Relating the phenomenological universe with the quantum and classical cosmological models in the tomographic approach

(a new way to investigate the properties of the universe and its origin)

QGSKY meeting

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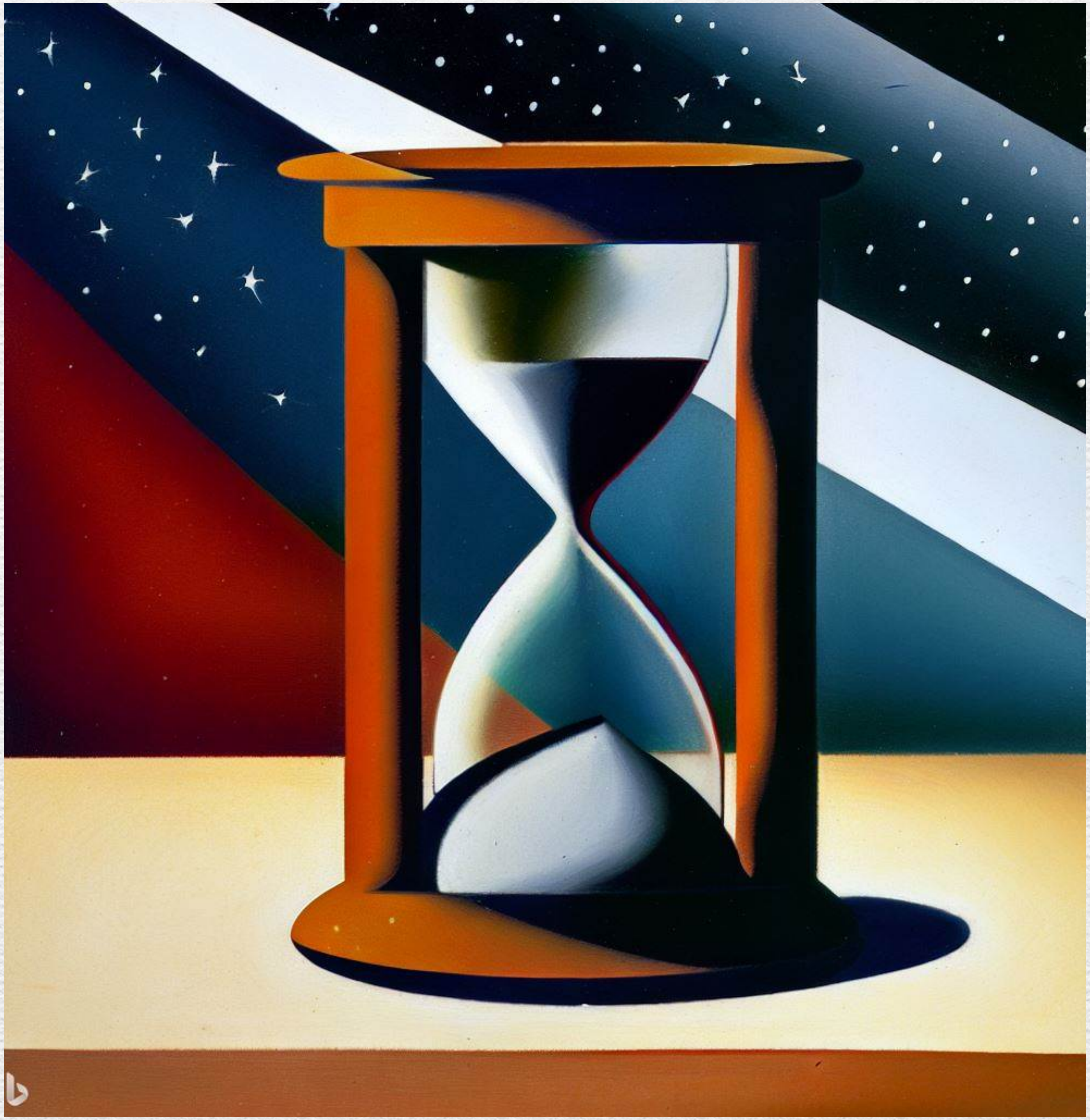

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Introduction

- In cosmology it would be interesting to be able to understand some problems related to the initial conditions of the universe
- Quantum cosmology gives a description of the initial state of the universe that general relativity cannot give, but a significant result would be to have a phenomenological confirmation of its predictions
- So it is necessary to have a tool to be able to describe the states of the quantum and classical universe. In recent years we have insisted a lot on the fact that this is achieved through the *tomographic description* of these states.
- In fact, through this description we are able to get more information on the *relationship* between the classical states. and the quantum states in their asymptotic limit.
- But a phenomenological definition of the tomogram of the universe is necessary to eventually highlight some aspects of the original quantum state that the current universe could have inherited.

Overview of the talk

- In this talk I try to outline four steps with which one can attempt to reconstruct the initial quantum state of the universe using the tomographic approach..
- First step I recall the definitions and the properties of the quantum and classical tomograms
- In the next step I recall some previous results for a universe with only the cosmological constant and show how these results can be generalised if more general potentials are considered
- In the third step I show how a tomogram can evolve over time through the transition probability function that can be defined using the propagators of quantum theory
- Finally, in the fourth step I introduce the definition of the phenomenological tomogram of the universe and show how through the comparison of this tomogram with the different models that can be obtained following the procedure of the third step, information on the structure of the quantum universe can be obtained.
- In the conclusions I discuss the future work to be done

First step

Definition of quantum and classical tomograms

Classical $$\mathscr{W}\left(X,\mu,\nu\right)=\int f(q,p)\delta(X-\mu q-\nu p)dqdp$$

Quantum

$$\mathscr{W}\left(X,\mu,\nu\right)=\int W(q,p)\delta(X-\mu q-\nu p)dqdp$$

$$\mathscr{W}\left(X,\mu,\nu\right)=\frac{1}{2\pi\hbar\,|\,\nu\,|}\left|\int\psi\left(y\right)\exp\left[i\left(\frac{\mu}{2\hbar\nu}y^2-\frac{X}{\hbar\nu}y\right)\right]dy\right|^2$$

$X=\mu q+\nu p$ $\quad P=-\nu q+\mu p$ $\quad \nu=s^{-1}\sin\theta$ $\quad \mu=s\cos\theta$

we can define quantum and classical states with the same
class of functions

Properties of the tomogram

- Tomograms are marginal probability functions for this reason the fundamental properties of a tomogram are
- Non negativity $\mathscr{W}(X,\mu,\nu) \geq 0$
- Normalized $\int \mathscr{W}\left(X,\mu,\nu\right) dX = 1$
- Homogeneity $\mathscr{W}(\lambda X, \lambda\mu, \lambda\nu) = \frac{1}{|\lambda|}\mathscr{W}(X,\mu,\nu)$
- These conditions guarantee their observability.

Uncertainty relations for quantum tomograms

$$\left[\int W(X,1,0)X^2dX - \left(\int W(X,1,0)XdX\right)^2\right]$$

$$\times\left[\int W(X,0.1)X^2dX - \left(\int W(X,0,1)XdX\right)^2\right] \geq \frac{1}{4}$$

Classical De Sitter model

(See Louko (1987,1988, Louko and Halliwell (1990))

$$ds^2 = -\frac{N^2}{q}d\tau^2 + \frac{q}{1-kr^2}dr^2 + qr^2 d\Omega_2$$

$$G_{ab} + \Lambda g_{ab} = 8\pi G T_{ab}$$

$$q = a^2$$

$$\frac{1}{4}\frac{1}{N^2}\frac{\dot{q}^2}{q} + \frac{1}{q} = \frac{8\pi G}{3}T_{00} + \frac{\Lambda}{3}$$

$$k = 1$$

$$\frac{\ddot{q}}{N^2} + \frac{1}{2}\frac{1}{N^2}\frac{\dot{q}^2}{q} - \frac{\dot{N}}{N^3}\dot{q} + \frac{2}{q} = -\frac{8\pi G}{3}T + \frac{4}{3}\Lambda\,.$$

$$q(t) = \Lambda^{-1}\cosh^2(\Lambda^{1/2}t) \qquad N = \sqrt{q}$$

Tomogram of the universe during the inflationary stage

- The classical distribution, we consider all the states allowed by the Hamiltonian. We don't choose any preferred "initial state". In a de Sitter universe distribution function the set of all the allowed states satisfying the Hamiltonian constraint is given by a Dirac delta function

$$f(q,p) = \delta\left(-4p^2 + \lambda q - 1\right)$$

- where $q = a^2$ and $p = -\dot{q}$ and λ is the cosmological constant, and the units are chosen appropriately.

How we calculate the classical tomogram

- Consequently the tomogram is obtained applying

$$\mathscr{W}_C\left(X,\mu,\nu\right) = \int \delta\left(-4p^2 + \lambda q - 1\right)\delta(X - \mu q - \nu p)\,dq\,dp$$

- applying the rules for the Dirac delta

- the classical tomogram of the de Sitter model is

$$\mathscr{W}_{class.} = \frac{1}{2|\mu|}\frac{1}{\left|\sqrt{\frac{\lambda^2\nu^2}{16\mu^2} + \frac{\lambda X}{\mu} - 1}\right|}$$

- The Wheeler-DeWitt equation is

$$\left(4\hbar^2\frac{d^2}{dq^2}+\lambda q-1\right)\psi(q)=0\,.$$

- On the other side quantum tomograms are combinations of the Airy functions. Their classical limit is obtained taking the limit $\hbar\lambda\to 0$. We can consider the case with $\hbar\neq 0$ and $\lambda\neq 0$, but $\lambda\ll 1$,

$$\mathscr{W}_{HH}(q,p)=\lim_{\hbar\lambda\to 0}\frac{2^{1/3}A^2}{\pi(\hbar\lambda)^{1/3}}Ai\left[\frac{4p^2-\lambda q+1}{(\hbar\lambda)^{2/3}}\right]\approx\frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}}\left|\cos\left(\frac{2}{3}\left|S\right|^{3/2}-\frac{\pi}{4}\right)\right|^2$$

$$\mathscr{W}_V(X,\mu,\nu)\approx\frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}}\times\left|e^{iS}\right|^2$$

$$S=\frac{1}{3\hbar\lambda}\left(1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right)$$

$$\mathscr{W}_{Linde}(q,p)=\lim_{\hbar\lambda\to 0}\frac{2^{1/3}A^2}{\pi(\hbar\lambda)^{1/3}}Bi\left[\frac{4p^2-\lambda q+1}{(\hbar\lambda)^{2/3}}\right]\approx\frac{1}{2|\mu|}\frac{1}{\left|1-\frac{\lambda X}{\mu}-\frac{\lambda^2}{16}\frac{\nu^2}{\mu^2}\right|^{1/2}}\left|\sin\left(\frac{2}{3}\left|S\right|^{3/2}-\frac{\pi}{4}\right)\right|^2$$

Some physical conclusions

- From these results we see that the tomographic representation of the quantum and classical states of the universe gives us some interesting results
- The first is that the decay of the cosmological constant drove the transition of the quantum universe into the classical one
- The second is that quantum fluctuations of spatial curvature generate the classical perturbations
- In certain models such as that of Hawking and Penrose, interference terms can be generated which should be observed in the spectrum of curvature perturbations
- Therefore this is a signature that one should look for to discriminate among the various models of the initial state of the universe

C.S. Tomographic analysis of quantum and classical de Sitter cosmological models
Int.J.Mod.Phys.D 28 (2019) 16, 2040009

General Wheeler-DeWitt equation for the tomograms

- General solutions can be found in an analytic way only for very few cases when the potential is very simple (linear, quadratic,…)

- However to study the transition from the quantum to the classical regimes of the universe it is sufficient to determine the tomograms of the asymptotic solution of the Wheeler-DeWitt equation (see C.S. Emergent classical universes from initial quantum states in a tomographical description
Int.J.Geom.Meth.Mod.Phys. 17 (2020) 11, 2050167 • e-Print: 2007.03726 [gr-qc])

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Let us first consider the classical Hamiltonian

$\mathscr{H} = \frac{1}{2}\left(-4p^2 + V(q)\right)$ and suppose that the Hamiltonian constraint is valid

$$g(q,p) = \delta\left(-4p^2 + V(q)\right)$$

$$g(q,p) = \frac{\delta\left(p - \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}\,|} + \frac{\delta\left(p + \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}\,|}$$

$$\delta(F(x)) = \frac{\delta(x - x_0)}{|F'(x_0)|} \qquad \textit{if } x_0 \textit{ is the only solution of equation } F(x) = 0$$

$$\delta(F(x)) = \Sigma_{i=1}^{n}\frac{\delta(x - x_i)}{|F'(x_i)|} \qquad \textit{if } F(x) = 0 \textit{ has } n \textit{ solutions } x_i\,.$$

but for sake of simplicity we consider in the following only the first case

Classical Tomogram

$$\mathscr{W}(X,\mu,\nu) = \int \left[\frac{\delta\left(p - \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}|} + \frac{\delta\left(p + \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}|} \right] \delta(X - \mu q - \nu p)\, dq\, dp$$

$$= \frac{1}{|\nu|} \int \left[\frac{\delta\left(\frac{X-\mu q}{\nu} - \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}|} + \frac{\delta\left(\frac{X-\mu q}{\nu} + \frac{\sqrt{V(q)}}{2}\right)}{|\sqrt{V(q)}|} \right] dq$$

$$= \frac{2}{|\nu|} \left(\frac{1}{\left| -\frac{2\mu\sqrt{V(q_{(1)})}}{\nu} - V'(q_{(1)}) \right|} + \frac{1}{\left| -\frac{2\mu\sqrt{V(q_{(2)})}}{\nu} + V'(q_{(2)}) \right|} \right)$$

Equations

$q_{(1)} = q_{(1)}(X, \mu, \nu)$ and $q_{(2)} = q_{(2)}(X, \mu, \nu)$ are the solutions of the equations

$$\frac{X - \mu q_{(1)}}{\nu} - \frac{\sqrt{V(q_{(1)})}}{2} = 0 \tag{1}$$

and

$$\frac{X - \mu q_{(2)}}{\nu} + \frac{\sqrt{V(q_{(2)})}}{2} = 0 \tag{2}$$

Cosmological quantum tomograms

$$4\hbar^2 \frac{d^2\psi(q)}{dq^2} - V(q)\psi(q) = 0$$

In the limit $\hbar \to 0$ we apply the WKB method

or $V(q) > 0$ the solution is

$$\psi(q) \approx \frac{A}{|V(q)|^{\frac{1}{4}}} e^{\pm \int^x \sqrt{V(y)} dy},$$

and for $V(q) < 0$

$$\psi(q) \approx \frac{A}{|V(q)|^{\frac{1}{4}}} e^{\pm i \int^q \frac{\sqrt{V(y)}}{\varepsilon} dy}.$$

where A is a normalization constant.

WKB approximation

$$\psi(X,\mu,\nu) = \int \frac{A}{|V(q)|^{\frac{1}{4}}} e^{\pm\frac{1}{2\hbar} i \int^{q} \frac{\sqrt{V(y)}}{\varepsilon} dy + i\frac{\mu q^2}{2\hbar\nu} - i\frac{Xq}{\nu}} dq$$

$$\psi(X,\mu,\nu) = \int \frac{A}{|V(q_0)|^{\frac{1}{4}}} \exp\left[\pm\frac{1}{2\hbar} i \int^{q_0} \sqrt{V(y)}dy + i\frac{\mu q_0^2}{2\hbar\nu} - i\frac{Xq_0}{\hbar\nu}\right.$$

$$\left. + \frac{i}{\hbar}\left(\pm\sqrt{V(q_0)} + \frac{\mu}{\nu}q_0 - \frac{X}{\nu}\right)(q-q_0) + \frac{i}{2\hbar}\left(\pm\frac{1}{2}\frac{V'(q_0)}{\sqrt{V(q_0)}} + \frac{\mu}{\nu}\right)(q-q_0)^2\right] dq$$

Laplace method

$$\psi(X,\mu,\nu) = \int \frac{A}{|V(q_0)|^{\frac{1}{4}}} \exp\left[\pm\frac{1}{2\hbar} i \int^{q_0} \sqrt{V(y)}dy + i\frac{\mu q_0^2}{2\hbar\nu} - i\frac{Xq_0}{\hbar\nu}\right.$$

$$\left. + \frac{i}{\hbar}\underbrace{\left(\pm\frac{\sqrt{V(q_0)}}{2} + \underbrace{\frac{\mu}{\nu}q_0 - \frac{X}{\nu}}_{p}\right)}_{=0 \text{ correlation condition}}(q-q_0) + \frac{i}{2\hbar}\left(\pm\frac{1}{2}\frac{V'(q_0)}{\sqrt{V(q_0)}} + \frac{\mu}{\nu}\right)(q-q_0)^2\right] dq$$

$$p = \frac{X - \mu q}{\nu}$$

which is equivalent to find the peaks of the function

$$\hat{\psi}(q) = \psi(q) \exp(i\frac{\mu q^2}{2\nu} - i\frac{Xq}{\nu})$$

We notice that the correlation equations are the same of equations 1) and 2) of the classical problem

Finally we obtain

$$\psi_1(X,\mu,\nu) = \left|\sqrt{\frac{4\pi\hbar}{V'(q_{(1)}) + 2\frac{\mu}{\nu}\sqrt{V(q_{((1))})}}}\right| \mathrm{e}^{\pm\left(\frac{i}{\hbar}\int^{q(1)} V(y)dy + \frac{i\mu q_{(1)}^2}{\hbar\nu} - \frac{iXq_{(1)}}{\hbar\nu}\right)}$$

$$\psi_2(X,\mu,\nu) = \left|\sqrt{\frac{4\pi\hbar}{-V'(q_{(2)}) + 2\frac{\mu}{\nu}\sqrt{V(q_{(2)})}}}\right| \mathrm{e}^{\pm\left(-\frac{i}{\hbar}\int^{q(2)} V(y)dy + \frac{i\mu q_{(2)}^2}{\hbar\nu} - \frac{iXq_{(2)}}{\hbar\nu}\right)}$$

We have the following wave functions

$$\psi_j^V(X,\mu,\nu) = \left| \sqrt{\frac{4\pi\hbar}{V'(q_{(j)}) + 2\frac{\mu}{\nu}\sqrt{V(q_{(j)})}}} \right| \mathrm{e}^{-\left(\frac{i}{\hbar}\int^{q(j)} V(y)dy + \frac{i\mu q_{(j)}^2}{\hbar\nu} - \frac{iXq_{(j)}}{\hbar\nu}\right)}$$

$$\psi_j^{HH}(X,\mu,\nu) = \left| \sqrt{\frac{4\pi\hbar}{V'(q_{(j)}) + 2\frac{\mu}{\nu}\sqrt{V(q_{(j)})}}} \right| \cos\left(\frac{1}{\hbar}\int^{q_{(j)}} V(y)dy + \frac{\mu q_{(j)}^2}{\hbar\nu} - \frac{Xq_{(j)}}{\hbar\nu}\right)$$

$$\psi_j^{L}(X,\mu,\nu) = \left| \sqrt{\frac{4\pi\hbar}{V'(q_{(j)}) + 2\frac{\mu}{\nu}\sqrt{V(q_{(j)})}}} \right| \sin\left(\frac{1}{\hbar}\int^{q_{(j)}} V(y)dy + \frac{\mu q_{(j)}^2}{\hbar\nu} - \frac{Xq_{(j)}}{\hbar\nu}\right)$$

$$j = 1,2$$

General solutions:

$$\psi(X,\mu,\nu) = c_1\,\psi_1(X,\mu,\nu) + c_2\,\psi_2(X,\mu,\nu)$$

$$\mathscr{W}(X,\mu,\nu) = \frac{1}{2\pi\hbar\,|\nu|}\left|\psi\right|^2 = \frac{2}{|\nu|}\left[c_1^2\left|\psi_1\right|^2 + c_2^2\left|\psi_2\right|^2 + 2c_1c_2\psi_1\psi_2\right]$$

Quantum tomograms that converge to the classical tomograms when there are no interference terms. This can happen only if we have $q_1 = q_2$ and

$$\psi(X,\mu,\nu) = \alpha\psi^V(X,\mu,\nu)$$

$$\mathscr{W}(X,\mu,\nu) = \frac{\alpha^2}{2\pi\hbar\,|\nu|}\left|\psi^V(X,\mu,\nu)\right|^2$$

The probability transition function

- $\Pi(X, \mu, \nu, t, X', \mu', \nu', t') =$
  $$= \frac{1}{4\pi} \int k^2 G\left(a + \frac{k\nu}{2}, y, t\right) G^*\left(a - \frac{k\nu}{2}, z, t\right)$$
  $$\times \exp\left[ik\left(X' - X + \mu a - \mu' \frac{y+z}{2}\right)\right]$$

- $$\frac{\partial \Pi}{\partial t} - \mu \frac{\partial \Pi}{\partial \nu} + i\left[V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} - i\nu \frac{\partial}{\partial X}\right) - V\left(-\left(\frac{\partial}{\partial X}\right)^{-1} \frac{\partial}{\partial \mu} + i\nu \frac{\partial}{\partial X}\right)\right]$$
  $$= \delta(\mu - \mu')\delta(\nu - \nu')\delta(X - X')\delta(t)$$

The probability transition function

- The evolution of the tomogram is given by the transition probability function which is the propagator of the tomogram.

- We can apply it to the initial tomograms to obtain a model for the present universe

- We obtain the total propagation probability associating to each cosmological epoch a function Π and exploiting the associative properties we can obtain a final Π_{final}

$$\Pi(X_3, \mu_3, \nu_3, t_3, X_1, \mu_1, \nu_1, t_1) = \int \Pi(X_3, \mu_3, \nu_3, t_3 X_2, \mu_2, \nu_2, t_2)\Pi(X_2, \mu_2, \nu_2, t_2, X_1, \mu_1, \nu_1, t_1) dX_2\, d\mu_2\, d\nu_2, t_2$$

$$\mathscr{W}_{final}(X, \mu, \nu, t) = \int \Pi_{final}(X, \mu, \nu, X', \mu', \nu', t)\mathscr{W}_{initial}(X', \mu', \nu') dX'\, d\mu'\, d\nu'$$

The tomogram of the universe

- Galaxies or better standard candles can be used as probes of the state of the universe to construct a statistical distribution $f_{universe}(q, p)$
- Where q is the expansion factor a or any function of it $a, a^2, \ln a, z$
- And p is the corresponding conjugate variable which is a function of $\dot{a}$ e.g. $6a\dot{a},\ H = \dfrac{\dot{a}}{a}$
- They can all be connected by canonical transformations
- The corresponding tomogram of the universe by applying the definition

$$\mathscr{W}_{phenomen.}\left(X, \mu, \nu\right) = \int f(q, p)_{universe}\delta(X - \mu q - \nu p)dqdp$$

- In our case this is nothing more than an alternative way to describe the statistical distribution. However it is necessary to compare the state of the universe with the classical and quantum theoretical models

The final step

- The final step is to compare the two tomograms $\mathscr{W}_{phenomen.}(X, \mu, \nu)$ and $\mathscr{W}_{final}(X, \mu, \nu)$, taking the single sectors with μ, ν fixed.
- For example if $\mu = \cos\phi, \nu = \sin\phi$, we should compare the one dimensional functions given angle by angle looking for the model that fits best with the corresponding histograms of $\mathscr{W}_{phenomen.}(X, \mu, \nu)$ given at each ϕ .
- Or applying the inverse Radon transform to $\mathscr{W}_{final}(X, \mu, \nu)$ we find the corresponding $f_{th.}(q, p)$ and compare it with the phenomenological $f_{universe}(q, p)$

Phenomenology of the initial condition in the words of J. Hartle*

- The observed features of the universe may not uniquely fix an initial condition but one should not exaggerate their weakness.
- The density matrix $\rho = I/Tr(I)$, where I is the unit matrix, is the unique representation of complete ignorance of the initial condition (i.e. no condition at all).
- But it also corresponds to infinite temperature in equilibrium $\rho \propto \exp(-H/kT))$ — an initial condition whose implication of infinite temperature today is obviously inconsistent with present observations.
- The entropy $S/k = -Tr(\rho \log \rho)$ is a measure of the missing information about the initial state in a density matrix ρ. Most of the entropy in the matter in the visible universe is in thecosmic background radiation, a number of order $S/k \approx 10^{80}$.
- As Penrose [4] has stressed, this is a large number, but infinitesimally small compared to the maximum possible value of $S/k \approx 10^{120}$ if all that matter composed a black hole.

*Quantum Cosmology: Problems for the 21st Century∗ James B. Hartle†Physics 2001, ed. by M. Kumar and in the Proceedings of the 10th Yukawa-Nishinomiya Symposium, November 7–8, 1996, Nishinomiya, Japan.

To do list

- Tomogram of the CMB considering its interaction with matter through a specific propagator Π
- Determine the potential $V(a)$ in presence of scalar fields and cosmological fluids
- How perturbations modify the cosmological tomogram (non homogeneity on lower scales)
- Extend this work to alternative theories of gravity
- Tomographic approach to loop quantum gravity and loop quantum cosmology (see for example Jason Berra-Montiel, Alberto Molgado, Tomography in Loop Quantum Cosmology, *Eur.Phys.J.Plus* 137 (2022) 2, 283 e-Print: 2104.09721 [gr-qc]

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Thank you for your attention!