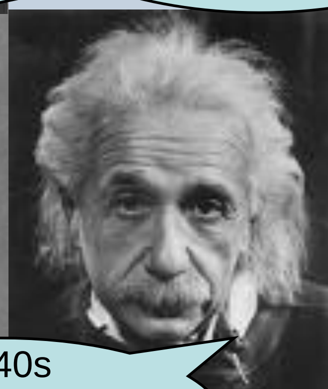
1923/1963

.. hotter!

1907/08

moving bodies appear cooler

$$T'(w) = T\,(1 - w^2)^{\alpha/2} \qquad \alpha = \begin{cases} +1 & \text{Planck, Einstein} \\ 0 & \text{Landsberg, van Kampen} \\ -1 & \text{Ott} \end{cases}$$



1940s

maybe ... not

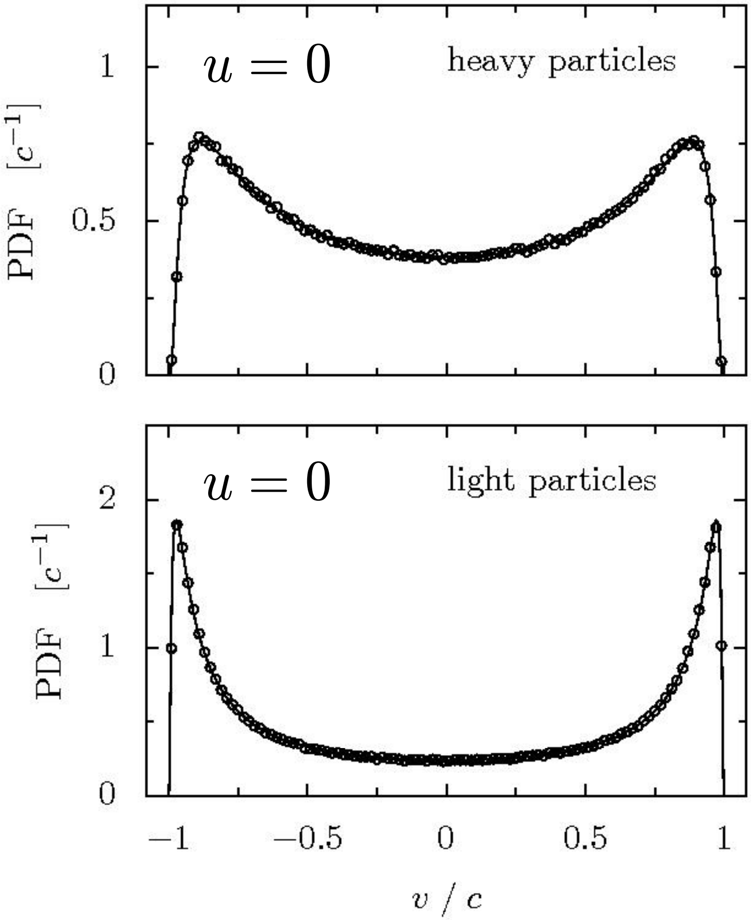
T=T' 1966-69

*CK Yuen, Amer. J. Phys. 38:246 (1970)*

# Jüttner Gas

$$f_{\mathrm{Maxwell}}(\vec{p}) = [\beta/(2\pi m)]^{d/2} \exp\left(-\beta p^2/2m\right)$$

$$f_{\mathrm{Jüttner}}(\vec{p}) = Z_d^{-1} \exp\left[-\beta_{\mathrm{J}}(m^2c^4 + p^2c^2)^{1/2}\right]$$




$$\langle \vec{p} \cdot \vec{v} \rangle = d k_{\mathrm{B}} \mathcal{T} = d/\beta_{\mathrm{J}}$$

statistical realtivistic temperature

$$T = \mathcal{T} = (k_{\mathrm{B}}\beta_{\mathrm{J}})^{-1}$$

J. Dunkel & P.H., Phys. Rep. **471**, 1-73 (2009)

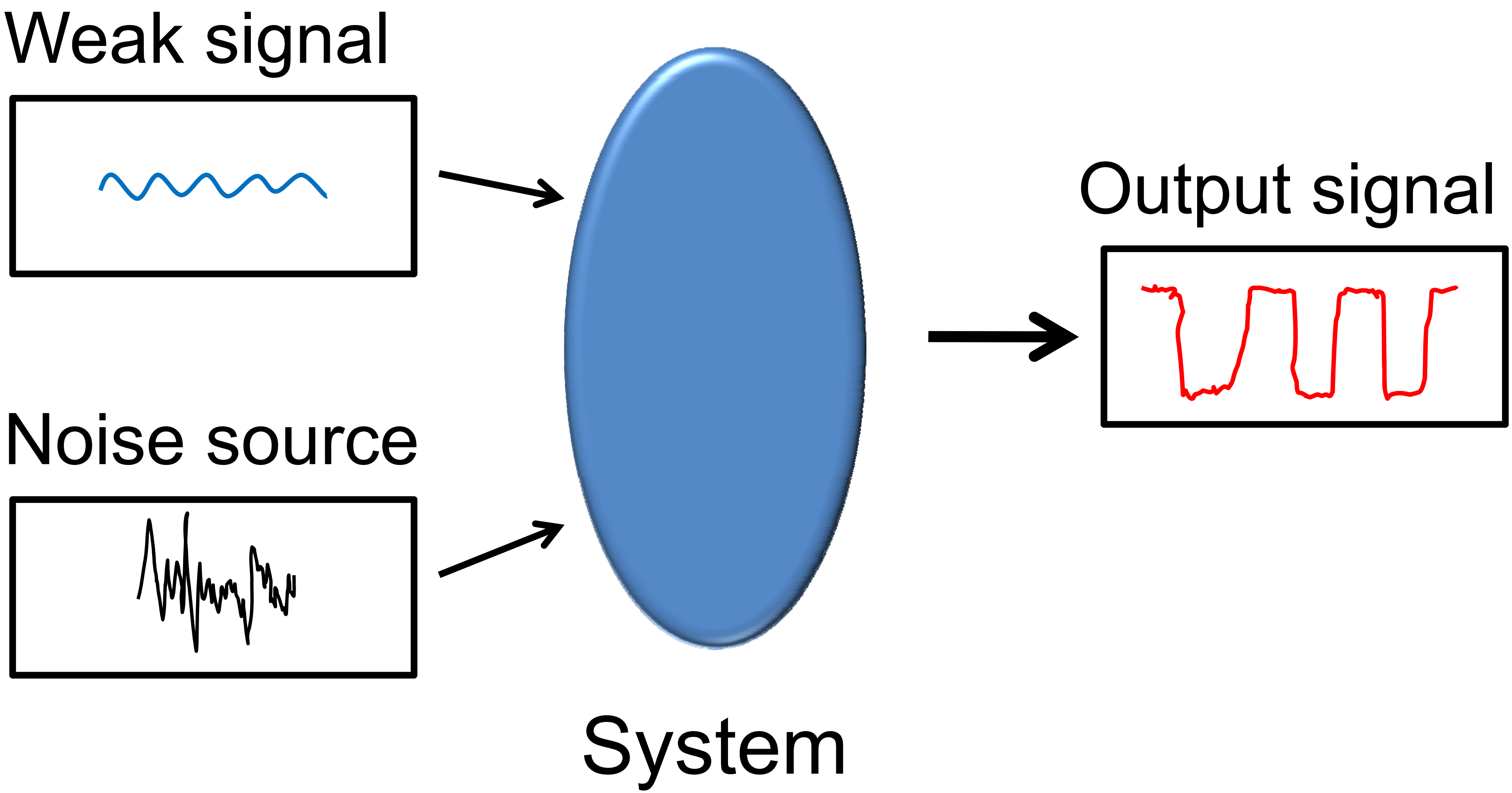
# Two prominent examples

Stochastic Resonance

Brownian Motors

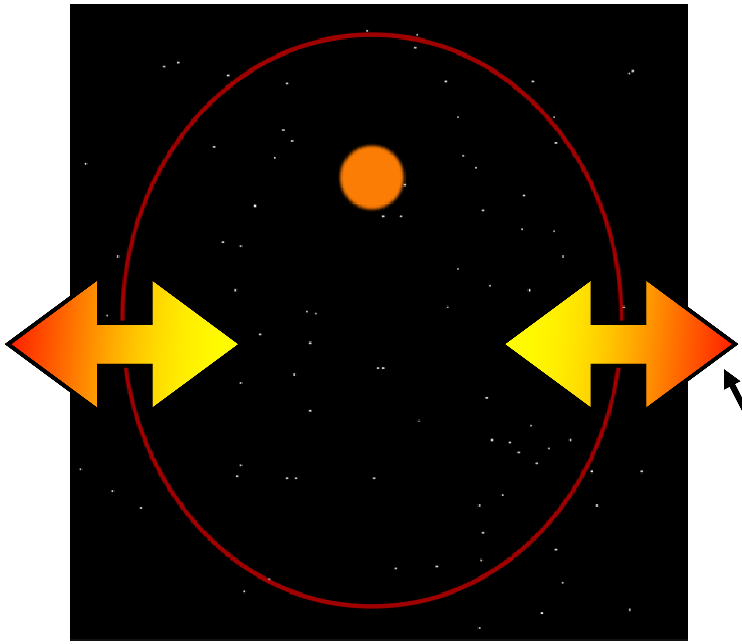
# Stochastic Resonance

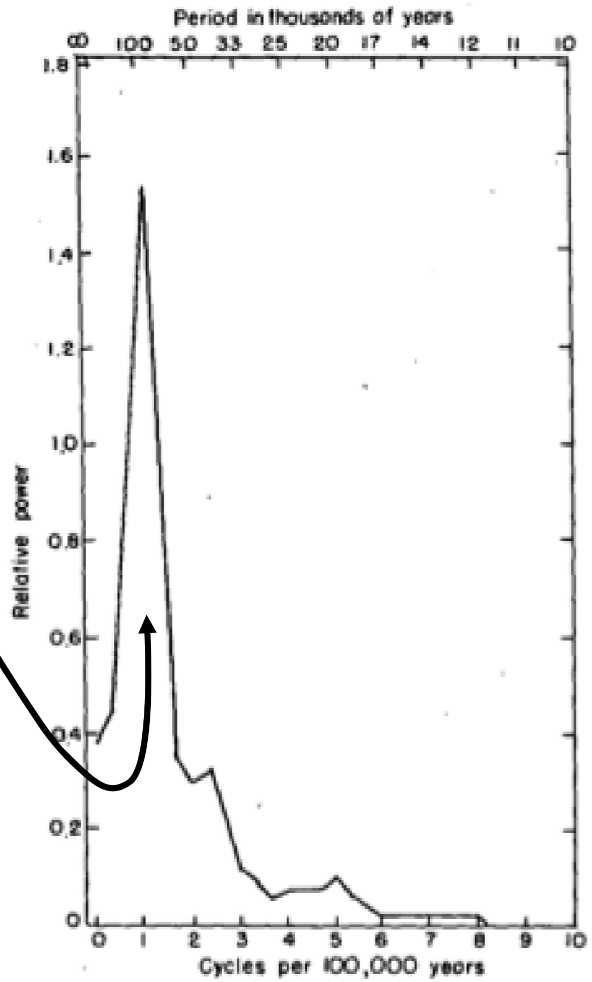
( in a nutshell )



# Why are the ice-ages so periodic ?

Milankowitch cycles:
Small changes in earth orbit eccentricity with 100k year periodicity





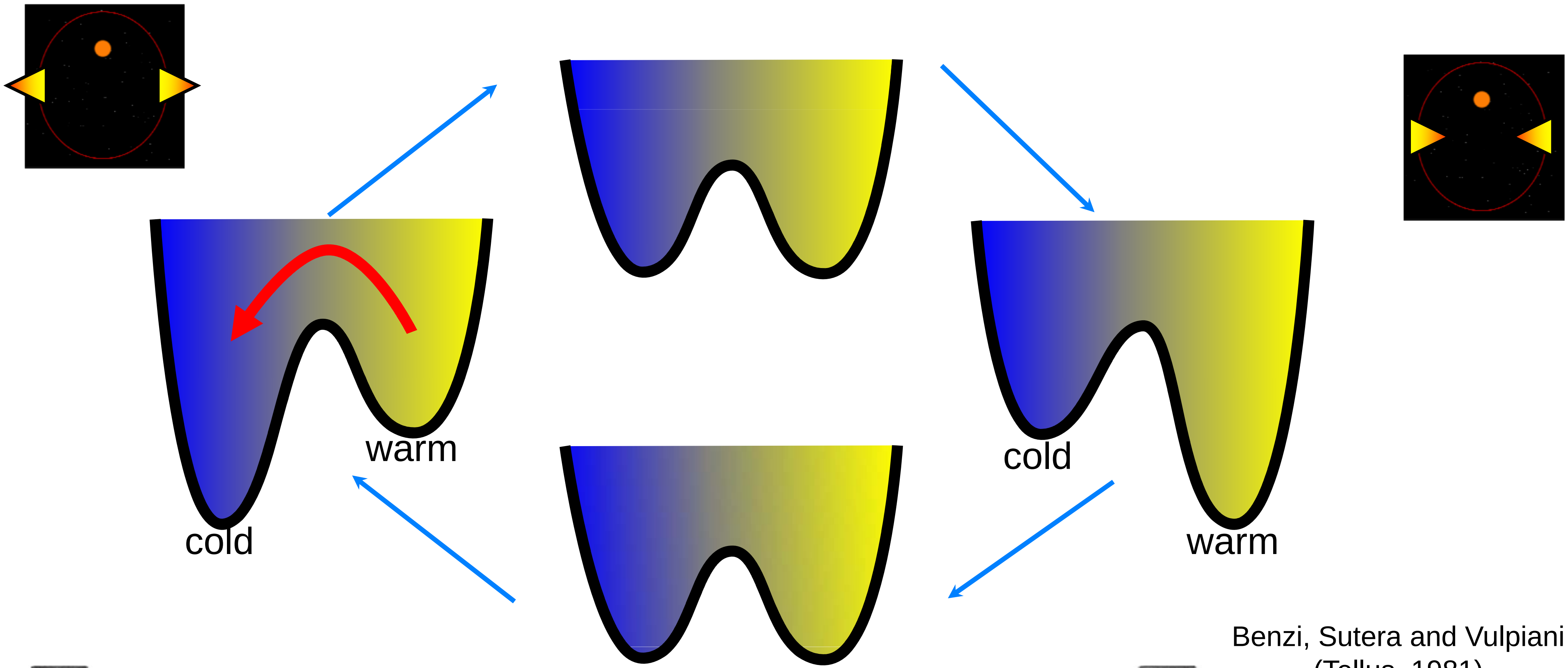

M. Milankowitch, Handbuch der Klimatologie I (1930)

Changes are small!
(<0.1% of solar constant)

What can amplify those small changes ?

# Milankowitch Cycles and Bistability

Climate “landscape”

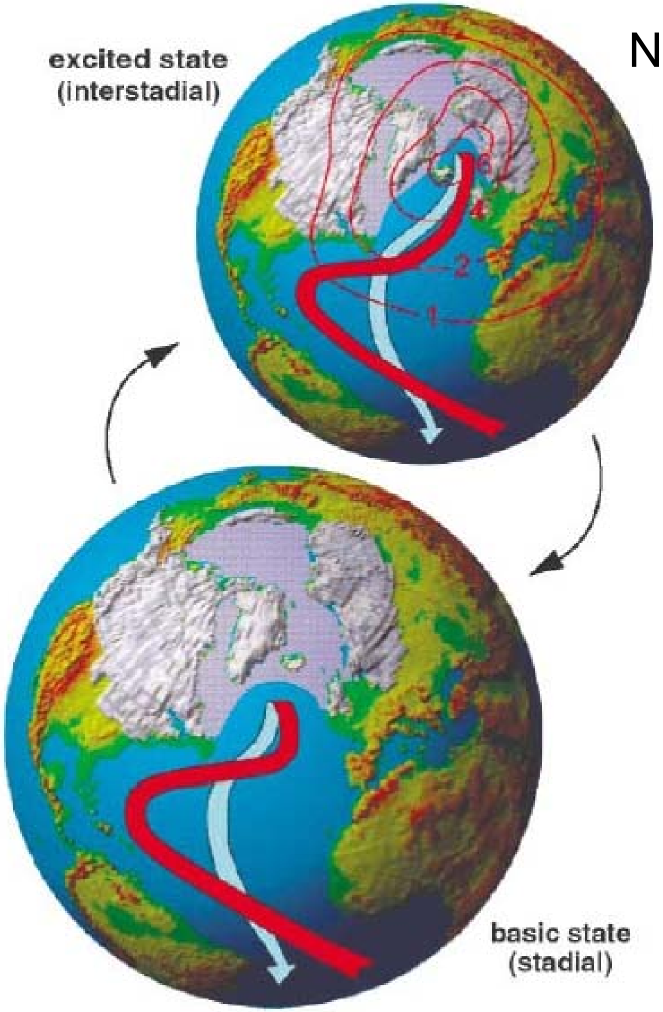


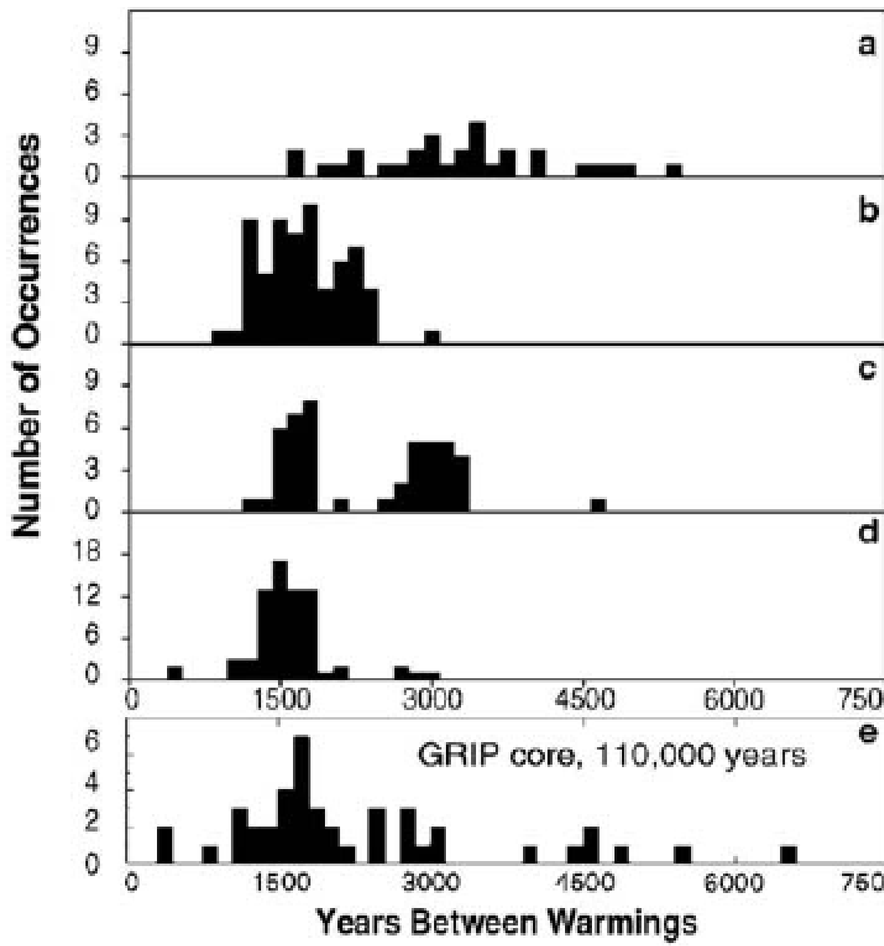

Benzi, Sutera and Vulpiani
(Tellus, 1981)
C. Nicolis and G. Nicolis
(Tellus, 1981)

- The 100ky cycles only bias the climate
- Fluctuations make climate switch
- small changes of conditions can have huge impact

# Occurrence of ice ages

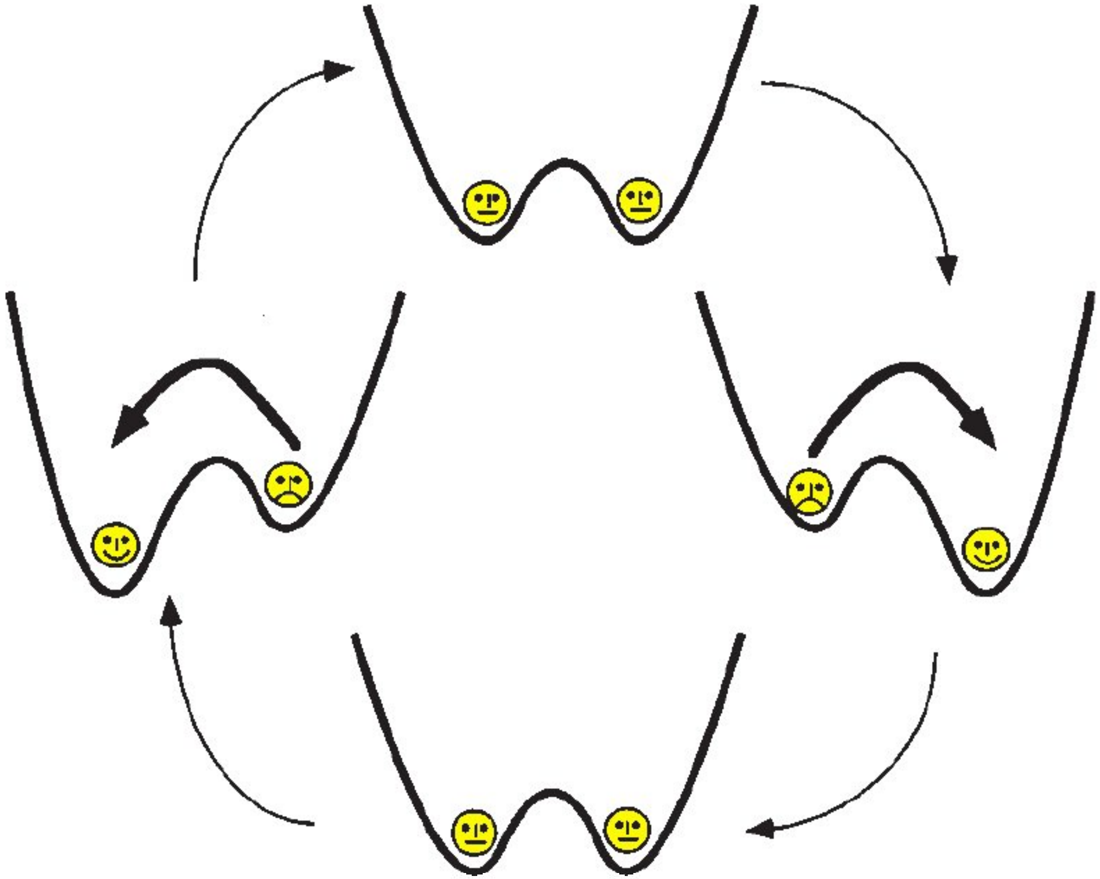
Noise and periodic of solar origin




A. Ganopolski, S. Rahmstorf, Phys. Rev. Lett. **88**, 038501 (2002)

# Noise-assisted synchronized hopping

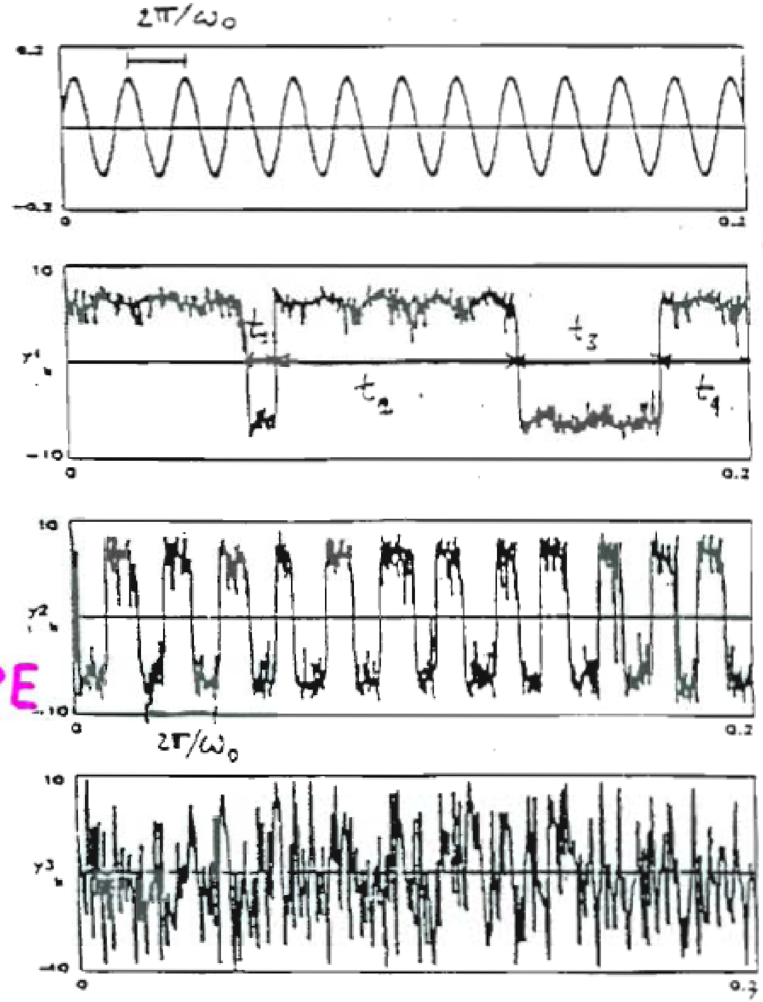


$$T_{\text{period}} \cong 2T_{\text{escape}}$$

# Synchronization

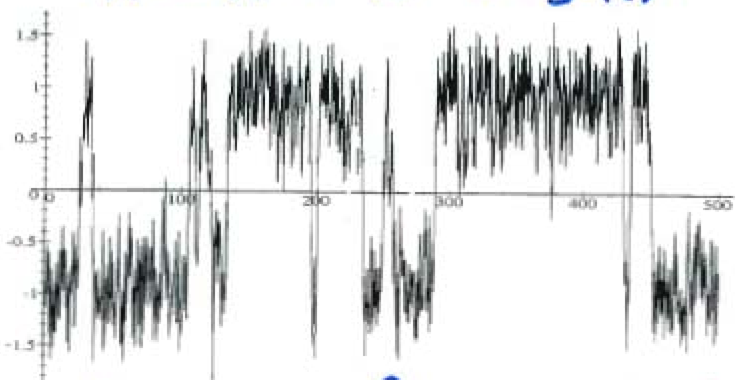
SIGNAL

$T_e \sim 2\Gamma^{-1}_{ESCAPE}$

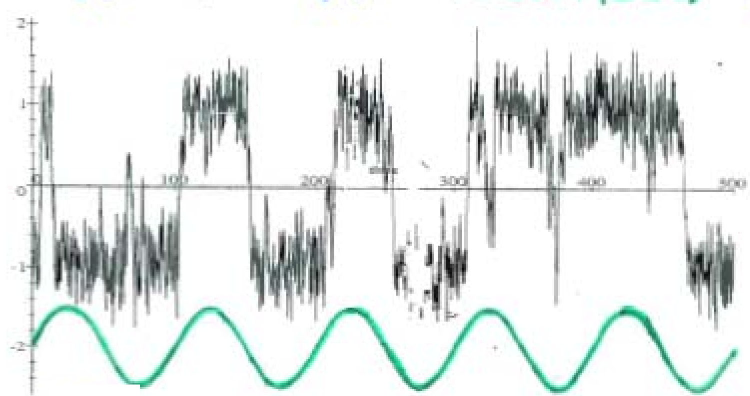


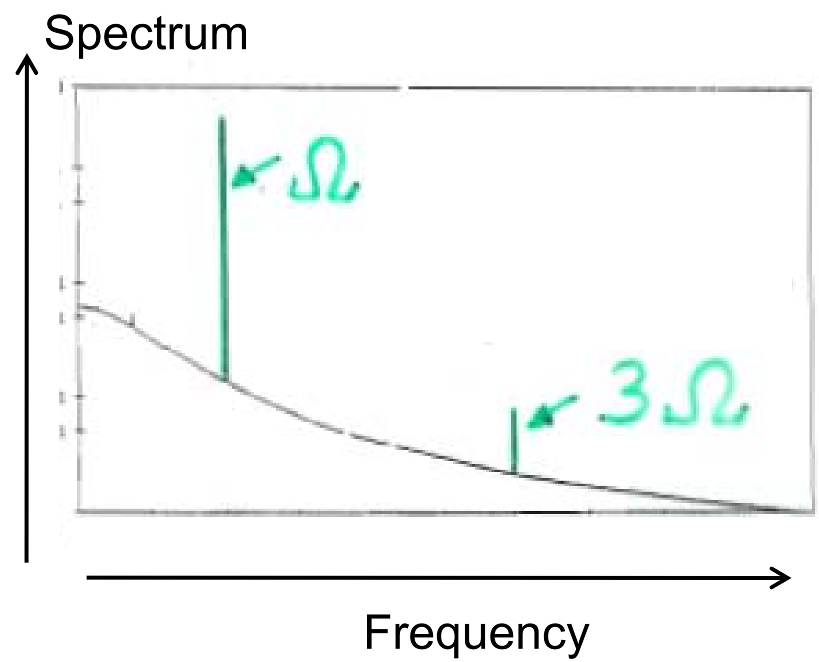
# Power spectral density

$$\dot{x} = x - x^3 + \xi(t)$$




$$\dot{x} = x - x^3 + A\sin(\Omega t) + \xi(t)$$



# Measuring SR

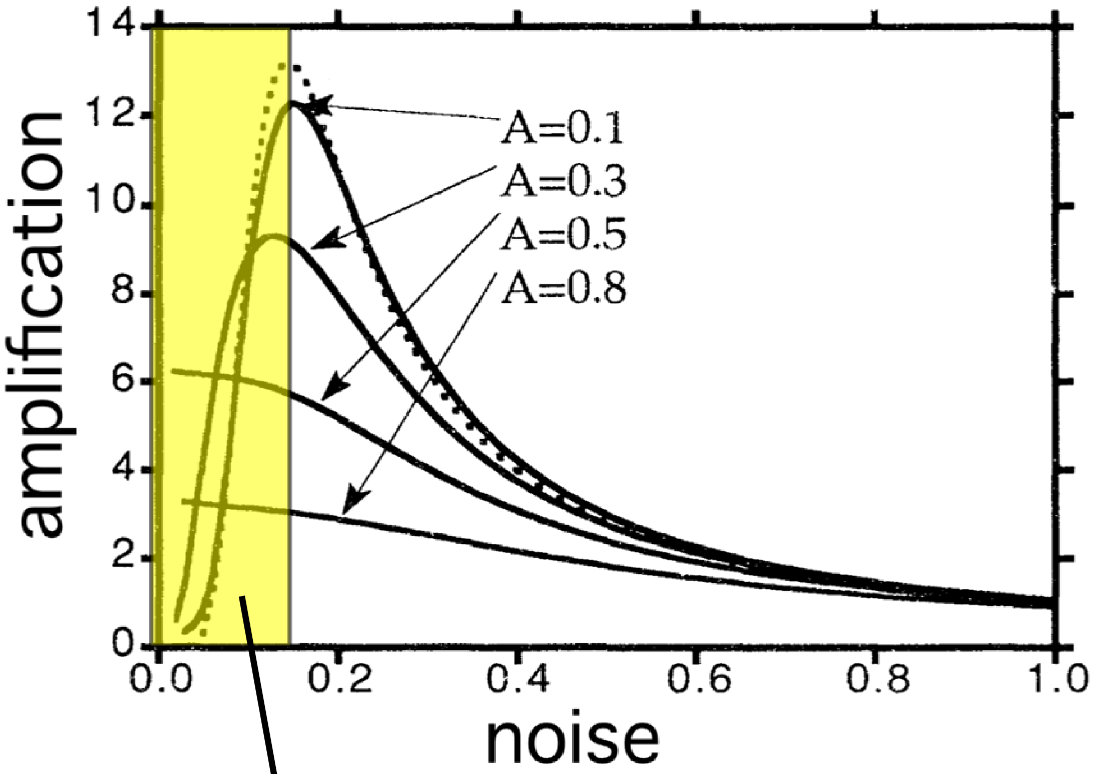
- Signal to noise ratio
- Spectral amplification
- mutual information
- cross-correlation: input $\leftrightarrow$ output
- peak area, (phase-) synchronization, ...

SR-reviews:


L. Gammaitoni, P. Hänggi, P. Jung, F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998)
P. Hänggi, ChemPhysChem **3**, 285 (2002)

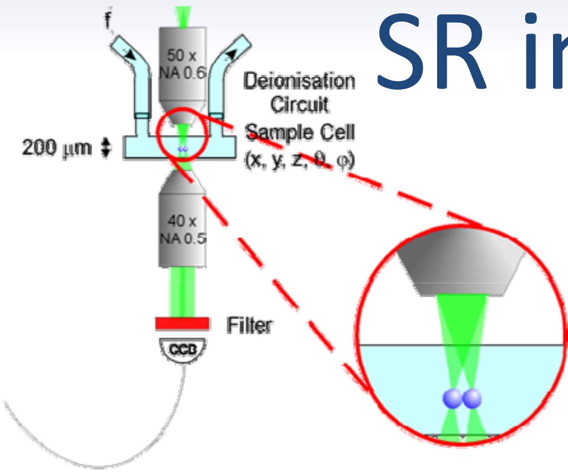
# Amplification of small signals by noise

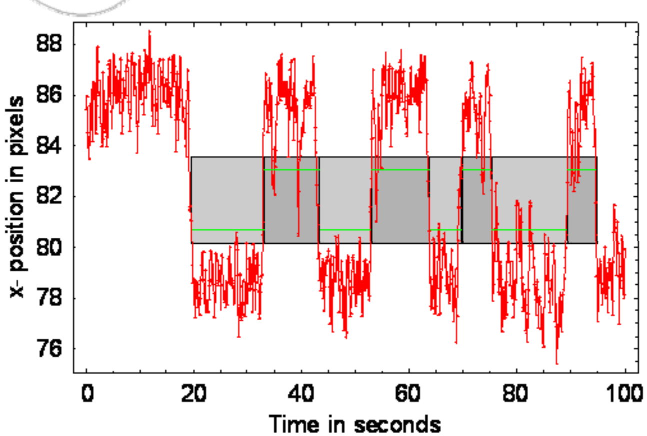
(P. Jung, P. Hänggi, Phys. Rev. A **44**, 8032 (1991))

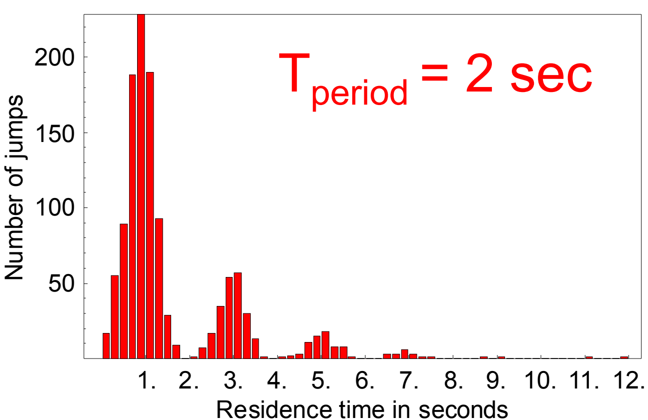


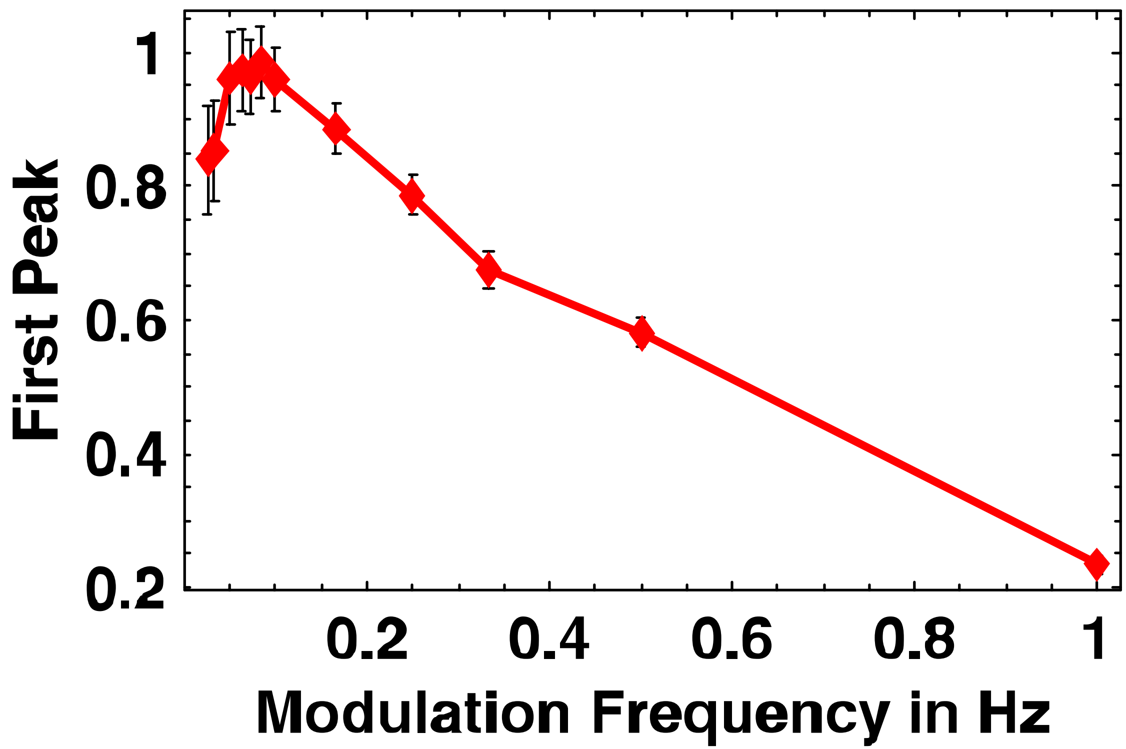

More noise , more signal !!

# SR in colloidal systems




D. Babic, C. Schmitt, I. Poberaj, C. Bechinger, Europhys. Lett. **67**, 158 (2004)

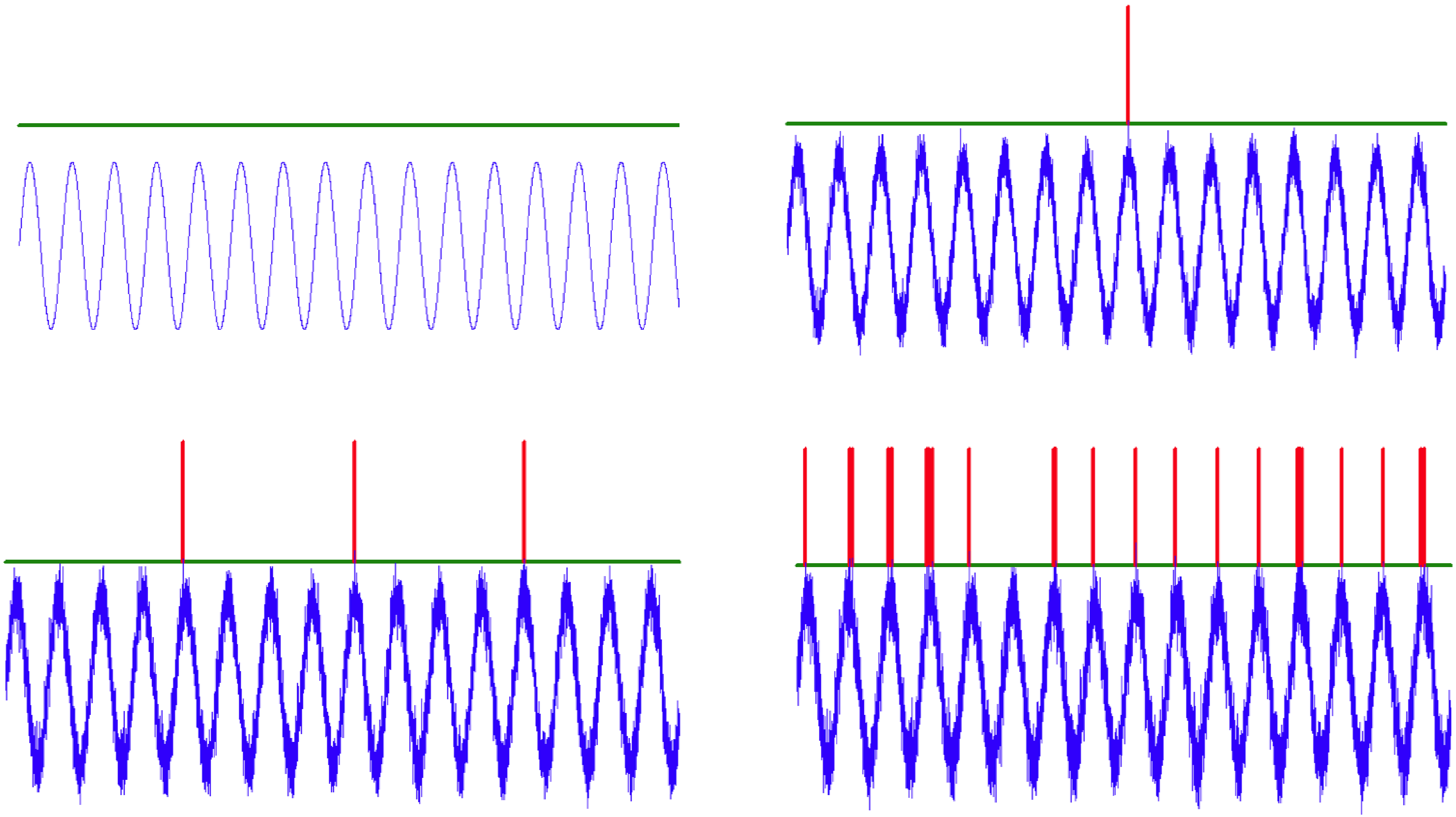
# SR - Ingredients

- ✓ Threshold system
- ✓ Weak ( subthreshold ) signal
- ✓ Noise

↓

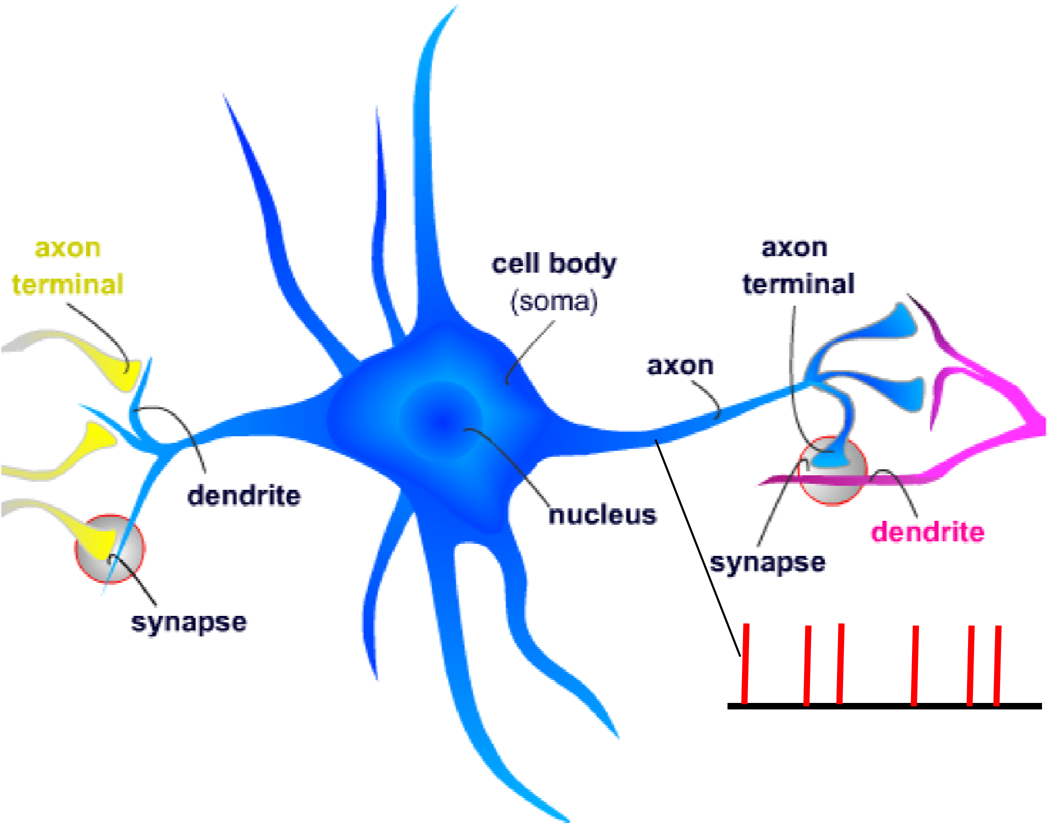
Anomalous amplification properties

# Thresholds and Stochastic Resonance



P. Jung, Phys. Rev. E50, 2513 (1994), F. Moss and L. Kiss, EPL, 29 (1995)

# Stochastic Resonance in Neurobiology




Input: currents at synapses

Processing: action potential if the sum of currents exceeds threshold

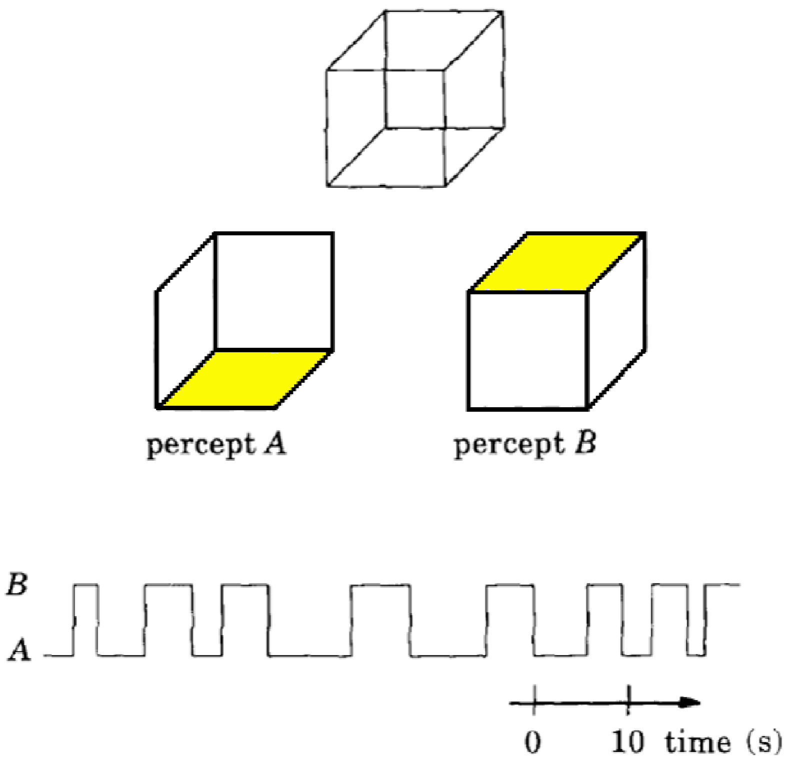
Output: electric pulses traveling down the axon

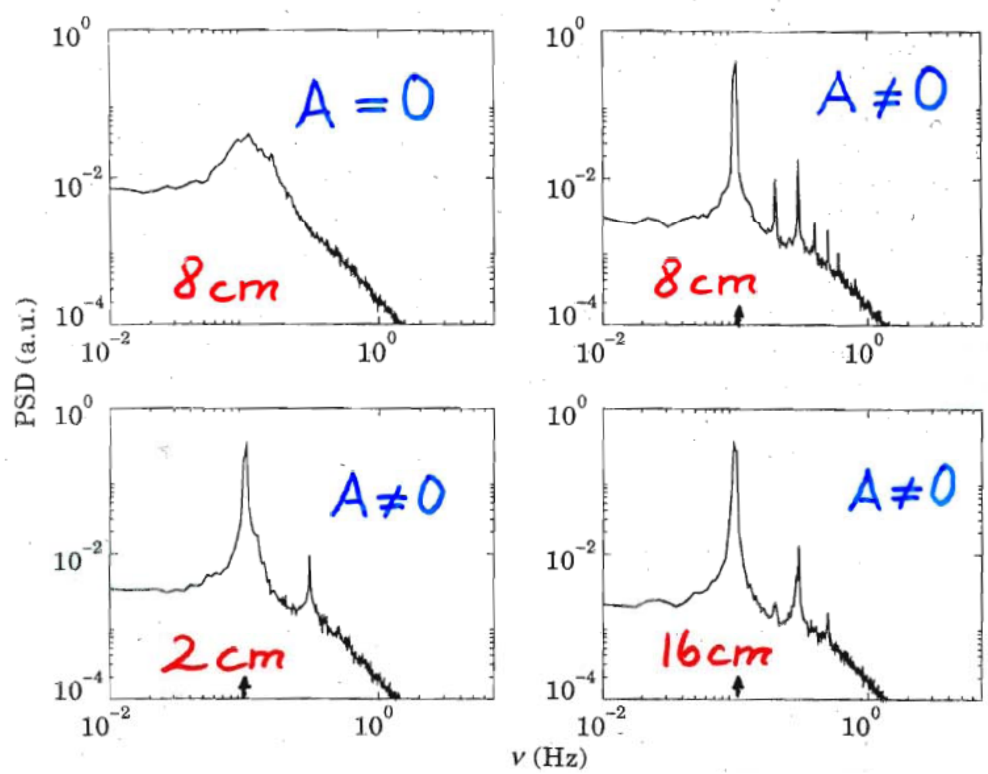
source: Consortium on Cognitive Science Instruction (CCSI)

Basic idea: Signals below threshold can be detected in the presence of additional noise

# SR in Visual Perception

Experiment:

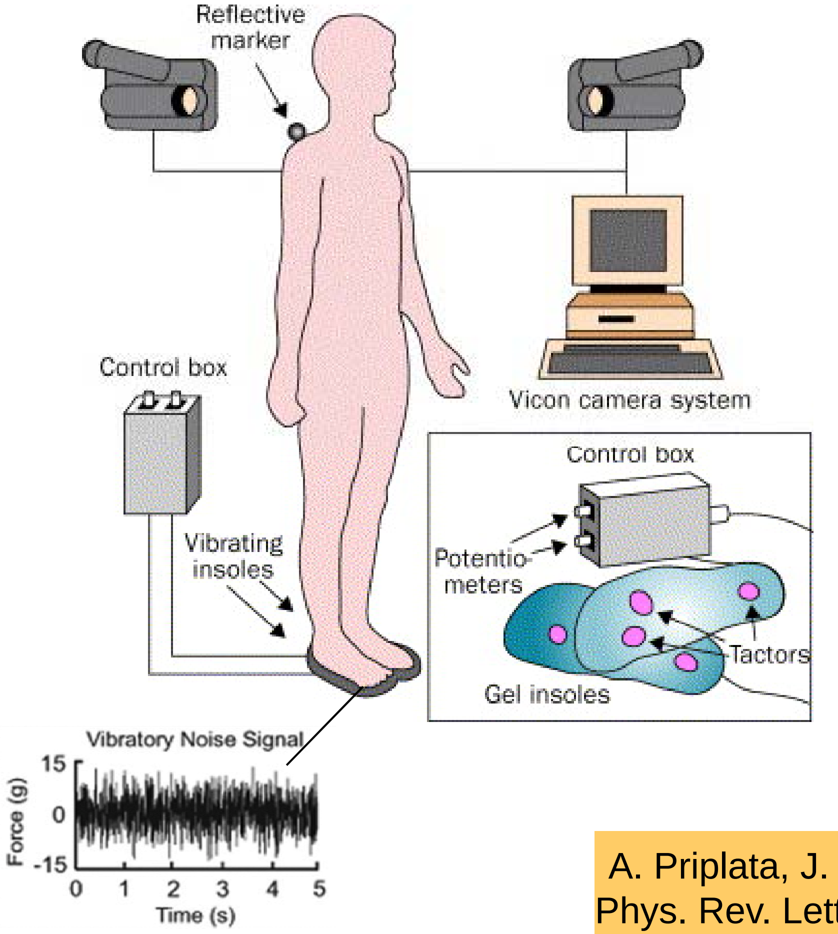



M. Riani, E. Simonotto, Nuovo Cimento D **17**, 903 (1995)

# SR and human posture control

Somatosensory function declines with age and in diabetic patients. Can additional noise help restore function?





Reduction in sway of person

A. Priplata, J. Niemi, M. Salen, J. Harry, L.A. Lipsitz and J.J. Collins Phys. Rev. Lett. 89 (2002)

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- Multiple sclerosis (MS)
- Stroke / skull-brain-trauma
- Cross-section paralysis
- Depression
- Pain
- …

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- Cross-section paralysis
- Depression
- Pain
- …

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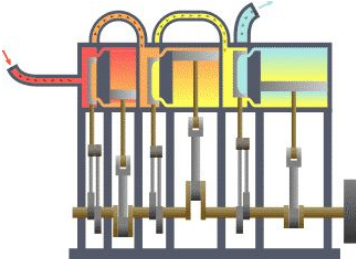
# SR trends

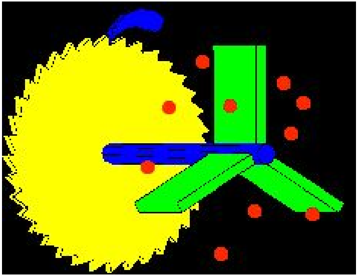
- Spatio – temporal SR
- Aperiodic SR
- Quantum SR

# Motors $\Longrightarrow$ Brownian motors

**Two heat reservoirs**

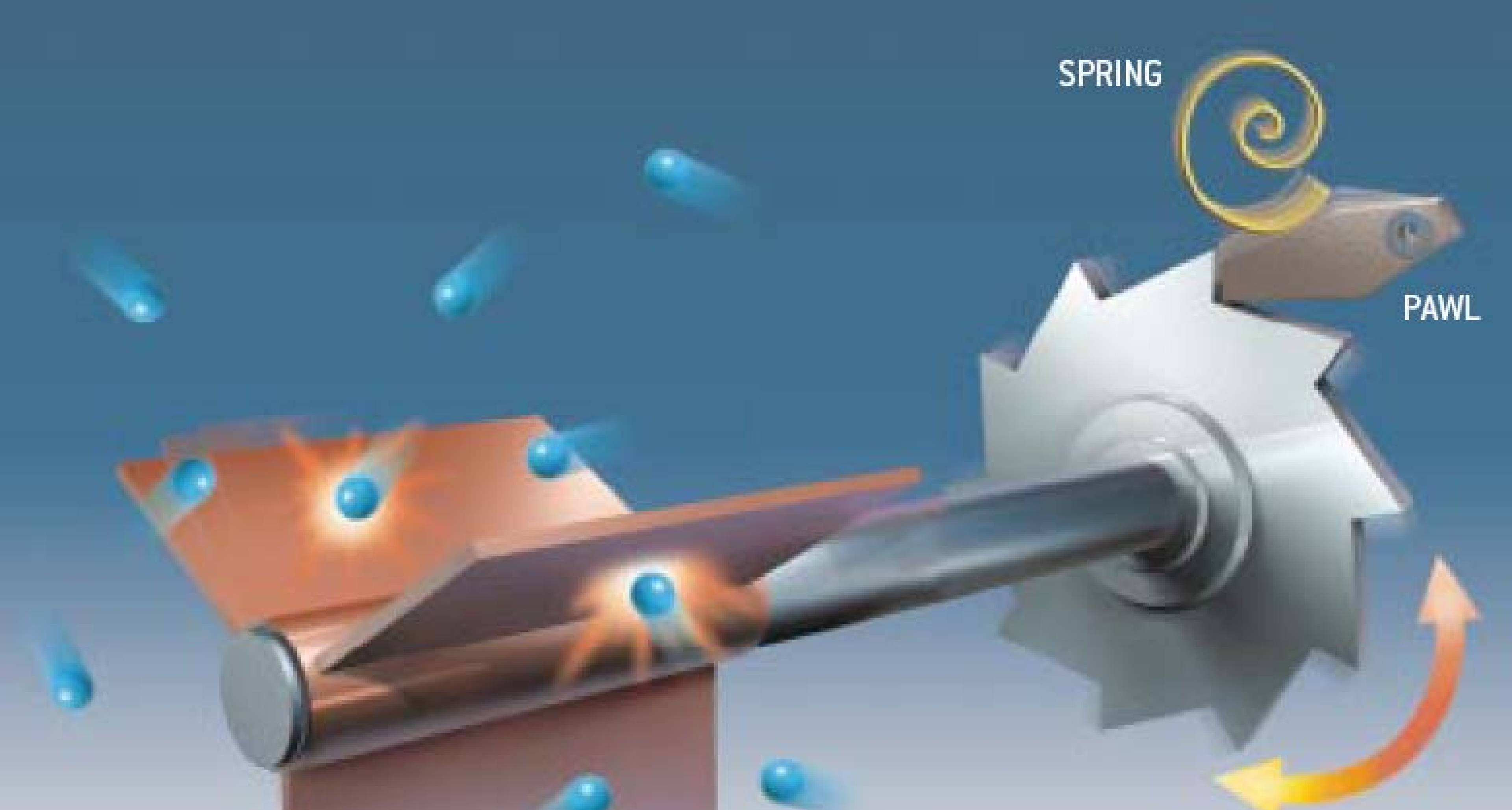


**One heat reservoir**



Perpetuum mobile of the second kind?

**NO !**

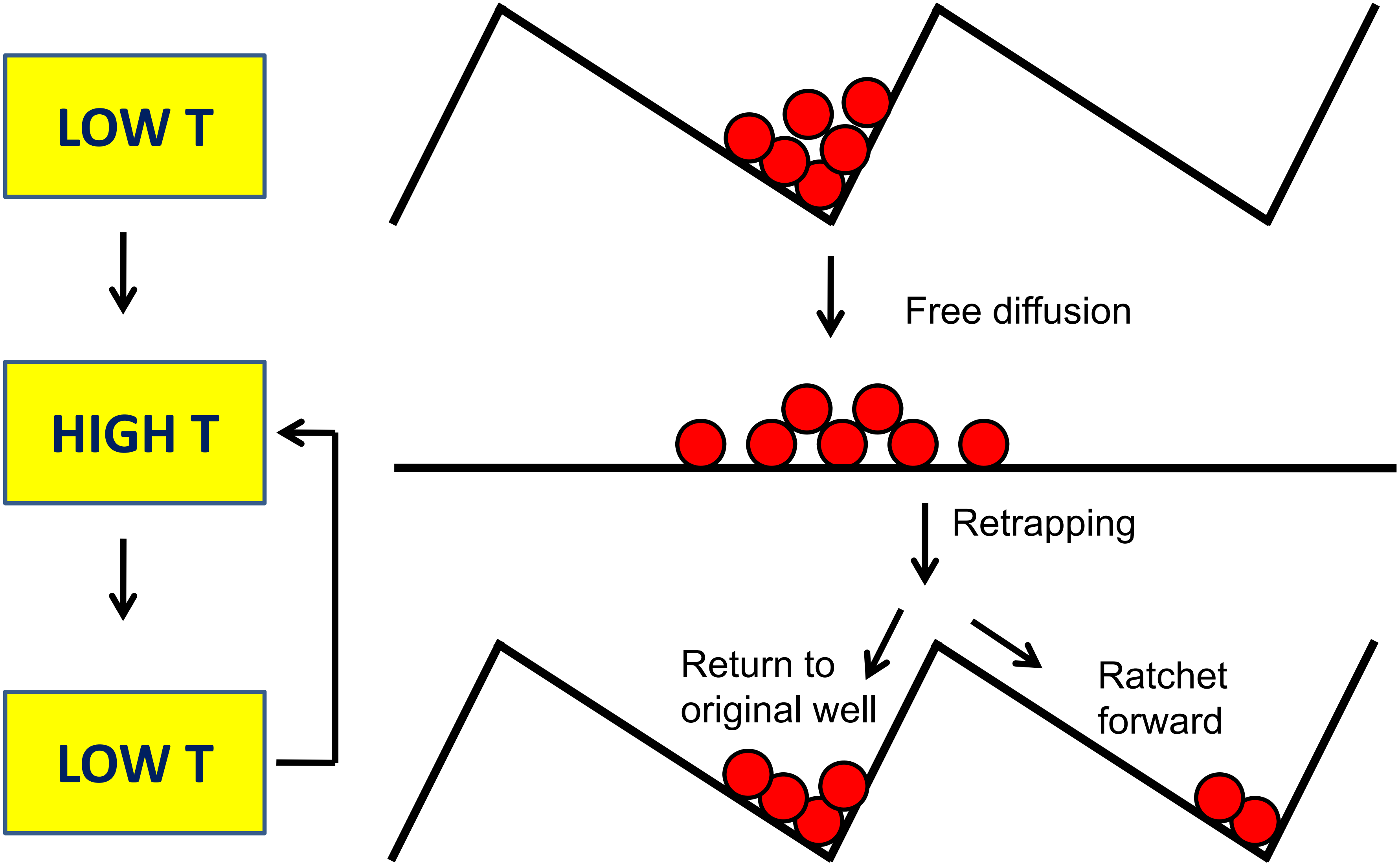



Source: Scientific American (2001)

# Brownian motor

Movie

# Temperature / Flashing Ratchet



# Brownian motors - Characteristics

- Noise & AC-Input → **DC-Ouput**
- Non-equilibrium Noise → **Directed Transport**
- **Current reversals**
- **Applications:**
  - **Novel pumps and traps for charged or neutral particles**
  - **Brownian diodes & transistors**

Ask not what physics can do for biology, ask what biology can do for physics

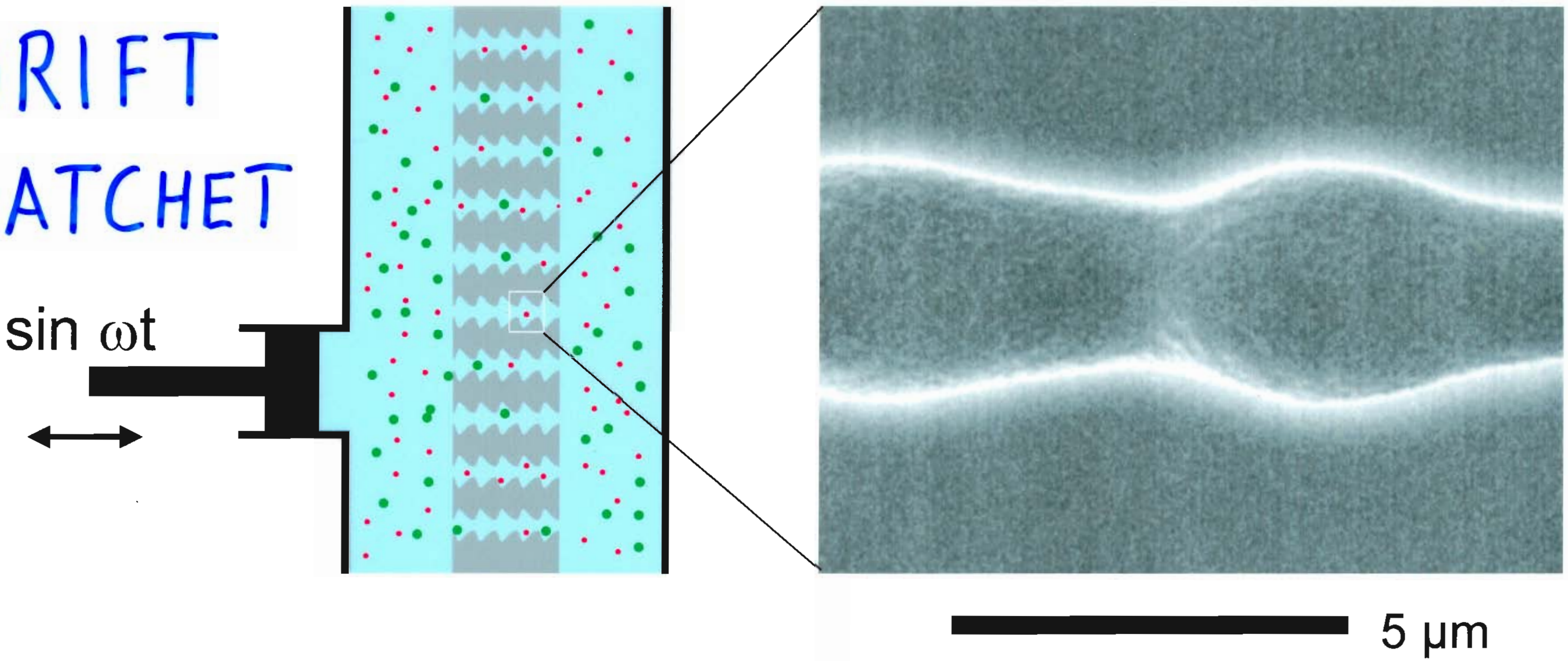
REVIEWS OF MODERN PHYSICS, VOLUME 81, JANUARY–MARCH 2009

Artificial Brownian motors: Controlling transport on the nanoscale

P.H. and F. Marchesoni

# Micro Pump based on Macroporous Silicon

DRIFT
RATCHET




CH. Kettner, P. Reimann, P. H., F. Müller, PHYS. REV. E 61: 312 (00)

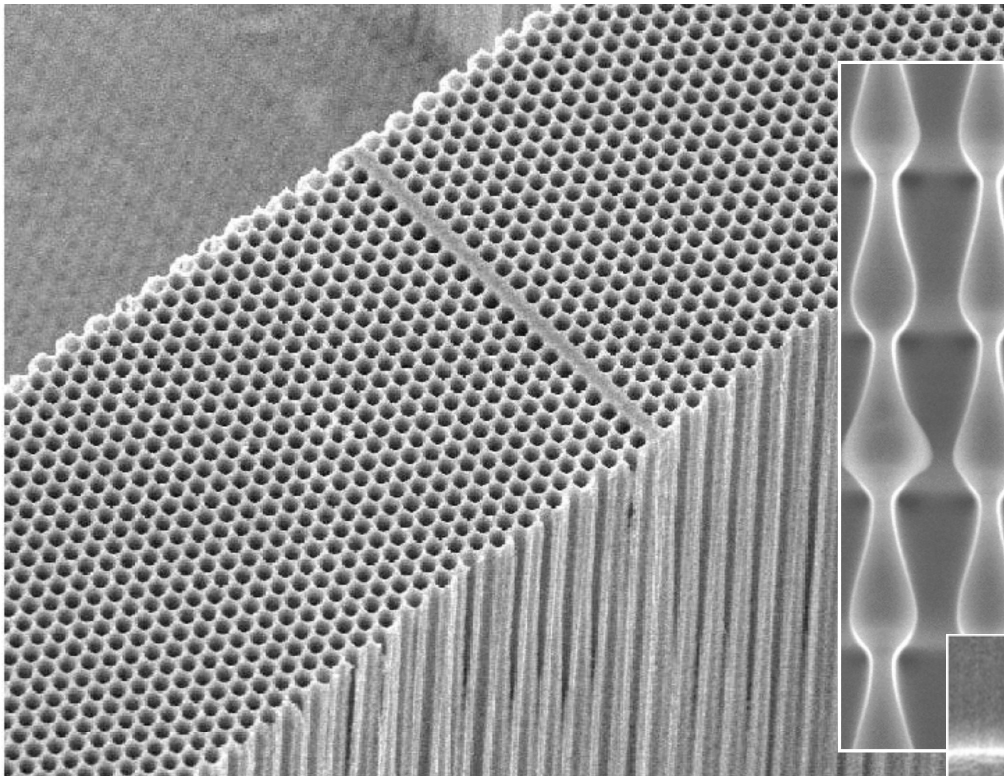
F. Müller, A. Birner, U. Gösele
MPI of Microstructure Physics, Halle/Saale, Germany

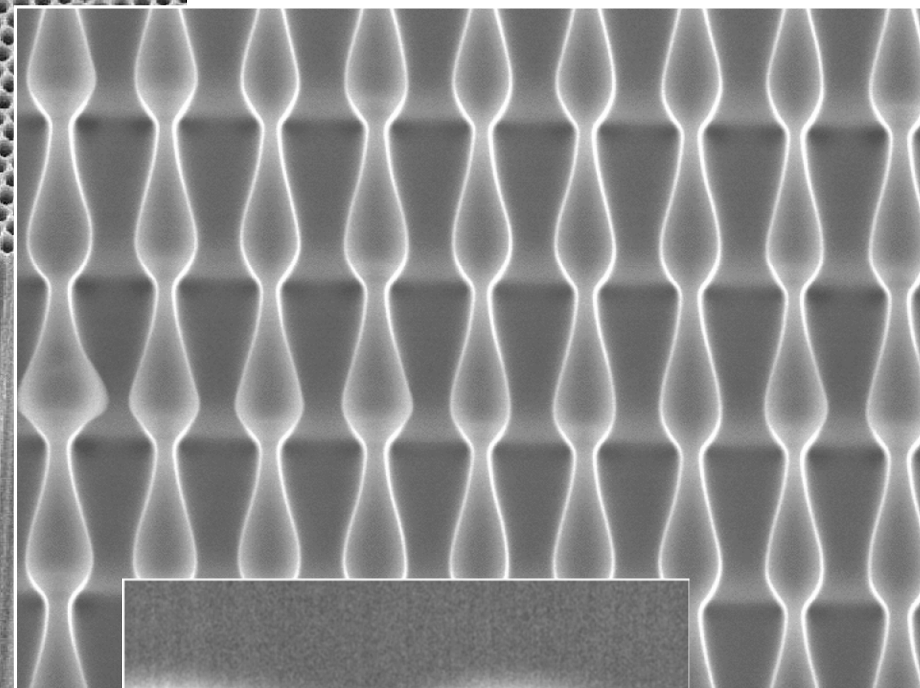
# LANGEVIN EQ. FOR BROWNIAN PARTICLES

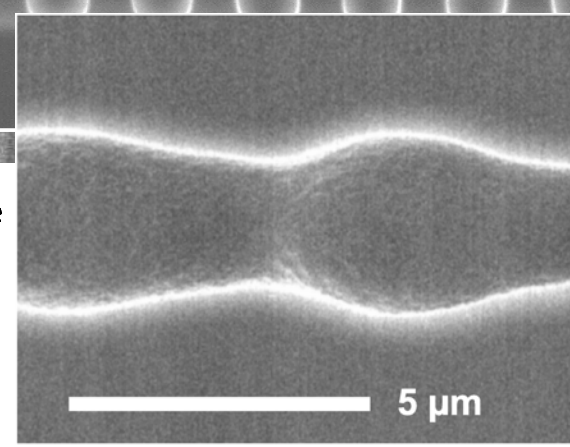
$$\dot{\vec{x}}(t) = \vec{V}(\vec{x}(t), t) + \sqrt{2D_{th}}\,\vec{\xi}(t)$$

$$D_{th} = \frac{k_B T}{\eta}$$

# Drift Ratchet - Device





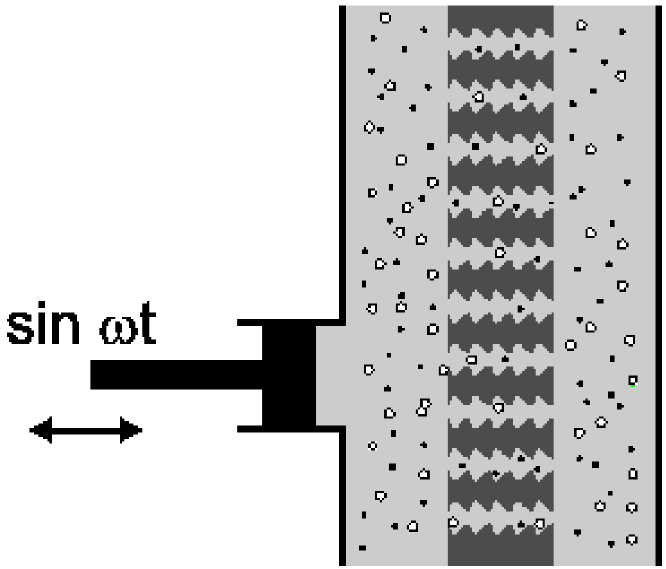



Source: F. Müller, MPI for microstructure physics, Halle

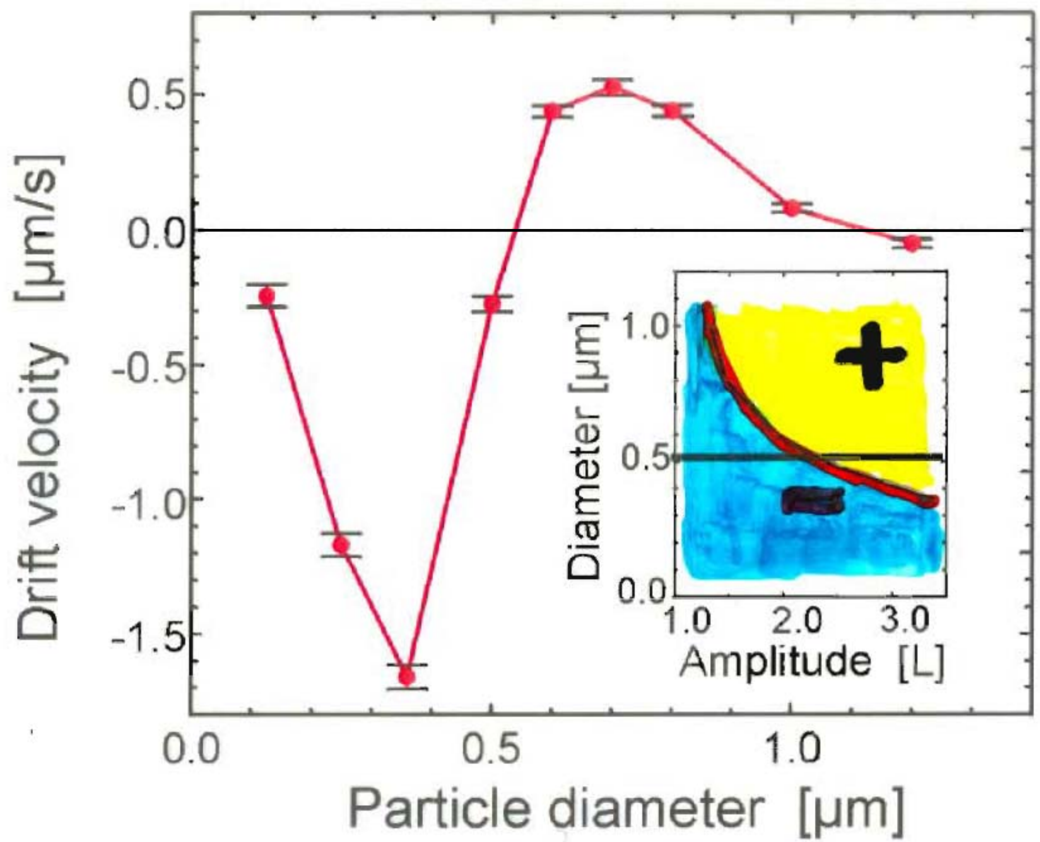
# Drift Ratchet - Theory

C. Kettner, P. Reimann, P. H., F. Müller, Phys. Rev. E **61**, 312 (2000)

Setup

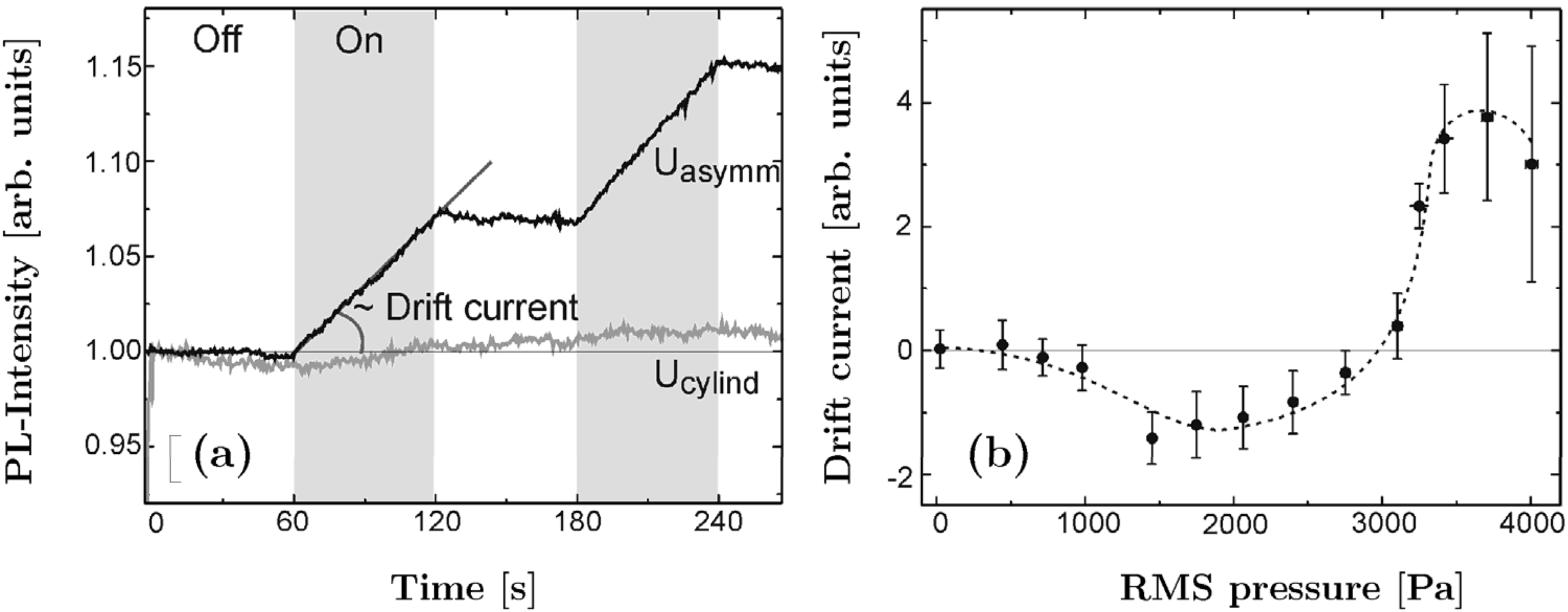



## Particle Separation



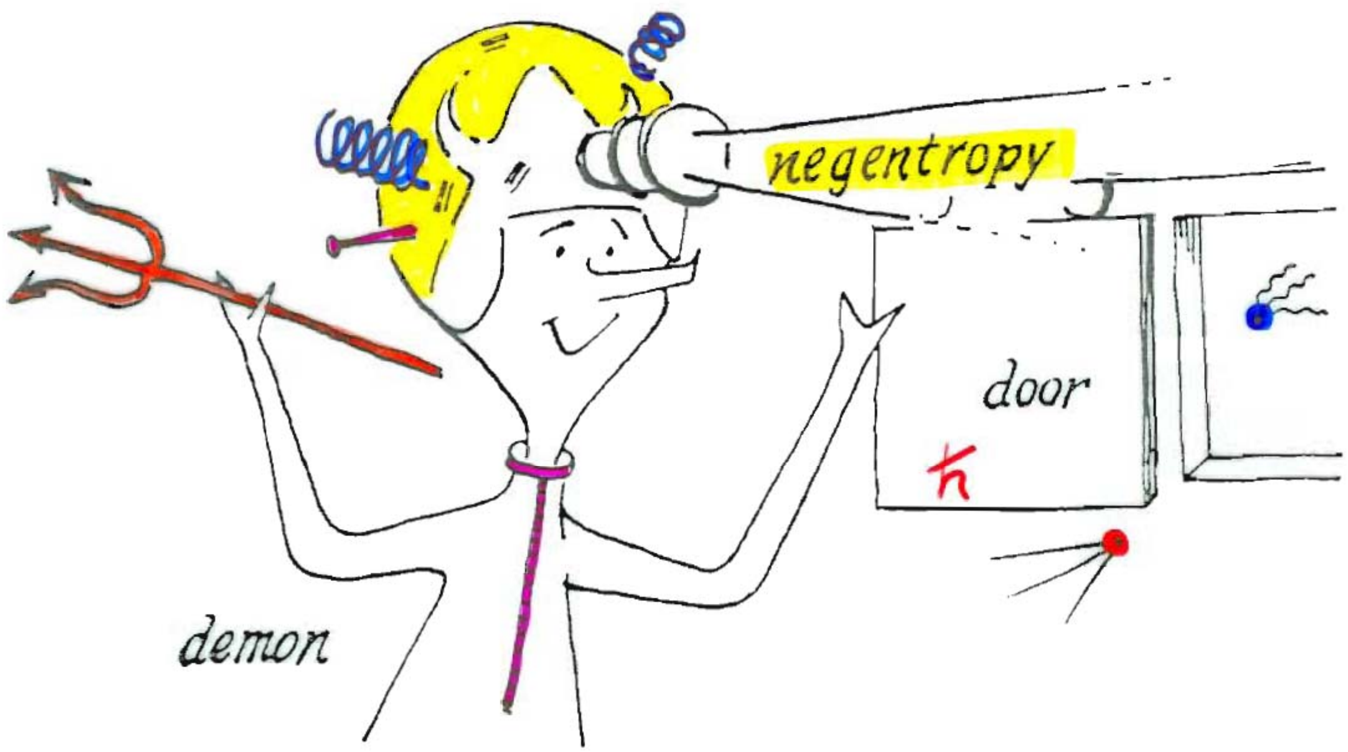
# Drift Ratchet – Experiment

S. Matthias, F. Müller, Nature **424**, 53 (2003)



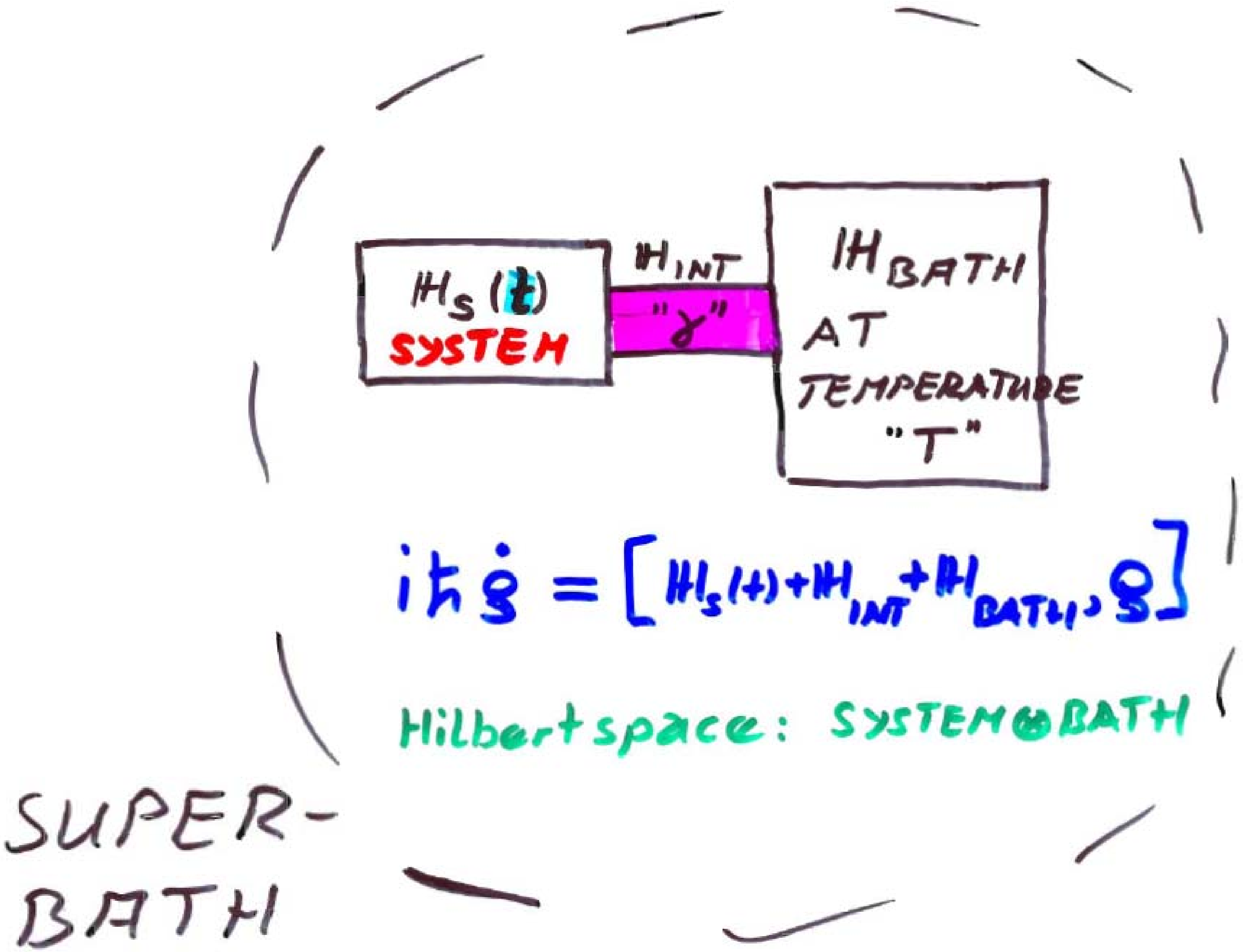
# Quantum Demon ?

A measurement → Increase information → Reduction of entropy




Source: H.S. Leff, *Maxwell's Demon* (Adam Hilger, Bristol, 1990)

# Quantum Brownian Motors



# Quantum-Langevin-equation

$$m\ddot{\mathbf{x}}(t) + \int_{-\infty}^{t} \gamma(t - t')\,\dot{\mathbf{x}}(t')\,\mathrm{d}t' + V'(\mathbf{x};t) = \boldsymbol{\xi}(t)$$

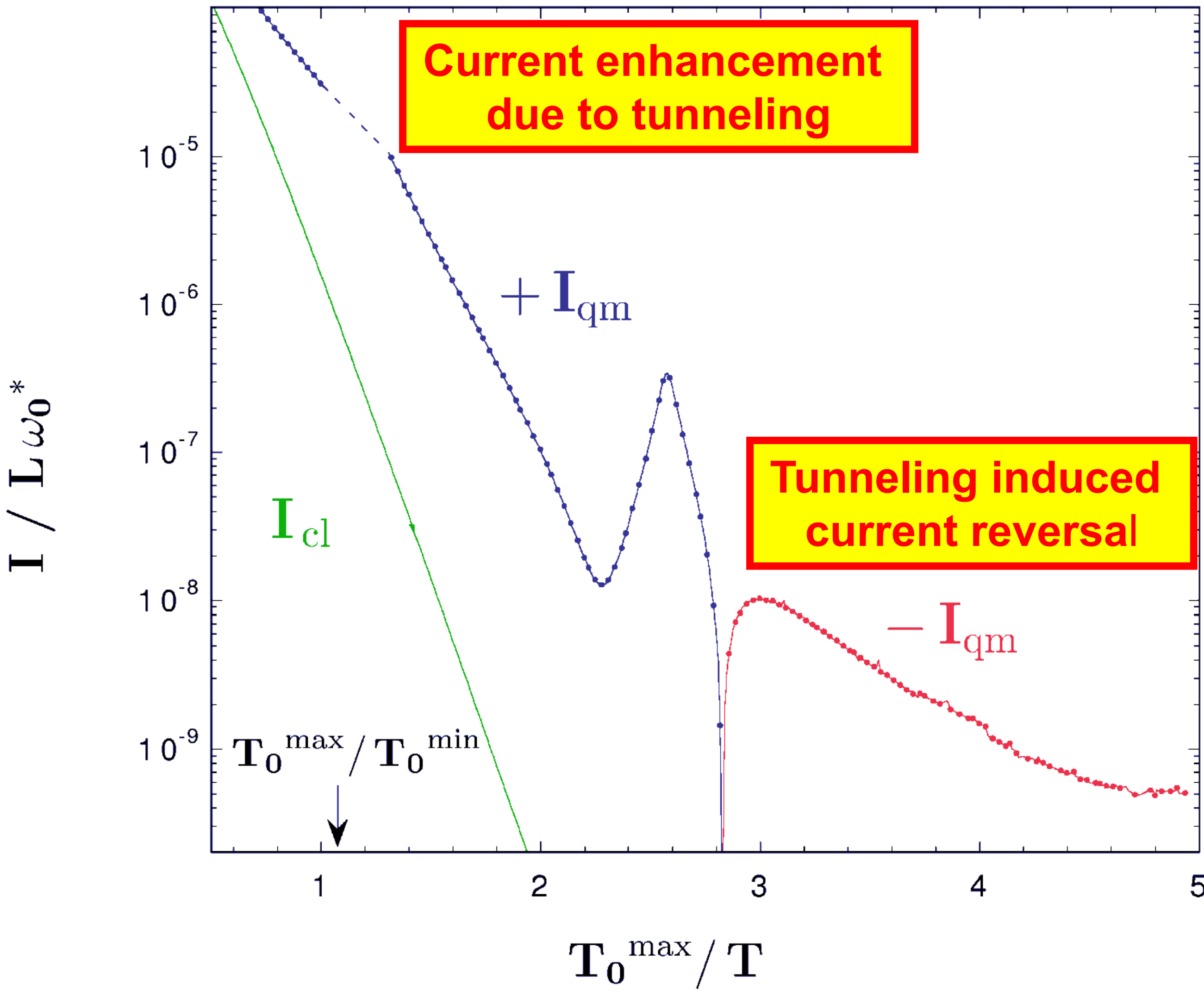
$$\frac{1}{2}\langle \boldsymbol{\xi}(t)\boldsymbol{\xi}(s) + \boldsymbol{\xi}(s)\boldsymbol{\xi}(t)\rangle_{\mathrm{bath}} = \frac{m}{\pi}\int_{0}^{\infty} \mathrm{Re}\hat{\gamma}(-i\omega + 0^{+})\,\hbar\omega \coth\left(\frac{\hbar}{2k_{\mathrm{B}}T}\right)\cos\left[\omega\left(t - s\right)\right]\,\mathrm{d}\omega$$

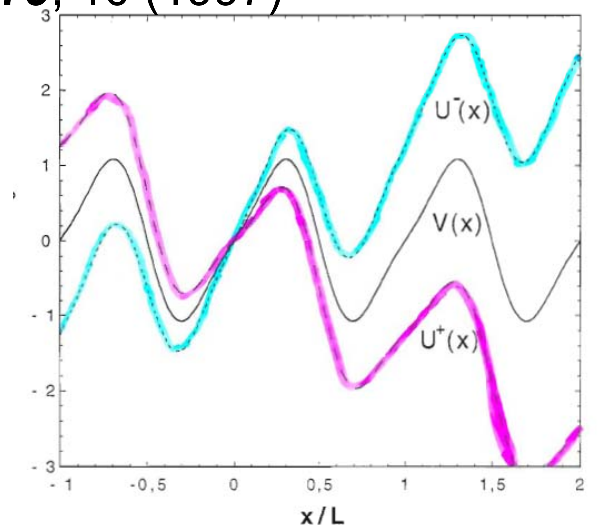
And:

$$[\boldsymbol{\xi}(t), \boldsymbol{\xi}(s)] = -i\hbar\cdots \neq 0$$

# Rocking Ratchet - Theory

P. Reimann, M. Grifoni, P. H., Phys. Rev. Lett. **79**, 10 (1997)



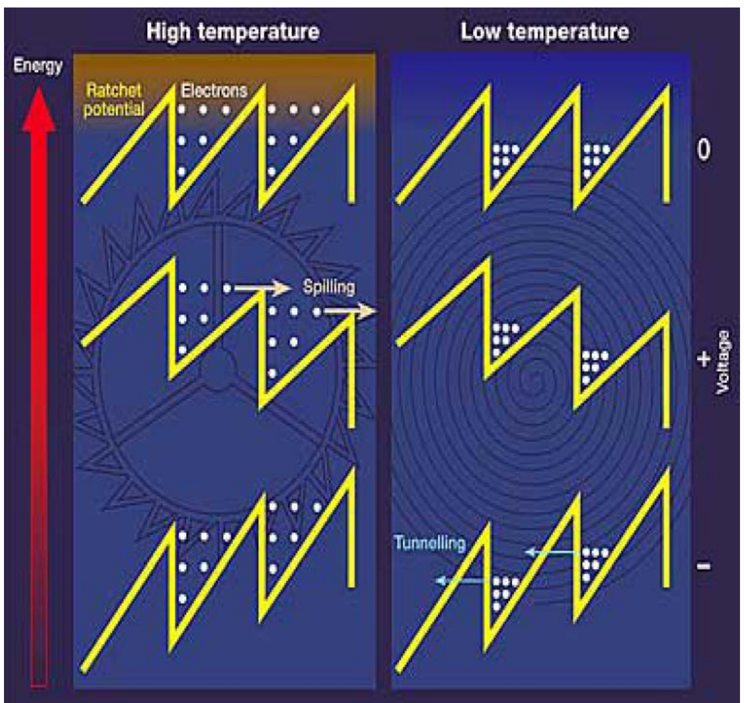

**Current enhancement due to tunneling**

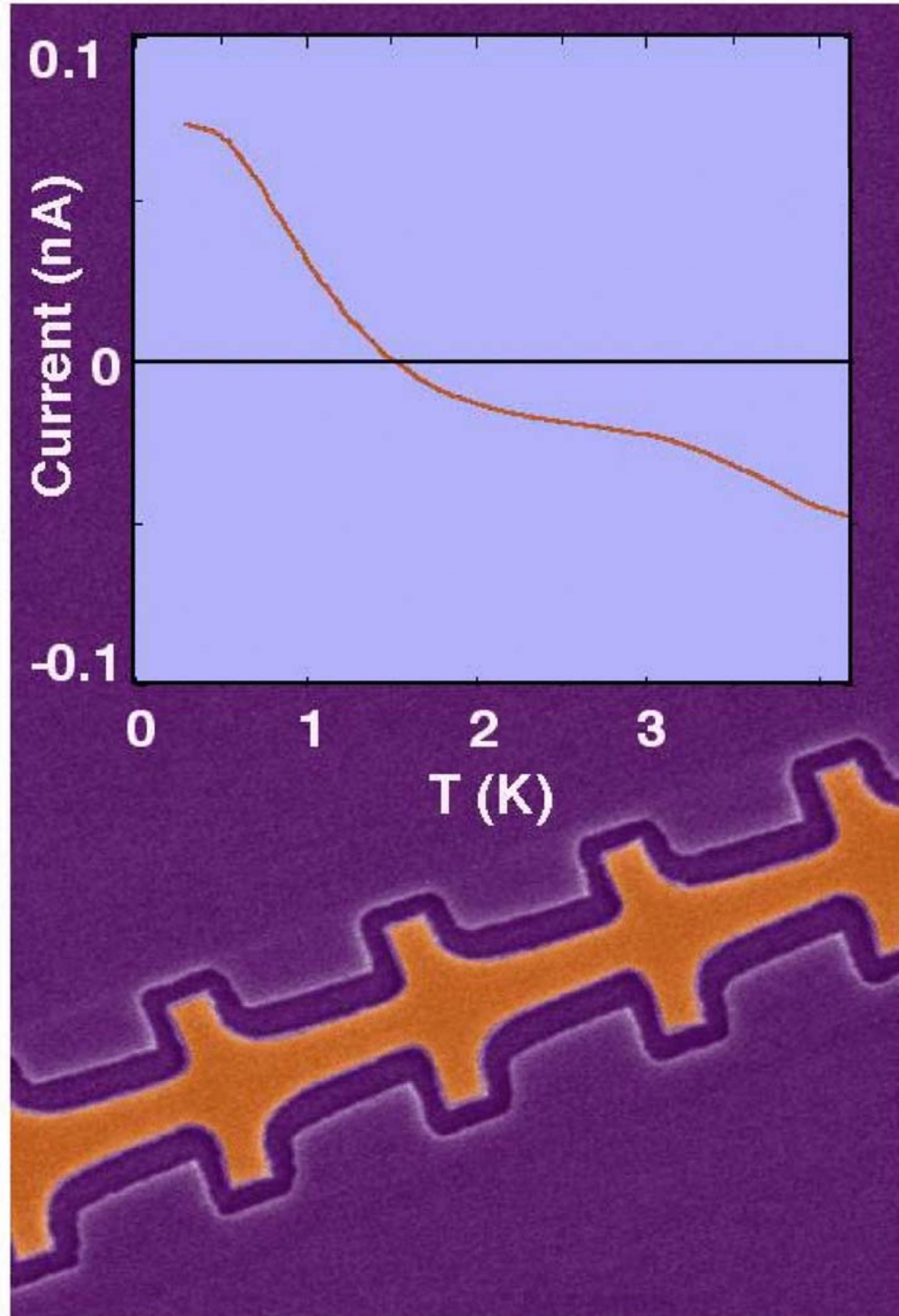
**Tunneling induced current reversal**

**Finite current for T→0**

# Rocking QM Ratchet – Experiment

H. Linke, *et al.*,
SCIENCE **286**, 2314 (1999)



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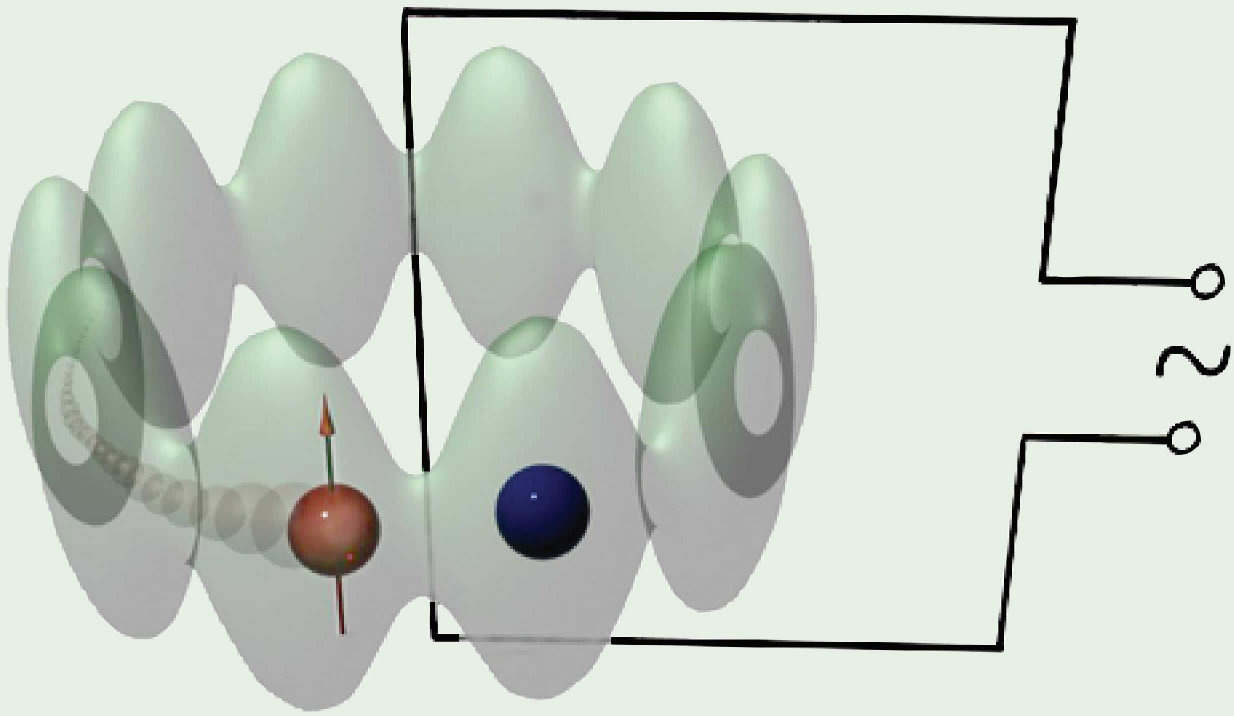


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American Physical Society

APS physics

Volume 102, Number 23

# Generalizations of Brownian Motion

# Brownian motion: Generalized Langevin-equation

Hamiltonian: $H_{\text{tot}} = H_{\text{sys}} + H_{\text{env}} + H_{\text{WW}}$

$$m\ddot{\mathbf{x}}(t) + \int_{-\infty}^{t} \gamma(t-t')\,\dot{\mathbf{x}}(t')\,\mathrm{d}t' + V'(\mathbf{x};t) = \boldsymbol{\xi}(t)$$

Asymptotically normal, anomalously fast, or anomalously slow
– via fractional Brownian motion –

$$\int_0^\infty \gamma(t)\,\mathrm{d}t = \begin{cases} \text{const} & \Rightarrow \text{normal} \\ 0 & \Rightarrow \text{superfast} \\ \infty & \Rightarrow \text{superslow} \end{cases}$$

Connection to the fractional Fokker-Planck-equation

# Confined Diffusion of Brownian particles: Entropic versus hydrodynamic interactions

## Hydrodynamic and entropic effects on colloidal diffusion in corrugated channels

Xiang Yang[a,b], Chang Liu[a,b], Yunyun Li[c,d], Fabio Marchesoni[c,e], Peter Hänggi[f,g], and H. P. Zhang[a,b,h,1]

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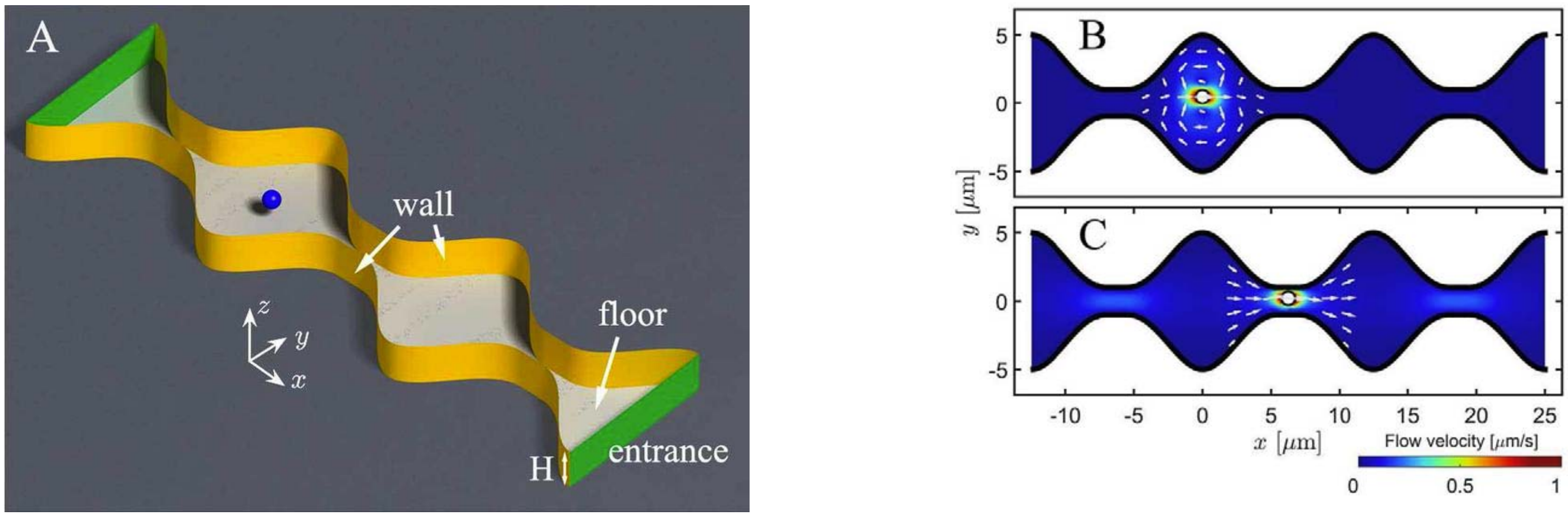
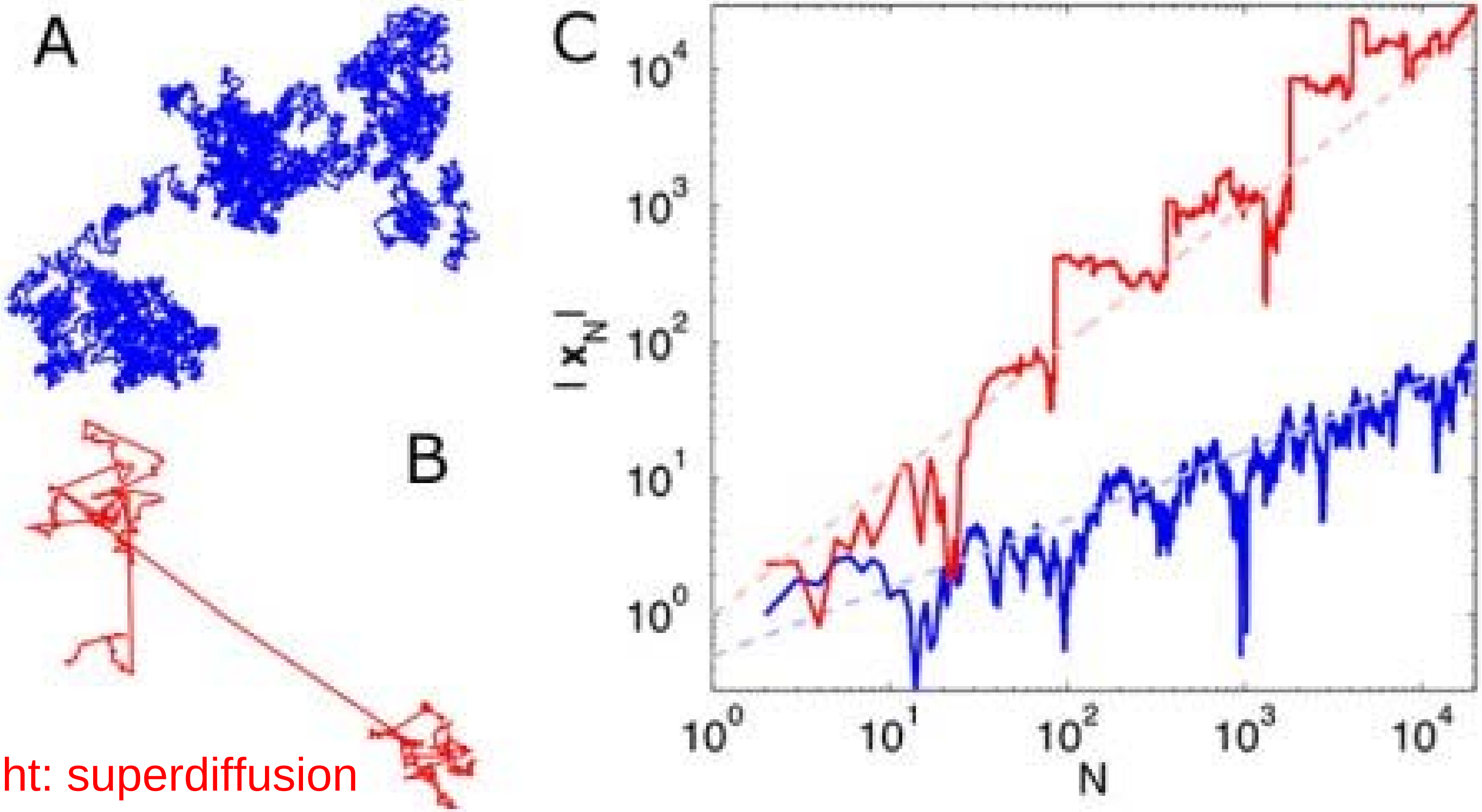
Edited by David A. Weitz, Harvard University, Cambridge, MA, and approved July 25, 2017 (received for review May 10, 2017)

**Proc. Natl. Acad. Sci. 114, 9564–9569 (2017)**



normal Brownian motion

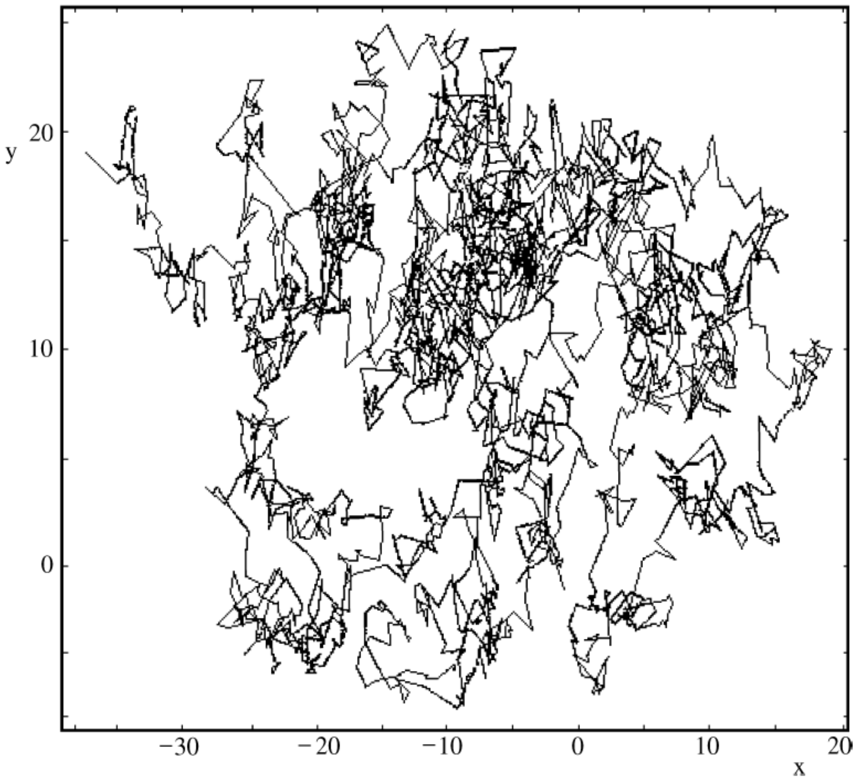



Levy flight: superdiffusion

# Mean squared displacement

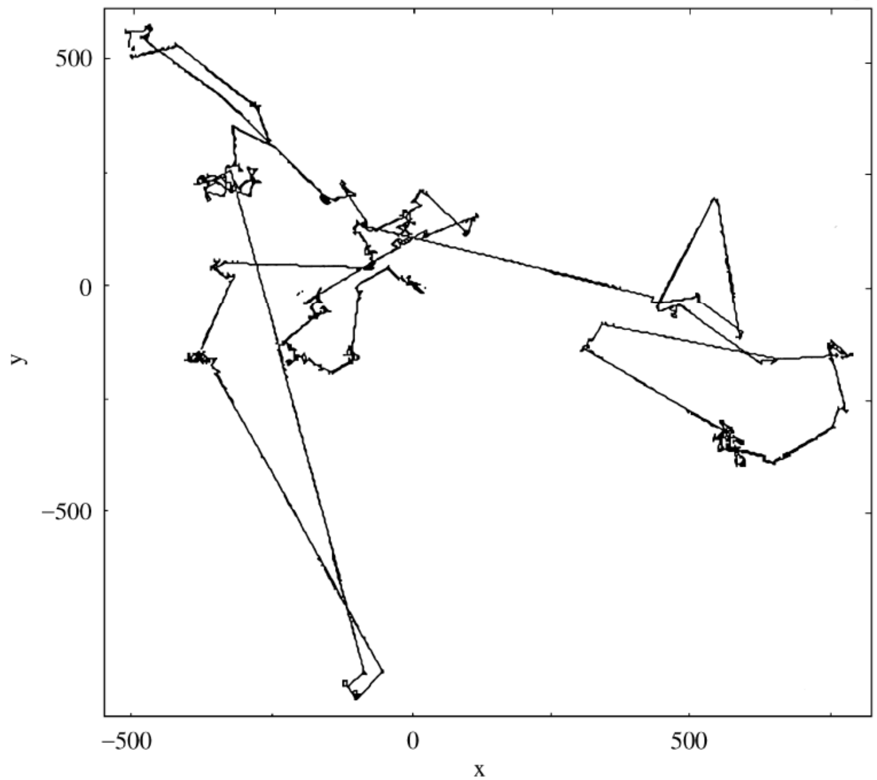
$$\langle x^2(t) \rangle \propto t^\alpha$$

Brownian movement $\alpha = 1$




Source: Physica A **282**, 13 (2000)

Lévy-Brownian movement $\alpha = \frac{4}{3}$




Source: Physica A **282**, 13 (2000)

# Fractional Fokker-Planck equation

Subdiffusion (α<1):

$$\frac{\partial P(x,t)}{\partial t} = \left[\frac{\partial}{\partial x}\frac{V'(x,t)}{\gamma_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2}\right] {}_0D_t^{1-\alpha} P(x,t)$$

Fat tails in the distribution of the residence times

Riemann-Liouville Operator

Superdiffusion (α>1):

$$\frac{\partial P(x,t)}{\partial t} = \left[\frac{\partial}{\partial x}\frac{V'(x,t)}{\gamma_\alpha} + K_\alpha \frac{\partial^{2/\alpha}}{\partial |x|^{2/\alpha}}\right] P(x,t)$$

Fat tails in the distribution of the jump lengths

Riesz-derivative

# Fractional Fokker-Planck equation

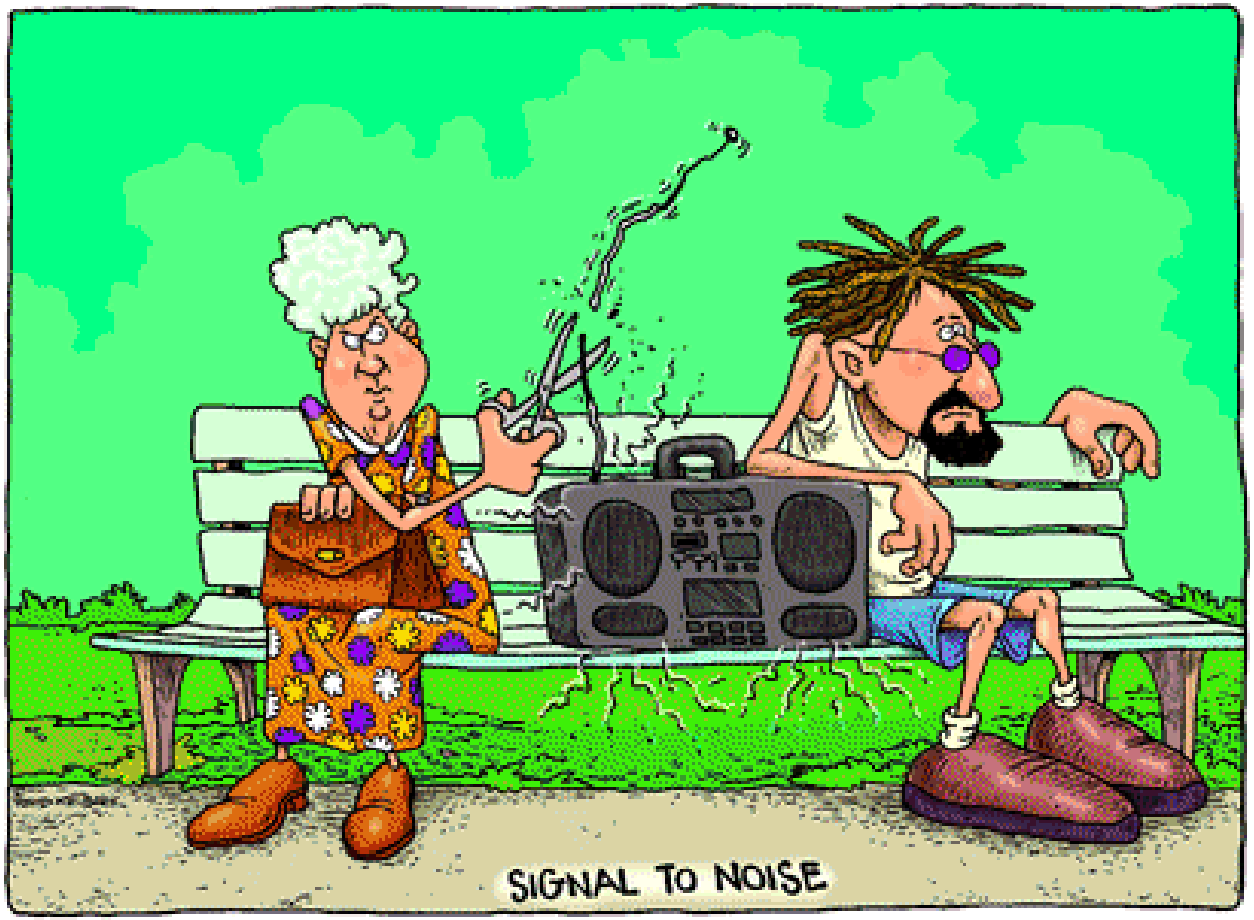
subdiffusive (α<1)

$$\frac{\partial P(x,t)}{\partial t} = \left[\frac{\partial}{\partial x}\frac{V'(x,t)}{\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2}\right] {}_0D_t^{1-\alpha} P(x,t)$$

Riemann-Liouville Operator

$${}_0D_t^{1-\alpha} f(t) = \frac{1}{\Gamma(\alpha)}\frac{d}{dt}\int_0^t \frac{f(t')}{(t-t')^{1-\alpha}}dt'$$

# Noise – always bad ?




Source: Agilent Technologies

# A QUESTION ?