Derivation of diffusion coefficients for the transport of charged particles in magnetic fields

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June 18, 2024

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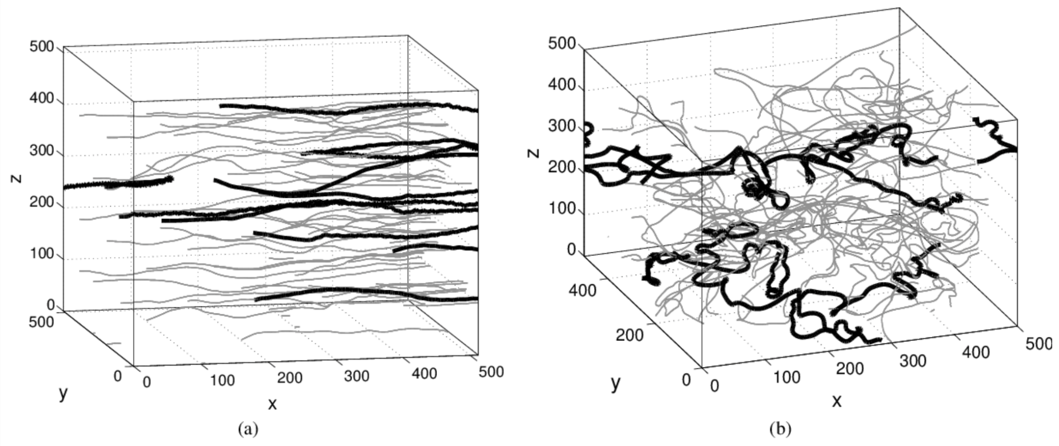
Cosmic-ray diffusion in turbulent magnetic field

Cosmic-ray diffusion in turbulent magnetic field

- $D_{ij}(t) \simeq \langle x_i x_j \rangle / \Delta t$
- MC simulations

Credits: Xu and Yan, ApJ 2013



Diffusion-tensor coefficients

- Spatial diffusion-tensor coefficients related to the velocity correlation function through a time integration [Taylor (1922); Green (1951); Kubo(1957)]:

$$D_{ij}(t) = \int_0^t \mathrm{d}t' \, \langle v_{0i} v_j(t') \rangle$$

- Aim: calculate $\langle v_{0i} v_i(t) \rangle$ given any turbulence $\delta \mathbf{B}$ on top of a mean field $\mathbf{B}_0$
- Equation of motion: $\partial v_i(t) / \partial t = \delta\Omega \, \epsilon_{ijk} v_j(t) \delta b_k(t) + \Omega_0 \, \epsilon_{ijk} v_j(t) \mathbf{b}_{0k}$
  - $\delta\Omega = c^2 Z|e|\delta B / E$: turbulent-related gyrofrequency
  - $\Omega_0 = c^2 Z|e| B_0 / E$: mean field-related gyrofrequency
  - $\delta b_k(t) \equiv \delta b_k(\mathbf{x}(t))$: k-th component of the magnetic field, expressed in units of δB, at the spatial coordinate $\mathbf{x}(t)$ of the particle at time t

Equation of motion – Auxiliary variable

- Mean field oriented in the z direction, $\mathbf{B}_0 = B_0 \mathbf{u}_z$
- Equation of motion: ${}^{\partial v_i(t)}/_{\partial t} = \delta\Omega\ \epsilon_{ijk} v_j(t) \delta b_k(t) + \Omega_0\ \epsilon_{ijk} v_j(t) \delta_{kz}$
- Expansion: infinite number of Dyson series mixing powers of δB and B_0
- Alternative: auxiliary variable $v_i(t) = R^{-1}_{ij}(\Omega_0 t) w_j(t)$, with R_{ij} the matrix elements of the rotation matrix around z [Plotnikov, Pelletier, & Lemoine, A&A 532 (2011) A68]
- Given that ${}^{\partial R^{-1}_{ij}(\Omega_0 t)}/_{\partial t} = \Omega_0 \epsilon_{ikz} R^{-1}_{kj}(\Omega_0 t)$,

$$\frac{\partial w_i(t)}{\partial t} = \delta\Omega\ R_{im}(\Omega_0 t) \epsilon_{mnk} \delta b_k R^{-1}_{nj}(\Omega_0 t) w_j(t)$$

- No additional input w.r.t. the pure turbulence case

Dyson series

- First iterative solution:
  $\langle w_i(t)\rangle = w_{i_0} + \delta\Omega\ \epsilon_{mnk} \int_0^t \mathrm{d}t' R_{im}(\Omega_0 t')\langle w_j(t')\delta b_k(t')\rangle R^{-1}_{nj}(\Omega_0 t')$
- Dyson series:

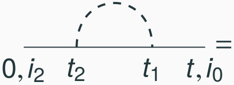
$$\langle v_{i_0}(t)\rangle = v_{0i_0} + \sum_{n=1}^{\infty} \delta\Omega^n\ \epsilon_{i_0 i_1 j_1}\epsilon_{i_1 i_2 j_2}\ldots\epsilon_{i_{n-1} i_n j_n} v_{0i_n} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \langle \delta b_{j_1}(t_1)\ldots\delta b_{j_n}(t_n)\rangle$$

- Wick theorem: expectation value in the integrand expressed in terms of all possible permutations of products of contractions of pairs of $\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle$
- Using the Ansatz $\langle \delta b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle = \varphi(t_{j_1} - t_{j_2})\,\delta_{i_1 i_2}/3$:

$$\langle w_{i_0}(t)\rangle = w_{0i_0} + \sum_{p=1}^{\infty}\left(\frac{\delta\Omega^2}{3}\right)^p w_{0i_{2p}} \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{2p-1}} \mathrm{d}t_{2p} \sum_{\text{pairings}} \prod_{j<\ell} \varphi(t_j - t_\ell)$$

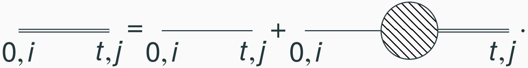
$$\left(R_{i_{j-1}k_j}(\Omega_0 t_j)R_{i_{\ell-1}k_j}(\Omega_0 t_\ell)R^{-1}_{m_j i_j}(\Omega_0 t_j)R^{-1}_{m_j i_\ell}(\Omega_0 t_\ell) - R_{i_{j-1}k_j}(\Omega_0 t_j)R_{i_{\ell-1}k_\ell}(\Omega_0 t_\ell)R^{-1}_{k_\ell i_j}(\Omega_0 t_j)R^{-1}_{k_j i_\ell}(\Omega_0 t_\ell)\right)$$

- Rules:
  - "free propagator": $u^{(0)}(t) = 1$
  - contraction of a pair $\langle b_{i_1}(t_{j_1})\delta b_{i_2}(t_{j_2})\rangle$ identified as an "interaction"
  - dashed lines $\rightarrow$ integrations over the ordered times crossing the continuous line
- For instance (with $0 \leq t_2 \leq t_1 \leq t \leq \infty$):



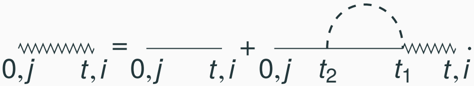
$$= \frac{\delta\Omega^2}{3}\int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \Big(R_{i_0k_1}(\Omega_0 t_1)R_{i_1k_1}(\Omega_0 t_2)R^{-1}_{m_1i_1}(\Omega_0 t_1)R^{-1}_{m_1i_2}(\Omega_0 t_2)$$
$$-R_{i_0k_1}(\Omega_0 t_1)R_{i_1k_2}(\Omega_0 t_2)R^{-1}_{k_2i_1}(\Omega_0 t_1)R^{-1}_{k_1i_2}(\Omega_0 t_2)\Big)\varphi(t_1 - t_2),$$

- Dyson series:

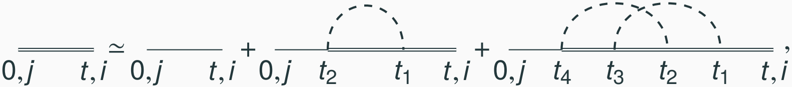


Partial summation of Dyson series

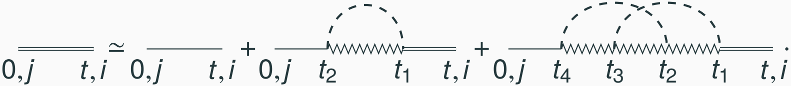
- Include all possible ways to have one nested loop in unconnected diagrams (Kraichnan propagator [Kraichnan, JMP 2 (1961) 124]):



- Include all possible nested and crossed diagrams:



- Approximation:



2-pt correlation function of the experienced magnetic field

- Formal expression:

$$\langle \delta b_i(t)\delta b_j(0)\rangle \simeq \iint \mathrm{d}k\mathrm{d}k' \, \langle \delta b_i(k)\delta b_j(k')\rangle\langle e^{\mathrm{i}k\cdot x(t)}\rangle$$

- Expansion of $\langle e^{\mathrm{i}k\cdot x(t)}\rangle$ as a Dyson series:

$$\langle e^{\mathrm{i}k\cdot x(t)}\rangle = \sum_{n\geq 0}\frac{\mathrm{i}^n}{n!}\int_0^t dt_1 \cdots \int_0^t dt_n \langle (k\cdot v(t_1))\ldots(k\cdot v(t_n))\rangle$$

- n-point correlation function substituted for the sum of all possible contractions of pairs:

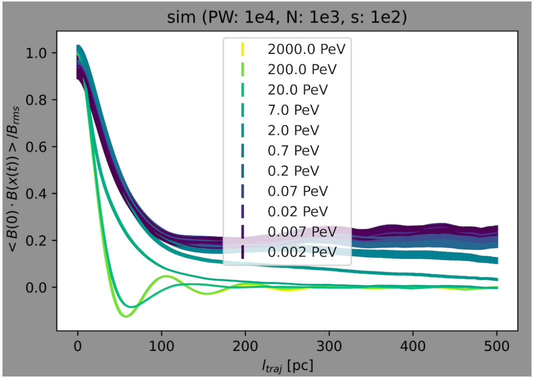
$$\langle (\mathbf{k}\cdot\mathbf{v}(t_1))(\mathbf{k}\cdot\mathbf{v}(t_2))\rangle \simeq \langle \mathbf{k}^2\rangle\langle \mathbf{v}(t_1)\cdot\mathbf{v}(t_2)\rangle \longrightarrow k^2c^2e^{-(t_1-t_2)/\xi(k)}$$

- Hence, with the heuristic expression $\xi(k) = \sqrt{\rho}/kc$,

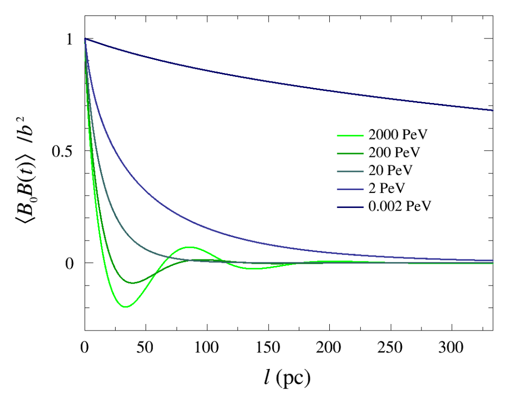
$$\langle e^{\mathrm{i}k\cdot x(t)}\rangle_0 \simeq 1-(kc)^2\int_0^t dt_1 \int_0^{t_1} dt_2 \, e^{-(t_1-t_2)/\xi(k)}\langle e^{\mathrm{i}k\cdot x(t-t_1)}\rangle_0$$

2-pt correlation function of the experienced magnetic field

- MC simulations
- Numerous features not accounted for in various approaches (e.g. QLT)



- Sum of unconnected diagrams
- Main features reproduced in gyro-resonant and high-rigidity regimes



Parallel diffusion

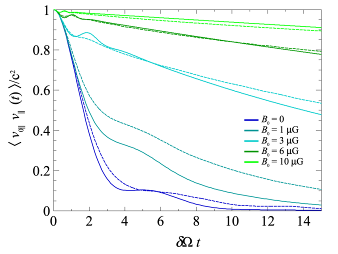
- First iteration:

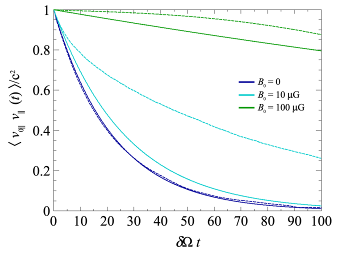
$$W_{0z}(s) = \frac{1}{s} - \frac{2\delta\Omega^2}{3}\frac{W_{0z}(s)}{s}\mathcal{L}\left[\varphi(x)\cos\Omega_0 x\right](s),$$

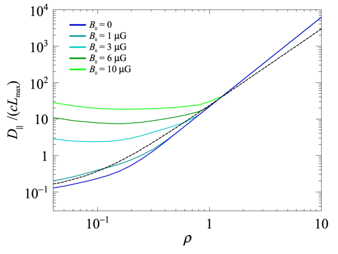
- Second iteration:

$$W_z(s) = \frac{1}{s} - \frac{2\delta\Omega^2}{3}\frac{W_z(s)}{s}\mathcal{L}\left[\varphi(x)\langle w_z(x)\rangle_0\cos\Omega_0 x\right](s)$$
$$+2\left(\frac{\delta\Omega^2}{3}\right)^2\frac{W_z(s)}{s}\mathcal{L}\Bigg[\varphi(x_1+x2)\varphi(x_2+x3)$$
$$\langle w_z(x_1)\rangle_0\langle w_z(x_2)\rangle_0\langle w_z(x_3)\rangle_0\cos\Omega_0(x_1-x_3)\Bigg](s).$$

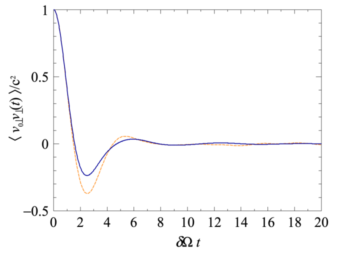
Parallel diffusion

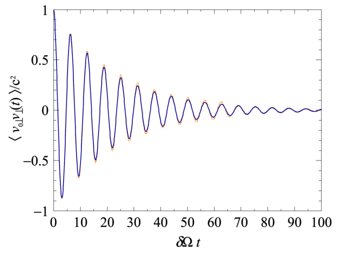


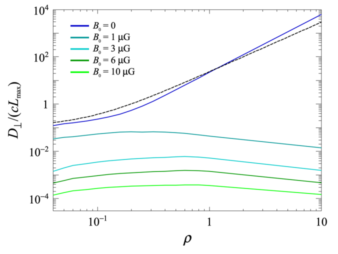




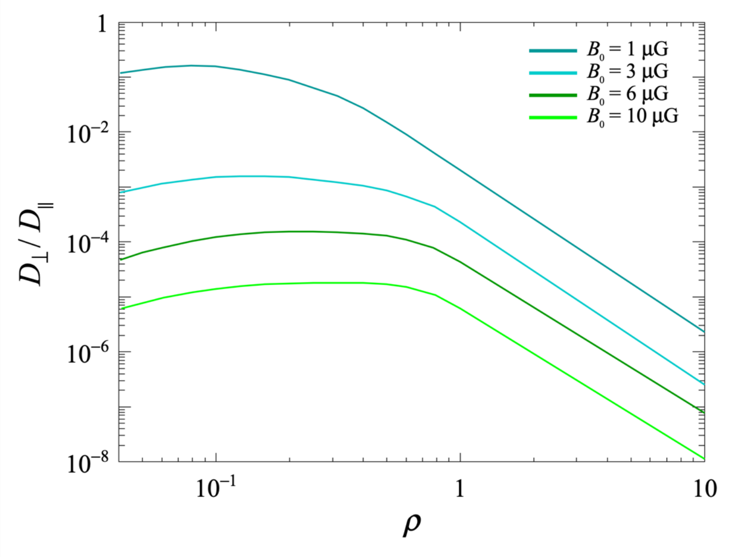
Perpendicular diffusion







“Benchmark plot”

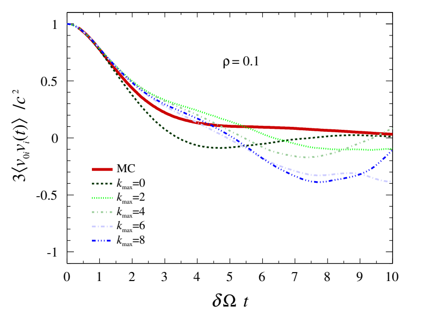


Conclusions

- Physical solution provided by partial summation schemes
- Convenient way to get dependencies in E, δB^2, B_0^2, etc. compared to heavy MC campaigns
- Main features captured in the gyro-resonant and high-rigidity regimes
- Details in Astrophys.J. 920 (2021) 2, 87 and arXiv:2406.00361 (to appear in Phys. Rev. D)

Backup

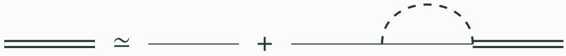
Truncation of the series?



- Summation of all terms up to some order $k_{\max}$?
- Series absolutely convergent for all t, very many terms required for $t > 3/(2\delta\Omega^2\tau)$ [e.g. Kraichnan, JMP 2 (1961) 124]
- $\longrightarrow$ combinatorics that determines the number of occurrence of each diagram rapidly non-tractable

Partial summation

- Simplest scheme: Bourret propagator [Bourret, NCim 26 (1962) 3833]
- → substitute the "mass operator" in the Dyson equation for unconnected contributions:



- → sum of unconnected diagrams
- NB: exact solution in the case of white-noise process (cancellation of nested/crossed diagrams) [Plotnikov, Pelletier, & Lemoine, A&A 532 (2011) A68]
- Decoupling in Laplace-transform space (change of variables $t = x + x_1 + x_2$, $t_1 = x_1 + x_2$, $t_2 = x_2$): $U^{(1)}(p) = p - \frac{2\delta\Omega^2}{3}\frac{U^{(1)}(p)}{p+\tau}$
- Inverse transform: $u(t) = \frac{A}{2B}e^{-At/6\tau}(e^{Ct/\tau} - 1)$, with $A = 3 + \sqrt{9 - 24\tau^2\delta\Omega^2}$, $B = A - 3$, and $C = \sqrt{1 - 8\tau^2\delta\Omega^2/3}$
- → nonphysical oscillations around 0 for $t > 3/(2\delta\Omega^2\tau)$